Mechanical metamaterials at the theoretical limit of isotropic elastic stiffness

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The significant performance of modern aerospace vehicles, deep-sea submersibles, high fuel efficiency automobiles, sports equipment, space telescopes, and a wide variety of other lightweighting applications[1], require that shape control be maintained under significant stress while mass is minimized. Materials such as bio-inspired hexagonal and square honeycombs, and lattice materials based upon repeating unit cells composed of webs or trusses[2], when made from materials of high elastic stiffness and low density[3], represent some of the lightest, stiffest and strongest materials available today[4]. These mechanical metamaterials have properties that are a function of their constituents as well as their mesoscale geometry[3,5–13], achieving combinations of properties that are otherwise unrivaled (**FIG.ED1**). Recent advances in 3-D printing and automated assembly now allow such complicated material geometries to be fabricated at low (and declining) cost. To identify the morphological features associated with high elastic performance we evaluate the manner in which strain energy distributes under load in a representative selection of material geometries taken from the literature. While theoretical bounds predict that mechanical metamaterials can outperform other systems, by potentially orders of magnitude, and over a wide range of property space, no material geometry has previously been identified that has this capability. Using finite elements, supported by analytical models, in a heuristic optimization scheme, we have identified the first and only known material geometry to achieve the theoretical bounds for isotropic stiffness. Previous work has mostly focused on truss networks and anisotropic honeycombs, both of which are unable to achieve the Hashin-Shtrikman (H-S) theoretical limit[14]. Stiff but well distributed networks of plates are required to efficiently transfer loads between neighboring members. Low-density mechanical metamaterials have many advantageous properties. Their mesoscale geometry can facilitate large crushing strains with high energy absorption[2,15,16], optical[17–20] and mechanically tunable acoustic band gaps[21], high thermal insulation[22], buoyancy, and fluid storage and transport. The relatively simple design that has been discovered can be manufactured through origami-like sheet folding[23] and bonding methods, providing access to many constituent material systems.

Only closed cell materials, composed of sheets (that exploit material constraint in two directions) rather than slender beams, are potentially capable of achieving the H-S upper bounds. It is shown here that periodic (ordered unit cell) foams, the closed cell analogue to lattice materials, are capable of reaching the H-S upper bounds for an isotropic material. The topology design principle recognizes that interconnectivity of material in closed cell geometries greatly reduces configurational entropy, increasing the strain energy storage substantially when compared to truss materials. In contrast, the beams in truss materials are able to activate the numerous modes of deformation that accommodate large bending strains. When loaded macroscopically, closed cell materials develop tensile membrane stresses that efficiently utilize material volume independent of the macroscopic loading direction[24]. This stretched material contributes maximally to the macroscopic stiffness[24,11]. The geometric complexity of ordered 3D closed cell materials has historically made their study and utilization impractical, from both an analytical predictive, and a manufacturing standpoint. However, modern analysis tools and manufacturing techniques now allow for their computational exploration and fabrication at relatively low (and declining) cost.

We employ a combination of qualitative and quantitative analysis in a heuristic optimization scheme that identifies novel geometries. We use strain energy distributions to identify the morphological features associated with high performance. A finite element (FE) method homogenization scheme, implemented in the FE code Abaqus CAE[25], is used to calculate the effective material properties. We obtain the structural efficiency by comparing these properties to a mass equivalent continuum material, in which stresses and strains are uniformly distributed. To verify the FE models, we calculate the strain energy in key geometries (see S.1.1) and use the results to derive moduli (S.1.2), finding the results to agree at the low density limit when flexural stiffness in negligible. We also show that strut based structures are fundamentally incapable of achieving the H-S upper bound (S.2). By assembling high-performance geometric features, materials with maximal stiffness and varying degrees of anisotropy, including one that achieves the theoretical upper H-S bound, can be designed.

Four closed, and two open cell geometries are investigated. The octet-truss[26] (OT) and an isotropic truss (IT) with maximum modulus[27] are representative of structurally efficient open cell materials, **Fig 2**. The closed cell cubic foam (CF) and octet-foam (OF) are found to store a maximum amount of strain energy but are highly anisotropic. A combination of the two (a cubic plus octet foam) inherits the maximum strain energy storage of the parent geometries, can be isotropic, and is able to achieve the H-S upper bound. We also study a quasi-random foam, formed from the Voronoi tessellation of a random seeding of space. These materials are largely representative of high-modulus two-phase cellular materials, allowing pragmatic population of elastic property – density space.

To characterize material properties, we subject FE models of the geometries to states of uniform macroscopic strain, using periodic boundary conditions, employing the method of *Danielson, Parks and Boyce* [28]. We consider only the elastic properties in the small strain limit, although nonlinear displacements and rotations are both allowed. The geometries either have cubic symmetry or are random and effectively isotropic (**FIG.2**) and therefore characterized by the Young's ($\bar{E}$), shear ($\bar{G}$) and bulk ($\bar{K}$) moduli of the cellular material. These are normalized by the respective modulus of the solid (E_S, G_S, and K_S), and plotted against the relative density, $(\bar{\rho}/\rho_S)$, where the bar notation is used for the effective properties of the cellular material, and the subscript *s* for the properties of the constituent material. The results are further normalized by the Voigt (anisotropic) bound, to highlight the efficiency of the material geometry. The relative density of each

structure was changed by varying the strut or web thickness while holding the length fixed. Plots of $(\bar{E}/E_S)/(\bar{\rho}/\rho_S)$, $(\bar{G}/G_S)/(\bar{\rho}/\rho_S)$ and $(\bar{K}/K_S)/(\bar{\rho}/\rho_S)$ versus relative density (**FIG.3**) evaluate the geometric efficiency of the six materials by comparing them to a mass equivalent continuum wherein stresses are uniformly distributed. At zero relative density, there is no defining geometry so that the plots in **FIG.3** are only meaningful for $(\bar{\rho}/\rho_S) > 0$. We use these results, in conjunction with distributions of strain energy, to identify high performance designs and the morphological features responsible for their high performance.

The fundamental performance of high modulus geometries can be quantified by consideration of their low density limiting stiffness. At very low densities, strut or web members are slender and have negligible bending stiffness. Their response is then solely determined by, and directly proportional to, the volume of material that deforms through stretching. The ordinate intercepts at zero relative density in **FIG.3**, which we denote S_E, S_G, and S_K for the plots of $(\bar{E}/E_S)/(\bar{\rho}/\rho_S)$, $(\bar{G}/G_S)/(\bar{\rho}/\rho_S)$ and $(\bar{K}/K_S)/(\bar{\rho}/\rho_S)$ respectively, quantify this volume, being the fractional modulus of an equivalent effective continuum. As relative density increases, 2nd and 3rd order contributions, associated with the bending of struts and plates respectively[11], become significant. In the case of the bulk modulus, the theoretical bounds are sensitive to the Poisson ratio, ν_S, of the constituent material. Macroscopic hydrostatic loads cause material to Poisson expand into voids rather straining purely hydrostatically. If the Poisson ratio is zero, $\nu_S = 0$, there is no driving force for material to expand into voids. Consequently, $0 \leq S_K \leq 0.67$ for $0 \leq \nu_S < 0.5$, while $0.47 \leq S_G \leq 0.60$, and $S_E = 0.52$, remains constant. In the cases of axial and shear loading this sensitivity to Poisson ratio is not as pronounced, and, in all cases, selection of a Poisson ratio value does not qualitatively affect the results, nor the ranking of performance.

As a metric for elastic performance, the stiffness can be quantified by Ω; the summation of the Young's modulus in the <100> and <110> directions, which is scaled by the properties of a material that achieves the theoretical bounds,

$$\Omega = \frac{\bar{E}+2\bar{G}(1-\bar{\nu})}{E_{HSU}+2G_{HSU}(1-\nu_{HSU})}. \quad (1)$$

Here the subscript *HSU* denotes the Hashin-Shtrikman upper bound specified for a two-phase system composed of a solid and void phase. A material that achieves the three theoretical bounds simultaneously will have $\Omega = 1$ (**FIG.4**). Despite the bounds being for isotropic materials, the strain energy storage in the highly anisotropic CF and OF topologies remain bounded by this limit (**FIG.4**). The three ordered closed cell materials have nearly identical, and essentially maximal, total stiffness, despite greatly varying degrees of anisotropy (**FIG.5**). In **Figure 4** it can be seen that, in terms of Ω, the combined CF+OF system (**FIG.2-c**) performs extremely well and is isotropic in the low density limit when $t_c/t_t = 8\sqrt{3}/9$ (see S.1.3), where t_c and t_t are the wall thickness of the cubic and octet subgeometries respectively. The relative wall thickness, t_c/t_t, can be altered to tailor the degree of isotropy, which varies slightly with relative density, without affecting total stiffness. For example, when $t_c/t_t = \sqrt{3}$ isotropy is achieved at $(\bar{\rho}/\rho_S) = 40\%$. The CF and OF by themselves are anisotropic but, by the measure Ω, perform no better than the combined CF+OF system. This shows that while directional stiffness can be traded for isotropy, a fundamental limit in performance exists for all cellular materials.

The manner in which strain energy is stored in structural metamaterials is most easily understood by examining the performance of the cubic foam topology (CF). Material in the CF is partitioned equally into three orthogonal cell walls that are each aligned with one of the basis vectors. When an axial stress is applied along any one of these directions, two of the three walls align with the principal stress directions, causing nearly uniform stress fields to develop in these members (**FIG.E1-b**). Loads are transmitted efficiently between neighboring unit cells due to the alignment and connectivity of members. The material is optimal, in that no increase in Young's modulus is possible through rearrangement of material. For the CF material, $S_E \approx 2/3$; roughly the fraction of material that is uniformly stressed through stretching by the principal stresses. A small additional contribution comes from the Poisson effect, and the slight stretching of the remaining transverse cell wall. The material in this wall can be transferred to the walls aligned with the principal stress, making the directional performance, $S_E = 1$, and the shear performance, $S_G \approx 0$, showing how it is possible to trade directional stiffness for isotropy. Under shear loading, only one cell wall aligns with the principal stress directions (**FIG.E2-b**), making $S_G = 1/3$ (**FIG.3**).

Under hydrostatic load, the response of the CF is similar to all other stiff closed cell materials (**FIG.E3**). As there is no opportunity for bending, these materials store strain energy nearly uniformly, and all essentially achieve the theoretical bound for bulk modulus (**FIG.3, right**). Small deviations in performance arise from inhomogeneities in the stress and strain energy fields associated with the intersection of cell faces, the amount of which varies amongst the designs and with relative density. In contrast to the closed cell geometries, and despite nearly uniform strain energy distributions, (**FIG.6-f**) the strut based geometries suffer in their performance due to the ability of struts to expand in the plane orthogonal to its axis, while material in plates is constrained to expand in only one direction that is normal to its plane.

The octet-foam (OF), similar to the CF, has a maximum amount of material aligned with the principal stresses when subject to one of the three fundamental modes of deformation (axial, shear and hydrostatic loading). Under shear loading, an equal amount of material is aligned with each of the two principal stress directions. All the cell walls respond through stretching in a self-similar manner (**FIG.E2-c**) so that further optimization is not possible. In order to achieve the Voigt bound, $(\bar{G}/G_S)(\bar{\rho}/\rho_S) = 1$ (**FIG.3**), every material point must support both principal stresses, not just one.

In a stiff isotropic stochastic closed cell material, about one third of the material is expected to be oriented preferentially to an arbitrary applied load[3] making $S_E = 1/3$. Our quasi-random (QR) materials are composed of only 12 and 20 unique cells, yet their average moduli match the predicted response of stiff fully stochastic materials[29] (**FIG.E4**). Networks of stiff but poorly aligned cell faces result in bending stresses and relatively poor performance. Although the individual cell faces are of uniform thickness, similar to the OF and CF materials, the inability for neighboring members to effectively transmit loads limits their efficiency.

There is a significant decrease in performance when cell walls are removed to form an analogous strut based geometry. For example, the OF outperforms the OT in terms of shear performance by nearly a factor of 3: $S_{G,OF}/S_{G,OT} = 2.8$, where $S_{G,OF}$ and $S_{G,OT}$ are the shear efficiencies of the OF and OT respectively. Compared

to the cubic foam, a cubic truss (CT) will have only one third of its struts aligned to an axial load, making $S_{E,CT} \approx 1/3$, and $S_{E,CF}/S_{E,CT} \approx 2.1$. When comparing strain energy distributions in the two isotropic ordered geometries, concentrations are much higher in the open cell geometry (**FIG.6**).

Through characterization of the CF and OF materials in the manner described above, we can take advantage of their unique properties. Their combination (**FIG.2-c**) can form a material that maintains maximal strain energy storage but has isotropic stiffness. Being composed of highly anisotropic sub-geometries, the relative wall thickness of the CF and OF walls must be chosen carefully so that in combination they generate an isotropic response. Isotropy is achieved, in the low-density limit, when the ratio of thickness of the CF to OF walls is $8\sqrt{3}/9$ (see S.1.2.3). This ratio can also be used as a design variable, allowing the local anisotropy to be functionally graded in a part, with the aim of producing highly optimized designs. The isotropic CF plus OF material has a total stiffness $\Omega > 94\%$ with $(\bar{\rho}/\rho_S) \leq 42\%$ (**FIG.4**). With the addition of fillets this can be increased even further, for example, with $r_f/L = 2\%$, $\Omega > 96\%$ for this same range, essentially achieving the theoretical upper bounds over the primary low-density range of interest for cellular materials; here L is the edge length of the cubic reference volume element (RVE) and r_f is the radius of the fillets. The addition of fillets affects the isotropy, requiring some optimization of the wall thickness to achieve isotropic properties.

A material with maximum isotropic elastic stiffness, as achieved in the combined CF+OF system, has obvious value as an engineering material for stiffness dominated aerospace and other lightweight vehicle designs. The strain energy, being of maximum average density, is well distributed under all states of loading (**FIG.6, a-c**). Under axial loading (**FIG.6-a**) nearly all the material is engaged in significant strain energy storage. Under shear (**FIG.6-b**, the two walls orthogonal to the principal stresses still deform almost entirely through bending and provide minimal contribution to the macroscopic stiffness; however, the theoretical bound is still achieved. Under hydrostatic loading (**FIG.6-c**) the behavior is similar to the other stiff closed cell materials (**FIG.E3**), with the deformations being nearly affine.

Motivated to fill property space, we can now posit fabricating the isotropic CF plus OF material using solid constituents that have useful properties. The materials plotted in **FIG.ED1** (listed in Table ED1) are a few of the many possible constituent materials. The lightest and stiffest material available is single crystal diamond. When used as the constituent, the CF+OF material bounds the upper end of Young's modulus versus density space in the metamaterials regime. While there is some room for optimization at higher relative densities (**FIG.4**), the result on a log scale exhibits a negligible difference in performance. The need for highly interconnected, aligned, yet well distributed material, combined with the symmetries present in the CF+OF material, indicates that it is close to, and may be, the best achievable topology. While it may be many years before methods for making this topology from diamond are developed, more accessible structures made from materials such as beryllium, aluminum matrix silicon carbide and carbon fiber polymer composites also have a very high stiffness. The beryllium CF+OF system likely forms a practical upper bound on property space given the difficulties in fabricating complex single crystal ceramic geometries. These systems have stiffnesses many times greater (at densities more than an order of magnitude lower) than existing stochastic foams. Their remarkable performance presents a challenging goal for novel manufacturing techniques.

When incorporated into applications, the CF+OF material can maximize stiffness-dominated performance, reduce energy storage in dynamic systems, and minimize the weight of material needed to achieve design goals. The relatively simple geometry, being composed of sheets of material, lends itself to fabrication by sheet folding[23], making fiber composites potential and attractive constituent materials. There are many promising fabrication techniques in addition to 3-D printing that may help facilitate the mass production of the CF+OF geometry. Snap-fit and brazing of members[2,6], expedited through automation, can potentially produce large scale metallic structures, while two photon lithography and subsequent pyrolysis can produce small scale ceramic structures with the potential for near theoretical strength [30]. With the continued development of manufacturing technology and light material systems, the use of complex cellular structural materials will inevitably become as common in engineered systems as those found in nature [3,31].

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Author Contributions J.B.B created the ideas, conceived of and designed the new material geometries, and the structural analysis, R.M.M. developed analytical models for strain energy and moduli, and, with H.N.G.W., contributed to refining the concepts, contextualizing the results, and providing critiques and assessments.

Author Information Reprints and permissions information available at www.nature.com/nature. Correspondence and requests for materials should be addressed to J.B.B. (berger@engineering.ucsb.edu). The CF+OF geometry has been included in a patent cooperation treaty (PCT/US2015/010458) by Nama Development, LLC (DE), which is owned by J.B.B.

Figure Legends:

Figure 1 | Material geometries. Representative topologies of six cellular structures studied, all with solid volume fractions, $(\bar{\rho}/\rho_S) = 0.2$. A coordinate system aligned with the axes oriented in the cubic directions is shown for the isotropic truss. This coordinate system is used in the cubic orientation for all systems depicted in this figure and is utilized to define the Young's and shear modulus and the Poisson ratio for the materials.

Figure 2 | Young's, shear and bulk modulus. The elastic stiffness of six material geometries, characterized by E, G, and K, the Young's shear and bulk modulus respectively. The Hashin-Shtrikman theoretical upper bounds for isotropic stiffness are plotted for each modulus. Only anisotropic materials can have stiffnesses in excess of the upper bounds. Open cell materials underperform closed cell materials by a large margin.

Figure 3 | Total stiffness. Structural performance in terms of elastic strain energy storage. A material that achieves the theoretical upper bound will have $\Omega \to 1$ as $(\bar{\rho}/\rho_S) \to 0$. Materials can achieve the bounds at low relative densities when stress and strain energy concentrations are limited to very small regions near cell edges. As relative density increases and members become thicker, non-uniformities become significant and an inability to achieve the theoretical bounds is observed, although, further optimization may be possible. There are three classes of material performance: maximally stiff closed cell materials, closed cell materials with geometric features that allow for bending, and ordered maximally stiff open cell materials.

Figure 4 | The Zener anisotropy ratio. Both the cubic+octet foam with $t_c/t_t = 8\sqrt{3}/9$ and the isotropic truss have $a \to 1$ as $(\bar{\rho}/\rho_S) \to 0$. The octet-foam and –truss both have nearly identical values of a, while the cubic foam is the most anisotropic. The anisotropy of the cubic+octet foam can be tailored by adjusting the ratio of wall thicknesses, t_c/t_t, where t_c and t_t are the wall thicknesses of the cubic and octet foam subgeometries respectively. For example, with $t_c/t_t = \sqrt{3}$ isotropy is achieved at $(\bar{\rho}/\rho_S) = 40\%$.

Figure 5 | Strain energy distributions. Axial (a and d), shear (b and e), and hydrostatic strain (c and f), in the cubic+octet foam (top row) and isotropic truss (bottom row). The local strain energy density, U_ε, is normalized by the macroscopic solid fraction strain energy density, $\overline{U}_\varepsilon$. The imposed strains are small, but exaggerated to reveal their nature. Both geometries have, $(\bar{\rho}/\rho_S) = 0.2$.

Extended Data Figure 1 | Property space of isotropic and nearly isotropic materials. Metamaterial geometries with suboptimal performance have been omitted. Theoretical bounds, $\Omega = 1$, limits the performance of all material systems and is defined by the highest performance possible for a two-phase system, achieved by the single crystal diamond and void system. The parameter Ω is defined in the text. The cubic+octet material can pragmatically bound property space when composed of materials with maximal properties, such as diamond, beryllium, fiber composites, and lightweight alloys. Fabrication techniques now limit our ability to achieve a wide and otherwise unoccupied region of property space.

Extended Data Figure 2 | Strain energy distributions from axial stress. Strain energy distributions in the geometries from FIG.1 subject to uniaxial stress; where U_ε is the local strain energy density, and $\overline{U}_\varepsilon$ is the average solid phase strain energy density. Macroscopic loads are transmitted through stiff networks of members aligned with the principal stress direction. Strains are small but scaled to reveal the nature of the deformations. The two-dimensional connectedness of material in closed cell geometries allows for the effective transmission of loads between neighboring members, facilitating materials that can achieve the theoretical bounds (a). Open cell and stochastic materials (bottom row) have significant strain energy concentrations.

Extended Data Figure 3 | Strain energy distributions from shear loading. Strains are scaled to highlight the nature of the deformations. The displacements in stiff closed cell materials (top row) are largely affine and absent of bending. Despite the identical alignment of material in the OT (f) and OF (c), the absence of membrane stress allows significant bending to take place in the open cell configuration. All geometries have, $(\bar{\rho}/\rho_S) = 0.2$.

Extended Data Figure 4 | Strain energy distributions from hydrostatic loading. Strains are scaled to highlight the nature of the deformations. In maximally stiff materials, the deformations are limited to the filling of void space through member swelling. The displacements are primarily affine and strain energy distributions nearly uniform. Poor alignment of neighboring cell walls in the QR material allows some bending to occur (d). All geometries have, $(\bar{\rho}/\rho_S) = 0.2$.

Extended Data Figure 5 | Moduli of Quasi-Random and stochastic foams. The normalized average Young's, shear, and bulk moduli of quasi-random foam finite element (FE) models plotted against relative density. Data are fit to third order polynomials, forced to go through the origin (0, 0) and the point (1, 1), corresponding to empty space and a dense solid respectively. Data are also fit to the Gibson and Ashby model for the stiffness of isotropic cellular materials. Experimental data for Young's modulus are taken from L.J. Gibson and M.F. Ashby[24].

Methods:

To assess geometric efficiency, we use a representative volume element (RVE) finite element (FE) modeling technique. Periodic boundary conditions (BC) allow us to model states of homogeneous macroscopic strain. Load states are imposed by prescribing the displacements of "virtual" nodes, and the principle of virtual work is used to calculate the stresses[28]. The details of this method are described by *Danielsson, Parks and Boyce*[28]. We report the Cauchy stress, which is equivalent to the First Piola-Kirchhoff stress when strains are small. Moduli are calculated through application of states of macroscopic strains consistent with states of uniaxial stress, pure shear, and pure dilatation (illustrated in FIGs.E1, E2, E3, respectively); here the coordinate axes are aligned with the edges of the cubic unit cells (**FIG.2**)

The commercial FE code Abaqus CAE[25] is used to generate model geometries and evaluate FE model solutions. Quadratic elements, fully integrated tetrahedral elements with ten nodes (C3D10) and 20-node brick elements (C3D20) are used. The imposed strains are small, however, nonlinear deformations are allowed using the *nlgeom* flag in Abaqus. The accuracy of the finite element (FE) method was assessed through direct comparison with experimental data, the predictions of analytical models, basic structural analysis, and against a more thorough study involving similar quasi-random closed cell foams[32] (see S.3).

As metrics for performance we use a suite of theoretical bounds, for Young's, shear and bulk modulus, developed by Hashin and Shtrikman[14] (H-S). The H-S bounds govern the stiffness of nearly isotropic multiphase materials. It is applied to foams and lattices by setting one of the phases in a two-phase system to have zero stiffness. The bounds then simplify to[11]:

$$\frac{K_{HSU}}{K_s} = \frac{4G_s(\bar{\rho}/\rho_s)}{4G_s+3K_s(1-\bar{\rho}/\rho_s)}, \tag{3}$$

$$\frac{G_{HSU}}{G_s} = \frac{(9K_s+8G_s)(\bar{\rho}/\rho_s)}{20G_s+15K_s-6(K_s+2G_s)(\bar{\rho}/\rho_s)} \tag{4}$$

The bound on Young's modulus is found by the application of isotropic linear elasticity on the preceding bounds to get,

$$E_{HSU} = \frac{9G_{HSU}K_{HSU}}{3K_{HSU}+G_{HSU}}, \tag{5}$$

where the subscript *HSU* denotes the H-S upper bound, the subscript *s* denotes the property of the constituent, and *E*, *G*, and *K*, are the Young's, shear, and bulk modulus respectively. The corresponding Poisson ratio is,

$$\nu_{HSU} = \frac{3K_{HSU}-2G_{HSU}}{2(3K_{HSU}+G_{HSU})}, \tag{6}$$

Quasi-random closed cell geometries are generated using Voronoi tessellation. A hard sphere seeding model is used to control the spacing between seed points and the resulting distribution in cell sizes. The sphere diameter is,

$$d = 2\alpha\left(\frac{3}{4\pi n}\right)^{1/3}, \tag{7}$$

where $\alpha = 0.55 - 0.6$ is the packing density of the hard spheres, and n is the number of unique cells in a unit cell. The initial seed is placed in the corner of a cubic RVE, this helps to insure the continuity of material in the unit cell geometry. Seed points are added randomly to the RVE. If a new seed causes overlap of the hard spheres it is removed and another attempt is made. If a maximum number of attempts is made in a single step the entire process is reinitialized and repeated until a solution is reached.

Macroscopic strains consistent with states of axial, shear and hydrostatic stress are applied to calculate the nine independent elastic constants for an orthotropic material. These are then averaged to calculate $\bar{E}$, $\bar{G}$ and $\bar{K}$. For cubically symmetric geometries only three elastic constants need to be calculated.

Input into our design scheme was in the form of material geometries taken from the literature. Geometries such as the body centered cubic (BCC) Kelvin foam[33], a similar geometry based instead on the spatial tessellation of space from the face centered cubic (FCC) seeding of space, the cubic foam, the octet-foam, as well as a variety of others, most of which have been omitted for the sake of brevity and clarity, were assessed using the analytical scheme provided in this work. Through inspection, we are able to identify the morphological features associated with high performance. It is through the identification and combination of fundamental geometries, in a genetic-type scheme, that we are able to produce novel geometries with extremal properties.

Data Availability:

Source data for finite element results in figures [2-5, and ED5] are provided with the paper.

Additional References:

32. Roberts, a. P. & Garboczi, E. J. Elastic moduli of model random three-dimensional closed-cell cellular solids. *Acta Mater.* **49,** 189–197 (2001).

33. Thomson, S. W. On the Division of Space with Minimum Partitional Area. *Acta Math.* **11,** 121–134 (1888).