Supplementary Information:

S.1 Supplementary Equations - Theoretical elastic moduli of closed cell foams

S.1.1 Strain Energy

S.1.1.1 Octet foam

Consider a unit cell of the octet foam as shown in **Figure 2(b)**, with the vertices in the unit cell at the corners of a cube of unit edge dimension (**FIG.2 (f)**), and with the walls composed of isotropic linear elastic material. To compute the resulting elastic strain energy, a strain is imposed on the unit cell, resulting in affine strain of the foam walls under plane stress conditions. The elastic moduli of the foam can then be deduced from this strain energy.

Consider first a uniaxial macroscopic strain imposed in the x_3 direction, **Figure 2(f)**, $\varepsilon_{33} = \varepsilon$, with all other components of the imposed strain being zero. Each plate forming a foam wall experiences uniform principal strains given by

$$\varepsilon_I = \frac{2}{3}\varepsilon$$

$$\varepsilon_{II} = 0 \tag{1}$$

$$\varepsilon_{III} = -\frac{2v_s}{3(1-v_s)}\varepsilon$$

where ε_I and ε_{II} are in the plane of the wall, ε_{III} is the through thickness strain thereby imposing plane stress conditions, and v_s is Poisson's ratio of the material composing the foam walls. With the principal stress σ_I aligned with the strain e_I, and σ_{II} aligned with ε_{II}, the through thickness stress in the wall, σ_{III}, is zero. Isotropic linear elasticity then provides the uniform principal stress components as

$$\sigma_I = \frac{E_s}{1+v_s}\left[\varepsilon_I + \frac{v_s}{1-2v_s}(\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})\right]$$

$$\sigma_{II} = \frac{E_s}{1+v_s}\left[\varepsilon_{II} + \frac{v_s}{1-2v_s}(\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})\right] \tag{2}$$

$$\sigma_{III} = 0$$

where, E_s is the Young's modulus of the solid wall material. The uniform strain energy density, U_s, in the material forming the wall is

$$U_s = \frac{1}{2}(\sigma_I \varepsilon_I + \sigma_{II} \varepsilon_{II}) = \frac{E_s}{2(1+v_s)(1-2v_s)}\left[(1-v_s)(\varepsilon_I^2 + \varepsilon_{II}^2) + v_s(2\varepsilon_I\varepsilon_{II} + \varepsilon_I\varepsilon_{III} + \varepsilon_{II}\varepsilon_{III})\right] \tag{3}$$

Using the strains given by Eq. (1), the strain energy density in the wall is then

$$U_s = \frac{2E_s\varepsilon^2}{9(1-v_s^2)} \tag{4}$$

Each wall has the same strain energy density, and since the relative density of the unit cell is $\rho_o = \overline{\rho}/\rho_s$, the total strain energy in the unit cell is

$$U_{oa} = \frac{2E_s\rho_o\varepsilon^2}{9(1-v_s^2)} \tag{5}$$

where the subscript o indicates the octet foam and the subscript a denotes axial strain.

Next we consider pure dilatation of the unit cell resulting in a volumetric strain e_V. In this case, the principal strains in the walls everywhere are

$$\varepsilon_I = \varepsilon_{II} = \frac{\varepsilon_V}{3}$$

$$\varepsilon_{III} = -\frac{2\nu_s \varepsilon_V}{3(1-\nu_s)}$$

(6)

where e_{III} is the through thickness strain maintaining plane stress in the wall. Using the same steps as above, the total strain energy in the unit cell is

$$U_{od} = \frac{E_s \rho_o \varepsilon_V^2}{9(1-\nu_s)}$$

(7)

Now consider a simple shear, with macroscopic strains

$$e_{12} = e_{21} = \frac{g}{2}$$

(8)

With this macroscopic strain, each wall in the unit cell has strains given by,

$$e_{xx} = \pm\frac{g}{3}$$

$$e_{yy} = 0$$

$$g_{xy} = g_{yx} = \pm\frac{g}{\sqrt{3}}$$

$$e_{zz} = -\frac{n_s}{1-n_s} e_{xx}$$

(9)

where e_{zz} is the through thickness strain ensuring plane stress in the wall, and the y axis is parallel to one of the wall intersections at the surface of the unit cell. In this case, the strain energy density in the wall is given by U_γ added to the result in Eq. (3) where

$$U_\gamma = \frac{E_s \gamma_{xy}^2}{4(1+\nu_s)}$$

(10)

accounts for the strain energy due to the shear strain and the result in Eq. (3) is computed with e_I replaced by e_{xx}, e_{II} by e_{yy} and e_{III} by e_{zz}. As a result, the total strain energy in the unit cell is given by,

$$U_{os} = \frac{E_s r_o (5 - 3n_s) g^2}{36(1 - n_s^2)}$$

(11)

S.1.1.2 Cubic foam

Similar considerations for the cubic foam, gives their strain energies

$$U_{ca} = \frac{E_s r_c e^2}{3(1 - n_s^2)}$$

(12)

for axial strain,

$$U_{cd} = \frac{E_s r_c e_V^2}{9(1-n_s)}$$

(13)

for pure dilatation, and

$$U_{cs} = \frac{E_s r_c g^2}{12(1+n_s)}$$

(14)

for simple shear, where $\rho_c = \bar{\rho}/\rho_s$ is the relative density of the cubic foam.

S.1.1.1 Cubic+octet foam

When we combine the octet and cubic foams to construct the unit cell shown in **Figure 2(c)**, the unit cell has strain energy at a given strain that is the sum of those of the octet and cubic foams. As a consequence, in uniaxial strain the cubic-octet foam has strain energies for its unit cell given by

$$U_{coa} = \frac{E_s(2r_o + 3r_c)e^2}{9(1-n_s^2)}$$

(15)

where the subscript *coa* indicates axial strain of the cubic-octet foam. Furthermore, pure dilatation produces a strain energy for the unit cell equal to

$$U_{cod} = \frac{E_s(r_o + r_c)e_V^2}{9(1-n_s)}$$

(16)

and simple shear gives

$$U_{cos} = \frac{E_s g^2}{36(1-n_s^2)}\left[r_o(5-3n_s) + 3r_c(1-n_s)\right]$$

(17)

S.1.2 Effective elastic properties of closed cell foams

The elasticity law for a cubic material with coordinate axes aligned with the cubic directions, **Figure 2(f)** is

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ g_{23} \\ g_{13} \\ g_{12} \end{Bmatrix}$$

(18)

where S_{ij} is the stress tensor, e_{ij} is the strain tensor, $g_{ij} = 2e_{ij}$, and C_{ij} is the elasticity matrix. In uniaxial strain with $e_{33} = e$ and other strain components zero, the strain energy density is thus

$$U_a = \frac{1}{2}C_{11}e^2$$

(19)

and is equal to the results in Eqs. (5) and (12) for the unit cell of unit volume for the octet and cubic foam respectively. Similarly, in pure dilatation with $e_{11} = e_{22} = e_{33} = e_V / 3$ and other strain components zero, the strain energy density is

$$U_d = \frac{1}{6}\left(C_{11} + 2C_{12}\right)e_V^2$$

(20)

and equal to the results in Eq. (7) & (13). In simple shear with $g_{12} = g_{21} = g$ and other strain components zero, the strain energy density is

$$U_s = \frac{1}{2}C_{44}g^2$$

(21)

and equal to the results in Eqs. (11) and (14). Furthermore, we can obtain moduli in the cubic coordinate system as follows. Young's modulus is given by

$$\bar{E}_i = \frac{\left(C_{11} - C_{12}\right)\left(C_{11} + 2C_{12}\right)}{C_{11} + C_{12}}$$

(22)

where i is o, c and co to denote the octet, cubic and cubic-octet foams respectively. Poisson's ratio is then

$$\bar{n}_i = \frac{C_{12}}{C_{11} + C_{12}}$$

(23)

The shear modulus is

$$\bar{G}_i = C_{44}$$

(24)

and the bulk modulus becomes

$$\bar{K}_i = \frac{1}{3}\left(C_{11} + 2C_{12}\right)$$

(25)

From the above results, we are able to calculate the components of the stiffness matrix for the octet and cubic closed cell foams, which are summarized in **Table S.1**.

Geometry:	C_{11}	C_{12}	C_{44}	$\bar{E}$	$\bar{G}$	$\bar{K}$	$\bar{v}$
octet foam	$\dfrac{4E_S\rho_o}{9(1-v_S^2)}$	$\dfrac{E_S\rho_o(1+3v_S)}{9(1-v_S^2)}$	$\dfrac{E_S\rho_o(5-3v_S)}{18(1-v_S^2)}$	$\dfrac{2E_S\rho_o}{5+3v_S}$	$\dfrac{G_S\rho_o(5-3v_S)}{9(1-v_S)}$	$\dfrac{2K_S\rho_o(1-2v_S)}{3(1-v_S)}$	$\dfrac{1+3v_S}{5+3v_S}$
cubic foam	$\dfrac{2E_S\rho_c}{3(1-v_S^2)}$	$\dfrac{E_S\rho_c v_c}{3(1-v_S^2)}$	$\dfrac{E_S\rho_c}{6(1+v_S)}$	$\dfrac{E_S\rho_c(2-v_S)}{3(2+v_S)(1-v_S)}$	$\dfrac{G_S\rho_c}{3}$	$\dfrac{2K_S\rho_c(1-2v_S)}{3(1-v_S)}$	$\dfrac{v_S}{2+v_S}$

Supplementary Information Table 1 | Elastic constants. Summary of the elastic properties of the closed cell octet and cubic foams.

The elastic constants and moduli of the combined cubic + octet foam, whose terms are too large to fit in a table, are

$$C_{11} = \frac{2E_s\left(2r_o + 3r_c\right)}{9\left(1 - n_s^2\right)}$$

$$C_{12} = \frac{E_s\left[3\left(r_o + r_c\right)\left(1 + n_s\right) - \left(2r_o + 3r_c\right)\right]}{9\left(1 - n_s^2\right)}$$

$$C_{44} = \frac{E_s\left[r_o\left(5 - 3n_s\right) + 3r_c\left(1 - n_s\right)\right]}{18\left(1 - n_s^2\right)}$$

(26)

and

$$\overline{E}_{co} = \frac{2\left(r_o + r_c\right)E_s\left[\left(1 - n_s\right)r_o + \left(2 - n_s\right)r_c\right]}{\left(1 - n_s\right)\left[\left(5 + 3n_s\right)r_o + 3\left(2 + n_s\right)r_c\right]}$$

$$\overline{n}_{co} = \frac{\left(1 + 3n_s\right)r_o + 3n_s r_c}{\left(5 + 3n_s\right)r_o + 3\left(2 + n_s\right)r_c}$$

$$\overline{G}_{co} = \frac{G_s\left[\left(5 - 3n_s\right)r_o + 3\left(1 - n_s\right)r_c\right]}{9\left(1 - n_s\right)}$$

$$\overline{K}_{co} = \frac{2\left(r_o + r_c\right)K_s\left(1 - 2n_s\right)}{3\left(1 - n_s\right)}$$

(27)

S.1.3 Isotropy

We note that neither the octet nor the cubic foams are isotropic. A condition that must be met for an isotropic material is

$$\frac{\overline{E}}{\overline{G}} + \frac{\overline{E}}{3\overline{K}} - 3 = 0$$

(28)

For the octet foam

$$\frac{\overline{E}_o}{\overline{G}_o} + \frac{\overline{E}_o}{3\overline{K}_o} - 3 = -\frac{24\left(1 + n_s\right)}{\left(5 + 3n_s\right)\left(5 - 3n_s\right)}$$

(29)

while the cubic foam provides

$$\frac{\overline{E}_c}{\overline{G}_c} + \frac{\overline{E}_c}{3\overline{K}_c} - 3 = \frac{4\left(1 + n_s\right)}{\left(2 + n_s\right)\left(1 - n_s\right)}$$

(30)

Given the limits on Poisson's ratio and moduli values, neither Eq. (29) nor Eq. (30) can be consistent with zero on the right hand side.

For the combined cubic+octet foam, the condition for isotropy is

$$\frac{\overline{E}_{co}}{\overline{G}_{co}} + \frac{\overline{E}_{co}}{3\overline{K}_{co}} - 3 = \frac{12\left(1 + n_s\right)\left(r_o + r_c\right)\left(3r_c - 2r_o\right)}{\left[\left(5 + 3n_s\right)r_o + 3\left(2 + n_s\right)r_c\right]\left[\left(5 - 3n_s\right)r_o + 3\left(1 - n_s\right)r_c\right]}$$

(31)

and we conclude that if $\rho_c = 2\rho_o / 3$ the right hand side of Eq. (31) will be zero and therefore that the resulting cubic-octet closed cell foam will be isotropic.

We now take a unit cell of the foam to have a volume L^3. It follows that

$$r_o = 4\sqrt{3}\frac{h_o}{L}$$

$$r_c = 3\frac{h_c}{L}$$

(32)

where h_o is the thickness of the walls forming the octet foam and h_c is that of the wall forming the cubic foam. Thus to achieve an isotropic cubic-octet closed cell foam in which $\rho_c = 2\rho_o/3$, we require a wall thickness ratio given by

$$\frac{h_c}{h_o} = \frac{8}{3\sqrt{3}} = 1.54$$

(33)

S.1.4 Low density moduli

From the results in the previous section, for $v_s = 0.3$ equations (27) and **Table S.1** provide the moduli summarized in **Table S.2**. For all three material geometries, the results agree very well with the finite element results in **Figure 3** where the relevant plots intercept the ordinate. Such a comparison is appropriate as the affine results best represent the behavior of closed cell foams with very thin walls in which bending stiffness plays a negligible role. For the isotropic cubic + octet foam, with $\rho_c = 2\rho_o/3$, the results lie just below the Hashin-Shtrikman upper bound for a foam in the $\overline{\rho}/\rho_s \to 0$ limit, thereby proving that an isotropic closed cell foam has been designed that achieves the maximum possible stiffness.

Geometry:	$\dfrac{\overline{E}/E_S}{\overline{\rho}/\rho_S}$	$\dfrac{\overline{G}/G_S}{\overline{\rho}/\rho_S}$	$\dfrac{\overline{K}/K_S}{\overline{\rho}/\rho_S}$
octet foam	0.34	0.65	0.38
cubic foam	0.70	0.33	0.38
Cubic+octet foam (isotropic)	0.50	0.52	0.38

Supplementary Information Table 2 | Normalized Moduli. Moduli of the cubic, octet and combined cubic + octet closed cell foams.

S.2 Supplementary Equations - Upper bound on stiffness of lattice structures

The results of *Gurtner and Durand*[27] can be used to investigate the maximum stiffness of lattice structures. Consider a 3-dimensional lattice structure composed of straight, slender members in which the stiffness of the joints is negligible; such a structure could be pin-jointed or simply composed of very slender beams who's bending stiffness near joints is negligible. The cross-sectional areas, *A*, and the joint to joint lengths, λ, of the members are taken to be diverse. We analyze a cube of the structure of volume L^3, which may be the entire structure or a repeating unit cell. The sum of the volumes of the lattice elements in the cube is ρL^3, where ρ is the relative density of the lattice structure, and the lattice elements are distributed in orientation in a manner characterized by $\omega(\theta,\phi)$. In this setting, r,θ,ϕ are spherical polar coordinates and $\rho L^3\omega(\theta,\phi)$ is the volume of all elements whose axes lies within the solid angle, $\sin\theta d\theta d\phi$. The arrangement of the lattice may thus be stochastic or it may be regular and periodic, with $\omega(\theta,\phi)$ in the latter case being represented as a set of Dirac delta functions and coefficients. The total volume of lattice elements within the representative cube is given by

$$\rho L^3 = \frac{\rho L^3\omega(\theta,\phi)\sin\theta}{4\pi}\int_0^\pi d\theta\int_0^{2\pi}d\phi$$

(34)

Therefore, the orientation distribution function $\omega(\theta,\phi)$ must satisfy the constraint

$$\frac{w(q,f)\sin q}{4p} \int_0^p dq \int_0^{2p} df = 1 \tag{35}$$

and thus, for a stochastic lattice is a probability distribution function. Unit vectors, $\underset{\sim}{n}$, can be defined parallel to each lattice element such that

$$\underset{\sim}{n} = \sin\theta\left(\underset{\sim}{e}_1 \cos\phi + \underset{\sim}{e}_2 \sin\phi\right) + \underset{\sim}{e}_3 \cos\theta \tag{36}$$

where $\underset{\sim}{e}_i$ are the unit base vectors for a Cartesian coordinate system x_i, and thus, in that coordinate system, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$ are the direction cosines of the unit vector $\underset{\sim}{n}$.

Displacement boundary conditions can be imposed at the perimeter of the representative cube that are compatible with the uniform strain, $\underset{\sim}{\varepsilon}$, within it. The strain energy within the cube is then defined to be

$$y = \frac{L^3}{2}\left\{ \begin{matrix} e_{11} & e_{22} & e_{33} & g_{23} & g_{13} & g_{12} \end{matrix} \right\}^T [C]\left\{ \begin{matrix} e_{11} & e_{22} & e_{33} & g_{23} & g_{13} & g_{12} \end{matrix} \right\} \tag{37}$$

where $\underset{\sim}{\gamma} = 2\underset{\sim}{\varepsilon}$ and $[C]$ is the symmetric effective elasticity matrix of the lattice structure within the cube. The matrix $[C]$ is thus a property of the exact solution to the deformation of the lattice within the representative cube.

To obtain an upper bound on measures of $[C]$, we impose an affine deformation at each joint within the representative cube where the affine deformation is compatible with the strain $\underset{\sim}{\varepsilon}$. The axial strain experienced by any given lattice element is thus

$$\varepsilon = \underset{\sim}{n}.\underset{\sim}{\varepsilon}.\underset{\sim}{n} \tag{38}$$

and the strain energy per unit volume of lattice elements is thus $E_s \varepsilon^2 / 2$. It follows that the total strain energy associated with the affine deformation of the joints is

$$y_a = \frac{rL^3 E_s e^2 w(q,f)\sin q}{8p}\int_0^p dq \int_0^{2p} df \tag{39}$$

The minimum potential energy principle requires that

$$y \pounds y_a \tag{40}$$

thereby providing the upper bound on measures of the elasticity matrix $[C]$.

Now consider a stochastic lattice with a uniform orientation distribution of both element cross-sections and lengths so that $w(q,f)$ = 1. Since this lattice is isotropic, without any favored orientation, Eq. (37) simplifies to

$$y = \frac{L^3}{2}\left(G\bar{g}^2 + Ke_V^2\right) \tag{41}$$

where G is the effective shear modulus of the lattice, K is its effective bulk modulus, the shear equivalent strain $\bar{g}$ is such that

$$\bar{g}^2 = 2e_{ij}e_{ij} \tag{42}$$

where $\underset{\sim}{e}$ is the deviatoric strain and e_V is the volume strain.

For an affine deformation on the joints that is compatible with the pure volume strain of magnitude e_V, it can be inferred that in Eq. (39) $e = e_V / 3$ for every

lattice element. With $w(q,f) = 1$ Eq. (39) gives

$$y^i_{ad} = \frac{1}{18} r L^3 E_s e_V^2$$

(43)

where the subscript *ad* denotes affine dilatation and the superscript *i* indicates an isotropic stochastic lattice. The bound from Eq. (40), (41) and (42) then provides

$$K \pounds \frac{1}{9} r E_s$$

(44)

This exercise can be repeated for a simple shear

$$\underset{\sim}{\varepsilon} = \frac{\gamma}{2}\left(\underset{\sim}{e_1}\underset{\sim}{e_2} + \underset{\sim}{e_2}\underset{\sim}{e_1}\right)$$

(45)

It follows that due to the affine deformation of the joints the axial strain in a lattice element is

$$e = \frac{g}{2} \sin 2f \sin^2 q$$

(46)

By inserting this into Eq. (39) and with $w(q,f) = 1$ it is found that

$$y^i_{as} = \frac{1}{30} r L^3 E_s g^2$$

(47)

where the subscript *as* denotes affine shear. The bound from Eq. (40), (41) and (47) then gives

$$G \pounds \frac{1}{15} r E_s$$

(48)

Now consider isotropic lattices with regular periodic layouts such that $w(q,f)\ ^1\ 1$ and the lattice is instead composed of a set of lattice elements with discrete orientations. The addition of identical lattices to a given cubic volume L^3, but without joining them at common nodes or intersections, creates a sufficiently dense set of intersecting lattices (but without common joints) that, by diverse rotations of the contributing lattices, the sum of all the functions $w(q,f)$ for all the lattices, divided by the number of intersecting lattices, is effectively unity. We note that the resulting combined function $w(q,f)$ is still discrete, but so densely fills solid angles that its behavior is just as if it were uniform and unit. Such an attribute is no different from that of a finite, stochastic lattice with uniform directional properties as studied above. Furthermore, the orientation of the lattices is sufficiently distributed such that lattice element lengths and cross-sectional areas are also uniformly distributed. The combination of intersecting lattices thus satisfies Eq. (44) and (48); however, each individual lattice has the same properties as any other, and thus, scaled by relative density, each individual lattice is bounded by the results in Eq. (44) and (48). Such bounds, stated by *Gurtner and Durand*[27], thus apply to any isotropic lattice composed of straight, slender elements where joint stiffness is negligible.

Now assume that a lattice can be constructed such that it has a regular periodic layout and is isotropic, as has been achieved by *Gurtner and Durand* [27]. If it achieves the bounds in Eq. (44) and (48), its Poisson's ratio is 0.25 and its Young's modulus is

$$E = \frac{1}{6} r E_s$$

(49)

as noted by *Gurtner and Durand* [27]. We note that the results in Figure 3 for the isotropic lattice, where the relevant plot intersects the ordinate, are slightly greater than the outcomes in Eq. (44), (48) and (49), consistent with the prediction of *Gurtner and Durand*[27]. The upper bounds are exceeded slightly, probably because of the stiffness of the joints in the slenderest lattice that we have computed.

S.3 Supplementary Methods - Finite element model verification

To verify that the FE model accurately predicts the stiffness of material geometries we compare our results to experimental data for Young's modulus for a variety

of cellular materials, and the corresponding analytical model for stiffness, from *Gibson and Ashby*[24] (G-A) (**FIG.E4**),

$$\frac{\bar{E}}{E_s} = \left(\frac{\bar{\rho}}{\rho_s}\right)^2 \phi^2 + \frac{\bar{\rho}}{\rho_s}(1 - \phi).$$

(50)

where ϕ is the fraction of material subject to bending and $(1 - \phi)$ the fraction subject to stretching. The linear term results from material subject to stretching, while the second order contribution is from material subject to bending. In this case, we assume the material is distributed so that one third of it is preferentially oriented toward an arbitrary applied axial stress, and thus responds through stretching, while the remainder of the material responds through bending ($\phi = 2/3$). We fit FE results to third order polynomials, forced to go through the origin (0, 0) and (1, 1), corresponding to empty space and a dense solid respectively. Excellent agreement is achieved between the G-A model and the polynomial curve fits at lower densities. In fact, when the G-A model is fit to FE data for Young's and shear moduli with relative densities lower than 25% the agreement is nearly perfect, R[2] > 0.999, and R[2] > 0.99 for bulk modulus. This, with slight modification, is the same G-A fit for stiff stochastic foams, notably the dense cluster of experimental data with $2\% \lesssim (\bar{\rho}/\rho_S) \lesssim 8\%$. Third order fits agree well with the FE data from *Roberts and Garboczi*[32], which involved larger unit cells, and are the average of five models each.

Closed form solutions for the Young's, shear, and bulk moduli of the cubic, octet and combined cubic + octet foam (S.1.4) agree very well with finite element results in the low density limit. It is also noted that FE models for the octet-truss agree with the original analysis [26] (not shown). These various comparisons all indicate that the FE boundary conditions and the mesh resolution used, produce results that are consistent with the existing literature and are quantitatively accurate, providing a foundation for our examination of new geometries.