Quadratic and Symmetric Bilinear Forms on Modules with Unique Base over a Semiring

Zur Izhakian, Manfred Knebusch, Louis Rowen

Received: September 4, 2015
Revised: December 3, 2015

Communicated by Ulf Rehmann

Abstract. We study quadratic forms on free modules with unique base, the situation that arises in tropical algebra, and prove the analog of Witt's Cancelation Theorem. Also, the tensor product of an indecomposable bilinear module (U, γ) with an indecomposable quadratic module (V, q) is indecomposable, with the exception of one case, where two indecomposable components arise.

2010 Mathematics Subject Classification: Primary 15A03, 15A09, 15A15, 16Y60; Secondary 14T05, 15A33, 20M18, 51M20
Keywords and Phrases: Semirings, (semi)modules, bilinear forms, quadratic forms, symmetric forms, orthogonal decomposition.

1. Overview

This paper is part of a program to understand the theory of quadratic forms over the max-plus algebra and related semirings that arise in several mathematical contexts. Our motivation comes from two sources, tropical mathematics and real algebra, which interact with each other. Since the first area is still in its nascent stage, for the reader's convenience, we provide a short overview of this mathematics and related subjects.

Consider the field $\mathbb{K}$ of *Puiseux series* over an algebraically closed field F of characteristic 0. The elements of $\mathbb{K}$ are of the form

$$f = \sum_{\tau \in \mathbb{Q}} c_\tau t^\tau,$$

where $c_\tau \in F$ and the powers of t are taken over well-ordered subsets of $\mathbb{Q}$. (In the literature one often takes $\mathbb{R}$ instead of $\mathbb{Q}$.)

Define the *order valuation* $v : \mathbb{K} \to \mathbb{Q}$ by

$$v(f) \ := \min\{\tau \in \mathbb{Q}_{\geq 0} \ : \ c_\tau \neq 0\}$$

for which the dominant term in f becomes $c_{v(f)}t^{v(f)}$ as $t \to 0$. Then v is a valuation, with residue field F, with respect to which $\mathbb{K}$ is complete and thus Henselian. By Hensel's lemma, $\mathbb{K}$ also is algebraically closed, and thus elementarily equivalent to F. Applying v takes us from $\mathbb{K}$ to the ordered group $\mathbb{Q}$, which can be viewed as a "max-plus" semiring (taking $-v$ instead of v), whose operations are "+" for multiplication and "sup" for addition. This process, called *tropicalization*, is explained in [15, 29]. The point of tropicalization is to simplify the combinatorics in algebraic geometry and linear algebra, and there has been considerable success in this direction in enumerative geometry.

One can tropicalize structures arising in linear algebra, such as quadratic forms, simply by replacing the classical addition and multiplication by the max-plus operations respectively, but then the classical theory does not go through because our new addition (max) does not have negatives.

Other important (non-tropical) semirings, where our below theory is relevant, occur in real algebra, such as the positive cone of an ordered field [4, p. 18] or a partially ordered commutative ring [5, p. 32]. A further application can be found in the algebra of groups over a splitting field, as described briefly at the end of this overview.

Recall that a (commutative) *semiring* is a set R equipped with addition and multiplication, such that both $(R, +, 0)$ and $(R, \cdot, 1)$ are abelian monoids with elements $0 = 0_R$ and $1 = 1_R$ respectively, and multiplication distributes over addition in the usual way. In other words, R satisfies all the properties of a commutative ring except the existence of negation under addition. We call a semiring R a *semifield*, if every nonzero element of R is invertible; hence $R \setminus \{0\}$ is an abelian group.

As in the classical theory, one considers bilinear and quadratic forms defined on (semi)modules over a semiring R, often a "supersemifield," in order to obtain more sophisticated "trigonometric" information, cf. [24, §2, §3].

On one hand, these semirings lack negation, thereby playing havoc even with the notion of the underlying bilinear form of a quadratic form. On the other hand, they have the pleasant property that free modules have "unique base," cf. Definition 1.2. Thus, our overall object is to classify quadratic forms over free modules having unique base, with applications to the supertropical setting.

For the reader's convenience, we recall some terminology and results from [22, §1-§4]. A *module* V over R (sometimes called a *semimodule*) is an abelian monoid $(V, +, 0_V)$ equipped with a scalar multiplication $R \times V \to V$, $(a, v) \mapsto av$, such that exactly the same axioms hold as customary for modules over a ring: $a_1(bv) = (a_1 b)v$, $a_1(v + w) = a_1 v + a_1 w$, $(a_1 + a_2)v = a_1 v + a_2 v$, $1_R \cdot v = v$, and $0_R \cdot v = 0_V = a_1 \cdot 0_V$ for all $a_1, a_2, b \in R$, $v, w \in V$. We write 0 for both 0_V and 0_R, and 1 for 1_R.

When considering modules over semifields, one encounters several versions of "base," as studied in depth in [21, §4 and §5.3]. Here we take the standard categorical version, and call an R-module V *free*, if there exists a family $(\varepsilon_i \mid i \in I)$ in V such that every $x \in V$ has a unique presentation $x = \sum_{i \in I} x_i \varepsilon_i$ with scalars

$x_i \in R$ and only finitely many x_i nonzero, and we call $(\varepsilon_i \mid i \in I)$ a *base* of the R-module V. Any free module with a base of n elements is clearly isomorphic to R^n, under the map $\sum_{i=1}^{n} x_i\varepsilon_i \mapsto (x_1, \dots, x_n)$.

Bilinear forms on V are defined in the obvious way, [21].

Definition 1.1. *For any module V over a semiring R, a* quadratic form on V *is a function $q : V \to R$ with*

$$q(ax) = a^2 q(x) \tag{1.1}$$

for any $a \in R$, $x \in V$, together with a symmetric bilinear form $b : V \times V \to R$ (not necessarily uniquely determined by q) such that for any $x, y \in V$

$$q(x + y) = q(x) + q(y) + b(x, y). \tag{1.2}$$

Every such bilinear form b will be called a companion *of q, and the pair (q, b) will be called a* quadratic pair *on V. We also call V a* quadratic module.

In this generality, it is difficult to describe quadratic forms adequately on free modules over an arbitrary semiring. However, our task becomes more manageable when we introduce the following condition.

Definition 1.2. *An R-*module with unique base *is a free R-module V in which any two bases $\mathfrak{B}$, $\mathfrak{B}'$ are projectively the same, i.e., we obtain the elements of $\mathfrak{B}'$ from those of $\mathfrak{B}$ by multiplying by units of R.*

Although this never happens for free modules of rank ≥ 2 over a ring, it turns out to be quite common in the context of tropical algebra (and also often in real algebra, as noted in Example 2.4.d).

Our main result, in §5, is an analog of Witt's cancelation theorem:

Theorem 5.9.*If W_1, W_1', W_2, W_2' are finitely generated quadratic or bilinear modules with unique base such that $W_1 \cong W_1'$ and $W_1 \perp W_2 \cong W_2 \perp W_2'$, then $W_2 \cong W_2'$ (where $\cong$ means "isometric").*

It actually is given in more general terms, where W_2 needs not be finitely generated.

When R is a ring, then a quadratic form q has just one companion, namely,

$$b(x, y) := q(x + y) - q(x) - q(y),$$

but if R is a semiring that cannot be embedded into a ring, this usually is not the case, and it is a major concern of quadratic form theory over semirings to determine all companions of a given quadratic form $q : V \to R$.

The first step in classifying quadratic forms is [22, Propositions 4.1 and 4.2], which lets us write a quadratic form q as the sum $q = \kappa + \rho$, where κ is *quasilinear* (and unique) in the sense that $\kappa(x + y) = \kappa(x) + \kappa(y)$, and ρ is *rigid* in the sense that it has a unique companion. Quasilinearity of a quadratic form q implies that, for any vector $x = \sum_{i \in I} x_i\varepsilon_i$ in V,

$$q(x) = \sum_{i \in I} x_i^2 q(\varepsilon_i), \tag{1.3}$$

i.e., q has *diagonal form* with respect to the base $(\varepsilon_i : i \in I)$.
Quasilinear forms follow aspects of the classical theory of quadratic forms, and satisfy a Cauchy-Schwartz inequality given in [24]. On the other hand, by [22, Theorem 3.5], the rigid forms are precisely those with $q(\varepsilon_i) = 0$ for all $i \in I$.
Our ultimate object being to classify quadratic forms over free modules with unique base, in this paper we study quadratic forms in terms of orthogonal decompositions of such forms into indecomposable forms, and then build them up again via tensor products of two symmetric bilinear forms and of a symmetric bilinear form with a quadratic form.
Let us turn now to the tools needed in proving Theorem 5.9.

1.1. Partial quasilinearity. We seldom require quasilinearity in its entirety, but the following partial version plays a major role in our consideration of orthogonal decompositions of quadratic modules.

Definition 1.3. *Given subsets S and T of V, we say that q is* quasilinear on $S \times T$ *if*

$$q(x + y) = q(x) + q(y).$$

for all $x \in S$, $y \in T$.

The following helpful fact is a special case of [22, Lemma 1.18]. (We write $S + S'$ for $\{s + s' : s \in S, s' \in S'\}$.)

Lemma 1.4. *Let S, S', T be subsets of V. If q is quasilinear on $S \times T$, $S' \times T$ and $S \times S'$, then q is quasilinear on $(S + S') \times T$.*

1.2. Disjoint orthogonality. In §3 we develop the notion of *(disjoint) orthogonality* of two given submodules W_1 and W_2 of a *quadratic R-module* (V, q) (endowed with a fixed quadratic form q), which means that $W_1 \cap W_2 = \{0\}$ and q is partially quasilinear on $W_1 \times W_2$. (Note that there is no direct reference to an underlying symmetric bilinear form.) When V has unique base, we look for orthogonal decompositions $V = W_1 \perp W_2$, and more generally $V = \underset{i \in I}{\perp} W_i$, where the W_i are *basic submodules* of V, i.e., are generated by subsets of a base $\mathfrak{B}$ of V.
We can choose a companion b of q (called "quasiminimal" companion) adapted to the notion of disjoint orthogonality, and then have an equivalence relation on the set $\mathfrak{B}$ at hands, which is generated by the pairs $(\varepsilon, \varepsilon')$ in $\mathfrak{B}$ with $\varepsilon \neq \varepsilon'$, $b(\varepsilon, \varepsilon') \neq 0$. By the use of this equivalence relation the indecomposable basic submodules of V (in the sense of disjoint orthogonality) can be described as follows.

Theorem 3.8.*Let $\{\mathfrak{B}_k \mid k \in K\}$ denote the set of equivalence classes in $\mathfrak{B}$ and, for every $k \in K$, let W_k denote the submodule of V having base $\mathfrak{B}_k$.*

(a) *Then every W_k is an indecomposable basic submodule of V and*

$$V = \underset{k \in K}{\perp} W_k.$$

(b) *Every indecomposable basic submodule U of V is contained in W_k, for some $k \in K$ uniquely determined by U.*

(c) *The modules W_k, $k \in K$, are precisely all the indecomposable basic orthogonal summands of V.*

In §4 we develop the analogous notion of disjoint orthogonality in a *bilinear R-module* (V, b) with respect to a fixed symmetric bilinear form b on V, and we show:

Theorem 4.9. *If b is a quasiminimal companion of a a quadratic module (V, q), then the indecomposable components of (V, q) coincide with the indecomposable components of (V, b).*

In §5, these decomposition theories yield the desired analog (Theorem 5.9) of Witt's cancelation theorem.

1.3. Tensor products. The last two sections of the paper are devoted to tensor products. Whereas tensor products of modules over general semirings can be carried out in analogy with the usual classical construction over rings, it requires the use of congruences, resulting in some technical issues dealt with in [7, Chap. 16], for example. But for free modules with unique base the construction can be carried out easily, since then one does not need to worry about well-definedness.

In §6 we construct the tensor product of two free bilinear R-modules over any semiring R, in analogy to the case where R is a ring, cf. [8, §2], [26, I, §5]. We then take the tensor product of a free bilinear R-module $U = (U, \gamma)$ with a free quadratic R-module $V = (V, q)$. A new phenomenon occurs here, in contrast to the theory over rings. It is necessary first to choose a so-called *balanced companion* b of q, which always exists, cf. [22, §1], but which usually is not unique. We then define the tensor product $U \otimes_b V$, depending on b, by choosing a so-called *expansion* $B : V \times V \to R$ of the quadratic pair (q, b) which is a (not necessarily symmetric) bilinear form B with

$$B(x, x) = q(x), \qquad B(x, y) + B(y, x) = b(x, y)$$

for all $x, y \in V$, cf. [22, §1] and then proceed essentially as in the case of rings, e.g. [26, Definition 1.51], [8, p. 51][1]. The resulting quadratic form $\gamma \otimes_b q$ does not depend on the choice of B but often depends on the choice of b. This is apparent already in the case $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where the matrix b is stored in the quadratic polynomial $\gamma \otimes_b q$, cf. Example 6.8 below.

In §7 we turn to the indecomposability of tensor products. For convenience, we assume that $R \setminus \{0\}$ is closed under multiplication and addition, implying by Theorem 2.3 that all free R-modules have unique base.

After obtaining partial results along the way, we arrive at the main result of this section, Theorem 7.16, which states that, discarding trivial situations and

[1]When R is a ring the "b" in the tensor product is not specified since q has only one companion.

excluding some pathological semirings, the tensor product of an indecomposable bilinear module (U, γ) with an indecomposable quadratic module (V, q) is again indecomposable, with the exception of one case, where two indecomposable components arise.

1.4. Applications. The remainder of this introduction discusses how quadratic forms over modules with unique base over semirings arise naturally in various contexts in mathematics. (The reader could skip directly on to the main theoretical results of this paper.)

1.4.1. *Quadratic forms over rings.* Supertropical semirings, to be defined below (cf. [25, 22]), establish a class of semirings over which every free module has a unique base. There is a way to pass from a quadratic form on a free module over a (commutative) ring R to quadratic forms on free modules over a supertropical semiring U. To explain this, we sketch the notion of *supertropicalization* of a quadratic form $q : V \to R$, obtained by a so-called *supervaluation* $\varphi : R \to U$. An *m-valuation* (= monoid valuation) on a ring R is a map $v : R \to M$ from R to a totally ordered abelian monoid $M = (M, \cdot\ , \leq)$, containing an absorbing element $0 = 0_M$ $(0 \cdot x = x \cdot 0 = 0)$ with $0 \leq x$ for all $x \in M$, which satisfies the following rules:

$$v(0) = 0, \quad v(1) = 1, \quad v(xy) = v(x)v(y),$$

and

$$v(x + y) \leq \max\{v(x), v(y)\} \tag{1.4}$$

for all $x, y \in M$. When $\Gamma := M \setminus \{0\}$ is a group, we call the m-valuation $v : R \to M$ a *valuation*. These are exactly the valuations as defined by Bourbaki [3] and studied, e.g., in [14] and [27, Ch. I], except that for Γ we have chosen the multiplicative notation instead of the additive notation. In this case $v^{-1}(0)$ is a prime ideal of R [loc. cit.]. When R is a field this forces $v^{-1}(0) = \{0\}$, and we return to Krull valuations.
Given an m-valuation $v : R \to M$, we equip M with the additive operation defined as

$$a + b := \max\{a, b\},$$

which makes M a *bipotent semiring*, i.e., a semiring M' in which $a + b \in \{a, b\}$ for all $a, b \in M'$. Conversely any bipotent semiring M' has a natural total order given by

$$a < b \quad \Leftrightarrow \quad a + b = b,$$

and can be viewed as a totally ordered abelian monoid with an absorbing element $0_{M'}$. Therefore, totally ordered monoids M with zero can be referred to as bipotent semirings (or bipotent semifields when $M \setminus \{0\}$ is a group). Viewed in this way, rule (1.4) reads

$$v(x + y) \leq v(x) + v(y). \tag{1.5}$$

This brings us into the realm of semirings. A semiring U is called *supertropical* if the following conditions hold:

- $e := 1_U + 1_U$ is idempotent (i.e., $2 \times 1 = 4 \times 1$),
- the *ghost ideal* $M = eU$ is a bipotent semiring,
- addition is defined in terms of the *ghost map* $a \mapsto ea$ and the ordering of M, as follows:

$$a + b = \begin{cases} a & \text{if } ea < eb; \\ b & \text{if } eb < ea; \\ ea & \text{if } ea = eb. \end{cases} \tag{1.6}$$

In particular $ea = 0$ implies $a = 0$ (take $b = 0$ in (1.6)). The elements of eU are called *ghost elements* and those of $U \setminus eU$ are called *tangible elements.* The zero element is regarded both as tangible and ghost. See [17, 18, 25] for the ideas behind this terminology.

A *supervaluation* on a ring R is a multiplicative map $\varphi : R \to U$ sending R into a supertropical semiring, such that $\varphi(0) = 0$, $\varphi(1) = 1$, and

$$e\varphi(x + y) \leq e\varphi(x) + e\varphi(y)$$

for all $x, y \in R$. The map $v := e\varphi : R \to M$, $x \mapsto e\varphi(x)$, is then an m-valuation, which as we say is *covered* by φ. For any given m-valuation $v : R \to M$, there usually is an extended hierarchy of supervaluations $\varphi : R \to U$ covering v (with $U \supset M$, $eU = M$, U varying) studied in [17, 18].

The supertropicalizations of a quadratic form $q : V \to R$ on a free R-module V are constructed by using a supervaluation $\varphi : R \to U$ as follows. We choose an ordered base $\mathcal{L}$ of V, say $\mathcal{L} = \{v_i : i \in I\}$ with $I = \{1, \dots, n\}$, and write q as a homogenous polynomial of degree 2

$$q\Big(\sum_{i=1}^{n} x_i v_i\Big) = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j, \tag{1.7}$$

with $\alpha_i = q(v_i)$, $\beta_{ij} = b(v_i, v_j)$, where b is the (unique) companion of q. We denote by U^n the free U-module consisting of all n-tuples in U. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard base of U, where each ε_i has i-th coordinate 1 and all other coordinates 0. Using a new set of variables $\lambda_1, \dots, \lambda_n$, we define

$$q^{\varphi}\Big(\sum_{i=1}^{n} \lambda_i \varepsilon_i\Big) := \sum_{i=1}^{n} \varphi(\alpha_i)\lambda_i^2 + \sum_{i<j} \varphi(\beta_{ij})\lambda_i\lambda_j \tag{1.8}$$

by applying φ to the coefficients of the polynomial (1.7).

We write $(U^{(I)}, q^{\varphi})$, or $(\widetilde{V}, \tilde{q})$ for short, for the *supertropicalization* of the quadratic module (V, q) with respect to the base $\mathcal{L}$. Since every U-module has a unique base, cf. §2, the base $\{\varepsilon_i : i \in I\}$ of $\widetilde{V}$ is unique up to permuting the ε_i and multiplying them by units of U (which are the invertible tangible elements of U). That is, the base $\mathcal{L}$ of V becomes "frozen" in the free quadratic module $(\widetilde{V}, \tilde{q})$ obtained from (V, q) by a kind of "degenerate scalar extension" $\varphi : R \to U$. $\{\varphi$ is multiplicative, but respects addition only weakly.$\}$ This central fact motivates our interest in supertropicalization.

One reason that we work with m-valuations in general, instead of just valuations covered by supervaluations, is that m-valuations which are not valuations often arise naturally in the context of commutative algebra as described in the paper [11] of Harrison and Vitulli. They construct so-called "V-valuations" (there named "formally finite" V-valuations). This construction has been complemented later by D. Zhang with somewhat dual "V^0-valuations" [33]. These constructions have been revised in [19, §1-§3], showing that any m-valuation on a ring can be coarsened both to a V-valuation and to a V^0-valuation, and also to a valuation in a minimal way.

In [12] Harrison and Vitulli, pursuing their idea of "infinite primes" (in the sense of classical number theory) from [11], construct $\mathbb{C}$-valued places on a field by a somewhat similar method. This construction has been extended by Valente and Vitulli in [31] to "preplaces" on a ring R, which are interpreted in [19] as multiplicative maps $\chi : R \to R'$ to a bipotent semiring R' such that $\chi(0) = 0$, $\chi(1) = 1$, and

$$\chi(x + y) \le c(\chi(x) + \chi(y))$$

for all $x, y \in R$, where c is a unit of R'. Such a map χ provides various supervaluations $\varphi : R \to U$ that cover V-valuations $v : R \to eU$ [19, §4]. Since the multiplicative monoids $eU \setminus \{0\}$ are cancellative, these V-valuations are true valuations. By a related method, supervaluations arise that cover V^0-valuations, which again are true valuations.

Although not all supervaluations can be constructed in this way, at least we gain a rich stock of m-valuations and supervaluations on a ring. Facing a problem on quadratic forms over a ring R, it may be a piece of art to address an appropriate supervaluation which fits best the supertropical framework. Much space is left for further study in this research direction.

1.4.2. *A surprise.* In an earlier version of this paper we considered quadratic forms over supertropical semirings, knowing already from [22, Theorem 0.9] that a free module over these semirings has unique base, and we obtained the results in §3-§7 for such quadratic forms. Only later did we realize that these results go through for any semiring R over which all free modules have unique base. As a consequence, supertropical semirings hardly appear explicitly in §3-§7. This paves the way for an extra application, which we now describe. Namely, take an algebra A with a bilinear form, whose orthogonal base generates a natural proper semiring of A.

1.4.3. *Table algebras.* A classical example is the set of characters of a finite group G over a field whose characteristic does not divide $|G|$; since the sum (resp. product) of characters is the character of the direct sum (resp. tensor product) of their underlying representations, we can restrict to the semiring of characters, which is a free module over $\mathbb{N}_0$. A similar situation arises for the center of the group algebra, which is a free module whose base is comprised of the sums of elements from conjugacy classes. These algebras have been generalized by Hoheisel [13] and Arad-Blau [1] as explained in the fine survey by Blau [2], where he defines Hoheisel algebras and table algebras. These have a distinguished base $\mathcal{L}$ that spans the sub-semialgebra A^+ that they generate over $\mathbb{R}^+$, so again A^+ is a free module over $\mathbb{R}^+$ (with unique base $\mathcal{L}$), and a natural framework in which to build quadratic forms.

2. R-modules with unique base and their basic submodules

We assume throughout this paper that V is a free R-module with unique base $\mathfrak{B}$. Accordingly, we begin by examining this property.

Remark 2.1. *Any change of base of the free module R^n is attained by multiplication by an invertible $n \times n$ matrix, so having unique base is equivalent to every invertible matrix in $M_n(R)$ being a generalized permutation matrix.*

Our interest in these modules stems from the following key fact.

Theorem 2.2 ([21, Corollary 5.25] and [22, Theorem 0.9]). *If R is a supertropical semiring, then every free R-module has unique base.*

More generally, one may ask, "What conditions on the semiring R guarantee that R^n has unique base, or equivalently, that every invertible matrix is a generalized permutation matrix?" The matrix question was answered in [30] and [6]. In their terminology, an "antiring" is a semiring R such that $R \setminus \{0\}$ is closed under addition. We prefer the terminology "lacks zero sums," since this property holds also for sums of squares in a real closed field, and "antiring" does not seem appropriate in that context.
Tan and Dolžna-Oblak classify the invertible matrices over these rings lacking zero sums. These are just the generalized permutation matrices when $R \setminus \{0\}$ also is closed under multiplication, which they call "entire" (the case in tropical mathematics), and more generally by [6, Theorem 1] (as interpreted in Theorem 2.5) when R is indecomposable, i.e., not isomorphic to a direct product $R_1 \times R_2$ of semirings.

Theorem 2.3 (cf. [6, §2, Corollary 3], an alternative proof given below). *If the set $R \setminus \{0\}$ is closed under addition and multiplication (i.e., $a + b = 0 \Rightarrow a = b = 0$, $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$), then every free R-module has unique base.*

In view of Remark 2.1, Theorem 2.3 follows from Dolžan and Oblak [6, §2, Corollary 3] using matrix arguments within a wider context extending work of Tan [30, Proposition 3.2], which in turn relies on Golan's book on semirings [9, Lemma 19.4].

Example 2.4. *Here are some instances where $R \setminus \{0\}$ is closed under addition and multiplication.*

a) *The "Boolean semifield" $\mathbb{B} = \{-\infty, 0\}$ (and thus subalgebras of algebras that are free modules over $\mathbb{B}$). This shows that our results pertain to "$\mathcal{F}_1$-geometry."*
b) *Rewriting the Boolean semifield instead as $\mathbb{B} = \{0, 1\}$ where $1 + 1 = 1$, one can generalize it to $\{0, 1, \dots, q\}$ $L = [1, q] := \{1, 2, \dots, q\}$ the "truncated semiring without 0" of [23, Example 2.14], where "$a + b''$ is defined to be the minimum of their sum and q.*

c) *Function semirings, polynomial semirings, and Laurent polynomial semirings over these semirings.*

d) *If F is a formally real field, i.e. -1 is not a sum of squares in F, then the subsemiring $R = \Sigma F^2$, consisting of all sums of squares in F, lacks zero sums. In fact R is a semifield; the inverse of a sum of squares*

$$a = x_1^2 + \cdots + x_r^2 \quad \text{is} \quad a^{-1} = \left(\frac{x_1}{a}\right) + \cdots + \left(\frac{x_r}{a}\right)^2.$$

Other than the trivial fact that every free R-module of rank 1 has unique base, all examples known to us of modules with unique base stem from Theorem 2.5, which is essentially [6, Theorem 1]:

Theorem 2.5 ([6, Theorem 1]). *Assume that R is an indecomposable semiring lacking zero sums. Then every free R-module has unique base.*

We now reprove Theorem 2.3 by a simple matrix-free argument in preparation for a reproof of the more general Theorem 2.5.

Proof of Theorem 2.3. Let V be a free R-module and $\mathfrak{B}$ a base of V. If $x \in V \setminus \{0\}$ is given, we have a presentation

$$x = \sum_{i=1}^{r} \lambda_i x_i$$

with $x_i \in \mathfrak{B}$ and $\lambda_i \in R \setminus \{0\}$. We call the set $\{x_1, \dots, x_r\} \subset \mathfrak{B}$ the *support* of x with respect to $\mathfrak{B}$ and denote this set by $\operatorname{supp}_{\mathfrak{B}}(x)$. Note that if $x, y \in V \setminus \{0\}$, then $x + y \neq 0$ and

$$\operatorname{supp}_{\mathfrak{B}}(x+y) = \operatorname{supp}_{\mathfrak{B}}(x) \cup \operatorname{supp}_{\mathfrak{B}}(y) \tag{2.1}$$

due to the assumption that $\lambda + \mu \neq 0$ for any $\lambda, \mu \in R \setminus \{0\}$. Also

$$\operatorname{supp}_{\mathfrak{B}}(\lambda x) = \operatorname{supp}_{\mathfrak{B}}(x) \tag{2.2}$$

for $x \in V \setminus \{0\}$, $\lambda \in R \setminus \{0\}$, due to the assumption that for $\lambda, \mu \in R \setminus \{0\}$ we have $\lambda\mu \neq 0$.

Now assume that $\mathfrak{B}'$ is a second base of V. Given $x \in \mathfrak{B}$, we have a presentation

$$x = \lambda_1 y_1 + \cdots + \lambda_r y_r$$

with $\lambda_i \in R \setminus \{0\}$ and distinct $y_i \in \mathfrak{B}'$. It follows from (2.1) and (2.2) that

$$\{x\} = \operatorname{supp}_{\mathfrak{B}}(x) = \operatorname{supp}_{\mathfrak{B}}(y_1) \cup \cdots \cup \operatorname{supp}_{\mathfrak{B}}(y_r).$$

This forces

$$\{x\} = \operatorname{supp}_{\mathfrak{B}}(y_1) = \cdots = \operatorname{supp}_{\mathfrak{B}}(y_r). \tag{2.3}$$

¿From this, we infer that $r = 1$. Indeed, suppose that $r \geq 2$. Then $y_1 = \mu_1 x$, $y_2 = \mu_2 x$ with $\mu_1, \mu_2 \in R \setminus \{0\}$. But this implies $\mu_2 y_1 = \mu_1 y_2$, a contradiction since y_1, y_2 are different elements of a base of V.

Thus $\{x\} = \operatorname{supp}_{\mathfrak{B}}(y)$ for a unique $y \in \mathfrak{B}'$, which means $y = \lambda x$ with $\lambda \in R \setminus \{0\}$. By symmetry we have a unique $z \in \mathfrak{B}$ and $\mu \in R \setminus \{0\}$ with $x = \mu z$. Then $x = \lambda\mu z$, whence $x = z$ and $\lambda\mu = 1$. Thus $\lambda, \mu \in R^*$ and $x \in R^* y$,

$y \in R^*x$. Of course, y runs through all of $\mathfrak{B}'$ if x runs through $\mathfrak{B}$, since both $\mathfrak{B}$ and $\mathfrak{B}'$ span the module V. $\square$

Proof of Theorem 2.5. Assume that $\mathfrak{B}$ and $\mathfrak{B}'$ are bases of V. Given $x \in \mathfrak{B}$, we write again

$$x = \lambda_1 y_1 + \cdots + \lambda_r y_r \tag{2.4}$$

with different $y_i \in \mathfrak{B}'$, $\lambda_i \in R \setminus \{0\}$. But now, instead of (2.3) we can only conclude that

$$\{x\} = \mathrm{supp}_{\mathfrak{B}}(\lambda_1 y_1) = \cdots = \mathrm{supp}_{\mathfrak{B}}(\lambda_i y_i). \tag{2.5}$$

Thus we have scalars $\mu_i \in R \setminus \{0\}$ such that

$$\lambda_i y_i = \mu_i x \qquad \text{for} \quad 1 \leq i \leq r. \tag{2.6}$$

Suppose that $r \geq 2$. Then we have for all $i, j \in \{1, \dots, r\}$ with $i \neq j$.

$$\mu_j \lambda_i y_i = \mu_j \mu_i x = \mu_i \mu_j x = \mu_i \lambda_j y_j.$$

Since the y_i are elements of a base, this implies $\mu_i \lambda_j = \mu_j \lambda_i = 0$ for $i \neq j$ and then

$$\mu_i \mu_j = 0 \qquad \text{for} \quad i \neq j. \tag{2.7}$$

On the other hand, we obtain from (2.4) and (2.6) that

$$x = \mu_1 x + \mu_2 x + \cdots + \mu_r x,$$

and then

$$1 = \mu_1 + \mu_2 + \cdots + \mu_r. \tag{2.8}$$

Multiplying (2.8) by μ_i and using (2.7), we obtain

$$\mu_i^2 = \mu_i. \tag{2.9}$$

Thus

$$R \cong R\mu_1 \times \cdots \times R\mu_r.$$

This contradicts our assumption that R is indecomposable.

We have proved that $r = 1$. Thus for every $x \in \mathfrak{B}$ there exist unique $y \in \mathfrak{B}'$ and $\lambda \in R$ with $x = \lambda y$. By the same argument as in the end of proof of Theorem 2.3, we conclude that $\mathfrak{B}$ is projectively unique. $\square$

Of course, if $R \setminus \{0\}$ is closed under multiplication, i.e., R has no zero divisors, then R is indecomposable. This also holds when R is supertropical (cf. [25, §3], [22, Definition 0.3]), since then for any two elements μ_1, μ_2 of R with $\mu_1 + \mu_2 = 1$ either $\mu_1 = 1$ or $\mu_2 = 1$. Thus, Theorem 2.5 generalizes both Theorems 2.2 and 2.3.

The following example reveals that Theorem 2.5 is the best we can hope for, in order to guarantee that every free R-module has unique base, as long as we stick to the natural assumption that R is a semiring lacking zero sums.

Example 2.6. *If R_0 is a semiring lacking zero sums, then $R := R_0 \times R_0$ also lacks zero sums. Put $\mu_1 = (1,0)$, $\mu_2 = (0,1)$. These are idempotents in R with $\mu_1\mu_2 = 0$ and $\mu_1 + \mu_2 = 1$. Now let V be a free R-module with base $\mathfrak{B} = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$, $n \geq 2$, choose a permutation $\pi \in S_n$, $\pi \neq 1$, and define*

$$\varepsilon'_i := \mu_1\varepsilon_i + \mu_2\varepsilon_{\pi(i)} \qquad (1 \leq i \leq n).$$

We claim that $\mathfrak{B}' := \{\varepsilon'_1, \dots, \varepsilon'_n\}$ is another base of V.
Indeed, V is a free R_0-module with base $(\mu_i\varepsilon_j \mid 1 \leq i \leq 2,\ 1 \leq j \leq n)$. We have

$$\mu_1\varepsilon'_i = \mu_1\varepsilon_i, \qquad \mu_2\varepsilon'_i = \mu_2\varepsilon_{\pi(i)},$$

and thus $(\mu_i\varepsilon'_j \mid 1 \leq i \leq 2,\ 1 \leq j \leq n)$ is a permutation of this base over R_0, i.e., regarded as a set, the same base. Thus certainly $\mathfrak{B}'$ spans V as R-module. Given $x \in V$, let $x = \sum_1^n a_i\varepsilon'_i$ with $a_i \in R$. We have

$$a_i = a_{i1}\mu_1 + a_{i2}\mu_2 \qquad \text{with} \qquad a_{i1} \in R_0,\ a_{i2} \in R_0,$$

whence

$$x = \sum_{i=1}^n a_{i1}(\mu_1\varepsilon_i) + \sum_{i=1}^n a_{i2}(\mu_2\varepsilon_{\pi(i)}).$$

This shows that the coefficients $a_{i1}, a_{i2} \in R_0$ are uniquely determined by x, whence the coefficients $a_i \in R$ are also uniquely determined by x. Our claim is proved.
Since $\operatorname{supp}_{\mathfrak{B}}(\varepsilon'_i)$ *has two elements if $\pi(i) \neq i$, $\mathfrak{B}'$ differs projectively from $\mathfrak{B}$. The base $\mathfrak{B}$ of the R-module V is not unique.*

3. Orthogonal decompositions of quadratic modules with unique base

Assume that V is an R-module equipped with a fixed quadratic form $q : V \to R$. We then call $V = (V, q)$ a *quadratic R-module.*

Definition 3.1.

(a) *Given two submodules W_1, W_2 of the R-module V, we say that W_1 is* disjointly orthogonal *to W_2, if $W_1 \cap W_2 = \{0\}$ and $q(x+y) = q(x)+q(y)$ for all $x \in W_1$, $y \in W_2$, i.e., q is quasilinear on $W_1 \times W_2$. (We say "orthogonal" for short, when it is clear a priori that $W_1 \cap W_2 = \{0\}$.)*

(b) *We write $V = W_1 \perp W_2$ if $V = W_1 \oplus W_2$ (as R-module) with W_1 disjointly orthogonal to W_2. We then call W_1 an* orthogonal summand *of W, and W_2 an* orthogonal complement *of W_1 in V.*

Caution. If $V = W_1 \perp W_2$, we may choose a companion b of q such that $b(W_1, W_2) = 0$, but note that it could well happen that the set of all $x \in V$ with $b(x, W_1) = 0$ is bigger than W_2, even if R is a semifield and $q|W_1$ is anisotropic (e.g., if q itself is quasilinear). Our notion of orthogonality does not refer to any bilinear form.

We now also define infinite orthogonal sums. This seems to be natural, even if we are originally interested only in finite orthogonal sums. Indeed, even if R is a semifield, a free R-module with finite base often has many submodules which are not finitely generated.

Definition 3.2. *Let $(V_i \mid i \in I)$ be a family of submodules of the quadratic module V. We say that V is* the orthogonal sum of the family (V_i), *and then write*

$$V = \underset{i\in I}{\perp} V_i,$$

if for any two different indices i, j the submodule V_i is disjointly orthogonal to V_j, and moreover $V = \bigoplus_{i\in I} V_i$.

N.B. Of course, then for any subset $J \subset I$, the module $V_J = \sum_{i\in J} V_i$ is the orthogonal sum of the subfamily $(V_i \mid i \in J)$; in short,

$$V_J = \underset{i\in J}{\perp} V_i.$$

We state a fact which, perhaps contrary to first glance, is not completely trivial.

Proposition 3.3. *Assume that we are given an orthogonal decomposition $V = \underset{i\in I}{\perp} V_i$. Let J and K be two disjoint subsets of I. Then the submodule $V_J = \underset{i\in J}{\perp} V_i$ of V is disjointly orthogonal to $V_K = \underset{i\in K}{\perp} V_i$, and thus*

$$V_{J\cup K} = V_J \perp V_K.$$

Proof. It follows from Lemma 1.4 above that for any three different indices i, j, k the form q is quasilinear on $V_i \times (V_j + V_k)$, and thus V_i is orthogonal to $V_j \perp V_k$. By iteration, we see that the claim holds if J and K are finite. In the general case, let $x \in V_J$ and $y \in V_K$. There exist finite subsets J', K' of J and K with $x \in V_{J'}$, $y \in V_{K'}$, and thus $q(x + y) = q(x) + q(y)$. This proves that V_J is orthogonal to V_K. $\square$

In the rest of this section, *we assume that V has unique base.*

Definition 3.4. *We call a submodule W of V* basic, *if W is spanned by $\mathfrak{B}_W := \mathfrak{B}\cap W$, and thus W is free with base $\mathfrak{B}_W$. Note that then we have a unique direct decomposition $V = W \oplus U$, where the submodule U is basic with base $\mathfrak{B} \setminus \mathfrak{B}_W$. W and U again are R-modules with unique base. We call U the* complement of W in V, *and write $U = W^c$.*

The theory of basic submodules of V is of utmost simplicity. All of the following is obvious.

Scholium 3.5.

(a) *We have a bijection $W \mapsto \mathfrak{B}_W := \mathfrak{B}\cap W$ from the set of basic submodules of V onto the set of subsets of $\mathfrak{B}$.*

(b) *If W_1 and W_2 are basic submodules of V, then also $W_1 \cap W_2$ and $W_1 + W_2$ are basic submodules of V, and*

$$\mathfrak{B}_{W_1 \cap W_2} = \mathfrak{B}_{W_1} \cap \mathfrak{B}_{W_2}, \qquad \mathfrak{B}_{W_1 + W_2} = \mathfrak{B}_{W_1} \cup \mathfrak{B}_{W_2}.$$

(c) *If W is a basic submodule of V, then as stated above,*

$$\mathfrak{B}_{W^c} = \mathfrak{B} \setminus \mathfrak{B}_W.$$

(d) *Finally, if $W_1 \subset W_2$ are basic submodules of V, then W_1 is basic in W_2 and $W_1^c \cap W_2$ is the complement of W_1 in W_2.*

Thus a basic orthogonal summand W of V has only one basic orthogonal complement, namely, W^c, equipped with the form $q|W^c$.

Definition 3.6. *If the quadratic module V has a basic orthogonal summand $W \neq V$, we call V* decomposable. *Otherwise we call V* indecomposable. *More generally, we call a basic submodule X of V* decomposable *if X is decomposable with respect to $q|X$, and otherwise we call X* indecomposable.

Our next goal is to decompose the given quadratic module V orthogonally into indecomposable basic submodules. Therefore, we choose a base $\mathfrak{B}$ of V (unique up to multiplication by scalar units). We then choose a companion b of q such that $b(\varepsilon, \eta) = 0$ for any two *different* $\varepsilon, \eta \in \mathfrak{B}$ such that q is quasilinear on $R\varepsilon \times R\eta$, cf. [22, Theorem 6.3]. We call such a companion b a *quasiminimal companion* of q.

Comment. In important cases, e.g., if R is supertropical or more generally "upper bound" (cf. [22, Definition 5.1]), the set of companions of q can be partially ordered in a natural way. The prefix "quasi" here is a reminder that we do not mean minimality with respect to such an ordering.

Lemma 3.7. *Let W and W' be basic submodules of V with $W \cap W' = \{0\}$. If b is any quasiminimal companion of q, then W is (disjointly) orthogonal to W' iff $b(W, W') = 0$.*

Proof. If $b(W, W') = 0$, then $q(x+y) = q(x) + q(y)$ for any $x \in W$ and $y \in W'$, which means by definition that W is orthogonal to W'. (This holds for any companion b of q.)
Conversely, if W is orthogonal to W', then for base vectors $\varepsilon \in \mathfrak{B}_W$, $\eta \in \mathfrak{B}_{W'}$ the form q is quasilinear on $R\varepsilon \times R\eta$ and thus $b(\varepsilon, \eta) = 0$. This implies that $b(W, W') = 0$. □

We now introduce the following equivalence relation on the set $\mathfrak{B}$. We choose a quasiminimal companion b of q. Given $\varepsilon, \eta \in \mathfrak{B}$, we put $\varepsilon \sim \eta$, iff either $\varepsilon = \eta$, or there exists a sequence $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r$ in $\mathfrak{B}$, $r \geq 1$, such that $\varepsilon = \varepsilon_0$, $\eta = \varepsilon_r$, and $\varepsilon_i \neq \varepsilon_{i+1}$, $b(\varepsilon_i, \varepsilon_{i+1}) \neq 0$ for $i = 0, \dots, r-1$.

Theorem 3.8. *Let $\{\mathfrak{B}_k \mid k \in K\}$ denote the set of equivalence classes in $\mathfrak{B}$ and, for every $k \in K$, let W_k denote the submodule of V having base $\mathfrak{B}_k$.*

(a) *Then every W_k is an indecomposable basic submodule of V and*

$$V = \underset{k\in K}{\perp} W_k.$$

(b) *Every indecomposable basic submodule U of V is contained in W_k, for some $k \in K$ uniquely determined by U.*

(c) *The modules W_k, $k \in K$, are precisely all the indecomposable basic orthogonal summands of V.*

Proof. (a): Suppose that W_k has an orthogonal decomposition $W_k = X \perp Y$ with basic submodules $X \neq 0$, $Y \neq 0$. Then $\mathfrak{B}_k$ is the disjoint union of the non-empty sets $\mathfrak{B}_X$ and $\mathfrak{B}_Y$. Choosing $\varepsilon \in \mathfrak{B}_X$ and $\eta \in \mathfrak{B}_Y$, there exists a sequence $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r$ in $\mathfrak{B}_k$ with $\varepsilon = \varepsilon_0$, $\eta = \varepsilon_r$ and $b(\varepsilon_{i-1}, \varepsilon_i) \neq 0$, $\varepsilon_{i-1} \neq \varepsilon_i$, for $1 \leq i \leq r$. Let s denote the last index in $\{1, \dots, r\}$ with $\varepsilon_s \in \mathfrak{B}_X$. Then $s < r$ and $\varepsilon_{s+1} \in \mathfrak{B}_Y$. But $b(X, Y) = 0$ by Lemma 3.7 and thus $b(\varepsilon_s, \varepsilon_{s+1}) = 0$, a contradiction. This proves that W_k is indecomposable. Since $\mathfrak{B}$ is the disjoint union of the sets $\mathfrak{B}_k$, we have

$$V = \bigoplus_{k\in K} W_k.$$

Finally, if $k \neq \ell$, then $b(W_k, W_\ell) = 0$ by the nature of our equivalence relation. Thus

$$V = \underset{k\in K}{\perp} W_k.$$

(b): Given an indecomposable basic submodule U of V, we choose $k \in K$ with $\mathfrak{B}_U \cap \mathfrak{B}_k \neq \emptyset$. Then $U \cap W_k \neq 0$. ¿From $V = W_k \oplus W_k^c$, we conclude that $U = (U \cap W_k) \oplus (U \cap W_k^c)$, and then have $U = (U \cap W_k) \perp (U \cap W_k^c)$ because W_k is orthogonal to W_k^c. Since U is indecomposable and $U \cap W_k \neq 0$, it follows that $U = U \cap W_k$, i.e., $U \subset W_k$. Since $W_k \cap W_\ell = 0$ for $k \neq \ell$, it is clear that k is uniquely determined by U.

(c): If U is an indecomposable basic orthogonal summand of V, then $V = U \perp U^c$. We have $U \subset W_k$ for some $k \in K$, and obtain $W_k = U \perp (U^c \cap W_k)$, whence $W_k = U$. □

Definition 3.9. *We call the submodules W_k of V occurring in Theorem 3.8 the* indecomposable components *of the quadratic module V.*

The following facts are easy consequences of the theorem.

Remark 3.10.

(i) *If U is a basic orthogonal summand of V, then the indecomposable components of the quadratic module $U = (U, q|U)$ are the indecomposable components of V contained in U.*

(ii) *If U is any basic submodule of V, then*

$$U = \underset{k\in K}{\perp} (U \cap W_k),$$

and every submodule $U \cap W_k \neq \{0\}$ is an orthogonal sum of indecomposable components of U.

4. Orthogonal decomposition of bilinear modules with unique base

We now outline a theory of symmetric bilinear forms analogous to the theory for quadratic forms given in §3. The bilinear theory is easier than the quadratic theory due the fact that, in contrast to quadratic forms, on a free module we do not need to distinguish between "functional" and "formal" bilinear forms cf. [22, §1]. As before, R is a semiring.

Assume in the following that V is an R-module equipped with a fixed symmetric bilinear form $b : V \times V \to R$. We then call $V = (V, b)$ a *bilinear R-module.* If X is a submodule of V, we denote the restriction of b to $X \times X$ by $b|X$.

Definition 4.1.

(a) *Given two submodules W_1, W_2 of the R-module V, we say that W_1 is* disjointly orthogonal *to W_2, if $W_1 \cap W_2 = \{0\}$ and $b(W_1, W_2) = 0$, i.e., $b(x, y) = 0$ for all $x \in W_1$, $y \in W_2$.*

(b) *We write $V = W_1 \perp W_2$ if W_1 is disjointly orthogonal to W_2 and moreover $V = W_1 \oplus W_2$ (as R-module). We then call W_1 an* orthogonal summand *of V and W_2 an* orthogonal complement *of W_1 in V.*

Definition 4.2. *Let $(V_i \mid i \in I)$ be a family of submodules of the bilinear module V. We say that V is* the orthogonal sum of the family (V_i), *and then write*

$$V = \perp_{i \in I} V_i,$$

if for any two different indices i, j the submodule V_i is disjointly orthogonal to V_j, and moreover $V = \bigoplus_{i \in I} V_i$.

In contrast to the quadratic case, the exact analog of Proposition 3.3 is now a triviality.

Proposition 4.3. *Assume that $V = \perp_{i \in I} V_i$. Let J and K be disjoint subsets of I. Then $V_J = \perp_{i \in J} V_i$ is disjointly orthogonal to $V_K = \perp_{i \in K} V_i$, and*

$$V_{J \cup K} = V_J \perp V_K.$$

In the following, *we assume again that V has unique base.* Then again a basic orthogonal summand W of V has only one basic orthogonal complement in V, namely, W^c equipped with the bilinear form $b|W^c$.

For X a basic submodule of V, we define the properties *"decomposable"* and *"indecomposable"* in exactly the same way as indicated by Definition 3.6 in the quadratic case.

We start with a definition and description of the "indecomposable components" of $V = (V, b)$ in a similar fashion as was done in §3 for quadratic modules. We choose a base $\mathfrak{B}$ of V and again introduce the appropriate equivalence relation on the set $\mathfrak{B}$, but now we adopt a more elaborate terminology than in §3. This will turn out to be useful later on.

DEFINITION 4.4. *We call the symmetric bilinear form b* alternate *if $b(\varepsilon, \varepsilon) = 0$ for every $\varepsilon \in \mathfrak{B}$.*

Comment. Beware that this does *not* imply that $b(x, x) = 0$ for every $x \in V$. The classical notion of an alternating bilinear form is of no use here since in the semirings under consideration here (cf. §2) $\alpha + \beta = 0$ implies $\alpha = \beta = 0$, whence $b(x + y, x + y) = 0$ implies $b(x, y) = 0$. An alternating bilinear form in the classical sense would be identically zero.

DEFINITION 4.5. *We associate to the given symmetric bilinear form b an alternate bilinear form b_{alt} by the rule*

$$b_{\mathrm{alt}}(\varepsilon, \eta) = \begin{cases} b(\varepsilon, \eta) & \text{if } \varepsilon \neq \eta \\ 0 & \text{if } \varepsilon = \eta \end{cases}$$

for any $\varepsilon, \eta \in \mathfrak{B}$.

LEMMA 4.6. *Let W and W' be basic submodules of V with $W \cap W' = \{0\}$. Then W is (disjointly) orthogonal to W' iff $b_{\mathrm{alt}}(W, W') = 0$.*

Proof. This can be seen exactly as with the parallel Lemma 3.7. Just replace in its proof the quasiminimal companion of q by b_{alt}. $\square$

DEFINITION 4.7.

(a) *A* path Γ *in $V = (V, b)$ of* length $r \geq 1$ *in $\mathfrak{B}$ is a sequence $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r$ of elements of $\mathfrak{B}$ with*

$$b_{\mathrm{alt}}(\varepsilon_i, \varepsilon_{i+1}) \neq 0 \qquad (0 \leq i \leq r - 1).$$

In essence this condition does not depend on the choice of the base $\mathfrak{B}$, since $\mathfrak{B}$ is unique up to multiplication by units, and so we also say that Γ is a path in V. *We say that the path* runs *from $\varepsilon := \varepsilon_0$ to $\eta := \varepsilon_r$, or that the path* connects *ε to η. A path of length 1 is called an* edge. *This is just a pair (ε, η) in $\mathfrak{B}$ with $\varepsilon \neq \eta$ and $b(\varepsilon, \eta) \neq 0$.*

(b) *We define an equivalence relation on $\mathfrak{B}$ as follows. Given $\varepsilon, \eta \in \mathfrak{B}$, we declare that $\varepsilon \sim \eta$ if either $\varepsilon = \eta$ or there runs a path from ε to η.*

It is now obvious how to mimic the theory of indecomposable components from the end of §3 in the bilinear setting.

SCHOLIUM 4.8. *Theorem 3.8 and its proof remain valid for the present equivalence relation on $\mathfrak{B}$. We only have to replace the quasiminimal companion b of q there by b_{alt} and to use Lemma 4.6 instead of Lemma 3.7. Again we denote the set of equivalence classes of $\mathfrak{B}$ by $\{\mathfrak{B}_k \mid k \in K\}$ and the submodule of V with base $\mathfrak{B}_k$ by V_k, and again we call the V_k the* indecomposable components *of V. Also the analog to Remark 3.10 remains valid.*

We state a consequence of the parallel between the two decomposition theories.

Theorem 4.9. *Assume that (V, q) is a quadratic module with unique base and b is a quasiminimal companion of q. The indecomposable components of (V, q) coincide with the indecomposable components of (V, b).*

Proof. The equivalence relation used in Theorem 3.8 is the same as the equivalence relation in Definition 4.7. ☐

We add an easy observation on bilinear modules.

Proposition 4.10. *Assume that (V, b) is a bilinear R-module with unique base. A basic submodule W of V is indecomposable with respect to b, iff W is indecomposable with respect to b_{alt}.*

Proof. The equivalence relation on $\mathfrak{B}$ just defined (Definition 4.7) does not change if we replace b by b_{alt}. ☐

5. Isometries, isotypical components, and a cancelation theorem

Let R be any semiring.

Definition 5.1.

(a) *For quadratic R-modules $V = (V, q)$ and $V' = (V', q')$, an* isometry *$\sigma : V \to V'$ is a bijective R-linear map with $q'(\sigma x) = q(x)$ for all $x \in V$. Likewise, if $V = (V, b)$ and (V', b') are bilinear R-modules, an* isometry *is a bijective R-linear map $\sigma : V \to V'$ with $b'(\sigma x, \sigma y) = b(x, y)$ for all $x, y \in V$.*

(b) *If there exists an isometry $\sigma : V \to V'$, we call V and V'* isometric *and write $V \cong V'$. We then also say that V and V' are in the same isometry class.*

In the following we study quadratic and bilinear R-modules with unique base on an equal footing.
It would not hurt if we supposed that the semiring R satisfies the conditions in Theorem 2.5, so that every free R-module has unique base, but the simplicity of all of the arguments in the present section becomes more apparent if we do not rely on Theorem 2.5.

Notation/Definition 5.2.

(a) *Let $(V_\lambda^0 \mid \lambda \in \Lambda)$ be a set of representatives of all isometry classes of indecomposable quadratic (resp. bilinear) R-modules with unique base of rank bounded by the cardinality of V, in order to avoid set-theoretical complications.*

(b) *If W is such an R-module, where $W \cong V_\lambda^0$ for a unique $\lambda \in \Lambda$, we say that W* has type λ *(or: W is* indecomposable of type λ*).*

(c) *We say that a quadratic (resp. bilinear) module $W \neq 0$ with unique base is* isotypical *of type λ, if every indecomposable component of V has type λ.*

(d) *Finally, given a quadratic (resp. bilinear) R-module with unique base, we denote the sum of all indecomposable components of V of type λ by V_λ and call the $V_\lambda \neq 0$ the* isotypical components of V.

The following is now obvious from §3 and §4 (cf. Theorem 3.8 and Scholium 4.8).

Proposition 5.3. *If V is a quadratic or bilinear R-module with unique base, then*

$$V = \mathop{\perp}_{\lambda \in \Lambda'} V_\lambda$$

with $\Lambda' = \{\lambda \in \Lambda \mid V_\lambda \neq 0\}$.

Since our notion of orthogonality for basic submodules of V is encoded in the linear and quadratic, resp. bilinear, structure of V, the following fact also is obvious, but in view of its importance will be dubbed a "theorem".

Theorem 5.4. *Assume that V and V' are quadratic (resp. bilinear) R-modules with unique bases and $\sigma : V \to V'$ is an isometry. Let $\{V_k \mid k \in K\}$ denote the set of indecomposable components of V.*

(a) *$\{\sigma(V_k) \mid k \in K\}$ is the set of indecomposable components of V'.*

(b) *If V_k has type λ, then $\sigma(V_k)$ has type λ, and so $\sigma(V_\lambda) = V'_\lambda$ for every $\lambda \in \Lambda$.*

Also in the remainder of the section, we assume that the quadratic or bilinear modules have unique base.

Definition 5.5. *Let $O(V)$ denote the group of all isometries $\sigma : V \to V$ (i.e., automorphisms) of (V, q), resp. (V, b). As usual, we call $O(V)$ the* orthogonal group *of V.*

Theorem 5.4 has the following immediate consequence.

Corollary 5.6. *Every $\sigma \in O(V)$ permutes the indecomposable components of V of fixed type λ, and so $\sigma(V_\lambda) = V_\lambda$ for every $\lambda \in \Lambda$.*
We have a natural isomorphism

$$O(V) \xrightarrow{1:1} \prod_{\lambda \in \Lambda'} O(V_\lambda),$$

sending $\sigma \in O(V)$ to the family of its restrictions $\sigma | V_\lambda \in O(V_\lambda)$.

Definition 5.7.

(a) *Let $\lambda \in \Lambda$. We denote the cardinality of the set of indecomposable components of V_λ by $m_\lambda(V)$, and we call $m_\lambda(V)$ the* multiplicity *of V_λ. $\{$N.B. $m_\lambda(V)$ can be infinite or zero.$\}$*

(b) *If $m_\lambda \in \mathbb{N}_0$ for every $\lambda \in \Lambda$, we say that V is* isotypically finite.

Theorem 5.8. *If V and V' are quadratic or bilinear R-modules with unique bases, then $V \cong V'$ iff $m_\lambda(V) = m_\lambda(V')$ for every $\lambda \in \Lambda$.*

Proof. This follows from Proposition 5.3 and Theorem 5.4. $\square$

We are ready for a main result of the paper.

Theorem 5.9. *Assume that W_1, W_2, W_1', W_2' are quadratic or bilinear modules with unique base and that W_1 is isotypically finite. Assume furthermore that $W_1 \cong W_1'$ and that $W_1 \perp W_2 \cong W_1' \perp W_2'$. Then $W_2 \cong W_2'$.*

Proof. For every $\lambda \in \Lambda$, clearly $m_\lambda(V) = m_\lambda(W_1) + m_\lambda(W_2)$ and $m_\lambda(V') = m_\lambda(W_1') + m_\lambda(W_2')$. Since $V \cong V'$, the multiplicities $m_\lambda(V)$ and $m_\lambda(V')$ are equal, and since $W_1 \cong W_1'$, the same holds for the multiplicities $m_\lambda(W_1')$. Since $m_\lambda(W_1) = m_\lambda(W_1')$ is finite, it follows that $m_\lambda(W_2) = m_\lambda(W_2')$. By Theorem 5.8 this implies that $W_2 \cong W_2'$. $\square$

Remark 5.10. *If the free R-module W_1 has finite rank, then certainly W_1 is isotypically finite. Thus Theorem 5.9 may be viewed as the analog of Witt's cancellation theorem from 1937* [32] *proved for quadratic forms over fields.*

The assumption of isotypical finiteness in Theorem 5.9 cannot be relaxed. Indeed if $m_\lambda(W_1)$ is infinite for at least one $\lambda \in \Lambda$, then the cancelation law becomes false. This is evident by Theorem 5.8 and the following example.

Example 5.11. *Assume that V is the orthogonal sum of infinitely many copies $V_1, V_2, \dots$ of an indecomposable quadratic or bilinear module V_0 with unique base. Consider the following submodules of V:*

$$\begin{aligned} W_1 &:= V_2 \perp V_3 \perp \cdots, & W_2 &:= V_1, \\ W_1' &:= V_3 \perp V_4 \perp \cdots, & W_2' &:= V_1 \perp V_2. \end{aligned}$$

Then $W_1 \perp W_2 = V = W_1' \perp W_2'$, and $W_1 \cong W_1'$. But W_2 is not isometric to W_2'.

6. Expansions and tensor products

Let $q : V \to R$ be a quadratic form on an R-module V. We recall from [22, §1] that, when V is free with base $(\varepsilon_i : i \in I)$, then q admits a (not necessarily unique) *balanced companion*, i.e., a companion $b : V \times V \to R$ such that $b(x,x) = 2q(x)$ for all $x \in V$, and that it suffices to know for this that $b(\varepsilon_i, \varepsilon_i) = 2q(\varepsilon_i)$ for all $i \in I$ [22, Proposition 1.7]. Balanced companions are a crucial ingredient in our definition below of a tensor product of a free bilinear module and a free quadratic module. They arise from "expansions" of q, defined as follows, cf. [22, Definition 1.9].

Definition 6.1. *A bilinear form $B : V \times V \to R$ (not necessarily symmetric) is an* expansion *of a balanced pair (q,b) if $B + B^t = b$, i.e.,*

$$B(x,y) + B(y,x) = b(x,y) \tag{6.1}$$

for all $x, y \in V$, and

$$q(x) = B(x,x) \tag{6.2}$$

for all $x \in V$. If only the form q is given and (6.2) *holds, we say that B is an* expansion *of q.*

As stated in the [22, §1], every bilinear form $B : V \times V \to R$ gives us a balanced pair (q, b) via (6.1) and (6.2), and, if the R-module V is free, we obtain all such pairs (q, b) in this way. But we will need a description of *all* expansions of (q, b) in the free case.

Construction 6.2. *Assume that V is a free R-module and $(\varepsilon_i \mid i \in I)$ is a base of V. When (q, b) is a balanced pair on V, we obtain all expansions $B : V \times V \to R$ of (q, b) as follows.*
Let $\alpha_i := q(\varepsilon_i)$, $\beta_{ij} := b(\varepsilon_i, \varepsilon_j)$ for $i, j \in I$. We have $\beta_{ij} = \beta_{ji}$. We choose a total ordering on I and for every $i < j$ two elements $\chi_{ij}, \chi_{ji} \in R$ with

$$\beta_{ij} = \chi_{ij} + \chi_{ji}, \qquad (i < j).$$

We furthermore put

$$\chi_{ii} := \alpha_i,$$

and define B by the rule

$$B(\varepsilon_i, \varepsilon_j) = \chi_{ij}$$

for all $(i, j) \in I \times I$.

In practice one usually chooses $\chi_{ij} = \beta_{ij}$, $\chi_{ji} = 0$ for $i < j$, i.e., takes the unique "triangular" expansion B of (q, b), cf. [22, §1], but now we do not want to depend on the choice of a total ordering of the base $(\varepsilon_i \mid i \in I)$. We used such an ordering above only to ease notation.

Tensor products over semirings in general require the use of congruences [10], but for free modules the basics can be done precisely as over rings, and we leave the formal details to the interested reader. We only state here that, given two free R-modules V_1 and V_2, with bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$, the R-module $V_1 \otimes_R V_2$ "is" the free R-module with base $\mathfrak{B}_1 \otimes \mathfrak{B}_2$, which is a renaming of $\mathfrak{B}_1 \times \mathfrak{B}_2$, writing $\varepsilon \otimes \eta$ for (ε, η) with $\varepsilon \in \mathfrak{B}_1$, $\eta \in \mathfrak{B}_2$. If

$$\mathfrak{B}_1 = \{\varepsilon_i \mid i \in I\}, \qquad \mathfrak{B}_2 = \{\eta_j \mid j \in J\}$$

and $x = \sum\limits_{i \in I} x_i \varepsilon_i \in V_1$ and $y = \sum\limits_{j \in J} y_j \eta_j \in V_2$, we define, as common over rings,

$$x \otimes y := \sum_{(i,j) \in I \times J} x_i y_j (\varepsilon_i \otimes y_j), \tag{6.3}$$

and this vector is independent of the choice of the bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$. If B_1 and B_2 are bilinear forms on V_1 and V_2 respectively, we have a well defined bilinear form on $V_1 \otimes_R V_2$, denoted by $B_1 \otimes B_2$, such that for any $x_i \in V_1$, $y_j \in V_2$ $(i, j \in \{1, 2\})$

$$(B_1 \otimes B_2)(x_1 \otimes x_2, y_1 \otimes y_2) = B_1(x_1, y_1) B_2(x_2, y_2). \tag{6.4}$$

If b_1 and b_2 are symmetric bilinear forms on V_1 and V_2 respectively, then $b_1 \otimes b_2$ is symmetric. Then we call the bilinear module $(V_1 \otimes_R V_2, b_1 \otimes b_2)$ the *tensor product of the bilinear modules* (V_1, b_1) and (V_2, b_2).
We next define the tensor product of a free bilinear and a free quadratic module. The key fact which allows us to do this in a reasonable way is as follows.

Proposition 6.3. *Let $\gamma : U \times U \to R$ be a symmetric bilinear form and (q, b) a balanced quadratic pair on V. Assume that B and B' are two expansions of (q, b). Then the bilinear forms $\gamma \otimes B$ and $\gamma \otimes B'$ on $U \otimes V$ yield the same balanced pair $(\tilde{q}, \tilde{b})$ on $U \otimes V$. We have $\tilde{b} = \gamma \otimes b$, whence for $u_1, u_2 \in U$, $v_1, v_2 \in V$,*

$$\tilde{b}(u_1 \otimes v_1, u_2 \otimes v_2) = \gamma(u_1, u_2) b(v_1, v_2). \tag{6.5}$$

Furthermore, for $u \in U$ and $v \in V$,

$$\tilde{q}(u \otimes v) = \gamma(u, u) q(v). \tag{6.6}$$

Proof. $\gamma \otimes B + (\gamma \otimes B)^t = \gamma \otimes B + \gamma^t \otimes B^t = \gamma \otimes B + \gamma \otimes B^t = \gamma \otimes (B + B^t) = \gamma \otimes b$. Also $\gamma \otimes B' + (\gamma \otimes B')^t = \gamma \otimes b$. Furthermore,

$$\begin{aligned} (\gamma \otimes B)(u \otimes v, u \otimes v) &= \gamma(u, u) B(v, v) \\ &= \gamma(u, u) q(v) \qquad = (\gamma \otimes B')(u \otimes v, u \otimes v) \end{aligned}$$

for any $u \in U$, $v \in V$. Together these equations imply

$$(\gamma \otimes B)(z, z) = (\gamma \otimes B')(z, z)$$

for any $z \in U \otimes V$. □

Definition 6.4. *We call $\tilde{q}$ the* tensor product *of the bilinear form γ and the quadratic form q* with respect to the balanced companion *b of q, and write*

$$\tilde{q} = \gamma \otimes_b q,$$

and we also write $\widetilde{V} = U \otimes_b V$ for the quadratic R-module $\widetilde{V} = (U \otimes V, \tilde{q})$.

Remark 6.5. *If q has only one balanced companion, we may suppress the "b" here, writing $\tilde{q} = \gamma \otimes q$. Cases in which this happens are: q is rigid, V has rank one, R is embeddable in a ring.*

Proposition 6.6. *If $U = (U, \gamma)$ has an orthogonal decomposition $U = \underset{i \in I}{\perp} U_i$, then*

$$U \otimes_b V = \underset{i \in I}{\perp} U_i \otimes_b V.$$

Proof. It is immediate that $(\gamma \otimes b)(U_i \otimes V, U_j \otimes V) = 0$ for $i \neq j$. □

We proceed to explicit examples. For this we need notation from [22, §1] which we recall for the convenience of the reader.

Assume that V is free of finite rank n and $\mathfrak{B}$ is a base of V for which we now choose a total ordering, $\mathfrak{B} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. Then we identify a bilinear form B on V with the $(n \times n)$-matrix

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & & \beta_{nn} \end{pmatrix}, \tag{6.7}$$

where $\beta_{ij} = B(\varepsilon_1, \varepsilon_j)$. In particular, a bilinear R-module (V, β) is denoted by a symmetric $(n \times n)$-matrix, namely its Gram matrix $b = (\beta_{ij})_{1\le i,j\le n}$, where $\beta_{ij} = \beta_{ji} = b(\varepsilon_i, \varepsilon_j)$.
Given a quadratic module (V, q), we choose a triangular expansion

$$B = \begin{pmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha_2 & \cdots & \alpha_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix} \tag{6.8}$$

of q and denote q by the triangular scheme

$$q = \begin{bmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ & \alpha_2 & \cdots & \alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix}, \tag{6.9}$$

so that q is given by the polynomial

$$q(x) = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j}^{n} \alpha_{ij} x_i x_j.$$

(Such triangular schemes have already been used in the literature when R is a ring, e.g. [28, I §2].) In the case that q is diagonal, i.e., all α_{ij} with $i < j$ are zero, we usually write instead of (6.8) the single row

$$q = [\alpha_1, \alpha_2, \ldots, \alpha_n]. \tag{6.10}$$

Analogously we use for a diagonal symmetric bilinear form b (i.e., $b(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$) the notation

$$b = \langle \beta_{11}, \beta_{22}, \ldots, \beta_{nn} \rangle. \tag{6.11}$$

We note that the quadratic form (6.9) has the balanced companion

$$b = \begin{pmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{12} & \alpha_2 & & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \cdots & & \alpha_n \end{pmatrix} \tag{6.12}$$

and (6.10), being diagonal, has the balanced companion

$$b = \langle 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n \rangle. \tag{6.13}$$

Example 6.7. *If $a_1, \ldots, a_n$, $c \in R$, then*

$$\langle a_1, \ldots a_n \rangle \otimes [c] = [a_1 c, \ldots, a_n c]. \tag{6.14}$$

This is evident from Proposition 6.6 and the rule $\langle a \rangle \otimes [c] = [ac]$ for one-dimensional forms which holds by (6.6). *In particular*

$$[a_1, \ldots, a_n] = \langle a_1, \ldots a_n \rangle \otimes [1]. \tag{6.15}$$

Example 6.8. *(As before, R is any semiring.) Assume that $V = (V, q)$ has dimension n, and take a base $\eta_1, \dots, \eta_n$ of V. Let*

$$(U, \gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with base $\varepsilon_1, \varepsilon_2$. We choose a balanced companion b of V, written as a symmetric $(n \times n)$-matrix $(b(\eta_i, \eta_j))$. We see by the use of the rules (6.5) *and* (6.6) *that*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_b q = \left[\begin{array}{c|c} 0 & b \\ \hline & 0 \end{array}\right] \tag{6.16}$$

written with respect to the base

$$\varepsilon_1 \otimes \eta_1, \dots, \varepsilon_1 \otimes \eta_n, \varepsilon_2 \otimes \eta_1, \dots, \varepsilon_2 \otimes \eta_n.$$

This example illustrates dramatically that in general *the tensor product of γ and q depends on the chosen balanced companion b of q*: tensoring q by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ produces the symmetric matrix of b.

Remark 6.9. *If γ_1 and γ_2 are bilinear forms on the same free R-module U, then the rules* (6.5) *and* (6.6) *imply for any $\lambda_1, \lambda_2 \in R$ that*

$$(\lambda_1\gamma_1 + \lambda_2\gamma_2) \otimes_b q = \lambda_1(\gamma_1 \otimes_b q) + \lambda_2(\gamma_2 \otimes_b q). \tag{6.17}$$

Example 6.10. *Using* (6.17) *with*

$$\gamma_1 = \langle a_1, a_2 \rangle, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda,$$

we obtain from Proposition 6.6 and Example 6.7 that

$$\begin{pmatrix} a_1 & \lambda \\ \lambda & a_2 \end{pmatrix} \otimes_b q = \left[\begin{array}{c|c} a_1 q & \lambda b \\ \hline & a_2 q \end{array}\right]. \tag{6.18}$$

Example 6.11. *Let*

$$q = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

with $a_{ij} \in R$ $(i < j)$. Then q is rigid (cf. [22, Proposition 3.4]*; no assumption on R is needed here). Furthermore, let*

$$\gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix}$$

with $\gamma_{ij} = \gamma_{ji} \in R$. Then we obtain by the rules (6.5) *and* (6.6) *that*

$$\gamma \otimes q = \begin{array}{|c|c|c|c|} \hline 0 & a_{12}\gamma & \cdots & a_{1n}\gamma \\ \hline & 0 & & a_{2n}\gamma \\ \hline & & \ddots & a_{n-1,n}\gamma \\ \hline & & & 0 \\ \hline \end{array} \tag{6.19}$$

More precisely, if the presentations of q *and* γ *above refer to ordered bases* $(\eta_1, \dots, \eta_n)$ *and* $(\varepsilon_1, \dots, \varepsilon_m)$*, respectively, then* (6.19) *refers to the ordered base*

$$(\varepsilon_1 \otimes \eta_1, \dots, \varepsilon_m \otimes \eta_1, \varepsilon_1 \otimes \eta_2, \dots, \varepsilon_m \otimes \eta_n).$$

We now consider the tensor product $\gamma \otimes [a] = \gamma \otimes_b [a]$, cf. Equation (6.10), where b is the unique balanced companion of $[a]$, (6.13). Our starting point is a definition which makes sense for any semiring R and any R-module U.

DEFINITION 6.12. *Let* $\gamma : U \times U \to R$ *be a symmetric bilinear form. The* norm form *of* γ *is the quadratic form* $n(\gamma) : U \to R$ *with*

$$n(\gamma)(x) := \gamma(x, x)$$

for any $x \in U$.

REMARK 6.13. *The norm form* $n(\gamma)$ *has the expansion* $\gamma : U \times U \to R$ *and the associated balanced companion* $\gamma + \gamma^{\mathrm{t}} = 2\gamma$*. The norm forms are precisely all the quadratic forms which admit a symmetric expansion. If* U *has a finite base* $\varepsilon_1, \dots, \varepsilon_n$*, then with respect to this base*

$$n(\gamma) = \begin{bmatrix} \gamma_{11} & 2\gamma_{12} & \cdots & 2\gamma_{1m} \\ & \gamma_{22} & & \\ & & \ddots & \vdots \\ & & & \gamma_{mm} \end{bmatrix}, \tag{6.20}$$

where $\gamma_{ij} := \gamma(\varepsilon_i, \varepsilon_j)$.

PROPOSITION 6.14. *Assume that* $U = (U, \gamma)$ *is a free bilinear* R*-module and* $a \in R$*. Then*

$$U \otimes [a] \cong (U,\ a\, n(\gamma)). \tag{6.21}$$

Proof. We realize the form $[a]$ as a quadratic module (V, q) with $V = R\eta$ free of rank 1 and $q(\eta) = a$. $\{q$ has the unique balanced companion $b : V \times V \to R$, with $b(\eta, \eta) = 2a.\}$ The form $\tilde{q} := \gamma \otimes q = \gamma \otimes_b q$ is given by

$$\tilde{q}(x \otimes \eta) = \gamma(x, x)a = (an(\gamma))(x).$$

The claim is obvious. $\square$

EXAMPLE 6.15. *Assume that* U *has base* $\varepsilon_1, \dots, \varepsilon_m$*. Let* $\gamma_{ij} := \gamma(\varepsilon_i, \varepsilon_j)$*. Then*

$$\gamma \otimes [a] \cong (a\gamma) \otimes [1],$$

and

$$\gamma \otimes [1] = \begin{bmatrix} \gamma_{11} & 2\gamma_{12} & \cdots & 2\gamma_{1n} \\ & \gamma_{22} & & \\ & & \ddots & \vdots \\ & & & \gamma_{mm} \end{bmatrix}, \tag{6.22}$$

where the right hand side refers to the base $\varepsilon_1 \otimes \eta$, $\varepsilon_2 \otimes \eta$, $\dots$, $\varepsilon_m \otimes \eta$.

At a crucial point in §7, we will need an explicit description of the tensor products $\gamma \otimes_b q$ with q indecomposable of rank 2. We start with a general fact.

Proposition 6.16. *Assume that γ is a symmetric bilinear form on a free R-module U and q_1, q_2 are quadratic forms on a free R-module V. Let b_1, b_2 be balanced companions of q_1 and q_2, respectively. Let $q := \lambda_1 q_1 + \lambda_2 q_2$ with $\lambda_1, \lambda_2 \in R$. Then $b := \lambda_1 b_1 + \lambda_2 b_2$ is a balanced companion of q, and*

$$\gamma \otimes_b q = \lambda_1(\gamma \otimes_{b_1} q_1) + \lambda_2(\gamma \otimes_{b_2} q_2). \tag{6.23}$$

This form has the balanced companion $\gamma \otimes b$ (as we know) and

$$\gamma \otimes b = \lambda_1(\gamma \otimes b_1) + \lambda_2(\gamma \otimes b_2). \tag{6.24}$$

Proof. An easy check by use of (6.5) and (6.6). □

Example 6.17. *We take a free module V with base η_1, η_2, and choose with respect to this base*

$$q_1 = \begin{bmatrix} a_1 & 0 \\ & a_2 \end{bmatrix} = [a_1, a_2], \qquad q_2 = \begin{bmatrix} 0 & c \\ & 0 \end{bmatrix}$$

with $a_1, a_2, c \in R$, $c \neq 0$, and the balanced companions

$$b_1 = \begin{pmatrix} 2a_1 & 0 \\ 0 & 2a_2 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.$$

Then

$$q := q_1 + q_2 = \begin{bmatrix} a_1 & c \\ & a_2 \end{bmatrix}$$

has the balanced companion

$$b := b_1 + b_2 = \begin{pmatrix} 2a_1 & c \\ c & 2a_2 \end{pmatrix}.$$

For

$$\gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix}$$

on a free module U with to the base $\varepsilon_1, \dots, \varepsilon_m$, we get

$$\gamma \otimes_{b_1} q_1 = \left[\begin{array}{c|c} a_1 n(\gamma) & 0 \\ \hline & a_2 n(\gamma) \end{array}\right], \qquad \gamma \otimes_{b_2} \begin{bmatrix} 0 & c \\ & 0 \end{bmatrix} = \left[\begin{array}{c|c} 0 & c\gamma \\ \hline & 0 \end{array}\right],$$

cf. (6.19), *and finally*

$$\gamma \otimes_b \begin{bmatrix} a_1 & c \\ & a_2 \end{bmatrix} = \left[\begin{array}{c|c} a_1 n(\gamma) & c\gamma \\ \hline & a_2 n(\gamma) \end{array}\right] \tag{6.25}$$

with respect to the base

$$\varepsilon_1 \otimes \eta_1, \dots, \varepsilon_m \otimes \eta_1, \varepsilon_1 \otimes \eta_2, \dots, \varepsilon_m \otimes \eta_2.$$

Remark 6.18. *¿From* (6.25) *and* (6.18)*, we obtain the useful formula*

$$\gamma \otimes_b \begin{bmatrix} a_1 & c \\ & a_2 \end{bmatrix} = \begin{pmatrix} a_1 & c \\ c & a_2 \end{pmatrix} \otimes_{2\gamma} n(\gamma)\ , \tag{6.26}$$

by use of Example 6.10 for the quadratic pair $(n(\gamma), 2\gamma)$.

From now on, we assume that V has unique base. {We do not need that U has unique base.}

Definition 6.19. *We call a companion b of q* faithful *if b is balanced and quasiminimal.*

Proposition 6.20. *Assume that b is a faithful companion of q, and that $V = W_1 \perp W_2$ is an orthogonal decomposition of V. Then, writing $U \otimes_b W_i$ instead of $U \otimes_{(b|W_i)} W_i$, we have*

$$U \otimes_b V = U \otimes_b W_1 \perp U \otimes_b W_2$$

for any bilinear R-module U.

Proof. $b(W_1, W_2) = 0$, since b is quasiminimal. It follows that

$$(\gamma \otimes b)(U \otimes W_1, U \otimes W_2) = 0.$$

Thus, $\tilde{q} = \gamma \otimes_b q$ is quasilinear on $(U \otimes W_1) \times (U \otimes W_2)$. □

Example 6.21. *Our assumption, that b is faithful, is necessary here. If $V = W_1 \perp W_2$, and b is balanced, but $b(W_1, W_2) \neq 0$, then*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_b V = \begin{bmatrix} 0 & b \\ & 0 \end{bmatrix}$$

is not the orthogonal sum of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_b W_1 \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_b W_2.$$

Example 6.22. *Let $q = [a_1, a_2, \dots, a_n]$ be a diagonal quadratic form. The diagonal symmetric bilinear form*

$$b := \langle 2q_1, \dots, 2a_n \rangle$$

is the unique faithful companion of q. For any bilinear R-module (U, γ), we have

$$\gamma \otimes_b q = \gamma \otimes [a_1] \perp \cdots \perp \gamma \otimes [a_n]. \tag{6.27}$$

Concerning the forms $\gamma \otimes [a_i]$, recall Proposition 6.14 and Example 6.15.

7. Indecomposability in tensor products

In this section, we assume for simplicity that $R \setminus \{0\}$ *is an entire semiring lacking zero sums*. So every free R-module has unique base (cf. Theorem 2.3), and R has no zero divisors. We discuss decomposability first in tensor products of (free) bilinear modules, later in tensor products of bilinear modules with quadratic modules.

Let $V_1 = (V_1, b_1)$ and $V_2 = (V_2, b_2)$ be indecomposable free (symmetric) bilinear modules over R, and let $V := V_1 \otimes V_2 = (V_1 \otimes V_2, b)$ with $b := b_1 \otimes b_2$. We take bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of the R-modules V_1, V_2 respectively and then have the base

$$\mathfrak{B} = \mathfrak{B}_1 \otimes \mathfrak{B}_2 := \{\varepsilon \otimes \eta \mid \varepsilon \in \mathfrak{B}_1, \eta \in \mathfrak{B}_2\}$$

of V. Our task is to determine the indecomposable components of V. First we discuss the "trivial" cases.

Remark 7.1. *Assume that V_1 has dimension (= rank) one, so $V_1 \cong \langle a \rangle$ with $a \in R$. If $a \neq 0$, then V is clearly indecomposable. If $a = 0$, then $b_1 \otimes b_2 = 0$, whence V is indecomposable only if also* $\dim V_2 = 1$. *Then $V = \langle 0 \rangle$.*

In all the following, we assume that $V_1 \neq \langle 0 \rangle$, $V_2 \neq \langle 0 \rangle$.
We resort to §4 to describe bases of the indecomposable components of $V = (V, b)$ as the classes in

$$\mathfrak{B} = \{\varepsilon \otimes \eta \mid \varepsilon \in \mathfrak{B}_1, \eta \in \mathfrak{B}_2\}$$

of an equivalence relation given by "paths", cf. Definition 4.7. So a path of length $r \geq 1$ in V, i.e., in $\mathfrak{B}$, is a sequence

$$\Gamma = (\varepsilon_0 \otimes \eta_0, \varepsilon_1 \otimes \eta_1, \dots, \varepsilon_r \otimes \eta_r) \tag{7.1}$$

with

$$b_1(\varepsilon_i, \varepsilon_{i+1}) b_2(\eta_i, \eta_{i+1}) \neq 0 \tag{7.2}$$

and

$$\varepsilon_i \neq \varepsilon_{i+1} \qquad \text{or} \qquad \eta_i \neq \eta_{i+1} \tag{7.3}$$

for $0 \leq i \leq r-1$.
Let us *first assume that both b_1 and b_2 are alternate*, whence also $b = b_1 \otimes b_2$ is alternate. Now condition (7.3) is a consequence of (7.2) and thus can be ignored. We read off from (7.2) that

$$\Gamma_1 = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r), \quad \Gamma_2 = (\eta_0, \eta_1, \dots, \eta_r) \tag{7.4}$$

are paths in V_1 and V_2 respectively of same length r. Conversely, given such paths Γ_1 and Γ_2, they combine to a path Γ of length r in V, as written in (7.1). {Here we use the assumption that R has no zero divisors.} We write

$$\Gamma = \Gamma_1 \otimes \Gamma_2. \tag{7.5}$$

We will speak of "cycles" in $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}$, in the following obvious way:

Definition 7.2. *Let $\mathfrak{C}$ be a base of a free bilinear R-module W.*

(a) *We denote the length of a path Γ in $\mathfrak{C}$ by $\ell(\Gamma)$.*
(b) *A* cycle *Δ in W with base point $\zeta \in \mathfrak{C}$ is a path $(\zeta_0, \zeta_1, \dots, \zeta_r)$ in $\mathfrak{C}$ with $\zeta_0 = \zeta_r = \zeta$. We say that the cycle Δ is* even *(resp.* odd*) if $\ell(\Delta)$ is* even *(resp.* odd*). We say that Δ is a* 2-cycle *if $\ell(\Delta) = 2$, whence $\Delta = (\zeta, \zeta', \zeta)$ with (ζ, ζ') an edge.*

Lemma 7.3. *Let $\varepsilon, \varepsilon' \in \mathfrak{B}_1$ and $\eta, \eta' \in \mathfrak{B}_2$. Let Γ_1 be a path from ε to ε' of length r and Γ_2 a path from η to η' of length s, and assume that $r \equiv s \pmod 2$. Then $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta'$.*

Proof. Assume, without loss of generality, that $s \geq r$, whence $s = r + 2t$ with $t \geq 0$. If $t = 0$, then $\Gamma_1 \otimes \Gamma_2$ is a path from $\varepsilon \otimes \eta$ to $\varepsilon' \otimes \eta'$ in V. If $t > 0$, we replace $\Gamma_1 = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r)$ by

$$\widetilde{\Gamma}_1 = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r, \varepsilon_{r-1}, \varepsilon_r, \ldots)$$

adjoining t copies of the 2-cycle $(\varepsilon_r, \varepsilon_{r-1}, \varepsilon_r)$ to Γ_1. Now $\widetilde{\Gamma}_1 \otimes \Gamma_2$ runs from $\varepsilon \otimes \eta$ to $\varepsilon' \otimes \eta'$. $\square$

Theorem 7.4. *Assume that both b_1 and b_2 are alternate (and $V_1 \neq \langle 0 \rangle$, $V_2 \neq \langle 0 \rangle$, as always).*

a) *If V_1 or V_2 contains an odd cycle, then $V_1 \otimes V_2$ is indecomposable.*
b) *Otherwise $V_1 \otimes V_2$ is the orthogonal sum of two indecomposable components.*

Proof. a): We assume that V_1 contains an odd cycle Δ with base point δ. Let $\varepsilon \otimes \eta$ and $\varepsilon' \otimes \eta'$ be different elements of $\mathfrak{B}$. We want to verify that $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta'$. We choose a path Γ_1 from ε to ε' in V_1 and a path Γ_2 from η to η' in V_2. If $\ell(\Gamma_1) \equiv \ell(\Gamma_2) \pmod 2$, then we know by Lemma 7.3 that $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta'$. Now assume that $\ell(\Gamma_1)$ and $\ell(\Gamma_2)$ have different parity. We choose a new path $\widetilde{\Gamma}_1$ from ε to ε' as follows: We first take a path H from ε to the base point δ of Δ, then we run through Δ, then we take the path inverse to H (in the obvious sense) from δ to ε, and finally we run through Γ_1. The length $\ell(\widetilde{\Gamma}_1)$ has different parity than $\ell(\Gamma_1)$ and thus the same parity as $\ell(\Gamma_2)$. We conclude again that $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta'$.

b): Now assume that both V_1 and V_2 contain only even cycles. This means that both in V_1 and V_2 all paths from a fixed start to a fixed end have length of the same parity. Given $\varepsilon \otimes \eta$ and $\varepsilon' \otimes \eta'$ in $\mathfrak{B}$, every path Γ from $\varepsilon \otimes \eta$ to $\varepsilon' \otimes \eta'$ has the shape $\Gamma_1 \otimes \Gamma_2$ with Γ_1 running from ε to ε', Γ_2 running from η to η', and $\ell(\Gamma_1) = \ell(\Gamma_2)$. Thus, if the paths from ε to ε' have length of different parity than those from η to η', then $\varepsilon \otimes \eta$ cannot be connected to $\varepsilon' \otimes \eta'$ by a path. But $\varepsilon \otimes \eta$ can be connected to $\varepsilon' \otimes \eta''$, where η'' arises from η' by adjoining an edge at the endpoint of η'. We fix some $\varepsilon_0 \in \mathfrak{B}_1$, and $\eta_0, \eta_1 \in \mathfrak{B}_2$ with $b_2(\eta_0, \eta_1) = 1$. Then every element of $\mathfrak{B}$ can be connected by a path to $\varepsilon_0 \otimes \eta_0$ or to $\varepsilon_0 \otimes \eta_1$, but not to both. V has exactly two indecomposable components. $\square$

Remark 7.5. *Assume again that b_1 and b_2 are alternate and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ both contain only even cycles. Let $\varepsilon, \varepsilon' \in \mathfrak{B}_1$ and $\eta, \eta' \in \mathfrak{B}_2$, and choose paths Γ_1 from ε to ε' and Γ_2 from η to η'. As the proof of Theorem 7.4.b has shown, $\varepsilon \otimes \eta$ and $\varepsilon' \otimes \eta'$ lie in the same indecomposable component of $V_1 \otimes V_2$ iff $\ell(\Gamma_1)$ and $\ell(\Gamma_2)$ have the same parity.*

There remains the case that b_1 or b_2 is not alternate.

Theorem 7.6. *Assume that b_1 is* not *alternate and – as before – that $V_1 = (V_1, b_1)$ and $V_2 = (V_2, b_2)$ are indecomposable. Then $(V_1 \otimes V_2, b_1 \otimes b_2)$ is indecomposable.*

Proof. Every path in $V := V_1 \otimes V_2$ with respect to $(b_1)_{\rm alt} \otimes (b_2)_{\rm alt}$ is also a path with respect to $b_1 \otimes b_2$, as is easily checked, and the paths in V_i with respect to b_i are the same as those with respect to $(b_i)_{\rm alt}$ $(i = 1, 2)$. Thus we are done by Theorem 7.4, except in the case that all cycles in V_1 and in V_2 are even. Then V has two indecomposable components W', W'' with respect to $(b_1)_{\rm alt} \otimes (b_2)_{\rm alt}$. The base

$$\mathfrak{B} = \mathfrak{B}_1 \otimes \mathfrak{B}_2 := (\varepsilon \otimes \eta \mid \varepsilon \in \mathfrak{B}_1, \eta \in \mathfrak{B}_2)$$

of $V_1 \otimes V_2$ is the disjoint union of sets $\mathfrak{B}'$, $\mathfrak{B}''$ which are bases of W' and W''. Any two elements of $\mathfrak{B}'$ are connected by a path with respect to $(b_1)_{\rm alt} \otimes (b_2)_{\rm alt}$, hence by a path with respect to $b_1 \otimes b_2$, and the same holds for the set $\mathfrak{B}''$.
We choose some $\rho \in \mathfrak{B}_1$ with $b_1(\rho, \rho) \neq 0$ and an edge (η_0, η_1) in $\mathfrak{B}''$. Since R has no zero divisors, it follows that $(\rho \otimes \eta_0, \rho \otimes \eta_1)$ is an edge in $\mathfrak{B}$ with respect to $b_1 \otimes b_2$. Perhaps interchanging W' and W'', we assume that $\rho \otimes \eta_0 \in \mathfrak{B}'$. Suppose that also $\rho \otimes \eta_1 \in \mathfrak{B}'$. Then there exists a path Γ in $\mathfrak{B}'$ with respect to $(b_1)_{\rm alt} \otimes (b_2)_{\rm alt}$ running from $\rho \otimes \eta_0$ to $\rho \otimes \eta_1$. Γ has the form $\Gamma_1 \otimes \Gamma_2$, with Γ_1 a cycle in V_1 with base point ρ, and Γ_2 a path in V_2 running from η_0 to η_1. We have $\ell(\Gamma_1) = \ell(\Gamma_2)$ and $\ell(\Gamma_2)$ is even. But there exists the path (η_0, η_1) from η_0 to η_1 of length 1. Since all paths in V_2 from η_0 to η_1 have the same parity, we infer that $\ell(\Gamma_2)$ is odd, a contradiction.
We conclude that $\rho \otimes \eta_1 \in \mathfrak{B}''$. The elements $\rho \otimes \eta_0 \in \mathfrak{B}'$ and $\rho \otimes \eta_1 \in \mathfrak{B}''$ are connected by a path with respect to $b_1 \otimes b_2$, and thus all elements of $\mathfrak{B}$ are connected by paths with respect to $b_1 \otimes b_2$. □

Turning to a study of indecomposable components of tensor products of bilinear and quadratic modules, we need some more terminology. Let $V = (V, q)$ be a free quadratic R-module and $\mathfrak{B}$ a base of V. We focus on balanced companions of q.

Definition 7.7.

(a) *We call a companion b of q* faithful *if b is* balanced and quasiminimal *(cf. §3 above), whence $b(\varepsilon, \varepsilon) = 2q(\varepsilon)$ for all $\varepsilon \in \mathfrak{B}$ and $b(\varepsilon, \eta) = 0$ for $\varepsilon \neq \eta$ in $\mathfrak{B}$ such that q is quasilinear on $R\varepsilon \times R\eta$.*

(b) *Given a balanced companion b of q, we define a new bilinear form b_f on V by the rule that, for $\varepsilon, \eta \in \mathfrak{B}$,*

$$b_f(\varepsilon, \eta) = \begin{cases} 0 & \text{if } \varepsilon \neq \eta \text{ and } q \text{ is quasilinear on } R\varepsilon \times R\eta, \\ b(\varepsilon, \eta) & \text{else.} \end{cases}$$

It is clear from [22, Theorem 6.3] that again b_f is a companion of q. By definition, this companion is quasiminimal. b_f is also balanced, since $b_f(\varepsilon, \varepsilon) = b(\varepsilon, \varepsilon) = 2q(\varepsilon)$ for all $\varepsilon \in \mathfrak{B}$, cf. [22, Proposition 1.7], and so b_f is faithful. We call b_f the *faithful companion of q associated to b.*

Theorem 7.8. *Assume that b is a balanced companion of q, and that W is a basic submodule of V. Then W is indecomposable with respect to q iff W is indecomposable with respect to b_f.*

Proof. This is a special case of Theorem 4.9, since $b_f|W = (b|W)_f$ is a quasi-minimal companion of $q|W$. $\square$

Definition 7.9.

(a) *We say that q is* diagonally zero *if $q(\varepsilon) = 0$ for every $\varepsilon \in \mathfrak{B}$.*

(b) *We say that q is* anisotropic *if $q(\varepsilon) \neq 0$ for every $\varepsilon \in \mathfrak{B}$.*

Remarks 7.10.

(i) *If q is diagonally zero, then q is rigid, cf.* [22, Proposition 3.4]. *Conversely, if q is rigid and the quadratic form* [1] *is quasilinear, i.e., $(\alpha+\beta)^2 = \alpha^2 + \beta^2$ for any $\alpha, \beta \in R$, then q is diagonally zero, as proved in* [22, Theorem 3.5].

(ii) *If q is anisotropic, then $q(x) \neq 0$ for every $x \in V \setminus \{0\}$. So our definition of anisotropy here coincides with the usual meaning of anisotropy for quadratic forms (which makes sense, say, for R a semiring without zero divisors and V any R-module).*

Definition 7.11. *In a similar vein, we call a symmetric* bilinear form *b on V* anisotropic *if $b(\varepsilon, \varepsilon) \neq 0$ for every $\varepsilon \in \mathfrak{B}$, and then have $b(x, x) \neq 0$ for every $x \in V \setminus \{0\}$.*

Note that, if b is a balanced companion of q, then b is anisotropic iff q is anisotropic.
Assume now that $U := (U, \gamma)$ is a free bilinear module, $V := (V, q)$ is a free quadratic module, and b is a balanced companion of q. Let

$$\widetilde{V} := (\widetilde{V}, \tilde{q}) := (U \otimes V, \gamma \otimes_b q).$$

We want to determine the indecomposable components of $\widetilde{V}$. Discarding trivial cases, we assume that $U \neq \langle 0 \rangle$, $V \neq [0]$.
We choose bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of the R-modules U and V, respectively, and introduce the subsets

$$\begin{aligned}
\mathfrak{B}_1^+ &:= \{\varepsilon \in \mathfrak{B}_1 \mid \gamma(\varepsilon, \varepsilon) \neq 0\},\\
\mathfrak{B}_1^0 &:= \{\varepsilon \in \mathfrak{B}_1 \mid \gamma(\varepsilon, \varepsilon) = 0\},\\
\mathfrak{B}_2^+ &:= \{\eta \in \mathfrak{B}_1 \mid q(\eta) \neq 0\},\\
\mathfrak{B}_2^0 &:= \{\eta \in \mathfrak{B}_1 \mid q(\eta) = 0\},
\end{aligned}$$

of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, and furthermore the basic submodules U^+, U^0, V^+, V^0 respectively spanned by these sets.

Lemma 7.12.

a) *If $\varepsilon \in \mathfrak{B}_1^+$, then the indecomposable components of the basic submodule $\varepsilon \otimes V := (R\varepsilon) \otimes V$ of $U \otimes V$ with respect to $\tilde{q}$ are the submodules $\varepsilon \otimes W$ with W running through the indecomposable components of V with respect to q.*

b) *If $\eta \in \mathfrak{B}_1^+$, then the indecomposable components of $U \otimes \eta := U \otimes (R\eta)$ with respect to $\tilde{q}$ are the modules $U \otimes \eta$ with U' running through the indecomposable components of U with respect to the norm form $n(\gamma)$ of γ (cf. Definition 6.12).*

Proof. This follows from the formulas $\tilde{q}(\varepsilon \otimes y) = \gamma(\varepsilon, \varepsilon)q(y)$ for $y \in V$ and $\tilde{q}(x \otimes \eta) = \gamma(x, x)q(\eta)$ for $x \in U$ (cf. (6.6)), since $\gamma(\varepsilon, \varepsilon) \neq 0$, $q(\eta) \neq 0$. □

In order to avoid certain pathologies concerning indecomposability in tensor products $U \otimes_b V$, we henceforth will assume that our semiring has the following property:

(NQL) For any a and c in $R \setminus \{0\}$ there exists some $\mu \in R$ with $a + \mu c \neq a$.

Clearly, this property means that every free quadratic module $\left[\begin{smallmatrix} a & c \\ & 0 \end{smallmatrix}\right]$ with $c \neq 0$ is *not quasilinear* on $(R\eta_1) \times (R\eta_2)$, where (η_1, η_2) is the associated base, whence the label "NQL".

Examples 7.13.

(a) *In the important case that R is supertropical the condition (*NQL*) holds iff all principal ideals in eR are unbounded with respect to the total ordering of eR. In particular, the "multiplicatively unbounded supertropical semirings" appearing in* [20, §7] *have* NQL.

(b) *If R is any entire semiring lacking zero sums, then the polynomial ring $R[t]$ in one variable (and so in any set of variables) has* NQL.

(c) *The polynomial function semirings over supersemirings appearing in* [25, §4] *have* NQL.

Lemma 7.14. *Assume that (V, q) is indecomposable. Let $a, c \in R \setminus \{0\}$. Then*

$$\begin{pmatrix} a & c \\ c & 0 \end{pmatrix} \otimes_b V = \left[\begin{array}{c|c} aq & cb \\ \hline & 0 \end{array}\right]$$

(cf. (6.19)*) is indecomposable.*

Proof. Let

$$(U, \gamma) = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix}$$

with respect to a base $\varepsilon_1, \varepsilon_2$ and assume for notational convenience that V has a finite base $\eta_1, \dots, \eta_n$. By Lemma 7.12.a, we have

$$\varepsilon_1 \otimes \eta_1 \sim \varepsilon_1 \otimes \eta_2 \sim \cdots \sim \varepsilon_1 \otimes \eta_n.$$

For given $\varepsilon_1 \otimes \eta_i$, $\varepsilon_2 \otimes \eta_j$ with $i \neq j$, $\gamma \otimes_b q$ has the value table

$$\begin{bmatrix} aq(\eta_i) & cb(\eta_i, \eta_j) \\ & 0 \end{bmatrix}.$$

Starting with $\varepsilon_2 \otimes \eta_j$, we find some η_i, $i \neq j$, with $b(\eta_i, \eta_j) \neq 0$, because (V, q) is indecomposable. Since R has NQL, it follows that $R(\varepsilon_i \otimes \eta_i) + R(\varepsilon_j \otimes \eta_j)$ is indecomposable with respect to $\tilde{q}$, whence $\varepsilon_1 \otimes \eta_i \sim \varepsilon_2 \otimes \eta_j$. Thus all $\varepsilon_k \otimes \eta_\ell$ are equivalent. □

Lemma 7.15. *Assume that $(U, n(\gamma))$ is indecomposable. Let $a, c \in R \setminus \{0\}$. Then the tensor product $U \otimes_b \begin{bmatrix} a & c \\ & 0 \end{bmatrix}$, taken with respect to $b = \begin{pmatrix} 2a & c \\ c & 0 \end{pmatrix}$, is indecomposable.*

Proof. By formula (6.26)

$$\gamma \otimes_b \begin{bmatrix} a & c \\ & 0 \end{bmatrix} = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix} \otimes_{2\gamma} n(\gamma).$$

Now Lemma 7.14 with $(V, q) := (U, n(\gamma))$ gives the claim. □

We are ready for the main result of this section. Recall that $U := (U, \gamma)$.

Theorem 7.16. *Assume that R has* NQL. *Assume furthermore that both $(U, n(\gamma))$ and the quadratic free module $V = (V, q)$ are indecomposable, and $U \neq \langle 0 \rangle$, $V \neq [0]$. Let b be a balanced companion of q. Then the quadratic module $U \otimes_b V := (U \otimes V, \gamma \otimes_b q)$ is indecomposable, except in the case that γ is alternate, q is diagonally zero, U and V contain only even cycles with respect to γ and b. Then $U \otimes_b V$ has exactly two indecomposable components, and these coincide with the indecomposable components of $U \otimes V$ with respect to $\gamma \otimes b$, and also with respect to $\gamma \otimes b_f$.*

Proof. Of course, indecomposability of $(U, n(\gamma))$ implies indecomposability of (U, γ). As before, let $\tilde{q} := \gamma \otimes_b q$. We distinguish three cases.

1) Assume that $V^+ \neq \{0\}$, i.e., there exist anisotropic base vectors in V. Our claim is that all elements of $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ are equivalent, whence $U \otimes_b V$ is indecomposable.
We choose $\eta_0 \in \mathfrak{B}_2^+$. By Lemma 7.12.b, the module

$$(U \otimes \eta_0, \tilde{q}) := (U \otimes \eta_0, \tilde{q} \mid U \otimes \eta_0)$$

is indecomposable, and thus all elements of $\mathfrak{B}_1 \otimes \eta_0$ are equivalent.
Let $\varepsilon \otimes \eta \in \mathfrak{B}_1 \otimes \mathfrak{B}_2$. We verify the equivalence of $\varepsilon \otimes \eta$ with some element of $\mathfrak{B}_1 \otimes \eta_0$, and then will be done. If $\gamma(\varepsilon, \varepsilon) \neq 0$, then by Lemma 7.12.a, all elements of $\varepsilon \otimes \mathfrak{B}_2$ are equivalent, whence $\varepsilon \otimes \eta \sim \varepsilon \otimes \eta_0$. Assume now that $\gamma(\varepsilon, \varepsilon) = 0$. Since (U, γ) is indecomposable, there exists some $\varepsilon' \in \mathfrak{B}_1$ with $c := \gamma(\varepsilon', \varepsilon) \neq 0$. Let $a := \gamma(\varepsilon', \varepsilon')$. We choose a base $\eta_1, \dots, \eta_n$ of V, assuming for notational convenience that V has finite rank. By Example 6.10,

$$(R\varepsilon' + R\varepsilon) \otimes_b V = \begin{bmatrix} aq & cq \\ & 0 \end{bmatrix}$$

with respect to the base $\varepsilon' \otimes \eta_1, \dots, \varepsilon' \otimes \eta_n, \varepsilon \otimes \eta_1, \dots, \varepsilon \otimes \eta_2$. Now Lemma 7.14 tells us that $(R\varepsilon' + R\varepsilon) \otimes_b V$ is indecomposable, whence all elements $\varepsilon \otimes \eta$, $\varepsilon' \otimes \eta'$ with $\eta, \eta' \in \mathfrak{B}_2$ are equivalent. In particular, $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta_0$.

2) Assume that $U^+ \neq \{0\}$, i.e., there exist an anisotropic base vector in U with respect to $n(\gamma)$. Our claim again is that all elements of $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ are equivalent, whence $U \otimes_b V$ is indecomposable. We choose $\varepsilon_0 \in \mathfrak{B}_1^+$, and then know by Lemma 7.12.a that all elements of $\varepsilon_0 \otimes \mathfrak{B}_2$ are equivalent.

Let $\varepsilon \otimes \eta \in \mathfrak{B}_1 \otimes \mathfrak{B}_2$ be given. We verify equivalence of $\varepsilon \otimes \eta$ with some element of $\varepsilon_0 \otimes \mathfrak{B}_2$, and then will be done. If $q(\eta) \neq 0$, then by Lemma 7.12.a all elements of $\mathfrak{B}_1 \otimes \eta$ are equivalent, and thus $\varepsilon \otimes \eta \sim \varepsilon_0 \otimes \eta$.
Hence, we may assume that $q(\eta) = 0$. Since (V, q) is indecomposable, there exists some $\eta' \in \mathfrak{B}_2$ with $c := b(\eta, \eta') \neq 0$. Let $a := q(\eta')$. Then

$$(R\eta' + R\eta, q) = \begin{bmatrix} a & c \\ & 0 \end{bmatrix}.$$

Let $b' := b|(R\eta' + R\eta) = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix}$. Then we see from (6.25) that

$$\gamma \otimes_{b'} \begin{bmatrix} a & c \\ & 0 \end{bmatrix} = \left[\begin{array}{c|c} an(\gamma) & c\gamma \\ \hline & 0 \end{array}\right].$$

By Lemma 7.15, this quadratic module is indecomposable, whence all elements $\varepsilon \otimes \eta$, $\varepsilon' \otimes \eta'$ with $\varepsilon, \varepsilon' \in \mathfrak{B}_1$ are equivalent. In particular, $\varepsilon \otimes \eta \sim \varepsilon_0 \otimes \eta'$.

3) The remaining case: $U = U^0$, and $V = V^0$, i.e., γ is alternate and q is diagonally zero. Now $(U \otimes V, \tilde{q})$ is rigid. By Theorem 7.8, the indecomposable components of $(U \otimes V, \tilde{q})$ coincide with those of $(U \otimes V, (\gamma \otimes b)_f)$. But $\tilde{q}$ has only one companion, whence $(\gamma \otimes b)_f = \gamma \otimes b = \gamma \otimes b_f$. Invoking Theorem 7.4, we see that the assertion of the theorem also holds in the case under consideration, where γ is alternate and b is diagonally zero. $\square$

In general, let $\{U_i \mid i \in I\}$ denote the set of indecomposable components of $(U, n(\gamma))$. Then

$$U \otimes_b V = \perp_{i \in I} U_i \otimes_b V$$

by Proposition 6.6, whence, applying Theorem 7.16 to each summand $U_i \otimes_b V$, we obtain a complete list of all indecomposable components of $U \otimes_b V$. In particular, if q is not diagonally zero, or if (V, b) contains an odd cycle, then the $U_i \otimes_b V$ themselves are the indecomposable components of $U \otimes_b V$.

References

[1] Z. Arad and H.I. Blau. On table algebras and applications to finite group theory, *J. of Alg.* 138:137–185, 1991.

[2] H.I. Blau. Table algebras, *European J. Combin.*, 30(6):1426–1455, 2009

[3] N. Bourbaki. *Alg. Comm. VI*, §3, No.1.

[4] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, Ergebnisse Math. Grenzgeb. 36, 1998.

[5] G.W. Brumfiel. *Partially ordered rings and semialgebraic geometry*, London Math. Soc. Lecture Notes 37, Cambridge Univ. Press, 1979.

[6] D. Dolžna and P. Oblak. Invertible and nilpotent matrices over antirings. *Lin. Alg. and Appl.*, 430(1):271–278, 2009.

[7] N. Durov. A new approach to Arakelov geometry, arXiv 0704.2030, 2007.

[8] R. Elman, N. Karpenko, and A. Merkurjev. *The Algebraic and Geometric Theory of Quadratic Forms*, Amer. Math. Soc. Colloquium Publications 56, Amer. Math. Soc., Providence, RI, 2008.

[9] J. Golan, *Semirings and their Applications*, Springer-Science + Business, Dordrecht, 1999. (Previously published by Kluwer Acad. Publ., 1999.)
[10] R. Harari. *Noncommutative Supertropical Semirings and Symbol Semialgebras*, Doctoral dissertation, Bar-Ilan University 2015.
[11] D.K. Harrison and M.A. Vitulli. V-valuations of a commutative ring I, *J. Algebra*, 126:264–292, 1989.
[12] D.K. Harrison and M.A. Vitulli. Complex-valued places and CMC subsets of a field, *Commun. in Algebra* 17:2529–2537, 1989.
[13] G. Hoheisel, Über Charaktere, *Monatsh. Math.* 48:448–456, 1939.
[14] R. Huber and M. Knebusch. On valuation spectra, *Contemp. Math.*, 155:167–206, 1994.
[15] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*, Oberwolfach Seminars, vol. 35, Birkhauser, 2007.
[16] Z. Izhakian. Tropical arithmetic and matrix algebra, *Commun. in Alg.*, 37(4):1445–1468, 2009.
[17] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical semirings and supervaluations. *J. of Pure and Appl. Alg.*, 215(10):2431–2463, 2011.
[18] Z. Izhakian, M. Knebusch, and L. Rowen. Dominance and transmissions in supertropical valuation theory, *Commun. in Algebra*, 41(7):2736–2782, 2013.
[19] Z. Izhakian, M. Knebusch, and L. Rowen. Monoid valuations and value ordered supervaluations, *Commun. in Alg.*, 43(8):3207-3248, 2015.
[20] Z. Izhakian, M. Knebusch, and L. Rowen. Minimal orderings and quadratic forms on a free module over a supertropical semiring, *Lin. Alg. and Appl.*, 507:420-461, 2016.
[21] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical linear algebra, *Pacific J. of Math.*, 266(1):43–75, 2013.
[22] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical quadratic forms I, *J. of Pure and Appl. Alg.*, 220(1):61–93, 2016. (Preprint at arXiv:1309.5729.v2, 2013.)
[23] Z. Izhakian, M. Knebusch, and L. Rowen. Layered tropical mathematics, *J. of Alg.*, 416:200–273, 2014.
[24] Z. Izhakian, M. Knebusch, and L. Rowen. *Supertropical Quadratic Forms II*, arXiv:1506.03404, 2015.
[25] Z. Izhakian and L. Rowen. Supertropical algebra. *Adv. in Math.*, 225(4):2222–2286, 2010.
[26] M. Knebusch. *Specialization of Quadratic and Symmetric Bilinear Forms*, Springer London, 2010.
[27] M. Knebusch and D. Zhang. *Manis Valuations and Prüfer Extensions. I. A New Chapter in Commutative Algebra*, Lecture Notes in Mathematics, 1791, Springer-Verlag, Berlin, 2002.
[28] M. Kneser. *Quadratische Formen*, Springer-Verlag, Berlin-Heidelberg, 2002.
[29] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, American Mathematical Society, 2015.

[30] Y. Tan. On invertible matrices over antirings. *Lin. Alg. and Appl.*, 423(2):428–444, 2007.
[31] K.G. Valente and M.A. Vitulli. Complex-valued places and CMC subsets of a ring, *Commun. in Algebra*, 18:3743–3757, 1990.
[32] E. Witt. Theorie der quadratischen Formen in beliebigen Körpen. *J. Reine Angew. Math.*, 176:31–44, 1937.
[33] D. Zhang. The m-valuation spectrum of a commutative ring, *Commun. in Algebra*, 30:2883–2896, 2002.

Zur Izhakian
Institute of Mathematics
University
of Aberdeen
AB24 3UE
Aberdeen
UK
zzur@abdn.ac.uk;
zzur@math.biu.ac.il

Manfred Knebusch
Department of Mathematics
NWF-I Mathematik
Universität Regensburg
93040 Regensburg
Germany
manfred.knebusch@
mathematik.uni-
regensburg.de

Louis Rowen
Department of Mathematics
Bar-Ilan
University
Ramat-Gan 52900
Israel
rowen@math.biu.ac.il