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Rings containing a field of characteristic zero

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Abstract. Let K be a field of characteristic zero, and let R be a ring containing K. Then either $R^{\times} = K^{\times}$ or K^{\times} is a subgroup of infinite index in R^{\times} .

This note was prompted by recent email exchanges with various colleagues. We use the term "ring" to mean a ring with identity, not necessarily commutative. For a ring R, denote by R^{\times} the multiplicative group of invertible elements of R.

Theorem. Let K be a field of characteristic zero, and let R be a ring containing K. Then either $R^{\times} = K^{\times}$ or K^{\times} is a subgroup of infinite index in R^{\times} .

Proof. Suppose, to the contrary, that $|R^{\times}:K^{\times}|=m>1$. Letting d=m!, there is a proper normal subgroup of index dividing d in R^{\times} , and contained in K^{\times} . Choose $s\in R^{\times}$, $s\not\in K^{\times}$. Then s^d is some element of K^{\times} . Let $n\in\mathbb{Z}\subseteq K$ with $n^d\neq s^d$. Then

$$(s-n)(s^{d-1} + ns^{d-2} + \dots + n^{d-1}) = s^d - n^d$$

is in K^{\times} , and hence s-n has a multiplicative inverse. So $s-n \in R^{\times}$, and hence $(s-n)^d \in K^{\times}$.

Choose d+1 different values of n with $n^d \neq s^d$, say n_0, \ldots, n_d . Then we have $(s-n_i)^d \in K^{\times}$ for $i=0,\ldots,d$. Expanding out these equations, and using the fact that we are in characteristic zero, the non-vanishing of the Vandermonde determinant then implies that $1, s, \ldots, s^d$ are all in K. But s was chosen not to be in K. This contradiction proves the theorem.



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