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Rings containing a field of characteristic zero

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Abstract. Let K be a field of characteristic zero, and let R be a ring containing K. Then either $R^\times = K^\times$ or $K^\times$ is a subgroup of infinite index in $R^\times$.

This note was prompted by recent email exchanges with various colleagues. We use the term “ring” to mean a ring with identity, not necessarily commutative. For a ring R, denote by $R^\times$ the multiplicative group of invertible elements of R.

Theorem. *Let K be a field of characteristic zero, and let R be a ring containing K. Then either $R^\times = K^\times$ or $K^\times$ is a subgroup of infinite index in $R^\times$.*

Proof. Suppose, to the contrary, that $|R^\times : K^\times| = m > 1$. Letting $d = m!$, there is a proper normal subgroup of index dividing d in $R^\times$, and contained in $K^\times$. Choose $s \in R^\times$, $s \notin K^\times$. Then s^d is some element of $K^\times$. Let $n \in \mathbb{Z} \subseteq K$ with $n^d \neq s^d$. Then

$$(s-n)(s^{d-1} + ns^{d-2} + \cdots + n^{d-1}) = s^d - n^d$$

is in $K^\times$, and hence $s - n$ has a multiplicative inverse. So $s - n \in R^\times$, and hence $(s-n)^d \in K^\times$.

Choose $d+1$ different values of n with $n^d \neq s^d$, say $n_0, \ldots, n_d$. Then we have $(s - n_i)^d \in K^\times$ for $i = 0, \ldots, d$. Expanding out these equations, and using the fact that we are in characteristic zero, the non-vanishing of the Vandermonde determinant then implies that $1, s, \ldots, s^d$ are all in K. But s was chosen not to be in K. This contradiction proves the theorem. □

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