Elastodynamic contact problem for coaxial penny-shaped cracks

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Abstract

The present paper is devoted to the solution of the three-dimensional fracture mechanics problem for linearly elastic, homogeneous and isotropic solid with two coaxial penny-shaped cracks under normally incident harmonic tension-compression wave with allowance for the contact interaction of cracks' adjoining edges.

The considered nonlinear problem with the Signorini unilateral constraints is solved by the method of boundary integral equations using the iteration algorithm, which is based on the variational principles of the elastodynamics. The dependence of the normalized stress intensity factor (opening mode) versus the mutual arrangement of cracks is studied for different wave numbers.

The present numerical results are compared with corresponding those obtained without allowance for the contact interaction of cracks' edges.

1 Statement of the problem

In the present paper we consider two coaxial penny-shaped cracks without any initial opening, which are contained in unbounded, isotropic, homogeneous and linearly elastic solid. The cracks are described by the middle surfaces

$$\Omega_1 = \{0 \le x_1 \le a\cos\beta,\ \ 0 \le x_2 \le a\sin\beta,\ \ \beta \in [0, 2\pi),\ \ x_3 = 0\},$$

$$\Omega_2 = \{0 \le x_1 \le a\cos\beta,\ \ 0 \le x_2 \le a\sin\beta,\ \ \beta \in [0, 2\pi),\ \ x_3 = c\}.$$

The solid is subjected to time-harmonic motion, the normally incident tension-compression wave is defined by

$$\Phi(\mathbf{x}, t) = \Phi_0 e^{i(k_1 x_3 - \omega t)},$$

where Φ_0 is the amplitude; $\omega = 2\pi/T$ is the angular frequency, T is the period of oscillation; $k_1 = \omega/c_1$ is the wave number; $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the longitudinal wave; λ and μ are the Lamé elastic constants, ρ is the mass density.

Under time-harmonic loading the adjoining cracks' edges come into contact. The nonvanishing contact forces arise in the time-dependent contact domain $\Omega_{cont}(t)$, which is unknown in advance. Thus the traction on the edges of the cracks has the form [1, 2]

$$\mathbf{p}(\mathbf{x},t) = \mathbf{p}^*(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t), \;\; \mathbf{x} \in \Omega := \Omega_1 \cup \Omega_2, \; t \in \mathcal{T} := [0;T],$$

where the load $\mathbf{p}^*(\mathbf{x},t)$, which is caused by the incident wave, is given by

$$\mathbf{p}^*(\mathbf{x},t) = \Big(0, 0, -k_1^2\Phi_0\big(\cos(k_1x_3)\cos(\omega t) + \sin(k_1x_3)\sin(\omega t)\big)\Big), \;\; \mathbf{x} \in \Omega, \; t \in \mathcal{T},$$

and $\mathbf{q}(\mathbf{x},t)$ is the vector of contact forces.

According to the normal incidence of the tension-compression wave, to solve the considered problem we have to define only the normal components of all mentioned vectors, because the tangential components are absent.

We impose the Signorini unilateral constraints for the normal components of the contact forces and the displacement discontinuity of the adjoining cracks' edges

$$[u_3(\mathbf{x},t)] \geq 0, \quad q_3(\mathbf{x},t) \geq 0, \quad [u_3(\mathbf{x},t)]q_3(\mathbf{x},t) = 0, \;\; \mathbf{x} \in \Omega, \; t \in \mathcal{T}. \tag{1}$$

2 Boundary integral equations

According the approach proposed in [1, 2], we expand the normal components of the displacement discontinuity and traction vectors into the trigonometric Fourier series

$$p_3(\mathbf{x},t) = \frac{p^0_{3,cos}(\mathbf{x})}{2} + \sum_{k=1}^{+\infty}\left(p^k_{3,cos}(\mathbf{x})\cos(\omega_k t) + p^k_{3,sin}(\mathbf{x})\sin(\omega_k t)\right),$$

$$[u_3(\mathbf{x},t)] = \frac{[u^0_{3,cos}(\mathbf{x})]}{2} + \sum_{k=1}^{+\infty}\left([u^k_{3,cos}(\mathbf{x})]\cos(\omega_k t) + [u^k_{3,sin}(\mathbf{x})]\sin(\omega_k t)\right),$$

where $\omega_k = 2\pi k/T$ and

$$p^k_{3,cos}(\mathbf{x}) = \frac{\omega}{\pi}\int\limits_0^T p_3(\mathbf{x},t)\cos(\omega_k t)dt, \quad p^k_{3,sin}(\mathbf{x}) = \frac{\omega}{\pi}\int\limits_0^T p_3(\mathbf{x},t)\sin(\omega_k t)dt,$$

$$[u^k_{3,cos}(\mathbf{x})] = \frac{\omega}{\pi}\int\limits_0^T [u_3(\mathbf{x},t)]\cos(\omega_k t)dt, \quad [u^k_{3,sin}(\mathbf{x})] = \frac{\omega}{\pi}\int\limits_0^T [u_3(\mathbf{x},t)]\sin(\omega_k t)dt.$$

The Fourier coefficients are related (see [1-7]) by the following complex-valued boundary integral system for $k = \overline{0, +\infty}$

$$p^k_{3,cos}(\mathbf{x}) - ip^k_{3,sin}(\mathbf{x}) =$$

$$= -\int\limits_{\Omega}\left(F^{Re}_{33}(\mathbf{x},\mathbf{y},\omega_k) + iF^{Im}_{33}(\mathbf{x},\mathbf{y},\omega_k)\right)\left([u^k_{3,cos}(\mathbf{y})] - i[u^k_{3,sin}(\mathbf{y})]\right)d\mathbf{y}, \tag{2}$$

where $F_{33}^{Re}(\mathbf{x},\mathbf{y},\omega_k)$ and $F_{33}^{Im}(\mathbf{x},\mathbf{y},\omega_k)$ are the real and imaginary components of the integral kernel $F_{33}(\mathbf{x},\mathbf{y},\omega_k)$, which can be obtained from the Green displacement tensor. The kernel has in the case on study the following form (see [3-7])

$$F_{33}(\mathbf{x},\mathbf{y},\omega_k)=\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}r^{-3}+\frac{\omega_k^2}{8\pi\mu}\left(\frac{\mu^2}{c_2^2}+\left(2\lambda^2+4\lambda\mu+3\mu^2\right)\frac{c_2^2}{c_1^4}\right)r^{-1}-$$

$$\sum_{n=4}^{+\infty}\frac{(-i\omega_k)^n(n-1)}{4\pi\mu n!(n+2)}\left(\frac{4\mu^2(n-1)}{c_2^n}+\left[\lambda^2 n(n+2)+4\lambda\mu(n+2)+12\mu^2\right]\frac{c_2^2}{c_1^{2+n}}\right)r^{n-3},$$

where $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$ is the distance between the observation point and the load point, $c_2=\sqrt{\mu/\rho}$ is the velocity of the transverse wave.

3 Numerical solution procedure

To solve the systems of boundary integral equations (2), we approximate the surface of cracks Ω by a set of plain polygonal elements $\Omega_j^h, j=\overline{1,N}$. We use the constant approximation within an element and the collocation method with collocation points at the centroid of the elements [8]. Then we obtain for all $k=\overline{0,+\infty}$ the following approximate system of complex-valued equations

$$p_{3,cos}^k(\mathbf{x}_j)-ip_{3,sin}^k(\mathbf{x}_j)=$$

$$=-\sum_{l=1}^{N}\int\limits_{\Omega_l^h}\left(F_{33}^{Re}(\mathbf{x}_j,\mathbf{y},\omega_k)+iF_{33}^{Im}(\mathbf{x}_j,\mathbf{y},\omega_k)\right)d\mathbf{y}\left([u_{3,cos}^k(\mathbf{y}_l)]-i[u_{3,sin}^k(\mathbf{y}_l)]\right), \quad (3)$$

where points $\mathbf{x}_j$ and $\mathbf{y}_j$ are located in the centroid of the element Ω_j^h.

Rewrite system (3) in the matrix form

$$\mathbf{F}_n^k\mathbf{U}_n^k=\mathbf{P}_n^k, \quad k=\overline{0,+\infty}, \quad (4)$$

where

$$\mathbf{F}_n^k=\begin{bmatrix}-\mathbf{F}_{33}^{k,Re} & -\mathbf{F}_{33}^{k,Im}\\ \mathbf{F}_{33}^{k,Im} & -\mathbf{F}_{33}^{k,Re}\end{bmatrix}, \quad \mathbf{U}_n^k=\begin{bmatrix}\mathbf{U}_{3,cos}^k\\ \mathbf{U}_{3,sin}^k\end{bmatrix}, \quad \mathbf{P}_n^k=\begin{bmatrix}\mathbf{P}_{3,cos}^k\\ \mathbf{P}_{3,sin}^k\end{bmatrix}$$

and

$$\mathbf{F}_{33}^{k,Re}=\begin{bmatrix}\int\limits_{\Omega_1^h}F_{33}^{Re}(\mathbf{x}_1,\mathbf{y},\omega_k)d\mathbf{y} & \int\limits_{\Omega_2^h}F_{33}^{Re}(\mathbf{x}_1,\mathbf{y},\omega_k)d\mathbf{y} & \dots & \int\limits_{\Omega_N^h}F_{33}^{Re}(\mathbf{x}_1,\mathbf{y},\omega_k)d\mathbf{y}\\ \int\limits_{\Omega_1^h}F_{33}^{Re}(\mathbf{x}_2,\mathbf{y},\omega_k)d\mathbf{y} & \int\limits_{\Omega_2^h}F_{33}^{Re}(\mathbf{x}_2,\mathbf{y},\omega_k)d\mathbf{y} & \dots & \int\limits_{\Omega_N^h}F_{33}^{Re}(\mathbf{x}_2,\mathbf{y},\omega_k)d\mathbf{y}\\ \vdots & \vdots & \ddots & \vdots\\ \int\limits_{\Omega_1^h}F_{33}^{Re}(\mathbf{x}_N,\mathbf{y},\omega_k)d\mathbf{y} & \int\limits_{\Omega_2^h}F_{33}^{Re}(\mathbf{x}_N,\mathbf{y},\omega_k)d\mathbf{y} & \dots & \int\limits_{\Omega_N^h}F_{33}^{Re}(\mathbf{x}_N,\mathbf{y},\omega_k)d\mathbf{y}\end{bmatrix},$$

$$\mathbf{F}_{33}^{k,Im} = \begin{bmatrix} \int\limits_{\Omega_1^h} F_{33}^{Im}(\mathbf{x}_1, \mathbf{y}, \omega_k) d\mathbf{y} & \int\limits_{\Omega_2^h} F_{33}^{Im}(\mathbf{x}_1, \mathbf{y}, \omega_k) d\mathbf{y} & \dots & \int\limits_{\Omega_N^h} F_{33}^{Im}(\mathbf{x}_1, \mathbf{y}, \omega_k) d\mathbf{y} \\ \int\limits_{\Omega_1^h} F_{33}^{Im}(\mathbf{x}_2, \mathbf{y}, \omega_k) d\mathbf{y} & \int\limits_{\Omega_2^h} F_{33}^{Im}(\mathbf{x}_2, \mathbf{y}, \omega_k) d\mathbf{y} & \dots & \int\limits_{\Omega_N^h} F_{33}^{Im}(\mathbf{x}_2, \mathbf{y}, \omega_k) d\mathbf{y} \\ \vdots & \vdots & \ddots & \vdots \\ \int\limits_{\Omega_1^h} F_{33}^{Im}(\mathbf{x}_N, \mathbf{y}, \omega_k) d\mathbf{y} & \int\limits_{\Omega_2^h} F_{33}^{Im}(\mathbf{x}_N, \mathbf{y}, \omega_k) d\mathbf{y} & \dots & \int\limits_{\Omega_N^h} F_{33}^{Im}(\mathbf{x}_N, \mathbf{y}, \omega_k) d\mathbf{y} \end{bmatrix},$$

$$\mathbf{U}_{3,cos}^k = \begin{bmatrix} [u_{3,cos}^k(\mathbf{y}_1)] \\ [u_{3,cos}^k(\mathbf{y}_2)] \\ \vdots \\ [u_{3,cos}^k(\mathbf{y}_N)] \end{bmatrix}, \quad \mathbf{U}_{3,sin}^k = \begin{bmatrix} [u_{3,sin}^k(\mathbf{y}_1)] \\ [u_{3,sin}^k(\mathbf{y}_2)] \\ \vdots \\ [u_{3,sin}^k(\mathbf{y}_N)] \end{bmatrix},$$

$$\mathbf{P}_{3,cos}^k = \begin{bmatrix} p_{3,cos}^k(\mathbf{x}_1) \\ p_{3,cos}^k(\mathbf{x}_2) \\ \vdots \\ p_{3,cos}^k(\mathbf{x}_N) \end{bmatrix}, \quad \mathbf{P}_{3,sin}^k = \begin{bmatrix} p_{3,sin}^k(\mathbf{x}_1) \\ p_{3,sin}^k(\mathbf{x}_2) \\ \vdots \\ p_{3,sin}^k(\mathbf{x}_N) \end{bmatrix}.$$

Observe that boundary integral equations (2) are hypersingular, since the integral kernel $F_{33}(\mathbf{x}, \mathbf{y}, \omega_k)$ behaves as r^{-3} when $\mathbf{x}$ tends to $\mathbf{y}$. Thus we have to consider the mentioned integrals in the sense of the Hadamard finite part. To calculate the coefficients of system (4) it is necessary to evaluate the following weakly singular and hypersingular integrals

$$J_1^{0,0}(\mathbf{x}, \Omega_j^h) = \int\limits_{\Omega_j^h} \frac{d\mathbf{y}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}\,, \tag{5}$$

$$J_3^{0,0}(\mathbf{x}, \Omega_j^h) = \int\limits_{\Omega_j^h} \frac{d\mathbf{y}}{\left(\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\right)^3}. \tag{6}$$

In [7, 9] it has been shown that the second Green theorem can be used for regularization of the divergent integrals. Following this approach, the divergent integrals (5) and (6) can be converted to regular curvilinear integrals of the first kind

$$J_1^{0,0}(\mathbf{x}, \Omega_j^h) = -\int\limits_{\partial\Omega_j^h} \frac{(x_1 - y_1)n_1(\mathbf{y}) + (x_2 - y_2)n_2(\mathbf{y})}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}} d\mathbf{y},$$

$$J_3^{0,0}(\mathbf{x}, \Omega_j^h) = \int\limits_{\partial\Omega_j^h} \frac{(x_1 - y_1)n_1(\mathbf{y}) + (x_2 - y_2)n_2(\mathbf{y})}{\left(\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\right)^3} d\mathbf{y},$$

where $\partial\Omega_j^h$ is the frontier of the element Ω_j^h.

In the case of polygonal boundary elements, these regular integrals can be easily evaluate by using following indefinite integrals

$$I_{1,0} = \int \frac{d\xi}{\sqrt{A\xi^2 + B\xi + C}} = \frac{1}{\sqrt{A}} Arsh \frac{2A\xi + B}{\sqrt{4AC - B^2}},$$

$$I_{3,0} = \int \frac{d\xi}{\left(\sqrt{A\xi^2 + B\xi + C}\right)^3} = \frac{2(2A\xi + B)}{(4AC - B^2)\sqrt{A\xi^2 + B\xi + C}}.$$

Finally, in order to determine the normal components of the contact forces and displacement discontinuity, that satisfy the constraints (1), we use the iteration algorithm, which is similar to the one introduced in [6, 10, 11].

4 Numerical examples

As a numerical example we consider the cracked solid with the Young elastic modulus $E = 200$ GPa, the Poisson ratio $\upsilon = 0.25$, the mass density $\rho = 7800 kg/m^3$.

Information on the distribution of the normal components of the displacement discontinuity and the contact forces vectors on the central sections of the cracks

$$S_1 = \{\mathbf{x} : -a \le x_1 \le a, x_2 = 0, x_3 = 0\},$$

$$S_2 = \{\mathbf{x} : -a \le x_1 \le a, x_2 = 0, x_3 = c\}$$

is presented in Figure 1 for the dimensionless wave number $k_2 a = 1.0$ and $c/a = 2$. Note that the Signorini constraints (1) clearly hold.

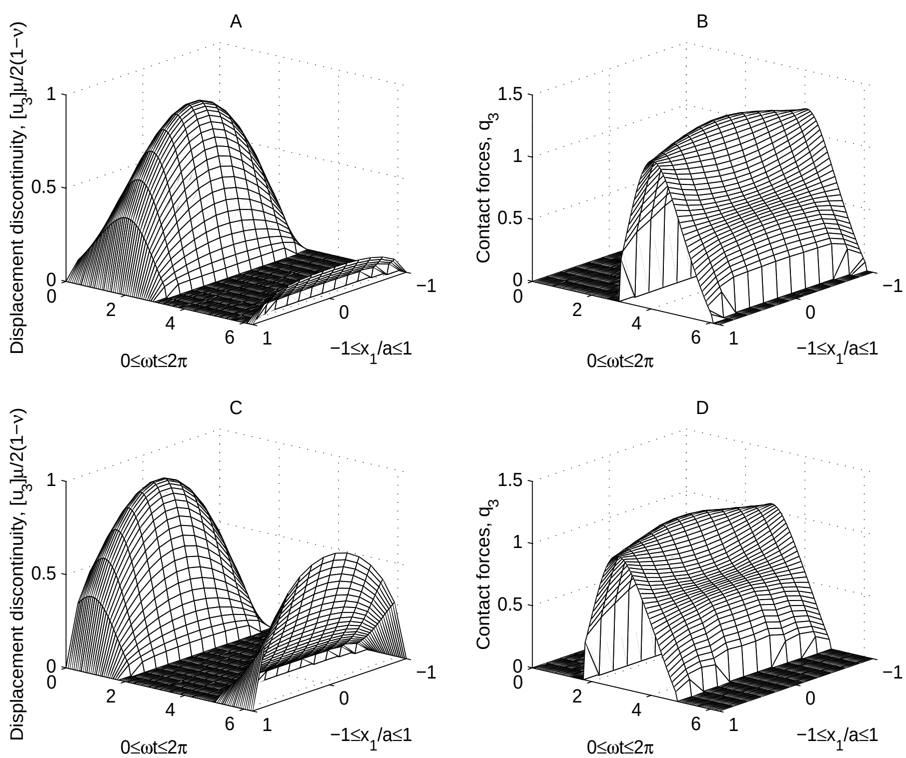

Figure 1. Normal displacement discontinuity and contact forces, A and B - the second crack, C and D - the first crack

Once the normal component of the displacement discontinuity of the cracks' edges has been calculated, the elastodynamic stress intensity factor (opening mode), which describes the remaining strength of the cracked body, can be evaluate by using the relation [12]

$$K_I(\mathbf{x},t) = \frac{\mu\sqrt{2\pi}}{4(1-\nu)} \lim_{\delta\to 0} \frac{[u_3(\mathbf{x},t)]}{\sqrt{\delta}},$$

where δ is the shortest distance from point $\mathbf{x}$ to the crack's front.

In solving the problem numerically, we use the value $[u_3(\mathbf{x},t)]$ obtained on the first element from the crack's front.

The dependence of the normalized stress intensity factor $|K_I^{max}(\mathbf{x})/\tilde{K}_I^{stat}|$ versus the dimensionless distance between cracks is presented in Figures 2 and 3 for two different dimensionless wave numbers. Here,

$$K_I^{max}(\mathbf{x}) = \max_{t\in\mathcal{T}} K_I(\mathbf{x},t),$$

and the static value for the unique penny-shaped crack is $\tilde{K}_I^{stat} = 2\sigma\sqrt{a/\pi}$.

In case of a penny-shaped crack and a normal loading, the maximal stress intensity factor $K_I^{max}(\mathbf{x})$ does not vary along the the crack's border. So, the present results are obtained at the points $(a,0,0)$ and $(a,0,c)$ for the first and the second cracks respectively.

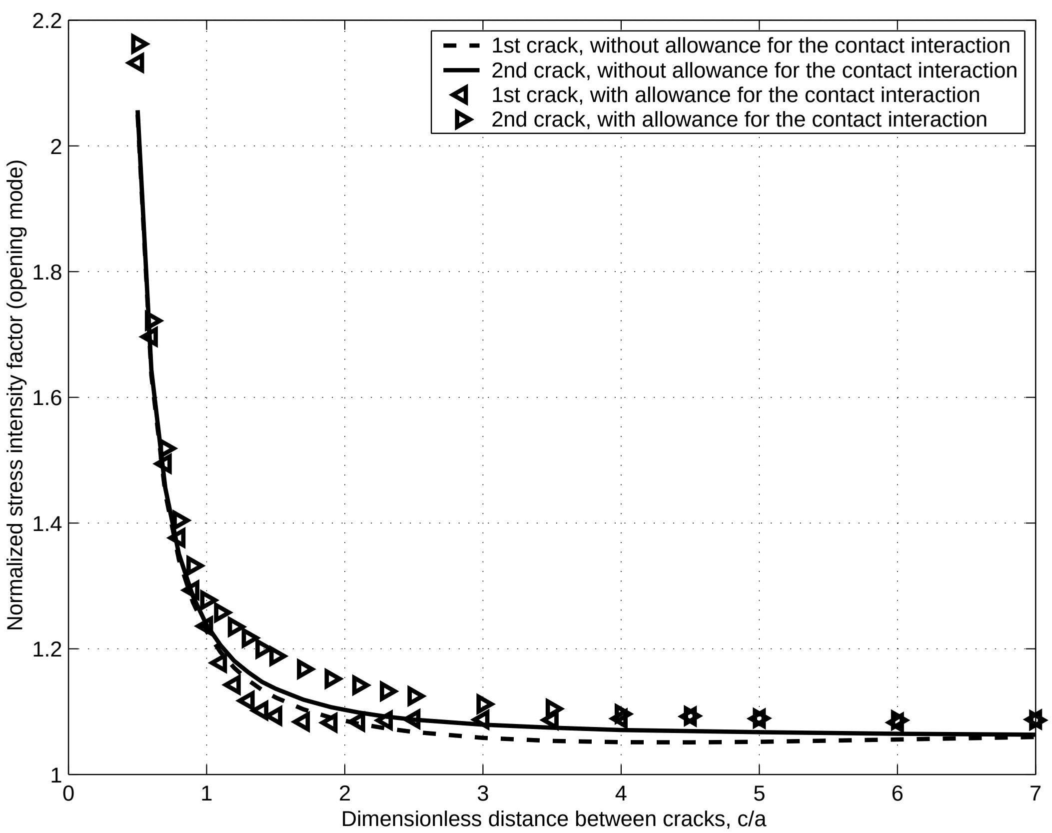

Figure 2. $|K_I^{max}/\tilde{K}_I^{stat}|$ versus c/a, the wave number $k_2a = 0.5$

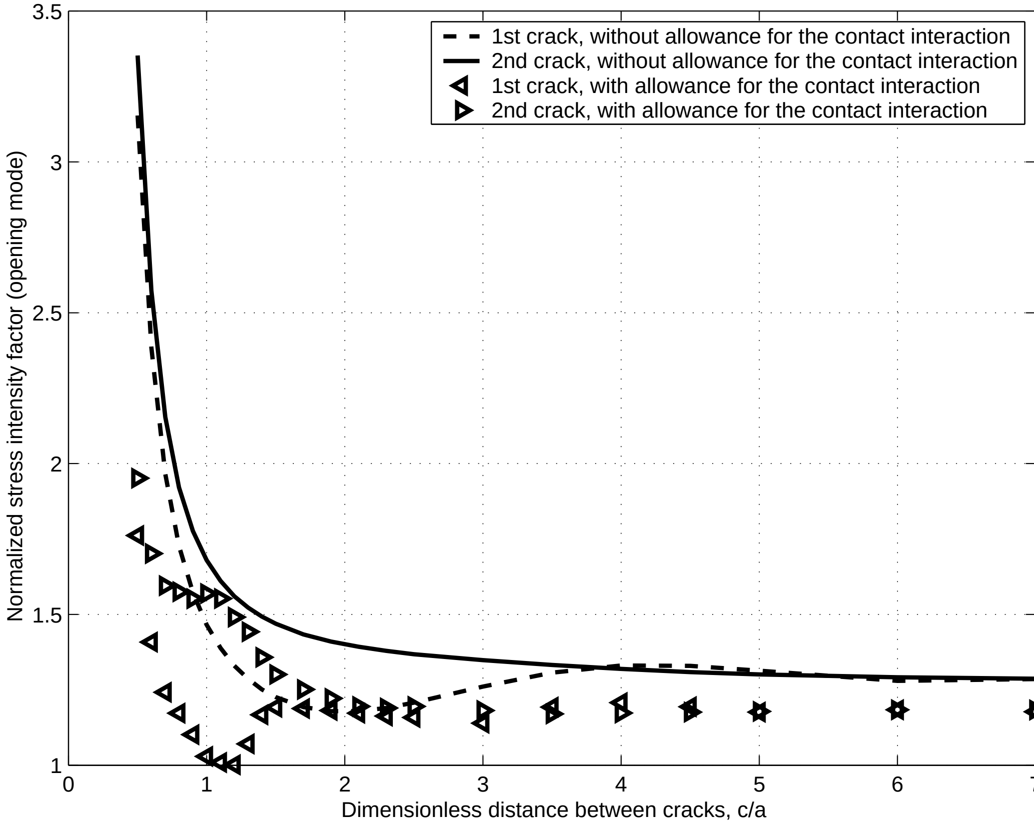

Figure 3. $|K_I^{max}/\tilde{K}_I^{stat}|$ versus c/a, the wave number $k_2a = 1.0$

In the covered frequency band the mutual influence of the cracks gets stronger as cracks approach each other, with allowance for the contact interaction as well as without allowance (for instance, see [1, 2, 12, 13]). On the contrary, for long distances the stress-strain state in the vicinity of both cracks tends to the stress-strain state for the solid with the unique penny-shaped crack.

The contact interaction of the cracks' adjoining edges considerably changes the solution. The distribution of the components of the stress-strain state becomes more complicated as the wave number increases. For relatively small distances and certain wave numbers the difference between results, which are obtained with allowance for the contact interaction and without, can achieve 50 percents.

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