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# Discussion Papers in Economics and Finance

## How to Measure Financial Market Efficiency?

A Multifractality-Based Quantitative Approach with an Application to the European Carbon Market

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## Abstract

This paper proposes a new measure for the evaluation of financial market efficiency, the so-called *intermittency coefficient*. This is a multifractality measure that can quantify the deviation from a random walk within the framework of the multifractal random walk model by Bacry et al. (2001b). While the random walk corresponds to the most genuine form of market efficiency, the larger the value of the intermittency coefficient is, the more inefficient a market would be. In contrast to commonly used methods based on Hurst exponents, the intermittency coefficient is a more powerful tool due to its well-established inference apparatus based on the generalised method of moments estimation technique. In an empirical application using data from the largest currently existing market for tradable pollution permits, the European Union Emissions Trading Scheme, we show that this market becomes more efficient over time. In addition, the degree of market efficiency is overall similar to that for the US stock market; for one sub-period, the market efficiency is found to be higher. While the first finding is anticipated, the second finding is noteworthy, as various observers expressed concerns with regard to the information efficiency of this newly established artificial market.

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# 1   Introduction

Ever since Maurice Kendall presented, back in 1953, his finding that economic time series appear to "be wandering" rather than show some predictable pattern (see Kendall (1953)), attempts have been made to either make sense of this feature or exploit it. The most famous attempt is certainly the establishment of a link between a virtually unpredictable random walk - the precise way of describing the feature of "wandering" - and the notion of efficient markets. Even though empirical tests for market efficiency based on this notion appear to be straightforward, these procedures are more complex than they might appear. First, testing for efficiency necessarily assumes some equilibrium model that defines normal behaviour. Second, the alternative hypothesis is vague; the mere finding that a particular market is inefficient does not reveal much about this market (see Fama (1998)). Third, due to the non-availability of corresponding empirical techniques, the vast majority of papers that study financial market efficiency employ qualitative measures of market efficiency; the most common approach is the variance ratio test originally developed by Lo and MacKinlay (1988). In addition, basic procedures, such as unit root tests and serial correlation tests, are also used. These methods only allow one to test whether a certain market is efficient or not efficient in a certain period.

Instead, Campbell et al. (1997) proposed the integrative use of different efficiency concepts or degrees (stages) of market efficiency. The benefit of this approach is that it allows one to compare financial markets with regard to their relative efficiency. The price of a security traded on an efficient financial market incorporates at all times all the relevant information relating to it; no information is disregarded on the market (Fama, 1965). At the heart of this definition is the notion that price movements in efficient markets are random and unpredictable, making it impossible to predict prices (or, equivalently, price changes) due to an information advantage. Once properly adjusted for risk, successive price changes are uncorrelated; however, they can satisfy certain nonlinear patterns of dependency. This is the martingale property of financial prices traded on an efficient market (Samuelson, 1965).

The martingale property is consistent with time-varying higher order return moments (e.g., volatility), which are predictable based on past information. In this paper, we focus on the weak form of market efficiency, capturing moments' predictability from past price changes. Campbell

et al. (1997) differentiate between different types of martingale efficiency, including the most genuine form of market efficiency, which is given by the random walk hypothesis. In this case, prices follow a random walk with i.i.d. returns, i.e., returns that lack any form of dependence.

Recent methodological advancements now make it possible to walk this path and compare different markets (either geographically separated markets or non-overlapping sub-periods of trading on the same market) with respect to their relative degree of efficiency. In this context, we call market A more efficient than market B if the dependence form of price changes generated on market A is weaker than that on market B. The underlying idea is to measure the intensity of dependence patterns present in the data by means of a comparison with the random walk case.[1]

This paper combines two topics of relevance for the modelling of financial markets – the evaluation of nonlinear patterns of dependence between returns and the multifractal property of returns – and notes the implications of multifractality for the efficiency of financial markets. The multifractal property was first confirmed in a financial context by Vassilicos et al. (1993) for quotation times and by Ghashghaie et al. (1996) and Vandewalle and Ausloos (1998) for price changes. It is a special type of nonlinear dependence manifested between returns with *different return periods* / along *different time scales* (minute, daily, monthly returns, etc.). We can also speak of scale-invariant returns. The presence of multifractality contravenes the random walk hypothesis, whereas the degree of multifractality can be used as a proxy for the degree of financial market efficiency.

In this paper, we formalise a new method for the evaluation of financial market efficiency based on the degree of nonlinear departure from the random walk benchmark using a multifractality measure, the so-called *intermittency coefficient* $\lambda^2$. Whereas alternative efficiency measures are computed model free, the measure introduced in this paper offers the advantage of a comprehensive model framework for financial volatility, the multifractal random walk (MRW) model by Bacry et al. (2001b), which makes forecast applications possible. Until now, numerous studies reported outstanding results for the multifractal modelling and forecasting of financial

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[1]In some recent publications, the deviation from a random walk is "quantified" by means of the value of the test statistic or the corresponding p-value for conventional random walk tests, such as the autocorrelation test and the variance ratio test. In this sense, the tests are applied in a quantitative way, but strictly speaking, these tests are nevertheless qualitative tests for market efficiency.

volatility (Calvet and Fisher, 2004; Lux, 2008; Bacry et al., 2008; Lux and Morales-Arias, 2010; Bacry et al., 2013; Lux et al., 2014), with recent applications for energy markets (Wang et al., 2016; Lux et al., 2016; Segnon et al., 2017) outperforming the widespread generalised autoregressive conditional heteroscedasticity (GARCH) class of models and stochastic volatility (SV) models.[2] The MRW model with $\lambda^2 = 0$ corresponds to Brownian motion, which is the continuous time counterpart of random walk behaviour. In this case, the MRW increments are (identically and) independently distributed, which corresponds to the most genuine form of market efficiency. By contrast, positive values of $\lambda^2$ correspond to a martingale form of efficiency with multifractal volatility. In this connection, the parameter $\lambda^2$ measures the deviation from the case of Brownian motion, i.e., the loss of financial market efficiency.

An empirical analysis of price data from the currently largest existing market for tradable pollution permits, the European Union Emission Trading Scheme (EU ETS), forms the second contribution of this paper. The permits are referred to as European Union Allowances (EUAs), the price of which is referred to as EUA price. Given the particularly short data samples available on this market we compute the empirical distribution function of the estimator for $\lambda^2$ and corresponding confidence intervals by means of Monte Carlo simulations. We test hypotheses about the value of $\lambda^2$ at the significance level of 5%. The main benefit of analysing the EU ETS market is that a clear ex ante expectation can be formulated with regard to the efficiency of this market and its development. This market is artificial in the sense that it has been designed by politicians and regulators based on economic reasoning. In one way or another, all financial markets are subject to regulatory interventions. This market, however, was developed from scratch; it did not evolve over time, and no formal or informal predecessors exist. In addition, participants in this market are mainly larger businesses from energy-intensive and energy producing sectors as well as financial institutions that represent smaller businesses covered by the market. Thus, the market participants should generally be experienced with financial trading. At the same time, there has been the opinion that the market participants would have to familiarise themselves with this market – for this very reason, the first phase of trading served as a so-called trial period. In light of these features, the public expectation is that the degree of

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[2]Another pertinent model of multifractal volatility is the Markov-switching multifractal model by Calvet and Fisher (2001).

efficiency of this market is generally increasing over time, as the market develops from a new to a more mature state. Several existing empirical studies have already come to this very conclusion: Charles et al. (2011), Niblock and Harrison (2013), Daskalakis (2013) and Ibikunle et al. (2013). This expectation is useful for the validation of our newly proposed method. The employment of additional data from US stock markets forms another benchmark for this empirical analysis.

The results obtained in this application can be summarised as follows. First, we can reject the random walk hypothesis $\lambda^2 = 0$ in favour of the alternative hypothesis of a martingale model with multifractal volatility at very small significance levels. Thus, the EU ETS is not perfectly efficient, which was obvious from the start, if we consider that i.i.d. returns constitute a theoretical model that cannot be satisfied in practice. Second, we find evidence of an improvement in informational efficiency of this market in the period 2013-2017 compared to 2008-2012. This finding is adequate for the general expectation about this market and confirms empirical findings obtained elsewhere; in addition, the estimated values for $\lambda^2$ are plausible, as they are generally in the range obtained for other financial markets. Third, we draw a parallel between the EU ETS and the US stock market, as represented by the Dow Jones Industrial Average stock market index, showing that these two markets are generally comparable in terms of informational efficiency. Our estimation results suggest that for one time period under consideration, the EU ETS is even more efficient than the US stock market. In light of the fact that the EU ETS is a newly established, artificial market, this result is worth highlighting.

A more conventional measure of multifractality is given by the so-called Hurst exponents. In contrast to this measure, the intermittency coefficient offers the advantage of an inference apparatus based on the generalised method of moments estimation technique with a well-established asymptotic theory. This is a very promising tool, all the more so considering that the present study is the first application with respect to the evaluation of financial market efficiency. Moreover, using a significance test for $\lambda^2$ makes it possible to test the null hypothesis of a perfectly efficient data-generating process (Brownian motion). By contrast, the evaluation of Hurst exponents is genuinely prone to errors due to its restricted estimation performance. To the best of our knowledge, financial market applications based on Hurst exponents disregard the exposition of significance levels, being of poor statistical relevance.

The remainder of this paper is organised as follows. Following a short presentation of the EU ETS in Section 2, Section 3 discusses methodological issues such as the intermittency coefficient, the Hurst exponents, the multifractal random walk model as well as the estimation procedures and several issues of statistical inference related to $\lambda^2$. Section 4 presents the empirical application; finally, Section 5 concludes this paper.

## 2 The European Union Emissions Trading Scheme

The EU ETS is, by far, the largest currently existing trading scheme for carbon emissions permits. Moreover, it is also the only existing multi-national scheme; other existing schemes are either on the national or regional level. Finally, this scheme has been in existence since 2005. Trading in this scheme is organised into so-called phases: 2005-2007 (Phase I), 2008-2012 (Phase II), and 2013-2020 (Phase III). Figure 1 displays price developments in Phase II and Phase III. Power generation and certain energy-intensive sectors from all EU countries as well as Norway, Iceland and Lichtenstein are covered by the scheme. The annual cap is approximately 2 billion tons of CO2; this cap is "melting" at an annual rate of 1.74% until 2030.[3]

Ever since trading in this scheme began, it has been subject to critical evaluations. Considerable attention has been focused on understanding the behaviour of prices for the pollution permits. The first studies appeared as early as 2008 and 2009; see Paolella and Taschini (2008) as well as Benz and Trueck (2009). Hintermann (2010) analyses the prices from a more structural point of view.

A number of more recent contributions compare price behaviour across the Phases; Creti et al. (2014) epitomise these research efforts.[4] It comes as no surprise that the informational efficiency of this market has also been analysed. Overall, this literature, of high relevance for this paper, finds that the EU ETS is becoming more efficient over time. Charles et al. (2011), for instance, find prices during Phase I to be predictable; in Phase II, however, the market is weak-form efficient, a conclusion also drawn by Montagnoli and de Vries (2011). Niblock and Harrison

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[3]A discussion of this system in detail is outside the scope of this paper. Interested readers are referred to overview articles such as Ellerman et al. (2015) and Hintermann and Gronwald (2016). Ellerman et al. (2010) provide a comprehensive discussion of carbon pricing in general.

[4]For an excellent literature overview, see Hintermann et al. (2016).
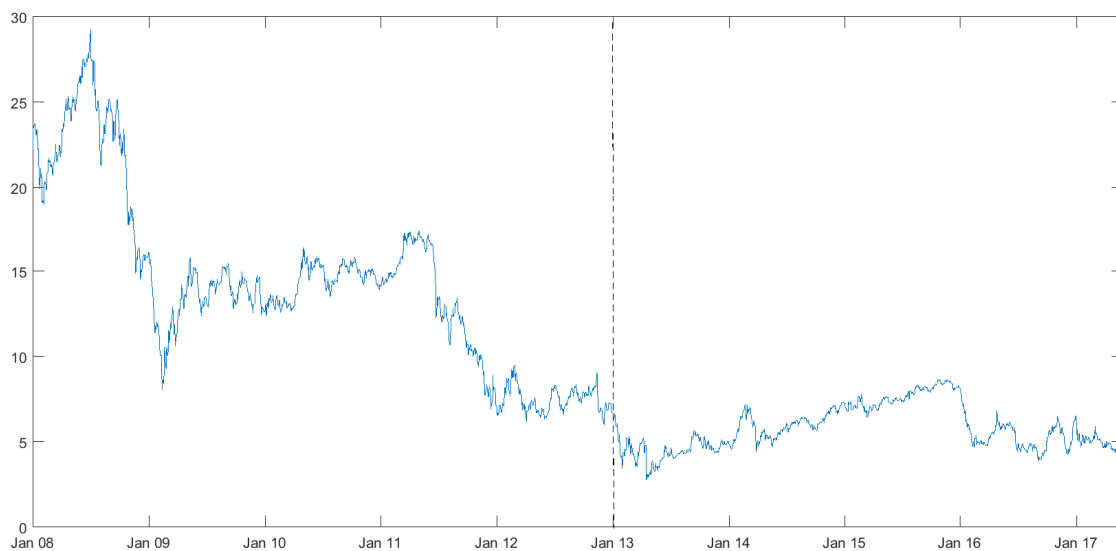
Figure 1: Permit prices in the EU ETS

(2013) offer some more refined insights: the authors only use Phase II data, but they split the sample into an early period (February 2008 - February 2010) and a late period (February 2010 - February 2012). They find that the early period is still characterised by predictability, while for the late period, no evidence of predictability is found. Daskalakis (2013) confirms these results overall; from 2010, the European carbon market is found to be weakly efficient.

These results are not just consistent, they are also in line with a plausible general expectation: the EU ETS is a newly established market. It can therefore be expected that this market was initially immature but that it would mature over time. The papers share another feature: they all apply qualitative tests for market efficiency. Montagnoli and de Vries (2011), Charles et al. (2011) and Niblock and Harrison (2013) use variance ratio tests, either the original procedure developed by Lo and MacKinlay (1988) or variants thereof; in addition, Niblock and Harrison (2013) use other conventional methods, such as standard unit root tests and serial correlation tests. Finally, Niblock and Harrison (2013) and Daskalakis (2013) employ trading rule profitability tests. A common feature of these procedures is that the null of market efficiency is tested against the unspecified alternative of market inefficiency. To the best of our knowledge, there are no quantitative studies on the efficiency of the carbon market. Zhuang et al. (2014) analyse the multifractality of carbon data with the Hurst exponents, but without any examination of market efficiency. Instead, the authors focus on the interplay between carbon and crude oil markets by using the multifractal detrended cross-correlation analysis (MF-DCCA) method.

## 3  Methodology

In this section the methodology used in this paper will be discussed in detail. Following an exposition of the concept of multifractality, the so-called multifractal random walk model is presented; this is followed by a discussion of estimation methods and performance issues.

### 3.1  Two measures of multifractality: Hurst exponents and the intermittency coefficient

Scale-invariant processes are stochastic processes with stationary increments, characterised by a "universal law", which holds in distribution and, under an appropriate transformation, over

different time scales, i.e., for different sampling intervals. For example, in the case of financial returns, the universal law would hold for different return periods (minute, daily, monthly returns, etc.). More precisely, the moments of a scale-invariant process $X(t)$, e.g., the moments of some logarithmic financial price, satisfy

$$E\left[|X(t) - X(t - \Delta t)|^q\right] = c_q \Delta t^{\zeta(q)} \tag{1}$$

for various sampling intervals $\Delta t$ and moment orders $q$, with $X(t) - X(t - \Delta t)$ the stationary process increments (fluctuations) and $c_q$ constant. It is important to note that return powers $|X(t) - X(t - \Delta t)|^q$ are measures of the extent of return fluctuations or volatility, which also allow us to speak of multifractal volatility. We refer readers to Bacry et al. (2008) for a detailed exposition on multifractality.

The function $\zeta(q)$, the so-called multifractal exponent, describes the complexity of the universal law expressed in Equation (1). According to the shape of $\zeta(q)$, we can distinguish between two types of scale-invariant processes. Fractal (uni-fractal, uni-scaling) processes display a linear multifractal exponent $\zeta(q) = Hq$ with constant Hurst exponent $H$. Correspondingly, the scaling property (1) is preserved under a unique transformation for all moment orders $q$, i.e., for all relevant magnitudes of fluctuations. This unique transformation is given by the constant $H$.

For example, fractional Brownian motion is a fractal process. For the case $H = 0.5$, this corresponds to Brownian motion with Gaussian increments. At the same time, this particular case represents the most genuine form of market efficiency, characterised by independent returns. Whereas $H$ is bounded between 0 and 1, values $H > 0.5$ are generated by persistent behaviours, with positive correlations. In this case, a positive or negative trend would be carried forward into the next period. Values $H < 0.5$ are related to an anti-persistent process, with trend reversals. Both cases describe an underlying structure in the data and are evidence of a smaller degree of market efficiency compared to the Brownian motion case.

Based on the evaluation of the Hurst exponent $H$, inference regarding market characteristics can be made. This was first proposed by Peters (1989), who estimated the Hurst exponent for S&P 500 monthly returns and 30-year treasury-bond returns, separately, using rescaled range

(R/S) analysis. Peters (1989) reported values $H > 0.5$ for these markets, suggesting persistent market behaviour, the so-called "memory effect". The author attributes this behaviour to investors' market sentiment, which is not immediately reflected in market prices but influences markets over the longer term.

However, empirical studies note that the behaviour of financial prices is more complex than the behaviour of fractal processes. Whereas (uni-)fractal processes are governed by a unique Hurst exponent $H$, the scaling property of financial returns holds for every moment $q$ under a different transformation $H(q)$, i.e., financial returns are characterised by different transformations $H(q)$ as a function of their magnitude. This allows us to determine a nonlinear multifractal exponent $\zeta(q) = H(q)q$ with a generalised family of Hurst exponents $H(q)$. In this case, we speak of multifractality (multi-scaling, anomalous scaling). More precisely, the multifractal exponent $\zeta(q)$ is strictly concave for multifractal processes (Lux and Segnon, 2013).

A significant number of publications have evaluated the degree of stock price deviations from a random walk using Hurst exponents (see, for example, Zhou et al. (2013); Rizvi et al. (2014); Arshad and Rizvi (2015); Anagnostidis et al. (2016)). Zunino et al. (2008) estimate the generalized Hurst exponents for 32 international stock market indices by means of multifractal detrended fluctuation analysis (MF-DFA). This method performs better than R/S analysis on small samples. They report higher multifractality degrees in emerging stock markets compared with developed stock markets. The MF-DFA has been also employed to evaluate the informational efficiency of foreign exchange markets (Ning et al., 2017) and of energy markets, in particular for oil trading (Jiang et al., 2014; Khediri and Charfeddine, 2015).[5]

An alternative measure of multifractality is the intermittency coefficient $\lambda^2$,

$$\lambda^2 := -\zeta''(0).$$

The intermittency coefficient is typically non-negative, $\lambda^2 \geq 0$, and increases with the degree

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[5]Khediri and Charfeddine (2015) use the DFA technique in a rolling window setting and find strong evidence of time-varying market efficiency in important energy markets. Other rolling window estimation approaches investigating the dynamic evolution of Hurst measures in time include Wang et al. (2010); Gu et al. (2013); Zhuang et al. (2015) and Anagnostidis et al. (2016). Additionally the Hurst method has been employed to identify the sources of multifractality in data (Gu et al., 2010; Plesoianu et al., 2012) or to investigate cross-correlations by means of MF-DCCA (Zhuang et al., 2014). Sukpitak and Hengpunya (2016) analyse the correlation between dynamic Hurst exponents and trading volume.

of multifractality, i.e., as the variation of dependence patterns in the volatility process becomes more pronounced. In the trivial case $\lambda^2 = 0$, multifractality disappears, and the process is fractal (cf. Bacry et al. (2008)).

The conventional evaluation of market efficiency focuses on the comparison between the estimated values of the Hurst exponents $H(q)$ for various $q$ and 0.5, the reference value in the random walk case. Moreover, one can determine the so-called degree of multifractality by the magnitude of the variation of $H(q)$ values as follows:

$$\Delta H = H\left(q_{min}\right) - H\left(q_{max}\right).$$

A high degree of multifractality $\Delta H$ is related to a rich (nonlinear) dependence structure and accounts for market inefficiency (Zunino et al., 2008; Gu et al., 2010; Wang and Wu, 2013).

This paper proposes a new method for the evaluation of market efficiency using the intermittency coefficient $\lambda^2$. We quantify the deviation of a price process from the random walk case by means of the deviation of the intermittency coefficient from the trivial case $\lambda^2 = 0$. This procedure is based on the so-called multifractal random walk model. The following subsection explains this model in greater detail.

## 3.2 The multifractal random walk model

Volatility models are martingale models capturing the dependence pattern in the volatility process for the purpose of volatility forecasts. Classic volatility models consider risk adjusted (zero mean) returns

$$r_t = \left[\ln\left(p_t\right) - \ln\left(p_{t-1}\right)\right] - \mu_t,$$

with $\mu_t = E_{t-1}\left[\ln\left(p_t\right) - \ln\left(p_{t-1}\right)\right]$ the conditional mean of the return series given the public information available at time $t - 1$. The focus is on the modelling of financial volatility $\sigma_t^2 = E_{t-1}\left[r_t^2\right]$ according to various specifications (e.g., GARCH, SV) within the following general framework:

$$r_t = \sigma_t u_t, \tag{2}$$

where $u_t$ is Gaussian white noise of mean 0 and variance $\sigma_u^2$ (Andersen et al., 2006). This construction reflects the economic ideas behind the efficient market hypothesis: the return fraction $\mu_t$ constitutes the fair payment expected in $t$, whereas $r_t$ is the supernormal profit or loss in $t$, which is impossible to predict systematically by market participants (cf. Fama (1965)).

Multifractal volatility models exploit the scaling properties of generalised volatility measures according to (1). For this purpose, they employ an additional variable, $\Delta t$, the sampling interval, which models the investment horizon. At the same time, the sampling interval $\Delta t$ is a measure of the representation accuracy. Accordingly, we construct the returns $r(t, \Delta t)$ between time $t-1$ and $t$, with sampling interval $\Delta t$, using only sampling intervals (time discretisation steps) not less than $\Delta t$. In other words, $r(t, \Delta t)$ takes into consideration only those investment decisions with investment horizons not less than $\Delta t$. Clearly, representation accuracy increases as the sampling interval $\Delta t$ decreases.

Multifractal returns $r(t)$ are defined as the limit process when $\Delta t \to 0^+$,

$$r(t) = \lim_{\Delta t \to 0^+} r(t, \Delta t) \qquad r(t, \Delta t) = \sum_{i=\left[\frac{t-1}{\Delta t}\right]+1}^{\left[\frac{t}{\Delta t}\right]} \sigma_{\Delta t}(i\,\Delta t)\, u_{\Delta t}(i\,\Delta t),$$

where $\sigma_{\Delta t}(t)$ constitutes the volatility process with sampling interval $\Delta t$ and $u_{\Delta t}(t)$ is Gaussian white noise with mean zero and variance $\Delta t$ (cf. Bacry et al. (2001a)). This discrete representation was generalised by Emmanuel Bacry and Jean-Francois Muzy in continuous time in a number of papers (Muzy and Bacry, 2002; Bacry and Muzy, 2003) generating the (log-normal) MRW returns in the limit of small sampling intervals $\Delta t \to 0^+$,

$$r(t) = \lim_{\Delta t \to 0^+} \int_{t-1}^{t} \sigma_{\Delta t}(u)\, dB(u), \tag{3}$$

with $\sigma_{\Delta t}(u)$ the volatility process

$$\sigma_\Delta t(u) = e^{\omega_{\Delta t}(u)} \tag{4}$$

and $dB(u)$ Gaussian white noise with mean 0 and variance $\sigma_{dB(u)}^2$.

The definition of the MRW volatility process in (4) is based on an auxiliary stationary

Gaussian process $\omega_{\Delta t}(t)$ independent of $dB(u)$ with mean

$$E\left[\omega_{\Delta t}(t)\right] = -\lambda^2 \left(\ln\left(\frac{T}{\Delta t}\right) + 1\right)$$

and autocovariance function

$$\gamma_{\omega_{\Delta t}}(h) := Cov\left[\omega_{\Delta t}(t), \omega_{\Delta t}(t+h)\right], \gamma_{\omega_{\Delta t}}(h) = \begin{cases} \lambda^2\left(\ln\left(\frac{T}{\Delta t}\right) + 1 - \frac{h}{\Delta t}\right), & 0 \leq h < \Delta t \\ \lambda^2 \ln\left(\frac{T}{h}\right), & \Delta t \leq h < T \\ 0, & h \geq T \end{cases},$$

with $0 \leq \lambda^2 < \frac{1}{2}$ and $T > 0$.

The MRW model provides a particularly parsimonious framework with only three parameters: the intermittency coefficient $\lambda^2$, the correlation length $T$ and the error term variance $\sigma^2_{dB(u)}$ (in short $\sigma^2$). Estimation of the MRW parameters can be done with the generalised method of moments (GMM) estimation technique using the mean and some various lags of the autocovariance function of the logarithmic absolute returns together with the return variance condition (Bacry et al., 2008, 2013; Sattarhoff, 2011).

In this paper, we employ the intermittency coefficient $\lambda^2$ as a measure of the efficiency of the market, which we use to analyse the EU ETS.

## 3.3 Estimation methods and performance issues

### 3.3.1 Hurst exponents

Both the Hurst exponents $H(q)$ and the intermittency coefficient $\lambda^2$ describe the shape of the multifractal exponent function $\zeta(q)$. Thus, they both quantify the degree of multifractality of a scale-invariant process. However, in practice, the estimation of Hurst exponents has proven unsatisfactory in finite samples. This can lead to erroneous results. In this sense, it is more convenient to employ the intermittency coefficient $\lambda^2$, which can be estimated robustly via GMM.

Typically, the estimation of both $H(q)$ and $\lambda^2$ can be accomplished from the estimation of the function $\zeta(q)$ in (1). Another popular approach for the estimation of $H(q)$ is R/S analysis,

which was introduced by Hurst (1951). A more robust generalised version of the latter method is given by the MF-DFA introduced by Kantelhardt et al. (2002). Each of these three methods is based on the fitting of a power law to the data. This is the so-called *scaling estimation* approach.

For this purpose, it is important to employ a loglinear transformation of the power law. For example, when using (1), we obtain the loglinear form

$$\ln \left( E \left[ |X \left( t \right) - X \left( t - \Delta t \right) |^q \right] \right) = \ln \left( c_q \right) + \zeta \left( q \right) \ln \left( \Delta t \right). \tag{5}$$

Then, one can fit (5) to the behaviour of the corresponding empirical moments and compute the least squares estimator $\widehat{H}(q)$, i.e., the scaling estimator for $H(q)$.

Numerous studies have proven the unreliability of scaling estimators for financial applications due to their high variance in finite samples. Lux (2004) computed scaling estimators from randomised financial data, which typically lack any temporal structure, including any multifractal characteristics. Nevertheless, the estimated values from randomised data are indistinguishable from those that come from genuine multifractal data. This result indicates the high fluctuation of small sample scaling estimates, leading to spurious detection of multifractality in the absence of true scaling. Gu et al. (2010) perform a similar exercise in order to determine the so-called "sources of multifractality". The authors construct shuffled data, which lack any temporal structure, and surrogate data, which are Gaussian distributed. The difference between the multifractality degrees of the original and transformed series is then used to quantify the influence of temporal patterns and fat tails on multifractality. However, the high multifractality degrees reported for the transformed data are doubtful and provide evidence of the unreliability of this estimation method. Ludescher et al. (2011) showed that additive noise, short-term correlations and periodicities, typically found in financial data, lead to considerable biases in the estimation of Hurst exponents. One can consult Lux and Segnon (2013) for a comprehensive survey on this subject.

A typical exercise in empirical applications is the comparison between the estimated value $\widehat{H}(q)$ and the reference value of 0.5 that characterises the random walk market hypothesis (Zunino et al., 2008; Gu et al., 2010; Wang and Wu, 2013). This is then interpreted as a "test"

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for the type of market efficiency, with estimated values different from 0.5 being interpreted as evidence for inefficient market structures, thus pointing to a rejection of the random walk hypothesis. Unfortunately, these kinds of studies are generally useless from a statistical point of view, as they do not provide any theoretical results on the distribution of the scaling estimator $\widehat{H}(q)$ and they do not investigate whether the Hurst exponents $H(q)$ of the data are *significantly* different from 0.5.

### 3.3.2 The intermittency coefficient

In this paper, we estimate the intermittency coefficient based on a well-established inference tool, the GMM technique. Under some regularity conditions, the GMM estimator for the MRW model is consistent and asymptotically normally distributed; see Bacry et al. (2013). Moreover Bacry et al. (2013) show within the framework of a Monte Carlo simulation study that the estimation of the intermittency coefficient $\lambda^2$ has very small bias and mean squared error (MSE) values, being also reliable in finite samples. Unfortunately, the normality of this estimator can be reached only for very large datasets with at least 16,000 points. In this paper, we employ the GMM estimation procedure in Sattarhoff (2011) using an additional moment condition. In this case, the asymptotic properties deliver a better approximation for the finite sample behaviour, with normally distributed estimates starting with approximately 2,000 data points.

However, given the short datasets on the relatively recently established EU ETS, we cannot employ the normal approximation for the scope of this paper. Instead, we compute the empirical distribution function of the estimator for $\lambda^2$ and corresponding confidence intervals by means of Monte Carlo simulations (cf. Bacry et al. (2013)).

By contrast, there are no theoretical results on the asymptotic properties (consistency, asymptotic distribution) of scaling estimators. Thomas Lux compares the scaling estimator and the GMM estimator in finite samples within the scope of a Monte Carlo simulation study and reports bias and MSE values for the estimation problem of the multifractal model of asset returns and the Markov-switching multifractal model of asset returns, respectively (Lux, 2003, 2008). This comparison shows that the GMM estimator significantly outperforms the scaling

procedure in finite samples.

The following significance (multifractality) test by Sattarhoff (2011) with the hypotheses

$$H_0 : \lambda^2 = 0 \qquad \text{vs.} \qquad H_1 : \lambda^2 > 0 \tag{6}$$

can also be interpreted as a test on the form of market efficiency. The test was designed for the MRW model, where $r(k)$ denote the MRW returns. Under the null hypothesis, the data-generating process is Brownian motion, i.e., the market is perfectly efficient, with the alternative being a martingale model with multifractal volatility. Sattarhoff (2011) shows that the test statistic $M$,

$$M := \sqrt{N} \cdot \frac{2\sqrt{2}}{\sqrt{\pi^2 - 4}} \cdot \left( \frac{\ln(m_\sigma)}{2} - m_\mu - \frac{\gamma + \ln(2)}{2} \right),$$

with

$$m_\sigma := \frac{1}{N} \sum_{k=1}^{N} r(k)^2 \qquad \text{and} \qquad m_\mu := \frac{1}{N} \sum_{k=1}^{N} \ln(|r(k)|),$$

converges in distribution to a standard normal variable under the null hypothesis.

# 4  Empirical application

Our dataset comprises approximately 9.5 years of daily EUA prices from January 2008 until May 2017. US stock market data (Dow Jones Industrial Average, DJIA), used as a benchmark, are taken from the identical period.[6] We computed the continuously compounded returns

$$r_t = \ln(p_t) - \ln(p_{t-1}).$$

We use the full sample as well as two subsamples: 2008-2012 (Phase II of the EU ETS) and 2013-2017 (currently running Phase III). Table 1 reports some descriptive statistics of returns, absolute returns and squared returns. We could reject the null hypothesis of no serial correlation using the Ljung-Box-Pierce statistic for each data series (significance level of 5%). For absolute

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[6]Source: For reasons of continuity, we use a linked data series consisting of the series ECBCS00 and ECFCS00 from Datastream. The DJIA data have been obtained from the Federal Reserve Bank of St. Louis.

and squared returns, this result is reinforced by the significant positive values over long lags of their empirical autocorrelation functions (see Figure 2). However, in the case of raw returns, the autocorrelations, although existent, are much less pronounced. According to the Shapiro-Wilk normality test (significance level of 5%) as well as the high kurtosis values, the distribution of our data deviates from normal. These findings are in line with the behaviour of the DJIA (see Table 1 and Figure 2).

| | EUA 2008-2017 | | | DJIA 2008-2017 | | |
|---|---|---|---|---|---|---|
| | Returns | Absolute | Squared | Returns | Absolute | Squared |
| Minimum | -0.422 | 0 | 0 | -0.082 | 0 | 0 |
| Maximum | 0.235 | 0.422 | 0.178 | 0.105 | 0.105 | 0.011 |
| Mean | $-5.944 \cdot 10^{-4}$ | 0.02 | $9.655 \cdot 10^{-4}$ | $1.956 \cdot 10^{-4}$ | 0.008 | $1.479 \cdot 10^{-4}$ |
| St. dev. | 0.031 | 0.023 | 0.004 | 0.012 | 0.009 | $5.182 \cdot 10^{-4}$ |
| Skewness | -0.886 | 4.144 | 29.688 | -0.072 | 3.387 | 11.665 |
| Kurtosis | 21.099 | 46.799 | 1159 | 13.278 | 22.490 | 193.636 |
| Shapiro-Wilk | 0.899* | 0.721* | 0.15* | 0.885* | 0.702* | 0.253* |
| Q(12) | 43.224* | 1047.725* | 109.189* | 57.351* | 3518.909* | 2722.756* |
| Q(20) | 60.091* | 1617.392* | 249.602* | 100.984* | 5087.237* | 3739.199* |

Table 1: Descriptive statistics (significance level: $^* = < 3 \cdot 10^{-5}$)

We estimated the MRW model for the EUA prices as well as for the DJIA reference data via GMM based on the moment conditions in Sattarhoff (2011). Table 2 displays our results. For both markets, the intermittency coefficient is found to differ significantly from zero at very small significance levels in all samples under consideration. At the same time, this implies that we could reject the null hypothesis of a perfectly efficient market with i.i.d. returns in favour of multifractal martingale prices. This result is obvious if we consider that i.i.d. returns constitute a theoretical model that cannot be satisfied in practice.

The results can be summarised as follows. For the full sample, we computed the estimated $\lambda^2$ value of 0.0415. The results for the two subsamples are worth noting: The estimated $\lambda^2$ value for the sample 2008-2012 is greater than the estimate for 2013-2017.[7] These point estimates suggest that our findings are in line with the literature on market efficiency of the EU ETS.

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[7]It is important to note that the value of the intermittency coefficient $\lambda^2$ is not necessarily related to the sample length. The intermittency coefficient captures the complexity of dependency patterns in the data, which can be, but does not necessarily have to be, a function of the sample length. Hence, both cases, in which the full sample estimate is larger or smaller than the subsample estimate, are possible without contradiction.
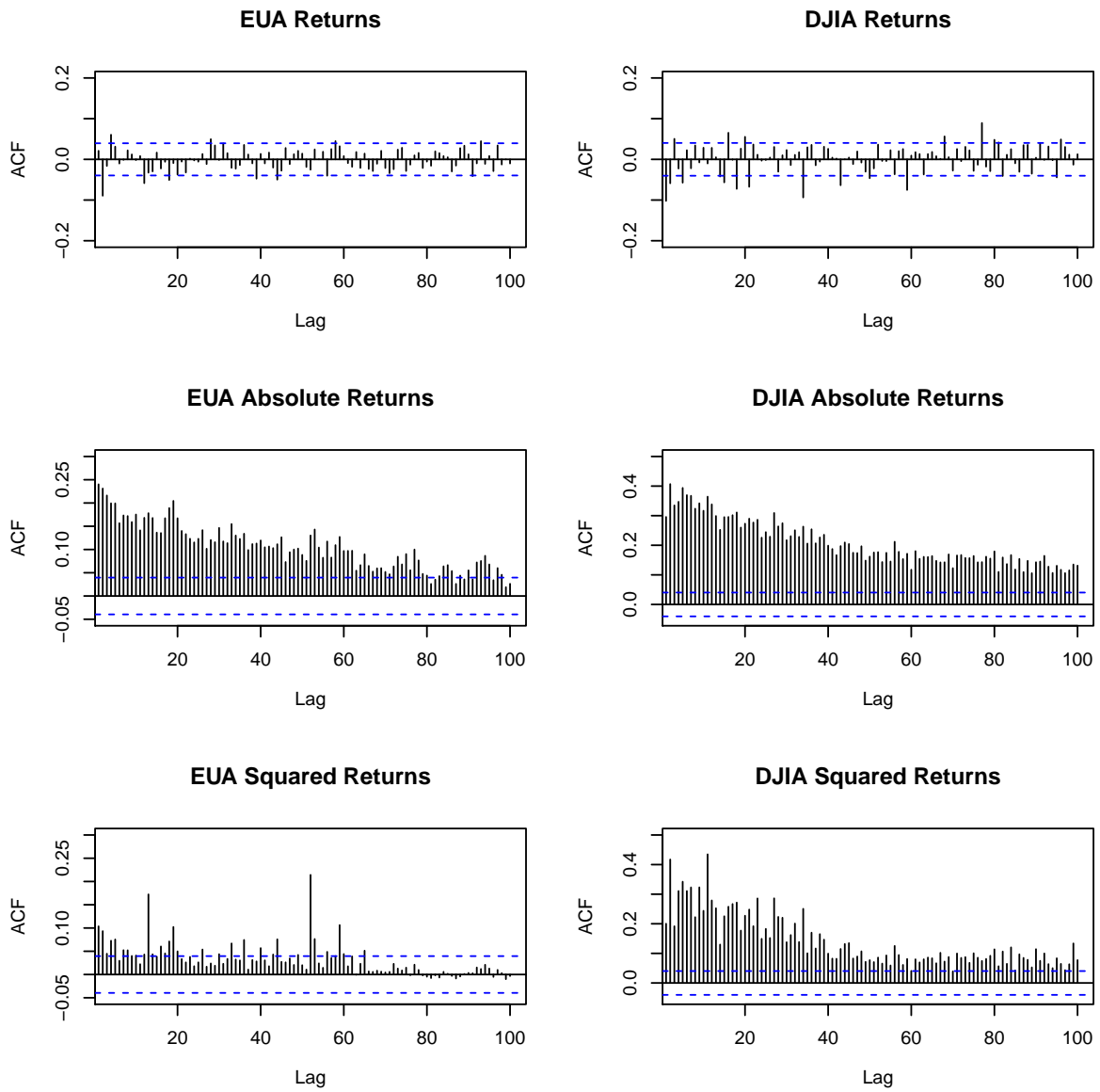
Figure 2: Autocorrelation functions for the EUA as well as the DJIA full sample data

| | EUA | | | DJIA | | |
|---|---|---|---|---|---|---|
| | 2008-2017 | 2008-2012 | 2013-2017 | 2008-2017 | 2008-2012 | 2013-2017 |
| | | no. of data points | | | no. of data points | |
| | 2,454 | 1,305 | 1,149 | 2,368 | 1,259 | 1,109 |
| | | $\lambda^2$ estimates | | | $\lambda^2$ estimates | |
| | 0.0415* | 0.0362* | 0.03* | 0.0439* | 0.0492* | 0.0316* |
| | | Monte Carlo c.i. ($\alpha = 0.1$) | | | Monte Carlo c.i. ($\alpha = 0.1$) | |
| $\lambda^2 = 0.02$ | | [0.005, 0.028] | [0.004, 0.029] | | [0.005, 0.028] | [0.004, 0.029] |
| $\lambda^2 = 0.025$ | | [0.009, 0.034] | [0.008, 0.034] | | [0.009, 0.034] | [0.007, 0.035] |
| $\lambda^2 = 0.03$ | | [0.013, 0.039] | [0.011, 0.04] | | [0.012, 0.04] | [0.01, 0.04] |
| $\lambda^2 = 0.04$ | | [0.019, 0.051] | [0.018, 0.051] | | [0.019, 0.051] | [0.017, 0.052] |

Table 2: Estimation results (significance level: $^* = < 0.001$) and approximate confidence intervals (c.i.) from Monte Carlo simulations. (For each parameter value $\lambda^2 \in \{0.02, 0.025, 0.03, 0.04\}$ and each sample length $N \in \{1305, 1149, 1259, 1109\}$, we generated 10,000 MRW series.)

To analyse this further, we will test in the following several hypotheses about the value of the parameter $\lambda^2$. Given the relatively short data samples, the normal asymptotics of the GMM estimator for $\lambda^2$ have not yet been reached. On account of this, we compute the empirical distributions of the GMM estimator from Monte Carlo simulations.

The estimated value of the intermittency coefficient reported in finance applications would typically vary in $\lambda^2 \in [0.01, 0.06]$ (Duchon et al., 2012). In the present study, we considered four parameter values $\lambda^2 \in \{0.02, 0.025, 0.03, 0.04\}$. For each data sample (2008-2012, 2013-2017) and for each value of $\lambda^2 \in \{0.02, 0.025, 0.03, 0.04\}$, we simulated 10,000 MRW series of the length of the sample and estimated the MRW.[8]

Based on the empirical distributions of the GMM estimator for $\lambda^2$ (see Figure A1), we computed approximate confidence intervals (Table 2). This allows us to test hypotheses of the form

$$H_0 : \lambda^2 \leq a \quad \text{vs.} \quad H_1 : \lambda^2 > a$$

with $a \in \{0.02, 0.025, 0.03, 0.04\}$. If $a = 0.025$, we may reject the null hypothesis for the data sample 2008-2012 at the significance level of 5% ($\lambda^2$ estimate $0.0362 \notin (0, 0.034)$). This suggests

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[8]In the simulations we set the error term variance $\sigma^2$ equal to its GMM estimate from the real data and chose a fairly large value for the correlation length $T$ of 20 years (Muzy et al., 2013).

Figure 3: Permit prices in the EU ETS and the estimated values for $\lambda^2$

that the value of $\lambda^2$ is greater than 0.025 in the first period. For 2013-2017, we could only reject the null hypothesis in the case $a = 0.02$ (significance level of 5%, confidence interval of $(0, 0.029)$), suggesting that $\lambda^2$ is greater than 0.02 in the second period. It is important to note that we could not reject the null hypothesis $\lambda^2 \leq 0.025$, i.e., the second data sample is consistent with values not greater than 0.025. In sum, these results indicate a presumably smaller value of $\lambda^2$ for the second time sample, which implies a higher degree of market efficiency during Phase III. Figure 3 further illustrates these results, with the higher multifractality degree in the years 2008-2012 capturing precisely the more pronounced "spikiness" in the data during this period.

To facilitate the understanding of these results, we undertake this exercise again using data from the DJIA stock market index. These findings can be summarised as follows: If $a = 0.03$, we may reject the null hypothesis for the data sample 2008-2012 at the significance level of 5% ($\lambda^2$ estimate $0.0492 \notin (0, 0.04)$). This suggests that the value of $\lambda^2$ is greater than 0.03 in the first period. For 2013-2017, we could only reject the null hypothesis in the case $a = 0.02$

(significance level of 5%, confidence interval of $(0, 0.029)$), suggesting that $\lambda^2$ is greater than 0.02 in the second period. It is important to note that we could not reject the null hypothesis $\lambda^2 \leq 0.03$, i.e., the second data sample is consistent with values not greater than 0.03, altogether we obtain a similar message: a presumable smaller value of $\lambda^2$ and, therefore, a higher degree of market efficiency during the second time sample.

Thus, our results indicate that the informational efficiency in the EU ETS and in the US stock market is similar overall. Our results for the first data sample 2008-2012 are certainly worth highlighting: in the case of the DJIA, we can reject the null hypothesis $\lambda^2 \leq 0.03$ at the significance level of 5%. However, for the EUA prices, we could not reject the null hypothesis $\lambda^2 \leq 0.03$, i.e., the EUA price data are consistent with values not greater than 0.03. This suggests that the EU ETS is more efficient than the US stock market during the first subsample 2008-2012.

In light of the fact that various observers have expressed concerns with regard to the informational efficiency of a newly established market such as the EU ETS, this result is remarkable. Among the possible explanations for this finding are, first, that the US stock market has been affected by the financial crisis of 2007-2008 to a larger extent than the EU ETS. Second, a market such as the US stock market is generally more exposed to external shocks, as it is affected by global macroeconomic and financial developments. By contrast, prices for pollution permits in the EU ETS are mainly determined by supply decisions for this market; these decisions, however, are made in advance by the European Commission. The demand for pollution permits is largely influenced by economic activity in the European Union. Some researchers also find that the EU ETS is a particularly political market; however, important political or regulatory decisions are made very infrequently; see, e.g., Sanin et al. (2015).

## 5 Conclusion

The question of whether financial markets are efficient has been addressed in a vast number of empirical studies. These concerted research efforts are more than justified, as financial markets provide essential information for investors and corporations. The notion of financial market efficiency employed today goes back to Fama (1965) and reflects the idea that all available

information is included in the observed prices and that there is no information left that could be used for the purpose of some systematic prediction. On the supposition of financial market efficiency, prices in financial markets should be a reliable indicator for company valuations, to name just one implication. Empirical procedures for testing financial market efficiency are usually focused on testing for non-predictability. At the same time the large majority of them show a considerable technical limitation: they are qualitative. In other words, for the most part, the empirical procedures only allow one to test whether a certain financial market is *efficient* or *not efficient*.

The recent history of financial markets includes various episodes of dramatic changes. One example is the emergence of automatic and high-frequency trading systems in financial markets. In addition, the behaviour of financial prices changes over time, often sparked by events such as the recent financial crisis of 2007-2008. Moreover, the market for crude oil is subject to the so-called financialisation, a considerable increase in liquidity in the oil futures market. Finally, new markets are emerging; examples include newly created markets for pollution permits such as the EU ETS as well as markets for newly created entities such as cryptocurrencies. It seems obvious that in relation to these changes, the distinction between efficient/inefficient is obsolete. Instead, it would be important to analyse whether stock markets are *more* or *less efficient* during a financial crisis, if the crude oil market is *more efficient* due to the financialisation and if newly emerged markets become *more efficient* over time.

This paper remedies this issue by proposing a quantitative measure for financial market efficiency. This measure is referred to as the intermittency coefficient and is based on the MRW model proposed by Bacry et al. (2001b). The intermittency coefficient allows one to measure the distance to a random walk, which represents a perfectly efficient market. It is worth highlighting that we provide a novel interpretation of this parameter with respect to the efficiency of financial markets, which forms the main contribution of this paper. The main advantage of this approach over commonly used methods based on Hurst exponents is that the intermittency coefficient is a more reliable tool due to its well-established inference apparatus based on the GMM estimation technique.

In an empirical application, we use data from the largest currently existing market for trad-

able pollution permits, the EU ETS. This market has been selected because a clear expectation can be formulated with regard to the results: it is a newly established market; thus, efficiency is expected to be low initially but would increase over time – a result confirmed in a number of papers. This paper's empirical application overall reconfirms these results, showing that the EU ETS becomes more efficient over time.

This clearly illustrates that our newly proposed method is reliable. However, our analysis yields additional detailed insights. In addition to studying whether a market becomes more efficient over time, our method also allows us to compare efficiency across markets. We find the degree of market efficiency of the EU ETS to be similar overall to that of the US stock market; for one sub-period, the efficiency of the EU ETS is even found to be more pronounced than that of the US stock market. These findings are noteworthy, as various observers have expressed concerns with regard to the informational efficiency of this newly established artificial market.

It is obvious that ample opportunities exist for future applications of our method, e.g., with respect to the effect of the financialisation mentioned above on the efficiency of oil futures markets, to name just one. In addition, a growing literature is analysing the efficiency of markets for cryptocurrencies such as Bitcoin. Finally, it is also possible to estimate intermittency coefficients from high frequency data (Duchon et al., 2012). It would be very interesting to investigate whether this estimation technique can improve the accuracy of financial market evaluations regarding efficiency.

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# A Appendix

Figure A1 displays the histograms of the $\lambda^2$ estimates from the simulated data. We scaled the histogram values so that the histogram area adds up to 1. This way, the histograms deliver empirical approximations for the density functions of the GMM estimator for $\lambda^2$. For reasons of comparison, we also plotted the normal distributions with the mean and the standard deviation of the $\lambda^2$ estimates. The empirical distributions deviate from normal.[9]

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[9]We could reject the null hypothesis of normal data with the Shapiro-Wilk normality test at the 5% significance level.

Figure A1: Histograms of $\lambda^2$ estimates. Each histogram outlines 10,000 estimates. The lines represent the normal distributions with the corresponding sample mean and sample standard deviation.
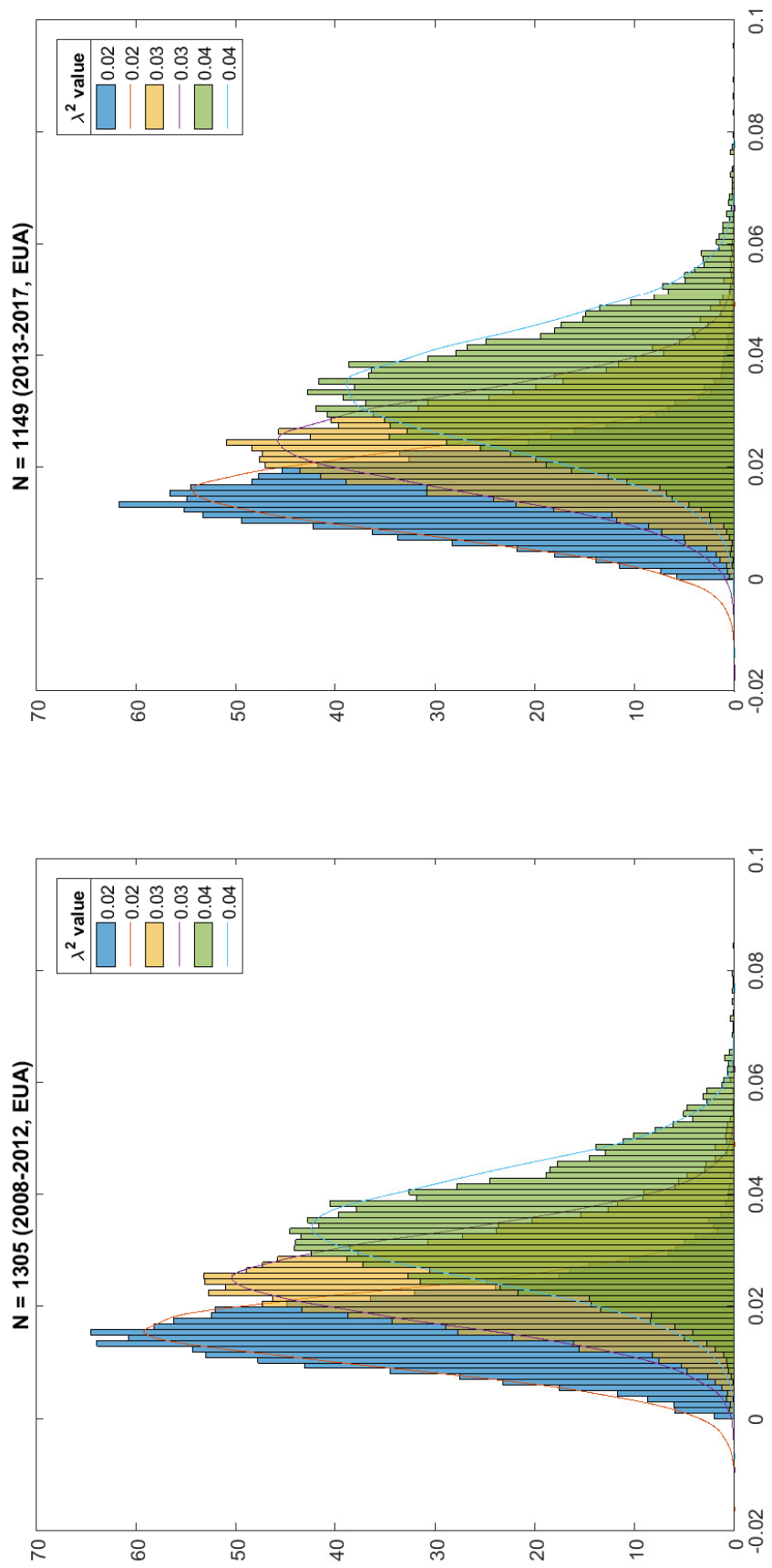
Figure A2: Histograms of $\lambda^2$ estimates. Each histogram outlines 10,000 estimates. The lines represent the normal distributions with the corresponding sample mean and sample standard deviation.