MODULES OF CONSTANT JORDAN TYPE WITH SMALL NON-PROJECTIVE PART

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ABSTRACT. Let E be an elementary abelian p-group of rank r and let k be an algebraically closed field of characteristic p. We prove that if M is a kE-module of stable constant Jordan type $[a_1]\dots[a_t]$ with $\sum_j a_j \le \min(r-1,p-2)$ then $a_1 = \dots = a_t = 1$. The proof uses the theory of Chern classes of vector bundles on projective space.

1. INTRODUCTION

Let $E = \langle g_1,\dots,g_r\rangle \cong (\mathbb{Z}/p)^r$ $(r \ge 2)$ be an elementary abelian p-group and let k be an algebraically closed field of characteristic p. Set $X_i = g_i - 1 \in kE$, and if $\alpha = (\lambda_1,\dots,\lambda_r) \in \mathbb{A}^r(k)$, set $X_\alpha = \sum_{i=1}^r \lambda_i X_i$. We say that a finite dimensional kE-module M has *stable constant Jordan type* $[a_1]\dots[a_t]$ (with $0 \le a_i \le p-1$) if for every $0 \ne \alpha \in \mathbb{A}^r(k)$ the Jordan canonical form of X_α on M has Jordan blocks of lengths $a_1,\dots,a_t$ together with some number (possibly zero) of blocks of length p. This concept was introduced by Carlson, Friedlander and Pevtsova [4] and investigated further in [1, 2, 3, 5, 6]. The following conjecture appears in section 9 of [4].

Conjecture 1.1. If M has stable constant Jordan type $[2][1]^j$ with $p \ge 5$ then $j \ge r-1$.

Our main theorem makes progress on this conjecture, but does not completely resolve it. For $p > r$ we provide a bound for j that differs by just one from the value given in the conjecture.

Theorem 1.2. *If M is a kE-module of stable constant Jordan type $[a_1]\dots[a_t]$ with*

$$\sum_j a_j \le \min(r-1,p-2)$$

then $a_1 = \dots = a_t = 1$.

Corollary 1.3. *If M has stable constant Jordan type $[2][1]^j$ with $p > r$ then $j \ge r-2$.*

Remark 1.4. The example $\Omega(k)$ of stable constant Jordan type $[p-1]$ shows that the condition on the prime cannot be weakened in Theorem 1.2. It is not clear whether the condition on the rank can be weakened by one, to fit better with Conjecture 1.1.

The main theorem has the following consequences for generic Jordan type, a notion investigated by Friedlander, Pevtsova and Suslin [7].

Corollary 1.5. *Let M be a kE-module of stable generic Jordan type $[a_1]\dots[a_t]$, and suppose that $p-2 \ge a = \sum_j a_j$. If some a_j is not equal to 1 then the locus of non-generic Jordan type of M has codimension at most a in affine space $\mathbb{A}^r(k)$.*

The proof of the main theorem uses the theory of Chern classes of vector bundles on projective space, and some congruences proved in Benson and Pevtsova [2].

2. Vector bundles on projective space

In this section we recall some notation and theorems from [2]. Let

$$\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \dots, Y_r],$$

where the Y_i are the coordinate functions on $\mathbb{A}^r(k)$ defined by $Y_i(X_j) = \delta_{ij}$ (Kronecker delta). Let $\mathcal{O}$ be the structure sheaf of $\mathbb{P}^{r-1}$, and let $\mathcal{O}(i)$ be its ith twist.

If M is a kE-module, we let

$$\widetilde{M} = M \otimes_k \mathcal{O},$$

a trivial sheaf of rank equal to the dimension of M, and

$$\widetilde{M}(i) = M \otimes_k \mathcal{O}(i)$$

its twists. Following Friedlander and Pevtsova [6], we define an operator

$$\theta = \theta_M \colon \widetilde{M}(j) \to \widetilde{M}(j+1)$$

by the formula

$$\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f.$$

We define sheaves $\mathcal{F}_i(M)$ $(1 \le i \le p)$ via the formula

$$\mathcal{F}_i(M) = \frac{\mathsf{Ker}(\theta) \cap \mathsf{Im}(\theta^{i-1})}{\mathsf{Ker}(\theta) \cap \mathsf{Im}(\theta^{i})}$$

as a subquotient of M.

The sheaf $\mathcal{F}_i(M)$ at a point $\bar{\alpha} \in \mathbb{P}^{r-1}(k)$ captures the sum of the socles of the Jordan blocks of length i for the action of X_α on M. The module M has constant Jordan type if and only if each $\mathcal{F}_i(M)$ is a vector bundle on $\mathbb{P}^{r-1}$. In this case, the rank of $\mathcal{F}_i(M)$ is equal to the number of the a_j that are equal to i.

Theorem 2.1 (Benson and Pevtsova [2])**.** *The trivial sheaf $\widetilde{M}$ has a filtration in which the filtered quotients are $\mathcal{F}_i(M)(j)$ for $0 \le j < i \le p$.*

Proof. This follows from Lemmas 2.2 and 2.3 of [2]. □

The Chow ring $A^*(\mathbb{P}^{r-1})$ is $\mathbb{Z}[h]/(h^r)$. If $\mathcal{F}$ is a vector bundle on $\mathbb{P}^{r-1}$ then its total Chern class is

$$c(\mathcal{F}) = c(\mathcal{F}, h) = 1 + c_1(\mathcal{F})h + \cdots + c_{r-1}(\mathcal{F})h^{r-1}$$

where $c_i(\mathcal{F})$ are the Chern numbers of $\mathcal{F}$.

Theorem 2.2 (Benson and Pevtsova [2])**.** *If $\mathcal{F}$ is a vector bundle of rank s on $\mathbb{P}^{r-1}$ then*

$$c(\mathcal{F})c(\mathcal{F}(1))\dots c(\mathcal{F}(p-1)) \equiv 1 - sh^{p-1} \pmod{(p, h^p)}$$

as elements of $\mathbb{Z}[h]/(h^r)$.

3. Proof of the main theorem

Proof of Theorem 1.2. Since $\widetilde{M}$ is a direct sum of copies of $\mathcal{O}$, using Theorem 2.1 we have

$$1 = c(\widetilde{M}) = \prod_{0\le j<i\le p} c(\mathcal{F}_i(M)(j)).$$

Dropping the terms with $i = p$, by Theorem 2.2 we deduce that

$$\prod_{0\le j<i\le p-1} c(\mathcal{F}_i(M)(j)) \equiv 1 \pmod{(p, h^{p-1})} \tag{3.1}$$

Let $a = \sum_j a_j$. Then the left hand side of this congruence is a polynomial of degree a. By hypothesis we have $a \le r-1$ and $a \le p-2$. Since $h^{r-1} \ne 0$, we may reduce modulo p and read this congruence as an equation in $\mathbb{F}_p[h]$. The units in this ring are the constant polynomials, so we deduce that all factors in the product are congruent to one modulo p. So if $2 \le i \le p-1$, both $c(\mathcal{F}_i(M))$ and $c(\mathcal{F}_i(M)(1))$ are congruent to one modulo p. Since

$$c_1(\mathcal{F}_i(M)(1)) = c_1(\mathcal{F}_i(M)) + \mathrm{rank}(\mathcal{F}_i(M))$$

this implies that for $2 \le i \le p-1$ the rank of $\mathcal{F}_i(M)$ is divisible by p. But $\mathrm{rank}(\mathcal{F}_i(M))$ is the number of a_j that are equal to i. Since $a < p$, we deduce that $\mathcal{F}_i(M) = 0$ for $2 \le i \le p-1$, so that each $a_j = 1$. □

Proof of Corollary 1.5. Let $V \subseteq \mathbb{A}^r(k)$ be the locus of non-generic Jordan type of M. If V has codimension larger than a then there is a linear subspace W of $\mathbb{A}^r(k)$ of dimension larger than a such that $V \cap W = \{0\}$. Corresponding to W there is a shifted subgroup of kE of rank larger than a on which M has constant Jordan type. The restriction of M to this shifted subgroup contradicts Theorem 1.2. □

4. On the boundary

In this section we investigate the case where M is a kE-module of stable constant Jordan type $[2][1]^{r-2}$, the case of Conjecture 1.1 not covered by Corollary 1.3, and we continue to assume that $p > r$.

Theorem 4.1. *If M is a module of stable constant Jordan type $[2][1]^{r-2}$ with $p > r$ and r odd then p is congruent to one modulo some prime factor of r.*

Proof. Let $\alpha = c_1(\mathcal{F}_2(M))$. Then $c_1(\mathcal{F}_2(M)(1)) = \alpha + 1$, and so the congruence (3.1) reads

$$(1 + \alpha h)(1 + (\alpha + 1)h)c(\mathcal{F}_1(M)) \equiv 1 \pmod{p, h^r}.$$

The left hand side is a polynomial of degree r, so it is equal to $1 - \beta h^r$ for some integer β. So we have the equation

$$(1 + \alpha h)(1 + (\alpha + 1)h)c(\mathcal{F}_1(M)) = 1 - \beta h^r$$

in $\mathbb{F}_p[h]$. By the factor theorem, both α and $\alpha + 1$ are rth roots of β modulo p, and so p does not divide α, $\alpha + 1$ or β. Furthermore, writing α^{-1} for the inverse of α modulo p, it follows that

$$(\alpha + 1)\alpha^{-1} = 1 + \alpha^{-1}$$

is an rth root of unity modulo p. Since it is not equal to one, it follows that for some prime factor q of r, there is a qth root of unity modulo p and so p is congruent to one modulo q. □

We illustrate this theorem in the following corollary.

Corollary 4.2. *If $r = 3$, $p \geq 5$, and M is a module of stable constant Jordan type $[2][1]$ then p is congruent to one modulo 3.*

So the smallest example that is not ruled out by the theorem is the case $r = 3$, $p = 7$, M of stable constant Jordan type $[2][1]$. The factorisation corresponding to this situation is either

$$(1 + h)(1 + 2h)(1 + 4h) \equiv 1 + h^3 \pmod 7,$$

so that

$$c_1(\mathcal{F}_2(M)) \equiv 1, \quad c_1(\mathcal{F}_2(M)(1)) \equiv 2, \quad c_1(\mathcal{F}_1(M)) \equiv 4 \pmod 7$$

or

$$(1 + 3h)(1 + 5h)(1 + 6h) \equiv 1 - h^3 \pmod 7,$$

so that

$$c_1(\mathcal{F}_2(M)) \equiv 5, \quad c_1(\mathcal{F}_2(M)(1)) \equiv 6, \quad c_1(\mathcal{F}_1(M)) \equiv 3 \pmod 7.$$

We have not been able to rule out a module of this type. Its restriction to a rank two subgroup would be a module of constant Jordan type $[2][1]$ that would be a counterexample to the following conjecture.

Conjecture 4.3. If M is an indecomposable kE-module of stable constant Jordan type $[2][1]^{r-1}$ with $p \geq 5$ then for some $n \in \mathbb{Z}$ we have either

$$M \cong \Omega^{2n}(kE/J^2(kE)) \quad \text{or} \quad M \cong \Omega^{2n}(\mathsf{Soc}^2(kE)).$$

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