

University of Aberdeen

## Fusion systems with Benson-Solomon components

Henke, Ellen; Lynd, Justin

*Publication date:*  
2018

*Document Version*  
Early version, also known as pre-print

[Link to publication](#)

*Citation for published version (APA):*  
Henke, E., & Lynd, J. (2018). *Fusion systems with Benson-Solomon components*. (arXiv).

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- ? Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- ? You may not further distribute the material or use it for any profit-making activity or commercial gain
- ? You may freely distribute the URL identifying the publication in the public portal ?

### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

# FUSION SYSTEMS WITH BENSON-SOLOMON COMPONENTS

ELLEN HENKE AND JUSTIN LYND

**ABSTRACT.** The Benson-Solomon systems comprise the one currently known family of simple exotic fusion systems at the prime 2. We show that if  $\mathcal{F}$  is a fusion system on a 2-group having a Benson-Solomon subsystem  $\mathcal{C}$  which is subintrinsic and maximal in the collection of components of involution centralizers, then  $\mathcal{C}$  is a component of  $\mathcal{F}$ , and in particular,  $\mathcal{F}$  is not simple. This is one part of the proof of a Walter's Theorem for fusion systems, which is itself a major step in a program for the classification of a wide class of simple fusion systems of component type at the prime 2.

## 1. INTRODUCTION

This paper is situated within a program to classify a large class of fusion systems of component type at the prime 2, and then use that result to rework and simplify the corresponding part of the classification of the finite simple groups. We refer to the survey article [Asc15] and the memoir [Asc16] for more details on Aschbacher's outline and first steps of this program. However, we take the opportunity to motivate and collect much of the background material in Section 2, and that section can serve as a detailed guide to the proof of the main theorem here for readers who are not familiar with the classification program.

The component-type portion of the classification of the finite simple groups is concerned with components in involution centralizers. A quasisimple group is a perfect group which is simple modulo its center, and a component is a subnormal quasisimple subgroup. A group is of component type if it has a component in the centralizer of one of its involutions. Proceeding by induction, one assumes that the components of involution centralizers in such a simple group are known, and then the objective is to show that the simple group itself is known.

A fusion system is a category  $\mathcal{F}$  with objects the subgroups of a fixed finite  $p$ -group  $S$ , with morphisms injective group homomorphisms between subgroups, and subject to two weak axioms. The standard example of a fusion system is the category  $\mathcal{F}_S(G)$  induced by a finite group  $G$  and one of its Sylow  $p$ -subgroups  $S$ , in which the morphisms are the conjugation maps induced by elements of  $G$ . A fusion system is saturated if it satisfies two more axioms which are easily seen to be satisfied in the standard example. However, there exist *exotic* fusion systems which satisfy the saturation axioms while not being the fusion system of any finite group in the above fashion. Among the most celebrated such examples are the Benson-Solomon systems  $\mathcal{F} = \mathcal{F}_{\text{Sol}}(q)$  at the prime 2, in which  $S$  is isomorphic to a Sylow 2-subgroup of  $\text{Spin}_7(q)$ ,  $q$  an odd prime power. This is a one parameter family of simple fusion systems (cf. Lemma 2.34) having a single

---

*Date:* June 7, 2018.

*2000 Mathematics Subject Classification.* Primary 20D20, Secondary 20D05.

*Key words and phrases.* fusion system, component, Benson-Solomon fusion system, involution centralizer.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 707758.

conjugacy class of involutions, and where the centralizer  $C_{\mathcal{F}}(z) \cong \mathcal{F}_S(\text{Spin}_7(q))$  for an appropriate involution  $z \in S$ . Currently the class of known simple fusion systems at the prime 2 consists of the Benson-Solomon systems together with fusion systems of finite simple groups.

As suggested implicitly above, many finite group theoretic constructions have been established in the context of saturated fusion systems, allowing one to speak of centralizers of subgroups, normal subsystems, quasisimple fusion systems, and so on. Some constructions, such as centralizers of subsystems, have not been defined in full generality.

We now define the terms necessary to state the main theorem. Precise definitions are given in Section 2. Fix a saturated fusion system  $\mathcal{F}$  over the 2-group  $S$ . Following Aschbacher, we denote by  $\mathfrak{C}(\mathcal{F})$  the collection of components of centralizers in  $\mathcal{F}$  of involutions in  $S$ , roughly speaking. The fusion system  $\mathcal{F}$  is said to be of *component type* if  $\mathfrak{C}(\mathcal{F})$  is nonempty. The E-balance Theorem in the form of the Pumpup Lemma (Subsection 2.4) allows one to define an ordering on  $\mathfrak{C}(\mathcal{F})$ , and thus obtain the notion of a *maximal* member of  $\mathfrak{C}(\mathcal{F})$ . For  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ ,  $\mathcal{I}(\mathcal{C})$  denotes the collection of involutions  $t$  such that there is a conjugate  $(\mathcal{C}^\alpha, t^\alpha)$  of  $(\mathcal{C}, t)$  with  $\mathcal{C}^\alpha$  a component of  $C_{\mathcal{F}}(t^\alpha)$ , roughly speaking. Finally a member  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$  is said to be *subintrinsic* in  $\mathfrak{C}(\mathcal{F})$  if there is  $\mathcal{L} \in \mathfrak{C}(\mathcal{C})$  such that  $Z(\mathcal{L}) \cap \mathcal{I}(\mathcal{L})$  is not empty. This means in particular that  $\mathcal{L}$  itself is in  $\mathfrak{C}(\mathcal{F})$ , as witnessed by some involution in the center of  $\mathcal{L}$ .

Our main theorem says that there is no simple fusion system  $\mathcal{F}$  containing a subintrinsic maximal member of  $\mathfrak{C}(\mathcal{F})$  isomorphic to a Benson-Solomon system.

**Theorem 1.1.** *Fix a saturated fusion system  $\mathcal{F}$  over a 2-group and a quasisimple subsystem  $\mathcal{C}$  of  $\mathcal{F}$ . Assume that  $\mathcal{C}$  is a subintrinsic, maximal member of  $\mathfrak{C}(\mathcal{F})$  isomorphic to a Benson-Solomon system. Then  $\mathcal{C}$  is a component of  $\mathcal{F}$ .*

Theorem 1.1 is situated within the proof of Walter’s Theorem for fusion systems [Asc17b], which is one of the major steps in the program. Assuming that each member of  $\mathfrak{C}(\mathcal{F})$  is a known quasisimple system, Walter’s Theorem implies that if the simple saturated 2-fusion system  $\mathcal{F}$  has a member  $\mathcal{C}$  of  $\mathfrak{C}(\mathcal{F})$  that is the 2-fusion system of a group of Lie type in odd characteristic and not too small, then either  $\mathcal{F}$  is the fusion system of a group of Lie type in odd characteristic, or  $\mathcal{F} \cong \mathcal{F}_{\text{Sol}}(q)$ .

A simple saturated 2-fusion system with an involution centralizer having a Benson-Solomon component would necessarily be exotic. Because of the subintrinsic condition, Theorem 1.1 does not immediately rule out the possibility of this happening. However, when combined with the information derived from Walter’s Theorem, it gives strong evidence that such a simple system does not exist. Later, we plan to apply Walter’s Theorem to treat the general case in which  $\mathcal{C}$  is not assumed subintrinsic in  $\mathfrak{C}(\mathcal{F})$ .

We now give an outline of the paper. Section 2 provides the requisite background material, most of which is due to Aschbacher, together with motivation coming from the group case. The proof of Theorem 1.1 begins in Section 3, where we show that a subintrinsic maximal Benson-Solomon component is necessarily a standard subsystem in the sense of Subsection 2.5. When combined with results of Aschbacher in [Asc16], this allows the consideration of a subsystem  $\mathcal{Q}$  which plays the role of the centralizer of  $\mathcal{C}$ , and with a little more work shows that the Sylow subgroup  $Q$  of  $\mathcal{Q}$  is either of 2-rank 1 or elementary abelian. Next, in Section 4, we handle the case in which  $Q$  is elementary abelian. In Section 5, we handle the case in which  $Q$  is quaternion using Aschbacher’s classification of quaternion fusion packets [Asc17a]. In the final Section 6 we handle the cyclic case and complete the proof of Theorem 1.1.

1.1. **Acknowledgements.** We would like to thank Michael Aschbacher for providing us with early copies of his preprints on the various steps in his program and for suggesting Lemma 2.7 to us.

## 2. PRELIMINARIES

2.1. **Local theory of fusion systems.** Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ . For general background on fusion systems, in particular for the definition of a saturated fusion system, we refer the reader to [AKO11, Chapter I]. In addition to the notations introduced there, we will write  $\mathcal{F}^f$  for the set of fully  $\mathcal{F}$ -normalized subgroups of  $S$ . Conjugation-like maps will be written on the right and in the exponent. In particular, if  $\mathcal{E}$  is a subsystem of  $\mathcal{F}$  over  $T$  and  $\alpha \in \text{Hom}_{\mathcal{F}}(T, S)$ , then  $\mathcal{E}^\alpha$  denotes the subsystem of  $\mathcal{F}$  over  $T^\alpha$  with  $\text{Hom}_{\mathcal{E}^\alpha}(P^\alpha, Q^\alpha) = \{\alpha^{-1} \circ \varphi \circ \alpha : \varphi \in \text{Hom}_{\mathcal{E}}(P, Q)\}$  for all  $P, Q \leq T$ .

We recall that, for any subgroup  $X$  of  $S$ , we have the normalizer and the centralizer of  $X$  defined. The normalizer  $N_{\mathcal{F}}(X)$  is a fusion subsystem of  $\mathcal{F}$  over  $N_S(X)$ , and the centralizer  $C_{\mathcal{F}}(X)$  is a fusion subsystem of  $\mathcal{F}$  over  $C_S(X)$ . These subsystems are not necessarily saturated, but if  $X$  is fully  $\mathcal{F}$ -normalized, then  $N_{\mathcal{F}}(X)$  is saturated, and if  $X$  is fully centralized, then  $C_{\mathcal{F}}(X)$  is saturated. Thus, we will often move from a subgroup of  $S$  to a fully  $\mathcal{F}$ -normalized (and thus fully  $\mathcal{F}$ -centralized) conjugate of this subgroup. In this context it will be convenient to use the following notation, which was introduced by Aschbacher.

**Notation 2.1.** For a subgroup  $X \leq S$ , denote by  $\mathfrak{A}(X)$  the set of morphisms  $\alpha \in \text{Hom}_{\mathcal{F}}(N_S(X), S)$  such that  $X^\alpha \in \mathcal{F}^f$ .

Throughout, we will use often without reference that  $\mathfrak{A}(X)$  is non-empty for every subgroup  $X$  of  $S$ . In fact, the following lemma holds.

**Lemma 2.2.** *If  $X \leq S$  and  $Y \in X^{\mathcal{F}} \cap \mathcal{F}^f$ , then there exists  $\alpha \in \mathfrak{A}(X)$  with  $X^\alpha = Y$ .*

*Proof.* See e.g. [AKO11, Lemma I.2.6(c)]. □

If  $x \in S$ , then we often write  $C_{\mathcal{F}}(x)$ ,  $N_{\mathcal{F}}(x)$  and  $\mathfrak{A}(x)$  instead of  $C_{\mathcal{F}}(\langle x \rangle)$ ,  $N_{\mathcal{F}}(\langle x \rangle)$  and  $\mathfrak{A}(\langle x \rangle)$  respectively. Similarly, we call  $x$  fully centralized (fully normalized), if  $\langle x \rangle$  is fully centralized (fully normalized respectively). If  $x$  is an involution, then the reader should note that  $C_{\mathcal{F}}(x) = N_{\mathcal{F}}(\langle x \rangle)$  and  $x$  is fully centralized if and only if  $\langle x \rangle$  is fully normalized.

Recall that a subgroup  $T$  of  $S$  is called *strongly closed* in  $\mathcal{F}$  if  $P^\varphi \leq T$  for every subgroup  $P \leq T$  and every  $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ . The following elementary lemma will be useful later on.

**Lemma 2.3.** *Let  $T$  be strongly closed in  $\mathcal{F}$  and suppose we are given two  $\mathcal{F}$ -conjugate subgroups  $U$  and  $U'$  of  $S$ . If  $T \leq N_S(U)$  and  $U'$  is fully normalized, then  $T \leq N_S(U')$ .*

*Proof.* By Lemma 2.2, there exists  $\alpha \in \mathfrak{A}(U)$  such that  $U^\alpha = U'$ . Then, as  $T$  is strongly closed,  $T = T^\alpha \leq N_S(U)^\alpha \leq N_S(U')$  and this proves the assertion. □

A subsystem  $\mathcal{E}$  of  $\mathcal{F}$  over  $T \leq S$  is called *normal* in  $\mathcal{F}$ , if  $\mathcal{E}$  is saturated,  $T$  is strongly closed,  $\mathcal{E}^\alpha = \mathcal{E}$  for every  $\alpha \in \text{Aut}_{\mathcal{F}}(T)$ , the Frattini condition holds, and a certain technical extra property is fulfilled (see [AKO11, Definition I.6.1]). Here the Frattini condition says that, for every  $P \leq T$  and every  $\varphi \in \text{Hom}_{\mathcal{F}}(P, T)$ , there are  $\varphi_0 \in \text{Hom}_{\mathcal{E}}(P, T)$  and  $\alpha \in \text{Aut}_{\mathcal{F}}(T)$  such that  $\varphi = \varphi_0 \circ \alpha$ .

Particularly important cases of normal subsystems include the normal subsystem  $O^p(\mathcal{F})$  of  $\mathcal{F}$  (cf. [AKO11, Theorem I.7.4]) and the (unique) smallest normal subsystem of  $\mathcal{F}$  over  $S$ , which is denoted by  $O^{p'}(\mathcal{F})$  (cf. [AKO11, Theorem I.7.7]).

Once normal subsystems are defined, there is then a natural definition of a subnormal subsystem. We will need the following lemma.

**Lemma 2.4.** *If  $\mathcal{E}$  is a subnormal subsystem of  $\mathcal{F}$  over  $T$ , then every subgroup of  $T$ , which is fully  $\mathcal{F}$ -normalized, is also fully  $\mathcal{E}$ -normalized.*

*Proof.* In the case that  $\mathcal{E}$  is normal in  $\mathcal{F}$ , this is [Asc08, Lemma 3.4.5]. The general case follows by induction on the subnormal length.  $\square$

Aschbacher [Asc11, Chapter 9] introduced components and the generalized Fitting subsystem  $F^*(\mathcal{F})$  of  $\mathcal{F}$ . By analogy with the definition for groups, a component is a subnormal subsystem of  $\mathcal{F}$  which is quasisimple. Here  $\mathcal{F}$  is called quasisimple if  $O^p(\mathcal{F}) = \mathcal{F}$  and  $\mathcal{F}/Z(\mathcal{F})$  is simple. By [Asc11, 9.8.2,9.9.1], the generalized Fitting subsystem of  $\mathcal{F}$  is the central product of  $O_p(\mathcal{F})$  and the components of  $\mathcal{F}$ . Moreover, for every set  $J$  of component of  $\mathcal{F}$ , there is a well-defined subsystem  $\Pi_{\mathcal{C} \in J} \mathcal{C}$ , which is the central product of the components in  $J$ . Writing  $E(\mathcal{F})$  for the central product of all components of  $\mathcal{F}$ ,  $F^*(\mathcal{F})$  is the central product of  $O_p(\mathcal{F})$  with  $E(\mathcal{F})$ . We will use the following lemma.

**Lemma 2.5.** *If  $\mathcal{C}$  is a component of  $\mathcal{F}$  over  $T$  then the following hold:*

- (a)  $\mathcal{C}$  is normal in  $F^*(\mathcal{F})$ .
- (b) If  $\gamma \in \text{Hom}_{\mathcal{F}}(T, S)$ , then  $\mathcal{C}^\gamma$  is a component  $\mathcal{F}$ .

*Proof.* By definition of a component,  $\mathcal{C}$  is subnormal and thus saturated. As mentioned above, by [Asc11, 9.8.2,9.9.1],  $F^*(\mathcal{F})$  is the central product of  $O_p(\mathcal{F})$  (more precisely  $\mathcal{F}_{O_p(\mathcal{F})}(O_p(\mathcal{F}))$ ) and the components of  $\mathcal{F}$ . It is elementary to check that each of the central factors in a central product of saturated fusion systems is normal. Hence, every component of  $\mathcal{F}$  is normal in  $F^*(\mathcal{F})$  and (a) holds.

For the proof of (b) let  $S_0 \leq S$  such that  $F^*(\mathcal{F})$  is a fusion system over  $S_0$ . The Frattini condition (applied to the normal subsystem  $F^*(\mathcal{F})$ ) says that we can factorize  $\gamma$  as  $\gamma = \gamma_0 \circ \alpha$  with  $\gamma_0 \in \text{Hom}_{F^*(\mathcal{F})}(T, S_0)$  and  $\alpha \in \text{Aut}_{\mathcal{F}}(S_0)$ . By (a),  $\mathcal{C}^{\gamma_0} = \mathcal{C}$  and thus  $\mathcal{C}^\gamma = \mathcal{C}^\alpha$ . As  $F^*(\mathcal{F})$  is a normal subsystem,  $\alpha$  induces an automorphism of  $F^*(\mathcal{F})$ . Thus,  $\mathcal{C}^\alpha$  is normal in  $F^*(\mathcal{F})$  as  $\mathcal{C}$  is normal in  $F^*(\mathcal{F})$ . So  $\mathcal{C}^\gamma = \mathcal{C}^\alpha$  is subnormal in  $\mathcal{F}$ . Hence,  $\mathcal{C}^\gamma$  is a component of  $\mathcal{F}$ , since  $\mathcal{C}^\gamma \cong \mathcal{C}$  is quasisimple.  $\square$

**Lemma 2.6.** *Let  $\mathcal{F}$  be a saturated fusion system which is the central product of saturated subsystems  $\mathcal{F}_1, \dots, \mathcal{F}_n$ . If  $\mathcal{C}$  is a component of  $\mathcal{F}$ , then there exists  $i \in \{1, 2, \dots, n\}$  such that  $\mathcal{C}$  is a component of  $\mathcal{F}_i$ .*

*Proof.* Assume that  $\mathcal{C}$  is a component of  $\mathcal{F}$  which, for all  $i = 1, \dots, n$ , is not a component of  $\mathcal{F}_i$ . Let  $\mathcal{C}$  be a subsystem on  $T \leq S$ , and let  $\mathcal{F}_i$  be a subsystem on  $S_i$  for  $i = 1, \dots, n$ . Since  $\mathcal{F}$  is the central product of  $\mathcal{F}_1, \dots, \mathcal{F}_n$ , each of the subsystems  $\mathcal{F}_1, \dots, \mathcal{F}_n$  is normal in  $\mathcal{F}$ . So for each  $i = 1, \dots, n$ , it follows from [Asc11, 9.6] and the assumption that  $\mathcal{C}$  is not a component of  $\mathcal{F}_i$  that  $T$  centralizes  $S_i$ . As  $S = \Pi_{i=1}^n S_i$ , this yields that  $T$  centralizes  $S$  and is thus abelian. Now [Asc11, 9.1] yields a contradiction to  $\mathcal{C}$  being quasisimple.  $\square$

The following lemma was suggested to us by Aschbacher. By a “known” finite simple group we mean a group isomorphic to one of the groups appearing in the statement of the classification of finite simple groups.

**Lemma 2.7.** *Let  $G$  be a finite group such that  $O(G) = 1$  and, for each component  $K$  of  $G$ ,  $K/Z(K)$  is a “known” finite simple group. Let  $S \in \text{Syl}_2(G)$  and let  $\mathcal{C}$  be a component of  $\mathcal{F}_S(G)$ . Then there exists a component  $K$  of  $G$  such that  $\mathcal{C} = \mathcal{F}_{S \cap K}(K)$ .*

*Proof.* Set  $\mathcal{F} = \mathcal{F}_S(G)$  and  $\mathcal{E} = \mathcal{F}_{S \cap F^*(G)}(F^*(G))$ . Suppose  $\mathcal{C}$  is a subsystem of  $\mathcal{F}$  on  $T$ . As  $F^*(G)$  is normal in  $G$ , the subsystem  $\mathcal{E}$  is normal in  $\mathcal{F}$  by [AKO11, Proposition I.6.2].

Assume first that  $\mathcal{C}$  is not a component of  $\mathcal{E}$ . Write  $J$  for the set of components of  $\mathcal{F}$  which are not a component of  $\mathcal{E}$ , and set  $\mathcal{D} = \prod_{\mathcal{C}' \in J} \mathcal{C}'$ . Then by [Asc11, 9.13],  $\mathcal{F}$  contains a subsystem  $\mathcal{D}\mathcal{E}$  which is the central product of  $\mathcal{D}$  and  $\mathcal{E}$ . As  $\mathcal{C} \in J$ , it follows in particular that  $\mathcal{E} \subseteq C_{\mathcal{F}}(T)$ . As  $\mathcal{E} = \mathcal{F}_{S \cap F^*(G)}(F^*(G))$ , it follows now from [HS15, Theorem B] that  $T \leq C_S(F^*(G)) \leq C_G(F^*(G)) = Z(F^*(G))$ . In particular,  $T$  is abelian, which by [Asc11, 9.1] yields a contradiction to  $\mathcal{C}$  being quasisimple. Thus, we have shown that  $\mathcal{C}$  is a component of  $\mathcal{E}$ .

As  $O(G) = 1$ ,  $F^*(G)$  is the central product of  $O_2(G)$  and the components of  $G$ . Thus,  $\mathcal{E}$  is a central product of  $\mathcal{F}_{O_2(G)}(O_2(G))$  and the subsystems of the form  $\mathcal{F}_{S \cap K}(K)$  where  $K$  is a component of  $G$ . Since  $\mathcal{C}$  is not the fusion system of a 2-group, it follows now from Lemma 2.6 that  $\mathcal{E}$  is a component of  $\mathcal{F}_{S \cap K}(K)$  for some component  $K$  of  $G$ . Set  $\bar{K} := K/Z(K)$ . As  $O(G) = 1$ , we have that  $Z(K) \leq S$  and  $Z(K)$  is contained in the centre of  $\mathcal{F}_{S \cap K}(K)$ . Moreover,  $\mathcal{F}_{S \cap K}(K)/Z(K) = \mathcal{F}_{\overline{S \cap K}}(\bar{K})$ . Recall that  $\bar{K}$  is a “known” finite simple group. So by [Asc17a, Theorem 5.6.18],  $\mathcal{F}_{\overline{S \cap K}}(\bar{K})$  is either simple or  $\overline{S \cap K}$  is normal in  $\bar{K}$ . As  $\mathcal{C}$  is a component of  $\mathcal{F}_{S \cap K}(K)$ , the image of  $\mathcal{C}$  in  $\mathcal{F}_{S \cap K}(K)/Z(K) = \mathcal{F}_{\overline{S \cap K}}(\bar{K})$  is a component of  $\mathcal{F}_{\overline{S \cap K}}(\bar{K})$ . So by [Asc11, 9.9.1], the fusion system  $\mathcal{F}_{\overline{S \cap K}}(\bar{K})$  is not constrained, and thus  $\overline{S \cap K}$  is not normal in  $\mathcal{F}_{\overline{S \cap K}}(\bar{K})$ . Hence,  $\mathcal{F}_{S \cap K}(K)/Z(K) = \mathcal{F}_{\overline{S \cap K}}(\bar{K})$  is simple. In particular,  $Z(K) = Z(\mathcal{F}_{S \cap K}(K))$ . As  $K$  is quasisimple, we have  $K = O^p(K)$ . Therefore, it follows from Puig’s hyperfocal subgroup theorem [Pui00, §1.1] and [AKO11, Corollary I.7.5] that  $O^p(\mathcal{F}_{S \cap K}(K)) = \mathcal{F}_{S \cap K}(K)$ . So  $\mathcal{F}_{S \cap K}(K)$  is quasisimple and thus, by [Asc11, 9.4], we have  $\mathcal{C} = \mathcal{F}_{S \cap K}(K)$ .  $\square$

**2.2. Automorphisms and extensions of fusion and linking systems.** At several points, we will need to be able to construct various extensions of fusion systems and to determine the structure of extensions where they arise. For example, if  $\mathcal{F}$  is a saturated fusion system over  $S$  and  $\mathcal{E}$  is a normal subsystem of  $\mathcal{F}$ , then we want to be able to construct certain subsystems of  $\mathcal{F}$  containing  $\mathcal{E}$  and determine their structure from the structure of  $\mathcal{E}$ . In the category of groups, this is a relatively painless process when the normal subgroup is quasisimple. However, in fusion systems there are technical difficulties that necessitate in many cases the consideration of linking systems associated to  $\mathcal{F}$  and  $\mathcal{E}$ .

We refer to [AKO11, Section III.4] or [AOV12] for the definition of an abstract linking system as used here, and for more details on automorphisms of fusion and linking systems. Fix a linking system  $\mathcal{L}$  for  $\mathcal{F}$  with object set  $\Delta$  and structural functors  $\delta$  and  $\pi$ , which we write on the left of their arguments. The group of automorphisms of  $\mathcal{F}$  is defined by

$$\text{Aut}(\mathcal{F}) = \{\alpha \in \text{Aut}(S) \mid \mathcal{F}^\alpha = \mathcal{F}\}.$$

Then  $\text{Aut}_{\mathcal{F}}(S)$  is normal in  $\text{Aut}(\mathcal{F})$ , and the quotient  $\text{Aut}(\mathcal{F})/\text{Aut}_{\mathcal{F}}(S)$  is denoted  $\text{Out}(\mathcal{F})$ .

An automorphism of  $\mathcal{L}$  is a functor  $\alpha: \mathcal{L} \rightarrow \mathcal{L}$  that commutes with the structural functors. Each automorphism of  $\mathcal{L}$  is indeed an automorphism of the category  $\mathcal{L}$  (and not merely a self-equivalence), and we shall write  $\text{Aut}(\mathcal{L})$  for the group of automorphisms of  $\mathcal{L}$ . There is always a *conjugation map*

$$c: \text{Aut}_{\mathcal{L}}(S) \longrightarrow \text{Aut}(\mathcal{L})$$

which sends an element  $\gamma \in \text{Aut}_{\mathcal{L}}(S)$  to the functor  $c_\gamma \in \text{Aut}(\mathcal{L})$  defined on objects by  $P \mapsto P^\gamma := P^{\pi(\gamma)}$ . For a morphism  $\varphi \in \text{Mor}_{\mathcal{L}}(P, R)$ , the map  $c$  sends  $\varphi$  to  $\varphi^\gamma$ , namely the morphism

$$\varphi^\gamma := \gamma^{-1}|_{P^\gamma, P} \circ \varphi \circ \gamma|_{R, R^\gamma} \in \text{Mor}_{\mathcal{L}}(P^\gamma, R^\gamma),$$

where, for example,  $\gamma|_{R, R^\gamma}$  is the *restriction* of  $\gamma$ , uniquely determined by the condition that  $\delta_{R, S}(1) \circ \gamma = \gamma|_{R, R^\gamma} \circ \delta_{R^\gamma, S}(1)$  in  $\mathcal{L}$ . The image of  $c$  in  $\text{Aut}(\mathcal{L})$  is a normal subgroup of  $\text{Aut}(\mathcal{L})$ , and

$$\text{Out}(\mathcal{L}) := \text{Aut}(\mathcal{L}) / \{c_\gamma \mid \gamma \in \text{Aut}_{\mathcal{L}}(S)\}$$

is the group of outer automorphisms of  $\mathcal{L}$ .

There are natural maps  $\tilde{\mu}: \text{Aut}(\mathcal{L}) \rightarrow \text{Aut}(\mathcal{F})$  and  $\mu: \text{Out}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{F})$  which, at least when  $\Delta = \mathcal{F}^c$ , fit into a commutative diagram

$$(2.8) \quad \begin{array}{ccccccccc} & & & 1 & & 1 & & 1 & & \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ Z(\mathcal{F}) & \xrightarrow{\text{incl}} & Z(S) & \longrightarrow & \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) & \longrightarrow & \varprojlim^1(\mathcal{Z}_{\mathcal{F}}) & \longrightarrow & 1 & \\ & \parallel & \downarrow \delta_S & & \downarrow \tilde{\lambda} & & \downarrow \lambda & & & \\ & & \text{Aut}_{\mathcal{L}}(S) & \xrightarrow{c} & \text{Aut}(\mathcal{L}) & \longrightarrow & \text{Out}(\mathcal{L}) & \longrightarrow & 1 & \\ & & \downarrow \pi_S & & \downarrow \tilde{\mu} & & \downarrow \mu & & & \\ 1 & \longrightarrow & \text{Aut}_{\mathcal{F}}(S) & \longrightarrow & \text{Aut}(\mathcal{F}) & \longrightarrow & \text{Out}(\mathcal{F}) & \longrightarrow & 1 & \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ & & 1 & & 1 & & 1 & & & \end{array}$$

with all rows and columns exact. Here,  $\widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}})$  is a group of 1-cocycles of the center functor defined on the orbit category of  $\mathcal{F}$ -centric subgroups, and  $\varprojlim^1 \mathcal{Z}_{\mathcal{F}}$  is the corresponding cohomology group; see [AKO11, Section III.5].

**Lemma 2.9.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$  with associated centric linking system  $\mathcal{L}$ , and suppose that  $\mu: \text{Out}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{F})$  is injective. Then  $\ker(\tilde{\mu}) = \{c_{\delta_S(z)} \mid z \in Z(S)\}$  consists of automorphisms of  $\mathcal{L}$  induced by conjugation by elements of  $Z(S)$ .*

*Proof.* By assumption on  $\mu$ , we see from (2.8) that  $\varprojlim^1(\mathcal{Z}_{\mathcal{F}}) = 0$  by the exactness of the third column. The assertion follows from exactness of the top row (2.8), together with commutativity of the square containing  $Z(S)$  and  $\text{Aut}(\mathcal{L})$ .  $\square$

In the next proof, we reference a *normal pair* of linking systems  $\mathcal{L} \trianglelefteq \mathcal{L}_1$  as defined in [AOV12, Definition 1.27]. We take the opportunity to write certain maps on the left-hand side of their arguments, which is more standard when working with linking systems as categories.

**Proposition 2.10.** *Let  $\mathcal{F}$  be a saturated fusion system over the 2-group  $S$ , and let  $\mathcal{F}_1$  be a saturated subsystem over  $S_1 \leq S$ . Assume that  $\mathcal{C}$  is a perfect normal subsystem of  $\mathcal{F}_1$  over  $T \in \mathcal{F}^f$  having associated centric linking system  $\mathcal{L}$  such that*

- (i)  $C_S(T) \leq S_1$ , and
- (ii)  $\mu: \text{Out}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{C})$  is injective.

Then  $C_S(T) = C_{S_1}(\mathcal{C})Z(T)$ .

*Proof.* By assumption,  $\mathcal{C}$  is normal in  $\mathcal{F}_1$ , so we may form the product system  $\mathcal{C}_1 := \mathcal{C}S_1$  in this normalizer, as in [Asc11, Chapter 8] or [Hen13]. Then  $O^2(\mathcal{C}_1) = O^2(\mathcal{C}) = \mathcal{C}$  since  $\mathcal{C}$  is perfect, so by [AOV12, Proposition 1.31(a)], there is a normal pair of linking systems  $\mathcal{L} \trianglelefteq \mathcal{L}_1$  associated to the pair  $\mathcal{C} \trianglelefteq \mathcal{C}_1$  in which  $\text{Ob}(\mathcal{L}_1) = \{P \leq S_1 \mid P \cap T \in \mathcal{C}^c\}$ . Note that not only is  $\mathcal{L}$  a subcategory of  $\mathcal{L}_1$ , but the structural functors  $\delta, \pi$  for  $\mathcal{L}$  are the restrictions of the functors for  $\mathcal{L}_1$  by definition of an inclusion of linking systems. Because of this, we write  $\delta, \pi$  also for the structural functors for  $\mathcal{L}_1$ .

Now by the definition of a normal pair of linking systems [AOV12, Definition 1.27(iii)], the conjugation map  $c: \text{Aut}_{\mathcal{L}}(T) \rightarrow \text{Aut}(\mathcal{L})$  lifts to a map  $\text{Aut}_{\mathcal{L}_1}(T) \rightarrow \text{Aut}(\mathcal{L})$ , which we also denote by  $c$ . So the existence of the pair  $\mathcal{L} \trianglelefteq \mathcal{L}_1$  allows one to define a homomorphism  $\nu: S_1 \rightarrow \text{Aut}(\mathcal{L})$  given by the composition  $S_1 \xrightarrow{\delta_T} \text{Aut}_{\mathcal{L}_1}(T) \xrightarrow{c} \text{Aut}(\mathcal{L})$ . This map has kernel

$$(2.11) \quad \ker(\nu) = C_{S_1}(\mathcal{C})$$

by [Sem15, Theorem A].

We can now prove the assertion. Clearly  $C_{S_1}(\mathcal{C})Z(T) \leq C_S(T)$ . For the reverse inclusion, fix  $s \in C_S(T)$ . Then  $\nu$  is defined on  $s$  by (i). The map  $\tilde{\mu}: \text{Aut}(\mathcal{L}) \rightarrow \text{Aut}(\mathcal{C})$  is more precisely defined by the equation  $t^{\tilde{\mu}(\varphi)} = \delta_T^{-1}(\delta_T(t)^{\varphi_T})$  for all  $\varphi \in \text{Aut}(\mathcal{L})$  and all  $t \in T$ . Using this for  $\varphi = \nu(s) = c_{\delta_T(s)}$ , we obtain for all  $t \in T$  that  $t^{\tilde{\mu}(\nu(s))} = \delta_T^{-1}(\delta_T(s)^{-1} \circ \delta_T(t) \circ \delta_T(s)) = \delta_T^{-1}(\delta_T(t^s)) = t^s = t$ , where the last equality uses  $s \in C_S(T)$ . The automorphism  $\tilde{\mu}(\nu(s)) \in \text{Aut}(\mathcal{C})$  is thus trivial. Hence by Lemma 2.9 and assumption on  $\mu$ ,  $\nu(s) = c_{\delta_T(z)} = \nu(z)$  for some  $z \in Z(T)$ . It follows that  $\nu(sz^{-1})$  is the identity on  $\mathcal{L}$ . Hence,  $sz^{-1} \in C_{S_1}(\mathcal{C})$  by (2.11), so  $s \in C_{S_1}(\mathcal{C})Z(T)$ , which completes the proof.  $\square$

In the situation where  $\mathcal{F}$  is realized by a finite group  $G$  with Sylow subgroup  $S$ , there are maps which compare certain automorphism groups of  $G$  with the automorphism groups of  $\mathcal{L}$  and  $\mathcal{F}$ . For example, there is a group homomorphism  $\tilde{\kappa}_G: \text{Aut}(G, S) \rightarrow \text{Aut}(\mathcal{L})$ , where  $\text{Aut}(G, S)$  is the subgroup of  $\text{Aut}(G)$  consisting of those automorphisms which normalize  $S$ . Then  $\tilde{\kappa}_G$  sends the image of  $N_G(S)$  to  $\text{Im}(c) \leq \text{Aut}(\mathcal{L})$ , and so induces a homomorphism  $\kappa: \text{Out}(G) \rightarrow \text{Out}(\mathcal{L})$ .

**Definition 2.12.** A finite group  $G$  with Sylow subgroup  $S$  is said to *tamely realize*  $\mathcal{F}$  if  $\mathcal{F} \cong \mathcal{F}_S(G)$  and the map  $\kappa: \text{Out}(G) \rightarrow \text{Out}(\mathcal{L})$  is split surjective. The fusion system  $\mathcal{F}$  is said to be *tame* if it is tamely realized by some finite group.

From work of Andersen-Oliver-Ventura and Broto-Møller-Oliver, the fusion systems of all finite simple groups at all primes are now known to be tamely realized by some finite group [AO16, Section 3.3]. To give one example of the importance of tameness for getting ahold of extensions of fusion systems of finite quasisimple groups, we mention the following result of Oliver that will be useful later.



**Theorem 2.13.** *Let  $\mathcal{F}$  be a saturated fusion system over the finite  $p$ -group  $S$  and let  $\mathcal{E}$  be a normal subsystem over the subgroup  $T \leq S$ . Assume that  $F^*(\mathcal{F}) = O_p(\mathcal{F})\mathcal{E}$  with  $\mathcal{E}$  quasisimple and that  $\mathcal{E}$  is tamely realized by the finite group  $H$ . Then  $\mathcal{F}$  is tamely realized by a finite group  $G$  such that  $F^*(G) = O_p(G)H$ .*

*Proof.* This is Corollary 2.5 of [Oli16]. □

**2.3. Components of involution centralizers.** Suppose now that  $\mathcal{F}$  is a saturated fusion system over a 2-group  $S$ . If  $\mathcal{F}$  is of component type, then in analogy to the group theoretical case, one wants to work with components of involution centralizers (or more generally with components of normalizers of subgroups of  $S$ ). In fusion systems, the situation is slightly more complicated than in groups, since only components of saturated fusion systems are defined. Therefore, we can only consider components of normalizers of *fully normalized* subgroups. It makes sense to work also with conjugates of such components. Following Aschbacher [Asc16, Section 6] we will use the following notation.

**Notation 2.14.** If  $\mathcal{C}$  is a quasisimple subsystem of  $\mathcal{F}$  over  $T$ , then define the following sets:

- $\mathcal{X}(\mathcal{C})$  is the set of subgroups or elements  $X$  of  $C_S(T)$  such that  $C_{\mathcal{F}}(X)$  contains  $\mathcal{C}$ .
- $\tilde{\mathcal{X}}(\mathcal{C})$  is the set of subgroups or elements  $X$  of  $S$  such that  $\mathcal{C}^\alpha$  is a component of  $N_{\mathcal{F}}(X^\alpha)$  for some  $\alpha \in \mathfrak{A}(X)$ .
- $\mathcal{I}(\mathcal{C})$  is the set of involutions in  $\tilde{\mathcal{X}}(\mathcal{C})$ .

If we want to stress that these sets depend on  $\mathcal{F}$ , we write  $\mathcal{X}_{\mathcal{F}}(\mathcal{C})$ ,  $\tilde{\mathcal{X}}_{\mathcal{F}}(\mathcal{C})$  and  $\mathcal{I}_{\mathcal{F}}(\mathcal{C})$  respectively. Moreover, we write  $\mathfrak{C}(\mathcal{F})$  for the set of quasisimple subsystems  $\mathcal{C}$  of  $\mathcal{F}$  such that  $\mathcal{I}(\mathcal{C})$  is nonempty.

**Lemma 2.15.** *Let  $\mathcal{C}$  be a quasisimple subsystem of  $\mathcal{F}$  over  $T$  and  $X \in \tilde{\mathcal{X}}(\mathcal{C})$ . Then for any  $\varphi \in \text{Hom}_{\mathcal{F}}(\langle X, T \rangle, S)$  the following hold:*

- (a) *If  $X^\varphi \in \mathcal{F}^f$ , then  $\mathcal{C}^\varphi$  is a component of  $N_{\mathcal{F}}(X^\varphi)$ .*
- (b) *We have  $X^\varphi \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$ .*

*Proof.* Assume first  $X^\varphi \in \mathcal{F}^f$ . Let  $\alpha \in \mathfrak{A}(X)$  such that  $\mathcal{C}^\alpha$  is a component of  $N_{\mathcal{F}}(X^\alpha)$ . By Lemma 2.2, there exists  $\beta \in \mathfrak{A}(X^\alpha)$  such that  $X^{\alpha\beta} = X^\varphi$ . Then  $N_S(X^\alpha)^\beta = N_S(X^\varphi)$  and  $\beta$  induces an isomorphism from  $N_{\mathcal{F}}(X^\alpha)$  to  $N_{\mathcal{F}}(X^\varphi)$ . So  $\mathcal{C}^{\alpha\beta}$  is a component of  $N_{\mathcal{F}}(X^\varphi)$ . As  $X^{\alpha\beta} = X^\varphi$ , the map  $\beta^{-1}\alpha^{-1}\varphi$  is a morphism in  $N_{\mathcal{F}}(X^\varphi)$ . Moreover  $\mathcal{C}^{\alpha\beta}$  is conjugate to  $\mathcal{C}^\varphi$  under  $\beta^{-1}\alpha^{-1}\varphi$ . Thus,  $\mathcal{C}^\varphi$  is a component of  $N_{\mathcal{F}}(X^\varphi)$  by Lemma 2.5. This proves (a). If we drop the assumption that  $X^\varphi \in \mathcal{F}^f$  and pick  $\alpha \in \mathfrak{A}(X^\varphi)$ , then applying (a) with  $\varphi\alpha$  in place of  $\varphi$  gives that  $(\mathcal{C}^\varphi)^\alpha = \mathcal{C}^{\varphi\alpha}$  is a component of  $N_{\mathcal{F}}(X^{\varphi\alpha})$ . This gives (b). □

**Lemma 2.16.** *Let  $\mathcal{C}$  be a quasisimple subsystem of  $\mathcal{F}$  over  $T$  and let  $X \in \tilde{\mathcal{X}}(\mathcal{C})$  be a subgroup of  $S$ . Suppose we are given  $Y \in \mathcal{F}^f$  satisfying  $[X, Y] \leq X \cap Y$  and  $\mathcal{C} \subseteq N_{\mathcal{F}}(Y)$ . Then  $X \in \tilde{\mathcal{X}}_{N_{\mathcal{F}}(Y)}(\mathcal{C})$ . In particular, if  $X$  has order 2, then  $\mathcal{C} \in \mathfrak{C}(N_{\mathcal{F}}(Y))$ .*

*Proof.* Let  $\beta \in \mathfrak{A}_{N_{\mathcal{F}}(Y)}(X)$  so that  $X^\beta \in N_{\mathcal{F}}(Y)^f$ . Let  $\alpha \in \mathfrak{A}(X^\beta)$ . Then by [Asc10, 2.2.1, 2.2.2], we have that  $Y^\alpha \in N_{\mathcal{F}}(X^{\beta\alpha})^f$ ,  $(N_S(Y) \cap N_S(X^\beta))^\alpha = N_S(Y^\alpha) \cap N_S(X^{\beta\alpha})$ , and  $\alpha$  induces an isomorphism from  $N_{N_{\mathcal{F}}(Y)}(X^\beta)$  to  $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$ . By Lemma 2.15(a), we have that  $\mathcal{C}^{\beta\alpha}$  is a component of  $N_{\mathcal{F}}(X^{\beta\alpha})$ . So by [Asc16, 2.2.5.2],  $\mathcal{C}^{\beta\alpha}$  is a component of  $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$ . As  $\alpha$  induces an isomorphism from  $N_{N_{\mathcal{F}}(Y)}(X^\beta)$  to  $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$ , this implies that  $\mathcal{C}^\beta$  is a component of  $N_{N_{\mathcal{F}}(Y)}(X^\beta)$ . This proves  $X \in \tilde{\mathcal{X}}_{N_{\mathcal{F}}(Y)}(\mathcal{C})$  and the assertion follows. □

**2.4. Pumping up.** Crucial in the classification of finite simple groups of component type is the Pump-Up Lemma, which leads to the definition of a maximal component. As we explain in more detail in the next subsection, such maximal components have very nice properties generically, which ultimately allow one to pin down the group if the structure of a maximal component is known.

The main purpose of this section is to state the Pump-Up Lemma for fusion systems. However, to give the reader an intuition, we briefly want to describe the Pump-Up Lemma for groups. Let  $G$  be a finite group. To avoid technical difficulties which do not play a role in the context of fusion systems, we assume that none of the 2-local subgroups of  $G$  has a normal subgroup of odd order. The results we state here are actually true for all almost simple groups, but to show this one would have to use the  $B$ -theorem whose proof is extremely difficult. Avoiding the necessity to prove the  $B$ -theorem is one of the major reasons why it is hoped that working in the category of fusion systems will lead to a simpler proof of the classification of finite simple groups.

Let  $t$  be an involution of  $G$ . If  $O(G) = 1$ , then the  $L$ -balance theorem of Gorenstein and Walter gives that  $E(C_G(t)) \leq E(G)$ , where  $E(G)$  denotes the product of the components of  $G$ . Further analysis shows that a component  $C$  of  $C_G(t)$  lies in  $E(G)$  in a particular way. Namely, either  $C$  is a component of  $G$ , or there exists a component  $D$  of  $G$  such that  $D = D^t$  and  $C$  is a component of  $C_D(t)$ , or there exists a component  $D$  of  $G$  such that  $D \neq D^t$  and  $C = \{dd^t : d \in D\}$  is the homomorphic image of  $D$  under the map  $d \mapsto dd^t$ . If one applies this property to the centralizer of a suitable involution  $a$  rather than to the whole group  $G$ , then one obtains the Pump-Up Lemma. More precisely, consider two commuting involutions  $t$  and  $a$  centralized by a quasisimple subgroup  $C$  which is a component of  $C_G(t)$ , and thus of  $C_{C_G(a)}(t)$ . The result stated above yields immediately that one of the following holds:

- (1)  $C$  is a component of  $C_G(a)$ .
- (2) There exists a component  $D$  of  $C_G(a)$  such that  $D = D^t$  and  $C$  is a component of  $C_D(t)$ .
- (3) There exists a component  $D$  of  $C_G(a)$  such that  $D \neq D^t$  and  $C = \{dd^t : d \in D\}$  is a homomorphic image of  $D$ .

This statement is known as the Pump-Up Lemma. If (2) holds then  $D$  is called a *proper pump-up* of  $C$ . The component  $C$  is called *maximal* if it has no proper pump-ups.

We now state a similar result for fusion systems, which was formulated by Aschbacher. Again, the statement is slightly more complicated than the statement for groups, since we need to pass from an involution  $a$  to a fully centralized conjugate of  $a$  for the centralizer to be saturated.

**Lemma 2.17** ([Asc16, 6.1.11]). *Let  $\mathcal{F}$  be a saturated fusion system over a 2-group  $S$  and let  $\mathcal{C}$  be a quasisimple subsystem of  $\mathcal{F}$  on  $T$ . Suppose we are given two commuting involutions  $t, a \in S$  such that  $t \in \mathcal{I}(\mathcal{C})$  and  $\langle t, a \rangle \in \mathcal{X}(\mathcal{C})$ . Fix  $\alpha \in \mathfrak{A}(a)$ . Set  $\bar{a} = a^\alpha$ ,  $\bar{t} = t^\alpha$  and  $\bar{\mathcal{C}} = \mathcal{C}^\alpha$ . Then one of the following holds:*

- (1) (trivial)  $\bar{\mathcal{C}}$  is a component of  $C_{\mathcal{F}}(\bar{a})$ , so  $a \in \mathcal{I}(\mathcal{C})$ ,
- (2) (proper) there is  $\zeta \in \text{Hom}_{C_{\mathcal{F}}(\bar{a})}(C_S(\langle \bar{a}, \bar{t} \rangle), C_S(\bar{a}))$  and a  $\bar{t}^\zeta$ -invariant component  $\mathcal{D}$  of  $C_{\mathcal{F}}(\bar{a})$  such that  $\bar{\mathcal{C}}^\zeta$  is a component of  $C_{\mathcal{D}(\bar{t}^\zeta)}(\bar{t}^\zeta)$ , and we have  $\bar{\mathcal{C}}^\zeta \neq \mathcal{D}$ ,
- (3) (diagonal) there is a component  $\mathcal{D}$  of  $C_{\mathcal{F}}(\bar{a})$  such that  $\mathcal{D} \neq \mathcal{D}^{\bar{t}}$ ,  $\bar{\mathcal{C}} \leq \mathcal{D}_0 := \mathcal{D}\mathcal{D}^{\bar{t}}$ , and  $\mathcal{C}$  is a homomorphic image of  $\mathcal{D}$ .

**Definition 2.18.** Let  $\mathcal{F}$  be a saturated 2-fusion system and  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ .

- Whenever the hypotheses of Lemma 2.17 occurs, and  $\mathcal{D}$  satisfies (2) of the conclusion, then  $\mathcal{D}$  is a *proper pump-up* of  $\mathcal{C}$ .
- $\mathcal{C}$  is called *maximal* (or a *maximal component*) if it has no proper pump-ups.

**2.5. Standard components.** We explain now in more detail how maximal components play a role in pinning down the structure of a finite simple group  $G$ , and in how far these ideas carry over to fusion systems. As in the previous subsection, we start by explaining the basic ideas for groups. For that, assume again that  $G$  is a finite group in which no involution centralizer has a non-trivial normal subgroup of odd order.

Write  $\mathfrak{C}(G)$  for the set of components of involution centralizers of  $G$ . Using the Pump-Up Lemma, one can choose  $C \in \mathfrak{C}(G)$  such that every element  $D \in \mathfrak{C}(G)$  which maps homomorphically onto  $C$  is maximal. For such  $C$ , Aschbacher’s component theorem says basically that, with some “small” exceptions, either  $C$  is a homomorphic image of a component of  $G$ , or the following two conditions hold:

- (C1’)  $C$  does not commute with any of its conjugates; and
- (C2’) if  $t$  is an involution centralizing  $C$ , then  $C$  is a component of  $C_G(t)$ .

Assuming that (C1’) and (C2’) hold and  $C/Z(C)$  is a “known” finite simple group, the structure of  $G$  is determined case by case from the structure of  $C$ . The problem of classifying  $G$  from the structure of such a subgroup  $C$  is usually referred to as a *standard form problem*. The key to solving such a standard form problem is that properties (C1’) and (C2’) imply that the centralizer  $C_G(C)$  is a tightly embedded subgroup of  $G$  and thus has (by various theorems in the literature) a very restricted structure if  $G$  is simple. Here a subgroup  $K$  of  $G$  of even order is called *tightly embedded* in  $G$  if  $K \cap K^g$  has odd order for any element  $g \in G - N_G(K)$ . A *standard subgroup* of  $G$  is a quasisimple subgroup  $C$  of  $G$  such that  $C$  commutes with none of its conjugates,  $K := C_G(C)$  is tightly embedded in  $G$ , and  $N_G(C) = N_G(K)$ . If  $C$  is a component of an involution centralizer which satisfies properties (C1’) and (C2’), then it is straightforward to prove that  $C$  is a standard subgroup. So if  $G$  is simple, then with some small exceptions, Aschbacher’s component theorem implies that there exists a standard subgroup  $C$  of  $G$ .

We will now explain the theory of standard components of fusion systems, which Aschbacher [Asc16] has developed roughly in analogy to the situation for groups as far as this seems possible. For the remainder of this subsection let  $\mathcal{F}$  be a saturated fusion system over a 2-group  $S$ , and let  $\mathcal{C}$  be a quasisimple subsystem of  $\mathcal{F}$  on  $T$ . The situation for fusion systems is significantly more complicated, most importantly since the definition of a standard component of a group involves a statement about its centralizer, and the centralizer of  $\mathcal{C}$  in  $\mathcal{F}$  is currently only defined in certain special cases. For example, Aschbacher has defined the normalizer and the centralizer of a component of a fusion system [Asc16, Sections 2.1 and 2.2]. In particular, if  $\mathcal{C}$  is a component of  $C_{\mathcal{F}}(t)$  for a fully centralized involution  $t$ , then  $C_{C_S(t)}(\mathcal{C})$  is defined inside  $C_{\mathcal{F}}(t)$ . If  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ , then this allows us to define a subgroup of  $S$  which centralizes  $\mathcal{C}$ , dependent on an involution  $t \in \mathcal{I}(\mathcal{C})$ .

**Notation 2.19** (cf. (6.1.15) in [Asc16]). If  $t \in \mathcal{I}(\mathcal{C})$  and  $\alpha \in \mathfrak{A}(t)$ , then define

$$P_{t,\alpha} := C_{C_S(t^\alpha)}(\mathcal{C}^\alpha) \cap C_S(t)^\alpha$$

and

$$Q_t := Q_{t,\alpha} = P_{t,\alpha}^{\alpha^{-1}}$$

By [Asc16, 6.6.16.1],  $Q_{t,\alpha} \leq C_S(t)$  is independent of the choice of  $\alpha$  and so  $Q_t$  is indeed well-defined. With this definition in place, one can formulate conditions on  $\mathcal{C}$  which roughly correspond to conditions (C1') and (C2'). If  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$  fulfills such conditions, then  $\mathcal{C}$  is called *terminal*. The precise definition is given below in Definition 2.21.

**Notation 2.20** (cf. (6.1.17) and (6.2.7) in [Asc16]).

- $\Delta(\mathcal{C})$  is the set of  $\mathcal{F}$ -conjugates  $\mathcal{C}_1$  of  $\mathcal{C}$  such that, writing  $T_1$  for the Sylow of  $\mathcal{C}_1$ , we have  $T_1^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$  and  $T^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$ .
- $\rho(\mathcal{C})$  is the set of pairs  $(t^\varphi, \mathcal{C}^\varphi)$  such that  $t \in \mathcal{I}(\mathcal{C})$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(\langle t, T \rangle, S)$ .
- $\rho_0(\mathcal{C})$  is the set of  $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$  such that all nonidentity elements of  $Q_{t_1}$  lie in  $\tilde{\mathcal{X}}(\mathcal{C}_1)$ .

By Lemma 2.15(b), we have  $t_1 \in \mathcal{I}(\mathcal{C}_1)$  for any  $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$ . In particular, in the definition of  $\rho_0(\mathcal{C})$ , the subgroup  $Q_{t_1}$  is well-defined.

**Definition 2.21.** A subsystem  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$  is called *terminal* if the following conditions hold:

- (C0)  $T \in \mathcal{F}^f$ ,
- (C1)  $\Delta(\mathcal{C}) = \emptyset$ , and
- (C2)  $\rho(\mathcal{C}) = \rho_0(\mathcal{C})$ .

In this definition, property (C2) corresponds roughly to property (C2') above. Moreover, assuming (C2), property (C1) should be thought of as roughly corresponding to property (C1') above.

Aschbacher proved a version of his component theorem for fusion systems [Asc16, Theorem 8.1.5]. Suppose  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$  is such that every  $\mathcal{D} \in \mathfrak{C}(\mathcal{F})$  mapping homomorphically onto  $\mathcal{C}$  is maximal. The component theorem for fusion systems states essentially that, with some small exceptions, either  $\mathcal{C}$  is the homomorphic image of a component of  $\mathcal{F}$ , or  $\mathcal{C}$  is terminal. This statement is similar to the statement of the component theorem in the group case. However, it is not clear that the centralizer of a terminal component is defined and “tightly embedded” in  $\mathcal{F}$ . This makes it more complicated to define standard subsystems. We will work with Aschbacher’s definition of a standard subsystem, which we state next.

**Definition 2.22.** The quasisimple subsystem  $\mathcal{C}$  of  $\mathcal{F}$  is called a *standard subsystem* of  $\mathcal{F}$  if the following four conditions are satisfied:

- (S1)  $\tilde{\mathcal{X}}(\mathcal{C})$  contains a unique (with respect to inclusion) maximal member  $Q$ .
- (S2) For each  $1 \neq X \leq Q$  and  $\alpha \in \mathfrak{A}(X)$ , we have  $\mathcal{C}^\alpha \trianglelefteq N_{\mathcal{F}}(X^\alpha)$ .
- (S3) If  $1 \neq X \leq Q$  and  $\beta \in \mathfrak{A}(X)$  with  $X^\beta \leq Q$ , then  $T^\beta = T$ .
- (S4)  $\text{Aut}_{\mathcal{F}}(T) \leq \text{Aut}(\mathcal{C})$ .

If  $\mathcal{C}$  satisfies conditions (S1),(S2),(S3), then  $\mathcal{C}$  is called *nearly standard*.

**Remark 2.23.** In the above definition, the first condition (S1) says essentially that the centralizer of  $\mathcal{C}$  in  $S$  is well-defined. Namely, the unique maximal member  $Q$  of  $\tilde{\mathcal{X}}(\mathcal{C})$  should be thought of as this centralizer. Given a standard subsystem  $\mathcal{C}$  of  $\mathcal{F}$ , this allows Aschbacher [Asc16, Definition 9.1.4] to define a saturated subsystem  $\mathcal{Q}$  of  $\mathcal{F}$  over  $Q$  which plays the role of the centralizer of  $\mathcal{C}$  in  $\mathcal{F}$ . More precisely,  $\mathcal{Q}$  centralizes  $\mathcal{C}$  in the sense that  $\mathcal{F}$  contains a subsystem which is a central product of  $\mathcal{Q}$  and  $\mathcal{C}$  (cf. [Asc16, 9.1.6.1]). Also, by [Asc16, 9.1.6.2],  $\mathcal{Q}$  is a tightly embedded as defined in the next subsection (cf. Definition 2.26). We will refer to  $\mathcal{Q}$  as the *centralizer* of  $\mathcal{C}$  in  $\mathcal{F}$ .

In general, it is difficult to get control of  $C_S(T)$  when  $T$  is the Sylow subgroup of a member  $\mathcal{C}$  of  $\mathfrak{C}(\mathcal{F})$ . However,  $C_S(T) \leq N_S(Q)$  when  $\mathcal{C}$  is standard. This gives much needed leverage, as is shown in the next lemma.

**Lemma 2.24.** *Let  $\mathcal{F}$  be a saturated fusion system over the 2-group  $S$ . Suppose  $\mathcal{C}$  is a standard subsystem of  $\mathcal{F}$  with centralizer  $\mathcal{Q}$  over  $Q$ . Let  $\mathcal{L}$  be a centric linking system associated to  $\mathcal{C}$ . If  $\mu: \text{Out}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{C})$  (see Section 2.2) is injective, then  $C_S(T) = QZ(T)$ .*

*Proof.* As  $Q$  is fully  $\mathcal{F}$ -normalized by [Asc16, 9.1.1],  $\mathcal{F}_1 := N_{\mathcal{F}}(Q)$  is a saturated fusion system over  $S_1 = N_S(Q)$ . Further, (S2) says that  $\mathcal{C}$  is normal in  $\mathcal{F}_1$ . Finally, by [Asc16, Proposition 5],  $\mathcal{Q}$  is normal in  $N_{\mathcal{F}}(T)$  and  $C_{N_S(Q)}(\mathcal{C}) = Q$ . In particular,  $C_S(T) \leq S_1 = N_S(Q)$ . Thus, if  $\mu$  is injective, then  $C_S(T) = C_{N_S(Q)}(\mathcal{C})Z(T) = QZ(T)$  by Proposition 2.10.  $\square$

When considering involution centralizer problems for fusion systems, we generally need to verify in each individual case that the terminal component we consider is standard. In contrast with the group case, this is not a straightforward task. Indeed, in some cases a terminal component is provably not standard. However, it develops that many of these technically difficult configurations are ultimately unnecessary to treat, because they arise for example in almost simple fusion systems that are not simple, or in wreath products of a simple system with an involution. Thus, we will usually need assumptions that go beyond supposing merely that the structure of the terminal component we consider is known, but also information about the embedding of that subsystem in the ambient system. Such stronger assumptions can be made if one classifies, as was proposed by Aschbacher, the simple “odd systems” rather than the simple fusion systems of component type (cf. [Asc16]). The hypothesis in Theorem 1.1 that  $\mathcal{C}$  is subintrinsic should be seen in this context.

**Definition 2.25.** Let  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ . Then  $\mathcal{C}$  is said to be *subintrinsic in  $\mathfrak{C}(\mathcal{F})$*  if there exists  $\mathcal{H} \in \mathfrak{C}(\mathcal{C})$  such that  $\mathcal{I}_{\mathcal{F}}(\mathcal{H}) \cap Z(\mathcal{H}) \neq \emptyset$ .

It follows fairly straightforwardly from results of Aschbacher that a subintrinsic Benson-Solomon component  $\mathcal{C}$  is terminal. Rather than use the component theorem for fusion systems, it is more convenient in our case to show that  $\mathcal{C}$  is terminal using [Asc16, Theorem 7.4.14], which is a major ingredient of the proof of the component theorem. As suggested above, a nontrivial amount of work is then required to go on and show that  $\mathcal{C}$  is standard; see Section 3.

**2.6. Tightly embedded subsystems and tight split extensions.** Recall from the previous subsection that a subgroup  $K$  of a finite group  $G$  is called *tightly embedded* if  $K$  has even order and  $K \cap K^g$  has odd order for every  $g \in G \setminus N_G(K)$ . This definition does not translate well to fusion systems as it is, but there exist suitable reformulations. It follows from Aschbacher [Asc16, 0.7.1] that a subgroup  $K$  of  $G$  of even order is tightly embedded if and only if the following two conditions hold:

- (T1')  $K$  is normalized by  $N_G(X)$  for every non-trivial 2-subgroup  $X$  of  $K$ .
- (T2') For every involution  $x$  of  $K$ ,  $x^G \cap K = x^{N_G(K)}$ .

If  $K$  is tightly embedded and  $Q$  is a Sylow 2-subgroup of  $K$ , then note furthermore that  $N_G(Q) \leq N_G(K)$  and  $N_G(K) = KN_G(Q)$  by a Frattini argument. This leads to a definition of tightly embedded subsystem of saturated fusion systems at arbitrary primes.

**Definition 2.26** (cf. [Asc16, Chapter 3]). Let  $\mathcal{F}$  be a saturated fusion system on a  $p$ -group  $S$ , and let  $\mathcal{Q}$  be a saturated subsystem of  $\mathcal{F}$  on a fully normalized subgroup  $Q$  of  $\mathcal{F}$ . Then  $\mathcal{Q}$  is *tightly embedded in  $\mathcal{F}$*  if it satisfies the following three conditions:

(T1) For each  $1 \neq X \in \mathcal{Q}^f$  and each  $\alpha \in \mathfrak{A}(X)$ ,

$$O^{p'}(N_{\mathcal{Q}}(X))^\alpha \text{ is normal in } N_{\mathcal{F}}(X^\alpha).$$

(T2) For each  $X \leq Q$  of order  $p$ ,

$$X^{\mathcal{F}} \cap Q = X^{\text{Aut}_{\mathcal{F}}(Q)\mathcal{Q}}$$

where  $X^{\text{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} := \{X\alpha\varphi : \alpha \in \text{Aut}_{\mathcal{F}}(Q), \varphi \in \text{Hom}_{\mathcal{Q}}(X\alpha, Q)\}$ .

(T3)  $\text{Aut}_{\mathcal{F}}(Q) \leq \text{Aut}(\mathcal{Q})$ .

When working with standard subsystems later on, we will need the following lemma on tightly embedded subsystems.

**Lemma 2.27.** *Let  $\mathcal{F}$  be a saturated fusion system on  $S$ , and suppose  $\mathcal{Q}$  is a tightly embedded subsystem of  $\mathcal{F}$  on an abelian subgroup  $Q$  of  $S$ . Then  $\mathcal{F}_Q(Q)$  is tightly embedded in  $\mathcal{F}$ .*

*Proof.* As  $Q$  is abelian, by Alperin's fusion theorem (cf. [AKO11, Theorem I.3.6]), the following holds:

(\*) The  $p$ -group  $Q$ , and thus the subsystem  $\mathcal{F}_Q(Q)$ , is normal in any saturated fusion system on  $Q$ .

Let  $1 \neq X \leq Q$  and  $\alpha \in \mathfrak{A}(X)$ . By (\*), we have  $Q = N_Q(X) \trianglelefteq N_{\mathcal{Q}}(X)$  and thus  $Q = O^{p'}(N_{\mathcal{Q}}(X))$ . As  $\mathcal{Q}$  is tightly embedded, it follows  $N_Q(X)^\alpha = Q^\alpha = O^{p'}(N_{\mathcal{Q}}(X))^\alpha \trianglelefteq N_{\mathcal{F}}(X^\alpha)$ . So (T1) holds for  $\mathcal{F}_Q(Q)$ .

Let  $X \leq Q$  be of order  $p$ . Again using (\*), we have  $Q \trianglelefteq \mathcal{Q}$ . So every morphism in  $\mathcal{Q}$  extends to an element of  $\text{Aut}_{\mathcal{Q}}(Q) \leq \text{Aut}_{\mathcal{F}}(Q)$ , and this implies  $X^{\text{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} = X^{\text{Aut}_{\mathcal{F}}(Q)}$ . Hence, as  $\mathcal{Q}$  is tightly embedded,  $X^{\mathcal{F}} \cap Q = X^{\text{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} = X^{\text{Aut}_{\mathcal{F}}(Q)} = X^{\text{Aut}_{\mathcal{F}}(Q)\mathcal{F}_Q(Q)}$ . This shows that (T2) holds for  $\mathcal{F}_Q(Q)$ . Clearly (T3) holds for  $\mathcal{F}_Q(Q)$ .  $\square$

To exploit the existence of standard subsystems, it is useful in many situations to study certain kinds of extensions involving tightly embedded subsystems. We summarize the main definitions:

**Definition 2.28.** Let  $\mathcal{F}_0$  be a fusion system on a 2-group  $S_0$ .

- A *split extension* of  $\mathcal{F}_0$  is a pair  $(\mathcal{F}, U)$ , where
  - $\mathcal{F}$  is a saturated fusion system over a 2-group  $S$ ,
  - $\mathcal{F}_0$  is normal in  $\mathcal{F}$ ,
  - $O^2(\mathcal{F}) = O^2(\mathcal{F}_0)$ , and
  - $U$  is a complement to  $S_0$  in  $S$ .
- The split extension  $(\mathcal{F}, U)$  is *tight* if  $\mathcal{F}_U(U)$  is tightly embedded in  $\mathcal{F}$ .
- A *critical split extension* is a tight split extension in which  $U$  is a four group.
- $\mathcal{F}_0$  is said to be *split* if there exists no nontrivial critical split extension of  $\mathcal{F}_0$ ; that is, for each such extension  $(\mathcal{F}, U)$ , the fusion system  $\mathcal{F}$  is the central product of  $\mathcal{F}$  with  $C_S(\mathcal{F}_0)$ .

Suppose  $\mathcal{F}$  is a saturated 2-fusion system and  $\mathcal{C}$  is a standard component with centralizer  $Q$  on  $Q$ . If  $\mathcal{C}$  is split, then by [Asc16, Theorem 8],  $\mathcal{C}$  is either a component of  $\mathcal{F}$ , or  $Q$  is elementary abelian, or the 2-rank of  $Q$  equals 1. We show in Lemma 2.38 that the Benson–Solomon fusion systems are split. So after showing that a component  $\mathcal{C}$  as in Theorem 1.1 is standard, we know that, unless  $\mathcal{C}$  is a component of  $\mathcal{F}$ , its centralizer  $Q$  in  $S$  is either elementary abelian or quaternion or cyclic. Accordingly, these are the cases we will treat.

**Lemma 2.29.** *Let  $\mathcal{C}$  be a quasisimple saturated fusion system over the 2-group  $T$ , and let  $(\mathcal{F}, U)$  be a critical split extension of  $\mathcal{C}$  over the 2-group  $S$ . Then*

- (a)  $\text{Aut}_{\mathcal{F}}(U) = 1$  and so  $N_{\mathcal{F}}(U) = C_{\mathcal{F}}(U)$ ; and
- (b)  $\langle u \rangle \in \mathcal{F}^f$  and  $C_{\mathcal{F}}(u) = C_{\mathcal{F}}(U)$  for each  $1 \neq u \in U$ .

*Proof.* By definition of critical split extension,  $U$  is a four subgroup of  $S$  tightly embedded in  $\mathcal{F}$  and a complement to  $T$  in  $S$ . Also,  $O^2(\mathcal{F}) = O^2(\mathcal{C}) = \mathcal{C}$ , as  $\mathcal{C}$  is quasisimple. Since  $O^2(\mathcal{F}) = \mathcal{C}$ , this means  $\text{hnp}(\mathcal{F}) = T$ . Since  $S/\text{hnp}(\mathcal{F}) \cong U$  is abelian, we see from [AKO11, Lemma I.7.2] that also  $\text{foc}(\mathcal{F}) = T$ . Thus,  $\text{Aut}_{\mathcal{F}}(U) = 1$ , since otherwise  $T \cap U = \text{foc}(\mathcal{F}) \cap U \geq [U, \text{Aut}_{\mathcal{F}}(U)] > 1$ , which is not the case. This proves the first assertion in (a), and the second then follows by definition of the normalizer and centralizer systems.

Now by definition of tight embedding,  $U$  is fully normalized in  $\mathcal{F}$ . Fix  $1 \neq u \in U$ . By (T2) and part (a), it follows that  $u^{\mathcal{F}} \cap U = \{u\}$ . However, (3.1.5) of [Asc16] says that  $\langle u \rangle$  has a fully  $\mathcal{F}$ -normalized  $\mathcal{F}$ -conjugate in  $U$ , so  $\langle u \rangle \in \mathcal{F}^f$ . Then taking  $\alpha$  to be identity in (T1), we see that  $U$  is normal in  $N_{\mathcal{F}}(\langle u \rangle) = C_{\mathcal{F}}(u)$ , so that  $C_{\mathcal{F}}(u) \leq N_{\mathcal{F}}(U) = C_{\mathcal{F}}(U)$  by (a). This completes the proof of (b), as the other inclusion is clear.  $\square$

**2.7. The fusion system of  $\text{Spin}_7(q)$  and  $\mathcal{F}_{\text{Sol}}(q)$ .** Our main references for  $\mathcal{F}_{\text{Sol}}(q)$  and for 2-fusion systems of  $\text{Spin}_7(q)$  are [LO02, LO05, COS08, AC10, HL17].

We aim to follow Section 4 of Aschbacher and Chermak fairly closely [AC10], except that it will be convenient to restrict the choice of the finite fields  $\mathbf{F}_q$  over which the systems in question are defined, and to make changes to notation so as to not conflict with our later choices. For example, we will later take  $T$  to be a Sylow 2-subgroup of  $H = \text{Spin}_7(q)$  and  $\mathcal{F}_{\text{Sol}}(q)$ , so we shall write a split maximal torus of  $H$  in a different way. For concreteness, we consider a fixed but arbitrary nonnegative integer  $l$ , and set  $q_l = 5^{2^l}$ .

Let  $\bar{\mathbf{F}}$  be an algebraic closure of the field with 5 elements (thus, we take  $p = 5$  in [AC10, Section 4]). Let  $\bar{H} = \text{Spin}_7(\bar{\mathbf{F}})$ , and  $\bar{T}$  a maximal torus of  $\bar{H}$ . Let  $\psi$  be a Frobenius endomorphism of  $\bar{H}$  which induces the 5-th power map on  $\bar{T}$ , and set  $\psi_l = \psi^{2^l}$ . Then as  $\bar{H}$  is of universal type,  $H := C_{\bar{H}}(\psi_l) = \text{Spin}_7(q_l)$  and  $C_{\bar{T}}(\psi_l)$  is the aforementioned split maximal torus. From [AC10, Lemmas 4.8, 4.9], the normalizer of  $C_{\bar{T}}(\psi_l)$  in  $H$  contains a Sylow 2-subgroup of  $H$ , which we fix and denote by  $T$  for the remainder.

Write  $\mathcal{H} := \mathcal{H}_l := \mathcal{F}_{\text{Spin}}(q_l)$  for the fusion system  $\mathcal{F}_T(H)$ . Fix also a fusion system  $\mathcal{C}$  over  $T$  isomorphic to a Benson-Solomon system  $\mathcal{C} := \mathcal{C}_l := \mathcal{F}_{\text{Sol}}(q_l)$  in such a way that  $C_{\mathcal{C}}(z) = \mathcal{H}$  where  $\langle z \rangle = Z := Z(T)$  is of order 2. We next set up notation for some common subgroups of  $T$ , and we recall the various parts of the set up appearing in [AC10, §4] that are needed later.

**Notation 2.30.** Set  $k = l + 2$ , let  $T_k = T \cap \bar{T}$  be the 2-power torsion subgroup of the maximal torus  $C_{\bar{T}}(\psi_l)$  of  $H$ , and let  $w_0 \in T$  be the element of order 2 fixed in [AC10, Lemma 4.3]. Thus,  $w_0$  inverts  $\bar{T}$  and is centralized by  $\psi_m$  for all  $m \geq 0$ . The 2-group  $T$  has a sequence of elementary abelian subgroups

$$1 < Z < U < E < A,$$

each of index 2 in the next, with  $Z = Z(T)$  as above,  $U$  the unique normal four subgroup of  $T$ ,  $E = \Omega_1(T_k)$ , and  $A = E\langle w_0 \rangle$ , an elementary abelian subgroup of order 16. We also set  $R = C_T(E) = T_k\langle w_0 \rangle = T_k A$ .

The following lemma collects a number of properties of these subgroups and their automorphism groups.

**Lemma 2.31.** *The following hold.*

- (a) For each  $2 \leq k_0 \leq k$ ,  $T_{k_0}$  is the unique homocyclic abelian subgroup of  $T$  of rank 3 and exponent  $2^{k_0}$ , and  $T_{k_0}$  is inverted by  $w_0$ .
- (b)  $T_k$  is  $\mathcal{C}$ -centric,  $T/T_k \cong C_2 \times D_8$ , and  $\text{Aut}_{\mathcal{C}}(T_k) \cong C_2 \times GL_3(2)$ .
- (c)  $R$  is characteristic in  $T$ .
- (d)  $A$  is an elementary abelian subgroup of  $T$  of maximum order, and so  $T$  has 2-rank 4.
- (e)  $\text{Aut}_{\mathcal{C}}(X) = \text{Aut}(X)$  for  $X \in \{Z, U, E, A\}$ , and  $\text{Out}_{\mathcal{C}}(R) \cong GL_3(2)$ .

*Proof.* By Lemma 4.9(b) of [AC10],  $T_2 \leq T_{k_0}$  is the unique homocyclic subgroup of  $T$  of rank 3 and exponent 4. Moreover,  $T_k = C_H(T_2) = C_{\bar{T}}(\psi_l)$  is of rank 3 and of exponent  $2^k$ . This shows that  $T_2$ , and more generally,  $T_{k_0} = \Omega_{k_0}(T_k)$  for  $2 \leq k_0 \leq k$  is the unique subgroup of  $T$  of its isomorphism type. Also,  $w_0$  inverts  $T_{k_0}$  by [AC10, Lemma 4.3(a)].

This completes the proof of (a). Again as  $T_k = C_H(T_2)$ , it follows that  $T_k$  is  $\mathcal{C}$ -centric. The second statement in part (b) follows from [AC10, Lemma 4.3(c)], while the third is the content of [AC10, Theorem 5.2]. Since  $T_k$  is characteristic in  $T$  by (a), also  $R = C_T(\Omega_1(T_k))$  is characteristic in  $T$ , which is the statement in (c).

Now as  $E = \Omega_1(T_k)$  is elementary abelian of order 8 by (a), and  $w_0$  inverts  $T_k$ , it follows that  $A$  is elementary abelian of order 16. There are no elementary abelian subgroups of  $T$  of rank 5 by [AC10, Lemma 7.9(a)], so (d) holds. We refer to Lemma 3.1 of [LO02] for the  $\mathcal{C}$ -automorphism groups of  $X \in \{Z, U, E, A\}$ , where  $A$  is denoted “ $E^*$ ”. Finally, as  $T_k$  is fully  $\mathcal{C}$ -normalized by (a) and as  $R/T_k$  is of order 2 and induces  $O_2(\text{Aut}_{\mathcal{C}}(T_k))$  on  $T_k$ , the restriction map  $\rho: \text{Aut}_{\mathcal{C}}(R) \rightarrow \text{Aut}_{\mathcal{C}}(T_k)$  is surjective by the Extension Axiom. Let  $\varphi \in \ker(\rho)$ . Then by [BLO03, Lemma A.8] and the first statement in (b),  $\varphi$  is conjugation by an element of  $Z(T_k) = T_k$ . It follows that  $\ker(\rho) = \text{Aut}_{T_k}(R)$  is of index 2 in  $\text{Inn}(R)$ . Thus,  $\text{Out}_{\mathcal{C}}(R) \cong \text{Aut}_{\mathcal{C}}(T_k)/O_2(\text{Aut}_{\mathcal{C}}(T_k)) \cong GL_3(2)$  by the last statement in (b).  $\square$

**Lemma 2.32.** *Let  $\mathcal{F} \in \{\mathcal{C}, \mathcal{H}\}$ , and let  $\mathcal{L}$  be the centric linking system for  $\mathcal{F}$ . Then the natural map  $\mu: \text{Out}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{F})$  is an isomorphism.*

*Proof.* This follows from [LO02, Lemma 3.2] and the obstruction sequence in [AKO11, Proposition 5.12] (that is, from (2.8) above).  $\square$

Our choice of  $q_l = 5^{2^l}$  is motivated by the next two lemmas, especially Lemma 2.33(a). Lemma 2.34 is not strictly needed for the sequel, but we feel it is helpful for context.

**Lemma 2.33.** *Let  $\mathcal{H}$  be the 2-fusion system of  $\text{Spin}_7(q)$  for some odd  $q$ , let  $l + 3$  be the 2-adic valuation of  $q^2 - 1$ , and set  $H = \text{Spin}_7(5^{2^l})$  as above. Then the following hold.*

- (a)  $\mathcal{H}$  is tamely realized by  $H$ .
- (b) With  $R$  as in Notation 2.30, each automorphism of  $H$  that normalizes  $T$  and centralizes  $R$  is conjugation by an element of  $E = Z(R)$ .

*Proof.* Let  $\mathcal{L}$  be the centric linking system for  $\mathcal{H}$ . Then the composition  $\text{Out}(H) \rightarrow \text{Out}(\mathcal{H})$  of  $\mu$  with  $\kappa$  (see Section 2.2) is an isomorphism with  $q_l = 5^{2^l}$  by [BMO16, Proposition 5.16]. Thus, part (a) follows from Lemma 2.32 and the definition of tame (Definition 2.12).

Set  $k = l + 2$  as before. For the sake of brevity, we make appeals to [BMO16, §5] also for (b). Note that by choice of  $q_l$ ,  $H$  satisfies Hypotheses 5.1(III.1) of that reference. Let  $\alpha$  be an automorphism of  $H$  that normalizes  $T$  and centralizes  $R$ . Since  $R \geq T_k$  (recall  $T_k$  is the 2-power torsion in  $\bar{T} \cap H$ ),  $\alpha$  centralizes  $T_k$ . Thus, by [BMO16, Lemma 5.9],  $\alpha \in \text{Inndiag}(H) =$



$\text{Inn}(H)\text{Aut}_{\bar{T}}(H)$ , and so there is  $h \in H$  and  $t \in \bar{T}$  such that  $\alpha$  is conjugation by  $ht$ . Then also  $h \in C_{\bar{H}}(T_k) = \bar{T}$ , with the last equality by [BMO16, Lemma 5.3(a)], so that  $ht \in \bar{T}$ . However,  $R$  contains the element  $w_0$  inverting  $\bar{T}$  by Lemma 2.31(a), and so it follows that  $ht \in \Omega_1(T_k) = Z(R)$ .  $\square$

**Lemma 2.34.** *The following hold.*

- (a) *The collection  $\{\mathcal{F}_{\text{Sol}}(5^{2^l}) \mid l \geq 0\}$  gives a nonredundant list of the isomorphism types of the 2-fusion systems  $\mathcal{F}_{\text{Sol}}(q)$  as  $q$  ranges over odd prime powers.*
- (b) *The collection  $\{\mathcal{F}_{\text{Spin}}(5^{2^l}) \mid l \geq 0\}$  gives a nonredundant list of the isomorphism types of the 2-fusion systems  $\mathcal{F}_{\text{Spin}}(q)$  as  $q$  ranges over odd prime powers.*

*Proof.* Part (a) is the content of [COS08, Theorem B]. For each odd prime power  $q$ , the fusion system of  $\text{Spin}_7(q)$  is isomorphic to some fusion system in the given collection by Lemma 2.33(a). Then (b) follows as a Sylow 2-subgroup of  $\text{Spin}_7(5^{2^l})$  has order  $2^{10+3l}$  by Lemma 2.31(a,b).  $\square$

The next lemma augments the results of [HL17] on automorphisms and extensions.

**Lemma 2.35.** *Let  $\mathcal{D}$  be a saturated fusion system over the 2-group  $S$  such that  $F^*(\mathcal{D}) = \mathcal{C} = \mathcal{F}_{\text{Sol}}(5^{2^l})$ . Then all involutions in  $S - T$  are  $\mathcal{D}$ -conjugate. If  $f \in S - T$  is an involution such that  $\langle f \rangle$  is fully  $\mathcal{D}$ -centralized, then  $O^2(C_{\mathcal{D}}(\langle f \rangle)) \cong \mathcal{F}_{\text{Sol}}(5^{2^{l-1}})$ .*

*Proof.* The almost simple extensions of  $\mathcal{C}$  were determined in [HL17]. By [HL17, Theorem 3.10, Theorem 4.3], we may fix a complement  $F$  to  $T$  in  $S$  such that  $F$  is cyclic of order  $2^{l_0}$  with  $1 \leq l_0 \leq l$ , and such that the conjugation action of  $F$  on  $T$  is the restriction of the conjugation action of a group of field automorphisms of  $H$  to  $T$ . We may therefore assume that  $|F| = 2$ , and that  $F$  is generated by the standard field automorphism  $f := \psi_{l-1}|_H$  fixed at the beginning of this subsection. By Lemma 2.33(a) and Theorem 2.13,  $C_{\mathcal{D}}(z)$  is the fusion system of the extension  $H\langle f \rangle$ , a semidirect product.

Let  $k = l+2$ , and let  $H_1 = N_{\bar{H}}(H)$ . Then  $H_1/Z(H) \cong \text{Inndiag}(H)$  by [GLS98, Lemma 2.5.9(b)], and hence  $H_1 = HN_{\bar{T}}(H)$ . As  $\text{Outdiag}(H)$  has order 2, we may fix  $t \in N_{\bar{T}}(H) - H$  with order  $2^{k+1}$  and powering to  $z$ , so that  $H_1 = H\langle t \rangle$ . Considered as an endomorphism of  $\bar{H}$ ,  $\psi_{l-1}$  normalizes  $H$  and  $\bar{T}$ , and thus induces an automorphism of  $H_1$ . Set  $g = \psi_{l-1}|_{H_1}$ , so that  $g$  has order 4, and  $g|_H = f$  has order 2. Set  $G_1 := H_1\langle g \rangle$ ,  $G := H\langle f \rangle$ ,  $\hat{G}_1 = G_1/C_{G_1}(H)$ , and  $\tilde{G} = G/Z(H)$ . Note that  $C_{G_1}(H) = \langle g^2, z \rangle$ . Since  $g^2$  centralizes  $H$ ,  $\langle g^2 \rangle$  is normal in  $H\langle g \rangle$ , and  $H\langle g \rangle/\langle g^2 \rangle \cong H\langle f \rangle$  via an isomorphism which sends  $g\langle g^2 \rangle$  to  $f$ . Hence, there is an isomorphism  $\hat{H}\langle \hat{g} \rangle \rightarrow \tilde{H}\langle \tilde{f} \rangle = \tilde{G}$  which is the identity on  $\hat{H} = H/Z(H) = \tilde{H}$  and which sends  $\hat{g}$  to  $\tilde{f}$ .

By [GLS98, Theorem 4.9.1(d)],  $\text{Inndiag}(H) \cong \hat{H}_1$  acts transitively on the involutions in  $\hat{H}\hat{g} - \hat{H}$ , and so each involution in  $\hat{H}\hat{g}$  is  $\hat{H}$ -conjugate to  $\hat{g}$  or  $\hat{g}^{\hat{t}}$ . As  $t^g = t^{\psi_{l-1}} = t^{5^{2^{l-1}}}$  and  $5^{2^{l-1}} - 1$  has 2-adic valuation  $l + 1 = k - 1$ , we see that there is an element  $u \in \langle t^{2^{k-1}} \rangle \leq H$  of order 4, such that  $[g, u] = 1$ ,  $u^2 = z$ , and  $t^g = tu$ . Then  $g^t = gu^{-1}$ , so that  $\hat{g}^{\hat{t}} = \hat{g}\hat{u}^{-1}$ . From the isomorphism  $\hat{H}\langle \hat{g} \rangle \cong \tilde{H}\langle \tilde{f} \rangle$ , we conclude that each involution in  $\tilde{H}\langle \tilde{f} \rangle$  is  $\tilde{H}$ -conjugate to either of  $\tilde{f}$  or  $\tilde{f}\tilde{u}^{-1}$ . However, the two preimages of  $\tilde{f}\tilde{u}^{-1}$  in  $H\langle f \rangle$  are  $fu^{-1}$  and  $fu = fu^{-1}z$ , both of which are of order 4 as  $[f, u] = 1$ . Thus, all four subgroups of  $H\langle f \rangle$  which are not contained in  $H$  and contain  $\langle z \rangle$  are  $H$ -conjugate. Since  $\langle f, z \rangle$  is such a four subgroup, it is enough to show that  $f$  is  $H$ -conjugate to  $fz$ . But  $f^s = fz$  where  $s = t^2 \in H$ . This completes the proof that all involutions in  $Hf - H$  are  $H$ -conjugate, and this implies the first part of the assertion. It then follows from the choice

of  $f$  that  $f$  is fully  $\mathcal{D}$ -centralized. Hence,  $O^2(C_{\mathcal{D}}(f)) \cong \mathcal{F}_{\text{Sol}}(5^{2^{l-1}})$  by [LO02, Proposition 3.3(d)], so the second assertion of the lemma follows from the first.  $\square$

**Lemma 2.36.** *Let  $P \in \mathcal{C}^e$  be an essential subgroup. Then either  $\text{Aut}_{\mathcal{C}}(P) = \text{Aut}_{\mathcal{H}}(P)$  or  $P = C_T(U)$ .*

*Proof.* Recall that an essential subgroup in a fusion system is in particular both centric and radical. In [LS17], the centric radical subgroups and their outer automorphism groups in  $\mathcal{H}$  and  $\mathcal{C}$  are explicitly tabulated. From Tables 1 and 4 there, the only outer automorphism groups having a strongly embedded subgroup are  $S_3$  and a Frobenius group of order  $3^2 \cdot 2$ . In all cases, either  $P$  is essential in  $\mathcal{H}$  and  $\text{Out}_{\mathcal{C}}(P) = \text{Out}_{\mathcal{H}}(P)$ , or  $P = C_T(U)$ .  $\square$

Both in the work of Levi-Oliver and Aschbacher-Chermak, the Benson-Solomon systems are exhibited by constructing a group  $K$  containing  $N_H(U)$  with index 3, and then defining  $\mathcal{F}_{\text{Sol}}(q)$  to be the fusion system on  $T$  generated by  $H$  and  $K$ . We will need slightly different generation statements in the process of showing that a subintrinsic maximal Benson-Solomon subsystem is standard.

**Lemma 2.37.** *The following hold.*

- (a)  $\mathcal{C}$  is generated by  $\mathcal{H}$  and  $N_{\mathcal{C}}(C_T(U))$ .
- (b)  $\mathcal{C}$  is generated by  $\mathcal{H}$  and  $N_{\mathcal{C}}(R)$ .

*Proof.* Part (a) follows from Lemma 2.36 and the Alperin-Goldschmidt fusion theorem [AKO11, Theorem I.3.6]. As  $U \leq E$ ,  $R = C_T(E) \leq C_T(U)$ . Further,  $T_k$  is abelian and weakly  $\mathcal{C}$ -closed by Lemma 2.31(a), hence also  $R = C_T(\Omega_1(T_k))$  is weakly  $\mathcal{C}$ -closed. It follows that  $N_{\mathcal{C}}(C_T(U)) \leq N_{\mathcal{C}}(R)$ , and so (b) follows from (a).  $\square$

We close this section by verifying that the Benson-Solomon systems are split. This allows one, via Theorem 8 of [Asc16], to severely restrict the Sylow subgroup of the centralizer of a Benson-Solomon standard subsystem.

**Lemma 2.38.**  *$\mathcal{C}$  is split.*

*Proof.* Let  $(\mathcal{F}, U)$  be a critical split extension of  $\mathcal{C}$ , where  $\mathcal{F}$  a saturated fusion system over  $S$ . Let  $\mathcal{L}$  be the centric linking system for  $\mathcal{C}$ . Note that  $S/C_S(\mathcal{C})T$  embeds in  $\text{Out}(\mathcal{L})$  by [Sem15, Theorem A], while  $\text{Out}(\mathcal{L})$  is cyclic of 2-power order by [HL17, Theorem 3.10]. Hence

$$(2.39) \quad U \cap C_S(\mathcal{C})T > 1.$$

Write  $U = \langle u, v \rangle$  with  $u \in U \cap C_S(\mathcal{C})T$ . Then either  $U \cap C_S(\mathcal{C}) > 1$ , or there exist elements  $c \in C_S(\mathcal{C})$  and  $1 \neq t \in T$  such that  $u = ct$ .

In the former case, i.e. if  $U \cap C_S(\mathcal{C}) > 1$ , we have by (T1) that  $U$  is normal in  $N_{\mathcal{F}}(U \cap C_S(\mathcal{C})) = \mathcal{F}$ , and hence that  $U \leq Z(\mathcal{F})$  by Lemma 2.29(a). Thus  $\mathcal{F}$  is the central product of  $U = C_S(\mathcal{C})$  and  $\mathcal{C}$ , as desired.

Consider the latter case. As  $C_S(\mathcal{C}) \cap T \leq Z(\mathcal{C}) = 1$  and  $u$  is an involution,  $t$  is an involution. Then, as  $\mathcal{C}$  has one class of involutions,  $u$  is  $\mathcal{F}$ -conjugate to  $cz \in Z(S)$ . However,  $\langle u \rangle$  is itself fully  $\mathcal{F}$ -centralized by Lemma 2.29(a), and so  $u \in Z(S)$ . As  $\langle v \rangle$  is fully  $\mathcal{F}$ -centralized and  $C_{\mathcal{F}}(u) = C_{\mathcal{F}}(v)$  by Lemma 2.29, we have  $v \in Z(S)$ . But then, using Lemma 2.32 to see that Proposition 2.10 applies, we have  $U \leq Z(S) = C_S(T) = C_S(\mathcal{C})Z(T)$  by that proposition applied with  $\mathcal{F}_1 = \mathcal{F}$ , so that  $C_S(\mathcal{C})T = UT = S$ . Thus,  $\mathcal{F}$  is the central product of  $C_S(\mathcal{C})$  with  $\mathcal{C}$  in this case as well.  $\square$

### 3. SUBINTRINSIC MAXIMAL BENSON-SOLOMON COMPONENTS

We assume the following hypothesis throughout this section.

**Hypothesis 3.1.** Let  $\mathcal{F}$  be a saturated fusion system over the 2-group  $S$ . Fix an odd prime power  $q$  and assume  $\mathcal{C} \cong \mathcal{F}_{\text{Sol}}(q)$  over  $T \in \mathcal{F}^f$  is maximal in  $\mathfrak{C}(\mathcal{F})$ . Let  $z$  be the involution in  $Z(T)$ , set  $\mathcal{H} = C_{\mathcal{C}}(z)$ , and suppose that  $z \in \mathcal{I}(\mathcal{H})$ . Assume that  $\mathcal{C}$  is not a component of  $\mathcal{F}$ . Fix  $t \in \mathcal{I}(\mathcal{C})$ .

The purpose of this section is to prove the following theorem.

**Theorem 3.2.** *Assume Hypothesis 3.1. Then  $\mathcal{C}$  is standard.*

*Proof.* This is the content of Lemmas 3.8 and 3.9 below. □

**Lemma 3.3.** *Let  $\alpha \in \text{Hom}_{\mathcal{F}}(T, S)$ . Then  $\mathcal{C}^\alpha$  is maximal in  $\mathfrak{C}(\mathcal{F})$ ,  $\mathcal{H}^\alpha = C_{\mathcal{C}^\alpha}(z^\alpha)$ ,  $z^\alpha \in \mathcal{I}(\mathcal{H}^\alpha)$ , and  $\mathcal{C}^\alpha$  is not a component of  $\mathcal{F}$ .*

*Proof.* By Lemma 2.5(b),  $\mathcal{C}^\alpha$  is not a component of  $\mathcal{F}$  as  $\mathcal{C}$  is not a component of  $\mathcal{F}$ . As  $T \in \mathcal{F}^f$ , it follows from [Asc16, 6.2.13] that  $\mathcal{C}^\alpha$  is maximal in  $\mathfrak{C}(\mathcal{F})$ . As  $\alpha$  induces an isomorphism from  $\mathcal{C}$  to  $\mathcal{C}^\alpha$ , we have  $\mathcal{H}^\alpha = C_{\mathcal{C}^\alpha}(z^\alpha)$ . Since  $z \in \mathcal{I}(\mathcal{H})$ , Lemma 2.15(b) gives  $z^\alpha \in \mathcal{I}(\mathcal{H}^\alpha)$ . □

**Lemma 3.4.** *The following hold.*

- (a)  $\mathcal{C}$  is terminal in  $\mathfrak{C}(\mathcal{F})$ .
- (b) For each  $\alpha \in \mathfrak{A}(t)$ ,  $\mathcal{C}^\alpha$  is normal in  $C_{\mathcal{F}}(t^\alpha)$ .

*Proof.* By [Asc16, 8.1.2.3], property (b) follows from (a). So we only need to show (a). By Hypothesis 3.1,  $\mathcal{C}$  is maximal and subintrinsic in  $\mathfrak{C}(\mathcal{F})$ . Since  $m(T) = 4$  by Lemma 2.31(d), Theorem 7.4.14 of [Asc16] shows that  $\Delta(\mathcal{C}) = \emptyset$ . It remains to verify the last condition of terminality. Let  $\varphi \in \text{Hom}_{\mathcal{F}}(\langle t, T \rangle, S)$  so that  $(t^\varphi, \mathcal{C}^\varphi) \in \rho(\mathcal{C})$ . We need to show that  $(t^\varphi, \mathcal{C}^\varphi) \in \rho_0(\mathcal{C})$ . In other words, fixing  $1 \neq a \in Q_{t^\varphi}$ , we need to show that  $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$ . Note that  $a \in \mathcal{X}(\mathcal{C}^\varphi)$ . So if  $\tilde{a}$  is the unique involution in  $\langle a \rangle$  and  $\tilde{a} \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$ , then  $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$  by [Asc16, 6.1.5]. So we may assume without loss of generality that  $a$  is an involution. Fix  $\alpha \in \mathfrak{A}(a)$ . It remains to show that  $\mathcal{C}^{\varphi\alpha}$  is a component of  $C_{\mathcal{F}}(a^\alpha)$  and thus  $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$ .

Note first that, by definition of  $Q_{t^\varphi}$ ,  $\mathcal{C}^\varphi \subseteq C_{\mathcal{F}}(a)$  and thus  $\mathcal{C}^{\varphi\alpha} \subseteq C_{\mathcal{F}}(a^\alpha)$ . By Lemma 2.15(b) applied with  $(\langle t \rangle, \varphi\alpha)$  in place of  $(X, \varphi)$ , we have  $t^{\varphi\alpha} \in \tilde{\mathcal{X}}(\mathcal{C}^{\varphi\alpha})$ . Moreover,  $[t^\varphi, a] = 1$  by definition of  $Q_{t^\varphi}$  and thus  $[t^{\varphi\alpha}, a^\alpha] = 1$ . Hence, Lemma 2.16 yields  $\mathcal{C}^{\varphi\alpha} \in \mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$ .

We will argue next that  $\mathcal{C}^{\varphi\alpha}$  is subintrinsic in  $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$ . By Lemma 3.3, we have  $C_{\mathcal{C}^{\varphi\alpha}}(z^{\varphi\alpha}) = \mathcal{H}^{\varphi\alpha}$  and  $z^{\varphi\alpha} \in \mathcal{I}(\mathcal{H}^{\varphi\alpha})$ . Recall that  $\mathcal{H}^{\varphi\alpha} \subseteq \mathcal{C}^{\varphi\alpha} \subseteq C_{\mathcal{F}}(a^\alpha)$ . In particular,  $[T^{\varphi\alpha}, a^\alpha] = 1$  and thus  $[z^{\varphi\alpha}, a^\alpha] = 1$ . Hence, by Lemma 2.16 applied with  $(\langle z^{\varphi\alpha} \rangle, \langle a^\alpha \rangle, \mathcal{H}^{\varphi\alpha})$  in place of  $(X, Y, \mathcal{C})$ , we have  $z^{\varphi\alpha} \in \mathcal{I}_{C_{\mathcal{F}}(a^\alpha)}(\mathcal{H}^{\varphi\alpha})$ . As  $z^{\varphi\alpha} \in Z(\mathcal{H}^{\varphi\alpha})$ , this implies that  $\mathcal{C}^{\varphi\alpha}$  is indeed subintrinsic in  $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$  as we wanted to prove.

As we have verified that  $\mathcal{C}^{\varphi\alpha}$  is a subintrinsic member of  $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$ , it follows now from [Asc17b, 1.9.2] applied with  $C_{\mathcal{F}}(a^\alpha)$  in the role of  $\mathcal{F}$  and with  $\mathcal{C}^{\varphi\alpha}$  in the role of  $\mathcal{M}$  that  $\mathcal{C}^{\varphi\alpha}$  is contained in some component of  $C_{\mathcal{F}}(a^\alpha)$ . Since  $\mathcal{C}^\varphi$  is maximal in  $\mathfrak{C}(\mathcal{F})$  by Lemma 3.3, it follows from Lemma 2.17 applied with  $(t^\varphi, \mathcal{C}^\varphi)$  in place of  $(t, \mathcal{C})$  that  $\mathcal{C}^{\varphi\alpha}$  is a component of  $C_{\mathcal{F}}(a^\alpha)$ . As argued above this shows (a). □

By Lemma 2.34, we can and will assume that  $q = 5^{2^l}$  for some  $l \geq 0$ . Moreover, for the remainder of this section, we will adopt Notation 2.30.

**Lemma 3.5.** *The following hold:*

- (a) *We have  $\text{Aut}_{\mathcal{F}}(R) = C_{\text{Aut}_{\mathcal{F}}(R)}(E) \text{Aut}_{\mathcal{C}}(R)$  and  $O_2(\text{Aut}_{\mathcal{F}}(R)) = C_{\text{Aut}_{\mathcal{F}}(R)}(E)$ .*
- (b) *We have  $N_S(R) = N_S(T) = C_S(E)T$ .*
- (c)  *$R$  is fully  $\mathcal{F}$ -normalized.*
- (d) *We have  $\text{Out}_{\mathcal{F}}(R) = O_2(\text{Out}_{\mathcal{F}}(R)) \times \text{Out}_{\mathcal{C}}(R)$  and  $O_2(\text{Aut}_{\mathcal{F}}(R)) = \text{Aut}_{C_S(E)}(R)$ . In particular,  $O^2(\text{Aut}_{\mathcal{F}}(R)) = O^2(\text{Aut}_{\mathcal{C}}(R))$ .*

*Proof.* Set  $C := C_{\text{Aut}_{\mathcal{F}}(R)}(E)$ . Observe that  $\text{Aut}_{\mathcal{F}}(R)/C$  embeds into  $\text{Aut}(E) \cong \text{GL}_3(2)$ . As  $\text{Aut}_{\mathcal{C}}(R)/\text{Inn}(R) \cong \text{GL}_3(2)$  and  $C_{\text{Aut}_{\mathcal{C}}(R)}(E) = \text{Inn}(R)$ , it follows that  $\text{Aut}_{\mathcal{F}}(R) = C \text{Aut}_{\mathcal{C}}(R)$ . By Lemma 2.31(a),  $T_k$  is homocyclic of rank 3 and exponent  $2^k$ . Clearly,  $T_k$  is characteristic in  $R$ . So for every  $1 \leq i < k$ , the map  $\Omega_{i+1}(T_k)/\Omega_i(T_k) \rightarrow \Omega_i(T_k)/\Omega_{i-1}(T_k)$ ,  $x\Omega_i(T_k) \mapsto x^2\Omega_{i-1}(T_k)$  is an isomorphism of  $\text{Aut}_{\mathcal{F}}(R)$ -modules. So in particular,  $C$  acts trivially on  $\Omega_{i+1}(T_k)/\Omega_i(T_k)$  for all  $1 \leq i < k$ . As  $|R/T_k| = 2$ ,  $C$  acts also trivially on  $R/T_k$ . Hence,  $C$  is a 2-group and thus contained in  $O_2(\text{Aut}_{\mathcal{F}}(R))$ . As  $E$  is an irreducible  $\text{Aut}_{\mathcal{F}}(R)$ -module, it follows  $C = O_2(\text{Aut}_{\mathcal{F}}(R))$ . This shows (a).

As  $R$  is characteristic in  $T$  by Lemma 2.31(c), we have  $N_S(T) \leq N_S(R)$ . By (a),  $C \text{Aut}_T(R)$  is the unique Sylow 2-subgroup of  $\text{Aut}_{\mathcal{F}}(R)$  containing  $\text{Aut}_T(R)$ . As  $\text{Aut}_S(R)$  is a 2-group containing  $\text{Aut}_T(R)$ , it follows  $\text{Aut}_S(R) \leq C \text{Aut}_T(R)$  and thus  $N_S(R) \leq C_S(E)T \leq C_S(z)$ . Let now  $x \in C_S(E) \leq C_S(z)$ . As  $z \in \mathcal{I}(\mathcal{H})$ , there exists  $\alpha \in \mathfrak{A}(z)$  such that  $\mathcal{H}^\alpha$  is a component of  $C_{\mathcal{F}}(z^\alpha)$ . Then  $x^\alpha \in C_S(E^\alpha) \leq C_S(z^\alpha)$  and  $(\mathcal{H}^\alpha)^{x^\alpha}$  is a component of  $C_{\mathcal{F}}(z^\alpha)$  by Lemma 2.5. So by [Asc11, 9.8.2], either  $\mathcal{H}^\alpha = (\mathcal{H}^\alpha)^{x^\alpha}$  or  $\mathcal{H}^\alpha$  and  $(\mathcal{H}^\alpha)^{x^\alpha}$  form a commuting product. In the latter case,  $E^\alpha = (E^\alpha)^{x^\alpha} \leq Z(\mathcal{H}^\alpha)$ , a contradiction to  $Z(\mathcal{H}^\alpha) = \langle z^\alpha \rangle$ . Hence,  $\mathcal{H}^\alpha = (\mathcal{H}^\alpha)^{x^\alpha}$  and thus  $(T^x)^\alpha = (T^\alpha)^{x^\alpha} = T^\alpha$ . This implies  $x \in N_S(T)$ . So we have shown that  $C_S(E) \leq N_S(T)$  and thus  $N_S(R) \leq C_S(E)T \leq N_S(T) \leq N_S(R)$ . This yields (b).

For the proof of (c), let  $\gamma \in \mathfrak{A}(R)$ . Recall from (b) that  $T \leq N_S(T) = N_S(R)$ . So in particular, as  $T \in \mathcal{F}^f$ , we have  $T^\gamma \in \mathcal{F}^f$  and  $N_S(T)^\gamma = N_S(T^\gamma)$ . Thus it follows from Lemma 3.3 that we can apply (b) with  $t^\gamma, z^\gamma, \mathcal{C}^\gamma$  and  $R^\gamma$  in place of  $t, z, \mathcal{C}$  and  $R$  to obtain  $N_S(R^\gamma) = N_S(T^\gamma)$ . This gives  $N_S(T^\gamma) = N_S(T)^\gamma = N_S(R)^\gamma \leq N_S(R^\gamma) = N_S(T^\gamma)$  and therefore  $N_S(R)^\gamma = N_S(R^\gamma)$ . Since  $R^\gamma$  is fully normalized, it follows that  $R$  is fully normalized. This shows (c). In particular, by the Sylow axiom,  $\text{Aut}_S(R) \in \text{Syl}_2(\text{Aut}_{\mathcal{F}}(R))$  and so  $C = O_2(\text{Aut}_{\mathcal{F}}(R)) \leq \text{Aut}_S(R)$ . Thus,  $C = \text{Aut}_{C_S(E)}(R)$ . By (b),  $[C_S(E), T] \leq C_S(E) \cap T = C_T(E) = R$ . Hence,  $[C, \text{Aut}_T(R)] \leq \text{Inn}(R)$ . As  $\text{Aut}_{\mathcal{C}}(R) = \langle \text{Aut}_T(R)^{\text{Aut}_{\mathcal{C}}(R)} \rangle$  and  $C$  is normalized by  $\text{Aut}_{\mathcal{C}}(R)$ , it follows  $[C, \text{Aut}_{\mathcal{C}}(R)] \leq \text{Inn}(R)$ . This together with (a) implies that (d) holds.  $\square$

**Lemma 3.6.** *There exists  $\sigma \in \mathfrak{A}(t)$  such that  $z^\sigma$  is fully  $\mathcal{F}$ -centralized, and  $T^\sigma \in \mathcal{F}^f$ .*

*Proof. Step 1:* We show that there exists  $\chi \in \mathfrak{A}(t)$  with  $T^\chi = T$ . For the proof let  $\alpha \in \mathfrak{A}(t)$ . As  $T \in \mathcal{F}^f$ , there exists  $\beta \in \mathfrak{A}(T^\alpha)$  with  $T^{\alpha\beta} = T$ . It follows from Lemma 3.4(b) that  $T^\alpha \trianglelefteq C_S(t^\alpha)$ , i.e.,  $C_S(t^\alpha) \leq N_S(T^\alpha)$ . Hence, as  $t^\alpha$  is fully centralized,  $t^{\alpha\beta}$  is fully centralized and  $\alpha\beta \in \mathfrak{A}(t)$ . So  $\chi := \alpha\beta$  has the required properties.

*Step 2:* We show the existence of  $\sigma$ . By Step 1, we can choose  $\chi \in \mathfrak{A}(t)$  with  $T^\chi = T$ . Let  $\gamma \in \mathfrak{A}(z)$ . As  $T^\chi = T$  and  $Z(T) = \langle z \rangle$ , we have  $z^\chi = z$  and  $N_S(T) \leq C_S(z)$ . By Lemma 3.4(b), we have  $\mathcal{C}^\chi \trianglelefteq C_{\mathcal{F}}(t^\chi)$  and thus  $C_S(t^\chi) \leq N_S(T^\chi) = N_S(T) \leq C_S(z)$ . Since  $t^\chi$  is fully centralized, it follows that  $t^{\chi\gamma}$  is fully centralized and  $\sigma := \chi\gamma \in \mathfrak{A}(t)$ . Similarly, as  $T^\chi = T \in \mathcal{F}^f$ , we conclude that  $T^\sigma = T^\gamma \in \mathcal{F}^f$ . By the choice of  $\gamma$ ,  $z^\sigma = z^\gamma$  is fully centralized.  $\square$

Recall that  $t$  centralizes  $T$ . So by Lemma 3.3 and Lemma 3.6, we may assume that  $t$  and  $z$  are fully centralized. Moreover, we set

$$V_R := RC_S(R) \text{ and } Q_0 := C_S(T).$$

**Lemma 3.7.** *The following hold.*

- (a) *We have  $\mathcal{H} \subseteq C_{\mathcal{F}}(Q_0)$ .*
- (b) *We have  $C_S(R) = EC_S(T)$ , and hence  $V_R = RC_S(T)$ .*
- (c)  *$N_{N_{\mathcal{F}}(R)}(V_R)$  is a constrained fusion system and  $N_{\mathcal{C}}(R) \subseteq N_{N_{\mathcal{F}}(R)}(V_R)$ .*
- (d) *Let  $G_R$  be a model for  $N_{N_{\mathcal{F}}(R)}(V_R)$  and  $N := C_{G_R}(V_R/R)$ . Then  $N_1 := \langle T^N \rangle = O^2(N)R$  is a model for  $N_{\mathcal{C}}(R)$ .*
- (e) *We have  $Q_0 = \langle z \rangle \times Q$  where  $Q = C_{Q_0}(N_1)$  with  $N_1$  as in (d).*
- (f) *If  $Q$  is as in (e), then  $Q$  is the unique largest subgroup of  $S$  centralized by  $\mathcal{C}$ . More precisely,  $\mathcal{C} \subseteq C_{\mathcal{F}}(Q)$  and  $X \leq Q$  for all  $X \leq S$  with  $\mathcal{C} \subseteq C_{\mathcal{F}}(X)$ .*
- (g) *If  $Q$  is as in (e), then  $Q$  is the unique largest member of  $\tilde{\mathcal{X}}(\mathcal{C})$ .*

*Proof.* We start by proving (a) and (b). Recall that  $E = Z(R)$ . As  $R \leq T$ , clearly  $EC_S(T) \leq C_S(R)$ , so for (b) we must show the other inclusion. Since  $z \in R$ , we have  $C_S(R) = C_{C_S(z)}(R) \leq C_S(z)$ . Now by our choice of notation,  $\langle z \rangle$  is fully  $\mathcal{F}$ -centralized, so  $C_{\mathcal{F}}(z)$  is a saturated fusion system on  $C_S(z)$ . By Hypothesis 3.1,  $\mathcal{H}$  is a component of  $C_{\mathcal{F}}(z)$ . The normalizer of a component is constructed in [Asc16, §2.1], and thus, we may form  $N_{C_{\mathcal{F}}(z)}(\mathcal{H})$  over the 2-group  $N_S(T) = N_{C_S(z)}(T)$ . By Lemma 3.5(b),  $C_S(R) \leq N_S(R) = N_S(T)$ , so we may form the product system  $\hat{\mathcal{H}} := \mathcal{H}C_S(R)$  as in [Asc11, Chapter 8] or [Hen13] in the normalizer  $N_{C_{\mathcal{F}}(z)}(\mathcal{H})$ . Thus  $\hat{\mathcal{H}}$  is a saturated subsystem of  $C_{\mathcal{F}}(z)$  with  $O^2(\hat{\mathcal{H}}) = O^2(\mathcal{H}) = \mathcal{H} = E(\hat{\mathcal{H}})$ . So  $\hat{\mathcal{H}}$  is a small extension of  $\mathcal{H}/Z(\mathcal{H})$  in the sense of [AO16, Definition 2.21]. By Lemma 2.33(a),  $\mathcal{H}$  is tamely realized by  $H := \text{Spin}_7(5^{2^l})$ , so that by [AO16, Lemma 2.22], there is an extension  $\hat{H} = HC_S(R)$  of  $H$  that tamely realizes  $\hat{\mathcal{H}}$ . By Lemma 2.33(b), each automorphism of  $H$  normalizing  $T$  and centralizing  $R$  is conjugation by an element of  $E$ . Hence,  $Q_0 \leq C_S(R) \leq EC_S(H) \leq EC_S(T)$ . This implies  $C_S(R) = EC_S(T)$  and  $Q_0 = C_E(T)C_S(H) = \langle z \rangle C_S(H) = C_S(H)$ . The first property gives (b), and the latter property yields (a).

Since  $R$  is fully normalized by Lemma 3.5(c),  $N_{\mathcal{F}}(R)$  is saturated. Note that  $V_R$  is weakly closed and thus fully normalized in  $N_{\mathcal{F}}(R)$ . So  $N_{N_{\mathcal{F}}(R)}(V_R)$  is saturated. Clearly  $N_{N_{\mathcal{F}}(R)}(V_R)$  is constrained, as  $V_R$  is a centric normal subgroup of this fusion system. We show next that  $N_{\mathcal{C}}(R) \subseteq N_{N_{\mathcal{F}}(R)}(V_R)$ . Let  $R \leq P \leq T$  and  $\varphi \in \text{Aut}_{N_{\mathcal{C}}(R)}(P)$ . By Alperin's fusion theorem [AKO11, Theorem I.3.6], it is enough to show that  $\varphi$  extends to an element of  $\text{Aut}_{\mathcal{F}}(PV_R)$  normalizing  $V_R$ . Let  $\alpha \in \mathfrak{A}(P)$  and observe that  $\varphi^\alpha \in \text{Aut}_{\mathcal{F}}(P^\alpha)$ . By (b),  $V_R = RC_S(T) \leq PC_S(P)$ . Thus  $V_R^\alpha \leq P^\alpha C_S(P^\alpha) \leq N_{\varphi^\alpha}$ . As  $P^\alpha$  is fully normalized, it follows from the extension axiom that  $\varphi^\alpha$  extends to a morphism  $\psi: P^\alpha V_R^\alpha \rightarrow S$  in  $\mathcal{F}$ . Note that  $R^{\alpha\psi} = (R^\alpha)^{\varphi^\alpha} = R^\alpha$  as  $R^\varphi = R$ . Since  $R$  is fully normalized and thus fully centralized, we have  $C_S(R)^{\alpha\psi} = C_S(R^{\alpha\psi}) = C_S(R^\alpha) = C_S(R)^\alpha$  and thus  $V_R^{\alpha\psi} = R^\alpha C_S(R)^\alpha = V_R^\alpha$ . So  $\psi \in \text{Aut}_{\mathcal{F}}(P^\alpha V_R^\alpha)$  extends  $\varphi^\alpha$  and normalizes  $V_R^\alpha$ . Hence,  $\hat{\varphi} := \psi^{\alpha^{-1}} \in \text{Aut}_{\mathcal{F}}(PV_R)$  extends  $\varphi$  and normalizes  $V_R$ . This proves (c).

Let now  $G_R$  and  $N$  be as in (d), and set  $N_1 := \langle T^N \rangle$ . (The model  $G_R$  for  $N_{N_{\mathcal{F}}(R)}(R)$  exists and is unique up to isomorphism by [AKO11, Proposition III.5.8]. Moreover,  $C_{G_R}(V_R) \leq V_R$ .) Note that  $S_0 := N_S(R) \in \text{Syl}_2(G_R)$ . By (b),  $[V_R, T] \leq R$ . As  $V_R$  and  $R$  are normal in  $G_R$ , it follows  $[V_R, \langle T^{G_R} \rangle] \leq R$  and thus  $N_1 \leq \langle T^{G_R} \rangle \leq N$ . Let  $P \leq T$  be essential in  $N_{\mathcal{C}}(R)$ . As

$\text{Out}_{\mathcal{C}}(R) \cong GL_3(2)$ , we observe that  $R \leq P$ ,  $P/R \cong C_2 \times C_2$  and  $\text{Out}_{N_{\mathcal{C}}(R)}(P) \cong GL_2(2) \cong S_3$ . In particular,  $\text{Aut}_{N_{\mathcal{C}}(R)}(P) = \langle \text{Aut}_T(P)^{\text{Aut}_{N_{\mathcal{C}}(R)}(P)} \rangle$ . Since  $\text{Aut}_{N_{\mathcal{C}}(R)}(P) \leq \text{Aut}_{G_R}(P)$  by (c), it follows that  $\text{Aut}_{N_{\mathcal{C}}(R)}(P) \leq \text{Aut}_{\langle T^{G_R} \rangle}(P) \leq \text{Aut}_N(P)$ . Now we conclude similarly that  $\text{Aut}_{N_{\mathcal{C}}(R)}(P) \leq \langle \text{Aut}_T(P)^{\text{Aut}_N(P)} \rangle \leq \text{Aut}_{N_1}(P)$ . As  $P$  was arbitrary, the Alperin–Goldschmidt Fusion Theorem yields that  $N_{\mathcal{C}}(R) \subseteq \mathcal{F}_{S_0 \cap N_1}(N_1)$ .

Note that  $N/C_N(R)$  embeds into  $\text{Aut}_{\mathcal{F}}(R)$ . As  $C_{G_R}(V_R) \leq V_R$ , and  $C_N(R)$  centralizes  $V_R/R$  and  $R$ ,  $C_N(R)$  is a normal 2-subgroup of  $N$ . So it follows from Lemma 3.5(d) that  $N/O_2(N) \cong \text{Out}_{\mathcal{C}}(R) \cong GL_3(2)$  and  $O_2(N) = C_N(E) \leq C_{S_0}(E)$ . Using Lemma 3.5(b), we conclude that  $O_2(N) \leq C_S(E) \leq N_S(T)$  and thus  $[O_2(N), T] \leq C_T(E) = R$ . Since  $O_2(N)$  and  $R$  are normal in  $N$ , this implies  $[O_2(N), N_1] \leq R$ . In particular, noting  $O_2(N_1) = O_2(N) \cap N_1$  and setting  $\bar{N} := N/R$ , it follows that  $\overline{O_2(N_1)}$  is abelian. Observe that  $T/(T \cap O_2(N)) = T/R$  is isomorphic to a Sylow 2-subgroup of  $GL_3(2)$ . Thus,  $TO_2(N)/O_2(N)$  is a Sylow 2-subgroup of  $N/O_2(N)$  and so  $TO_2(N)$  is a Sylow 2-subgroup of  $N$ . In particular,  $TO_2(N_1) = (TO_2(N)) \cap N_1$  is a Sylow 2-subgroup of  $N_1$ . Note that  $T \cap O_2(N_1) = T \cap O_2(N) = C_T(E) = R$ . Thus  $\bar{T}$  is a complement to  $\overline{O_2(N_1)}$  in the Sylow 2-subgroup  $\overline{TO_2(N_1)}$  of  $\bar{N}_1$ . So by a Theorem of Gaschütz [KS04, Theorem 3.3.2], there exists a complement  $\bar{N}_0$  of  $\overline{O_2(N_1)}$  in  $\bar{N}_1$ . We choose a preimage  $N_0$  of such a complement  $\bar{N}_0$  with  $R \leq N_0 \leq N_1$ . As  $N/O_2(N) \cong GL_3(2)$  is simple, we have  $N = O_2(N)N_1 = O_2(N)N_0$ . Since  $O_2(N) \cap N_0 = O_2(N_1) \cap N_0 = R$  and  $O_2(N)$  is centralized by  $\bar{N}_1$ , it follows  $\bar{N} = \overline{O_2(N)} \times \bar{N}_0$ . In particular,  $N_0 = O^2(N)R$  is normal in  $G_R$ . As  $N_{\mathcal{C}}(R) \subseteq \mathcal{F}_{S_0 \cap N_1}(N_1) \subseteq \mathcal{F}_{S_0 \cap N}(N)$ , we have  $\text{hnp}(N_{\mathcal{C}}(R)) \leq \text{hnp}(\mathcal{F}_{S_0 \cap N}(N)) \leq O^2(N)$ . Hence  $T = \text{hnp}(N_{\mathcal{C}}(R))R \leq O^2(N)R = N_0$ . In particular,  $N_0 = N_1$ ,  $O_2(N_1) = R$ ,  $T \in \text{Syl}_2(N_1)$  and  $N_1/R \cong GL_3(2)$ . We show next that  $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$ . We have seen already that  $N_{\mathcal{C}}(R) \subseteq \mathcal{F}_T(N_1)$ . If  $P$  is essential in  $\mathcal{F}_T(N_1)$ , then it follows from  $N_1/R \cong GL_3(2)$  that  $R \leq P \leq T$ ,  $P/R \cong C_2 \times C_2$  and  $\text{Out}_{N_1}(P) \cong GL_2(2)$ . As  $GL_2(2) \cong \text{Out}_{N_{\mathcal{C}}(R)}(P) \leq \text{Out}_{N_1}(P)$ , it follows  $\text{Aut}_{N_1}(P) = \text{Aut}_{\mathcal{C}}(P)$ . Hence, we have  $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$ . Since  $C_{N_1}(O_2(N_1)) \leq N_1 \cap C_N(E) = N_1 \cap O_2(N) = O_2(N_1)$ , we conclude that  $N_1$  is a model for  $N_{\mathcal{C}}(R)$ . This completes the proof of (d).

We consider now the action of  $N_1/R \cong GL_3(2)$  on  $U_R := C_S(R) = C_{V_R}(R)$ . Note that  $E = Z(R)$  is central in  $U_R$  and recall that  $U_R = EC_S(T)$  by (b). In particular,  $U_R/\Phi(C_S(T))$  is elementary abelian and thus  $\Phi(U_R) \leq \Phi(C_S(T))$ . If  $E \cap \Phi(U_R)$  were non-trivial, then we would have  $E \leq \Phi(U_R)$  as  $N_1$  acts irreducibly on  $E$ . So it would follow that  $E \leq C_S(T)$  contradicting  $E \not\leq Z(T)$ . This shows that  $E \cap \Phi(U_R) = 1$ . Set

$$\widetilde{U}_R = U_R/\Phi(U_R).$$

As  $\widetilde{U}_R = \widetilde{E}\widetilde{C}_S(T)$  is elementary abelian, there is a complement to  $\widetilde{E}$  in  $\widetilde{U}_R$  which lies in  $\widetilde{C}_S(T)$ . So by a Theorem of Gaschütz [KS04, Theorem 3.3.2], applied in the semidirect product  $N_1 \ltimes \widetilde{U}_R$ , there exists a complement  $\widetilde{Q}$  to  $\widetilde{E}$  in  $\widetilde{U}_R$  which is normalized by  $N_1$ . We choose the preimage  $Q$  of  $\widetilde{Q}$  such that  $\Phi(U_R) \leq Q \leq U_R$ .

As  $[U_R, N_1] \leq [V_R, N] \leq R$ , we have  $[Q, N_1] \leq [U_R, N_1] \leq U_R \cap R = Z(R) = E$ . In particular,  $[\widetilde{Q}, N_1] \leq \widetilde{Q} \cap \widetilde{E} = 1$ . So  $[Q, N_1] \leq \Phi(U_R) \cap E = 1$ . Recalling  $Q_0 = C_S(T)$ , we conclude  $Q \leq C_{Q_0}(N_1)$ . Observe that  $Q$  has index 2 in  $Q_0 = C_S(T)$  as  $\widetilde{E} \cap \widetilde{C}_S(T) = \langle \widetilde{z} \rangle$  has order 2. Hence, since  $[z, N_1] \neq 1$ , it follows  $Q = C_{Q_0}(N_1)$  and  $Q_0 = \langle z \rangle \times Q$ . This proves (e).

By (a),  $Q_0$  centralizes  $\mathcal{H}$ , and by Lemma 2.37(b), we have  $\mathcal{C} = \langle \mathcal{H}, N_{\mathcal{C}}(R) \rangle$ . So if  $X \leq Q_0 = C_S(T)$ , then  $X$  contains  $\mathcal{C}$  in its centralizer if and only if it contains  $N_{\mathcal{C}}(R)$  in its centralizer. As  $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$  by (d) and  $Q$  is centralized by  $N_1$ , clearly every subgroup of  $Q$  contains  $N_{\mathcal{C}}(R)$  in its centralizer. Fix  $X \leq C_S(T)$  with  $N_{\mathcal{C}}(R) \subseteq C_{\mathcal{F}}(X)$ . To complete the proof of (f), we need to show that  $X \leq Q$ . To prove this let  $\Theta$  be the set of all pairs  $(Y, \varphi)$  such that  $RX \leq Y \leq V_R$ ,  $\varphi \in \text{Aut}_{\mathcal{F}}(Y)$ ,  $[Y, \varphi] \leq R$ ,  $\varphi|_X = \text{id}_X$ , and  $\varphi|_R \in \text{Aut}_{\mathcal{C}}(R)$  has order 7. As  $\text{Aut}_{\mathcal{C}}(R)/\text{Inn}(R) \cong \text{GL}_3(2)$ , there exists an element  $\varphi_0$  of order 7 in  $\text{Aut}_{\mathcal{C}}(R)$ . As  $N_{\mathcal{C}}(R) \subseteq C_{\mathcal{F}}(X)$ ,  $\varphi_0$  extends to an automorphism  $\varphi \in \text{Aut}_{\mathcal{F}}(RX)$  with  $\varphi|_X = \varphi_0$ , and for such  $\varphi$  we have  $(RX, \varphi) \in \Theta$ . Thus  $\Theta \neq \emptyset$  and we may fix  $(Y, \varphi) \in \Theta$  such that  $|Y|$  is maximal. Assume first that  $Y = V_R$ . Then  $\varphi$  is a morphism in  $N_{N_{\mathcal{F}}(R)}(V_R)$  and thus realized by conjugation with an element of  $G_R$ . Recall that  $H_1 = O^2(H)R$  is normal in  $G_R$  and contains  $T$ . Hence,  $Q = C_{V_R}(H_1)$  is normal in  $G_R$  and thus  $\varphi$ -invariant. As  $[V_R, \varphi] \leq R$  by definition of  $\Theta$ , it follows  $[Q, \varphi] \leq R \cap Q = 1$ . As  $U_R = EC_S(T) = EQ$  and  $\varphi|_R$  acts fixed-point-freely on  $E^{\#}$ , it follows  $Q = C_{U_R}(\varphi)$ . By definition of  $\Theta$ , we have  $\varphi|_X = \text{id}_X$  and thus  $X \leq C_{U_R}(\varphi) = Q$ . So  $X \leq Q$  if  $Y = V_R$ .

Assume now  $Y < V_R$ . Recall from above that  $N_{\mathcal{F}}(R)$  is saturated. So we can fix  $\alpha \in \mathfrak{A}_{N_{\mathcal{F}}(R)}(Y)$ . Then  $\varphi^\alpha \in \text{Aut}_{\mathcal{F}}(Y^\alpha)$  and  $[Y^\alpha, \varphi^\alpha] \leq R$  as  $[Y, \varphi] \leq R$  by definition of  $\Theta$ . Recall also that  $\varphi|_R \in \text{Aut}_{\mathcal{C}}(R)$  has order 7. By Lemma 3.5(d), we have  $O^2(\text{Aut}_{\mathcal{F}}(R)) = O^2(\text{Aut}_{\mathcal{C}}(R))$ . So we can conclude that  $\varphi^\alpha|_R = (\varphi|_R)^\alpha \in O^2(\text{Aut}_{\mathcal{F}}(R))^\alpha = O^2(\text{Aut}_{\mathcal{F}}(R)) \leq \text{Aut}_{\mathcal{C}}(R)$ . As  $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$  by (d), there exists thus  $n \in N_1$  with  $\varphi^\alpha|_R = c_n|_R$ . Set  $\psi := c_n|_{V_R} \in \text{Aut}_{\mathcal{F}}(V_R)$ . As  $N_1 \leq N$ , we have  $[V_R, \psi] \leq R$ . In particular, as  $R \leq Y^\alpha \leq V_R$ , we have  $(Y^\alpha)^\psi = Y^\alpha$ . Thus,  $\chi := (\psi|_{Y^\alpha})^{-1} \circ \varphi^\alpha \in \text{Aut}_{\mathcal{F}}(Y^\alpha)$  is well-defined. Observe also that  $\chi|_R = \text{id}_R$  and  $[Y^\alpha, \chi] \leq R$ , as  $[Y^\alpha, \psi] \leq [V_R, \psi] \leq R$  and  $[Y^\alpha, \varphi^\alpha] \leq R$ . So  $\chi$  is an element of  $C_{\text{Aut}_{\mathcal{F}}(Y^\alpha)}(R) \cap C_{\text{Aut}_{\mathcal{F}}(Y^\alpha)}(Y^\alpha/R)$ , which is a normal 2-subgroup of  $\text{Aut}_{N_{\mathcal{F}}(R)}(Y^\alpha)$ . Since  $Y^\alpha \in N_{\mathcal{F}}(R)^f$ , the Sylow axiom yields that  $\text{Aut}_{N_S(R)}(Y^\alpha)$  is a Sylow 2-subgroup of  $\text{Aut}_{N_{\mathcal{F}}(R)}(Y^\alpha)$ . Hence, there exists  $s \in N_{C_S(R)}(Y^\alpha)$  with  $\chi|_{Y^\alpha} = c_s|_{Y^\alpha}$ . So  $\varphi^\alpha = \psi|_{Y^\alpha} \circ c_s|_{Y^\alpha}$  extends to  $\rho = \psi \circ c_s|_{V_R} \in \text{Aut}_{\mathcal{F}}(V_R)$ . Since  $[V_R, \psi] \leq R$ , the automorphism  $\rho$  acts on  $V_R/R$  in the same way as  $c_s|_{V_R}$ . So writing  $m$  for the order of  $s$ , we have  $[V_R, \rho^m] \leq R$ . Moreover,  $\rho^m$  extends  $(\varphi^\alpha)^m$ . Since  $Y < V_R$ , we have  $Y < W := N_{V_R}(Y)$ . Note that  $R \leq Y^\alpha \leq W^\alpha \leq V_R$ , so  $[W^\alpha, \rho^m] \leq [V_R, \rho^m] \leq R$  and  $\rho^m|_{W^\alpha} \in \text{Aut}_{\mathcal{F}}(W^\alpha)$ . Therefore,  $\hat{\varphi} := (\rho^m|_{W^\alpha})^{\alpha^{-1}} \in \text{Aut}_{\mathcal{F}}(W)$  with  $[W, \hat{\varphi}] \leq R^{\alpha^{-1}} = R$ . Moreover,  $\hat{\varphi}|_R = (\varphi|_R)^m \in \text{Aut}_{\mathcal{C}}(R)$  has order 7, as  $\varphi|_R \in \text{Aut}_{\mathcal{C}}(R)$  has order 7 and  $m$  is a power of 2. Moreover,  $\hat{\varphi}|_X = (\varphi|_X)^m = \text{id}_X$  as  $\varphi|_X = \text{id}_X$ . This shows  $(W, \hat{\varphi}) \in \Theta$ . As  $|W| > |Y|$  and  $(Y, \varphi) \in \Theta$  was chosen such that  $|Y|$  is maximal, this is a contradiction. So we have shown that  $Y = V_R$ . As argued before, this yields  $X \leq Q$  and thus shows (f).

It remains to prove (g). By (f),  $\mathcal{C} \subseteq C_{\mathcal{F}}(Q)$  and  $X \leq Q$  for every  $X \leq C_S(T)$  with  $\mathcal{C} \subseteq C_{\mathcal{F}}(X)$ . In particular,  $X \leq Q$  for every  $X \in \tilde{\mathcal{X}}(\mathcal{C})$ . Moreover,  $t \in Q$  and  $Q \in \mathcal{X}(\mathcal{C})$ . As  $t \in \mathcal{I}(\mathcal{C}) \subseteq \tilde{\mathcal{X}}(\mathcal{C})$ , it follows thus from [Asc16, 6.1.5] that  $Q \in \tilde{\mathcal{X}}(\mathcal{C})$ . This shows (g).  $\square$

**Lemma 3.8.**  $\mathcal{C}$  is nearly standard.

*Proof.* By Lemma 3.4(a),  $\mathcal{C}$  is terminal in  $\mathfrak{C}(\mathcal{F})$ . By Lemma 3.7(g), the collection  $\tilde{\mathcal{X}}(\mathcal{C})$  has a unique maximal member. Hence,  $\mathcal{C}$  is nearly standard by [Asc16, Proposition 7].  $\square$

**Lemma 3.9.**  $\text{Aut}_{\mathcal{F}}(T) \leq \text{Aut}(\mathcal{C})$ .

*Proof.* Let  $\alpha \in \text{Aut}_{\mathcal{F}}(T)$  and note that  $\alpha \in C_{\mathcal{F}}(z)$ . Recall that  $z$  was chosen to be fully normalized. Thus,  $\mathcal{H}$  is a component of  $C_{\mathcal{F}}(z)$  as  $z \in \mathcal{I}(\mathcal{H})$ . It follows from [Asc11, 9.7] that there is a unique component of  $C_{\mathcal{F}}(z)$  with Sylow group  $T$ , so that  $\mathcal{H}^\alpha = \mathcal{H}$  by Lemma 2.5(b). Since  $T$  is fully

$\mathcal{F}$ -normalized by Hypothesis 3.1,  $\alpha$  extends to an automorphism  $\tilde{\alpha}$  of  $Q_0T = C_S(T)T = C_S(R)T$  with the last equality by Lemma 3.7(b). From Lemma 2.31(c),  $R$  is characteristic in  $T$ , so we have that  $R^\alpha = R$ , and hence that  $\tilde{\alpha}$  normalizes  $C_S(R)$ . Thus,  $\alpha \in N_{N_{\mathcal{F}}(V_R)}(R)$ , a model for which is, by definition,  $G_R$ . We may therefore choose  $g \in N_{G_R}(T)$  such that  $\alpha = c_g|_T$ . As  $H := C_{G_R}(V_R/R)$  is a normal subgroup of  $G_R$ ,  $g$  leaves invariant  $O^2(H)R = \langle T^H \rangle$ , which is a model for  $N_{\mathcal{C}}(R)$  by Lemma 3.7(d), whence  $\alpha$  normalizes  $N_{\mathcal{C}}(R)$ . Thus,  $\alpha \in \text{Aut}(\langle \mathcal{H}, N_{\mathcal{C}}(R) \rangle) = \text{Aut}(\mathcal{C})$ , the equality coming from the generation statement of Lemma 2.37(b), and now the assertion follows as  $\alpha$  was chosen arbitrarily.  $\square$

#### 4. THE CENTRALIZER OF $\mathcal{C}$

We operate for the remainder of the paper under the following hypothesis, although we will sometimes state it again for emphasis.

**Hypothesis 4.1.** Suppose  $\mathcal{F}$  is a saturated 2-fusion system on  $S$  and  $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$  is a standard subsystem of  $\mathcal{F}$  over  $T \in \mathcal{F}^f$ . Assume  $\mathcal{C} \cong \mathcal{F}_{\text{Sol}}(g)$  and  $\mathcal{C}$  is not a component of  $\mathcal{F}$ . Write  $\mathcal{Q}$  for the centralizer of  $\mathcal{C}$  (cf. Remark 2.23), and let  $Q$  be the Sylow group of  $\mathcal{Q}$ .

**Lemma 4.2.** *One of the following holds.*

- (a)  $Q$  is elementary abelian, or
- (b)  $Q$  is of 2-rank 1.

*Proof.* This is a direct consequence of Hypothesis 4.1, Lemma 2.38, and [Asc16, Theorem 8].  $\square$

**Proposition 4.3.** *Assume Hypothesis 4.1. Then  $Q$  has 2-rank 1.*

*Proof.* Write  $Z(T) = \langle z \rangle$ . The subsystem  $\mathcal{Q}$  is tightly embedded by [Asc16, 9.1.6.2]. Assume that  $Q$  has 2-rank larger than 1. Then by Lemma 4.2,  $Q$  is elementary abelian and  $|Q| > 2$ . Moreover, by Lemma 2.27,  $\mathcal{F}_Q(Q)$  is tightly embedded in  $\mathcal{F}$ . By [Asc16, 9.4.11], we can fix  $P \in \mathcal{Q}^{\mathcal{F}}$  such that  $P \leq N_S(Q)$  and  $P \neq Q$ . By [Asc16, 3.1.8], we have

$$P \cap Q = 1.$$

As  $\mathcal{C}$  is standard, we have  $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$ . In particular, we can form the product  $\mathcal{C}P$  inside of  $N_{\mathcal{F}}(Q)$ . As  $Q$  is normal in  $N_{\mathcal{F}}(Q)$ , we have  $Q \notin P^{\mathcal{C}P}$ . Furthermore, if  $\alpha \in \text{Hom}_{\mathcal{C}P}(P, TP)$  then  $\alpha$  induces the identity on  $PT/T$  by the construction of  $\mathcal{C}P$  in [Hen13] and since  $P \cong Q$  is abelian. So  $TP = TP^\alpha$ . Hence, replacing  $P$  by a suitable  $\mathcal{C}P$ -conjugate of  $P$ , we may assume

$$P \in (\mathcal{C}P)^f.$$

Then by [Asc16, Theorem 3.4.2],  $\mathcal{F}_P(P)$  is tightly embedded in  $\mathcal{C}P$ .

By [HL17, Theorem 3.10],  $\text{Out}(\mathcal{C})$  is cyclic. Note that  $N_S(Q)$  induces automorphisms of  $\mathcal{C}$  via conjugation as  $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$ . Moreover, the elements of  $N_S(Q)$  inducing inner automorphisms of  $\mathcal{C}$  are precisely the elements in  $TC_S(T)$ . Thus,  $N_S(Q)/TC_S(T)$  is cyclic. By Lemma 2.24 and Lemma 2.32,  $C_S(T) = \langle z \rangle Q$  and so  $TC_S(T) = TQ$ . Since  $P \cong Q$  is elementary abelian, it follows  $P \cap (TQ) \neq 1$ . Let  $1 \neq x \in P \cap (TQ)$  and write  $x = uv$  with  $u \in T$  and  $v \in Q$ . Note that  $u$  and  $v$  commute. As  $x$  is an involution, it follows that  $u$  and  $v$  have order at most 2. If  $u = 1$  then  $x = v \in P \cap Q$  contradicting  $P \cap Q = 1$ . Hence  $u$  is an involution. Let  $\alpha \in \text{Hom}_{\mathcal{C}P}(C_{TP}(x), TP)$  such that  $x^\alpha \in (\mathcal{C}P)^f$ . We proceed now in several steps to reach a contradiction.

*Step 1:* We show that  $x^\alpha \in C_S(T)$  and  $x^\alpha = zv^\alpha$  with  $v^\alpha \in Q$ . For the proof note first that, as  $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$ , we have  $T \trianglelefteq N_S(Q)$  and thus  $Z(T) = \langle z \rangle \trianglelefteq N_S(Q)$ . Hence,  $z$  is central in



$N_S(Q)$  and thus fully centralized in  $\mathcal{CP}$ . As  $u \in T$  is an involution and all involutions in  $T$  are by [LO02, Theorem 2.1]  $\mathcal{C}$ -conjugate, the element  $u$  is  $\mathcal{CP}$ -conjugate to  $z$ . Hence, there exists  $\varphi \in \text{Hom}_{\mathcal{CP}}(C_{TP}(u), TP)$  such that  $u^\varphi = z$ . Note that  $x, v \in C_{TP}(u)$ , since  $x = uv$  and  $u$  and  $v$  commute. We obtain  $x^\varphi = zv^\varphi$ , where  $v^\varphi \in Q \leq C_S(T)$ , as  $v \in Q$  and  $\varphi$  is a morphism in  $N_{\mathcal{F}}(Q)$ . Since  $z \in Z(T)$ , it follows  $T \leq C_S(x^\varphi)$ . Recall that  $\alpha$  was chosen such that  $x^\alpha \in (\mathcal{CP})^f$ . Thus, using Lemma 2.3, we can conclude that  $T \leq C_S(x^\alpha)$  and so  $x^\alpha \in C_S(T)$ . Note that  $u, v \in C_{TP}(x)$  as  $u$  and  $v$  commute. Moreover, since  $\alpha$  is a morphism in  $N_{\mathcal{F}}(Q)$ , we have  $v^\alpha \in Q \leq C_S(T)$ . So  $x^\alpha = u^\alpha v^\alpha$  and  $u^\alpha = x^\alpha (v^\alpha)^{-1} \in C_S(T)$ . As  $T$  is strongly closed in  $N_{\mathcal{F}}(Q)$ , we have  $u^\alpha \in T$  and thus  $u^\alpha \in Z(T) = \langle z \rangle$ . As  $u \neq 1$ , it follows  $u^\alpha = z$  and  $x^\alpha = zv^\alpha$  with  $v^\alpha \in Q$ . This completes Step 1.

*Step 2:* We show  $C_{\mathcal{C}}(z) \subseteq C_{\mathcal{CP}}(x^\alpha)$ . For the proof, we may assume that  $x^\alpha \neq z$ . By definition of  $Q$ , we have  $\mathcal{C} \subseteq C_{\mathcal{F}}(Q)$ . By Step 1,  $x^\alpha \in Q\langle z \rangle$ . Therefore  $C_{\mathcal{C}}(z) \subseteq C_{\mathcal{F}}(Q\langle z \rangle) \subseteq C_{N_{\mathcal{F}}(Q)}(x^\alpha)$ . Let  $R \in C_{\mathcal{C}}(z)^{fc}$  and let  $\chi \in \text{Aut}_{C_{\mathcal{C}}(z)}(R)$  be an arbitrary element of odd order. Then  $\chi$  extends to some  $\hat{\chi} \in \text{Aut}_{N_{\mathcal{F}}(Q)}(R\langle x^\alpha \rangle)$  with  $(x^\alpha)^{\hat{\chi}} = x^\alpha$ . The order of  $\hat{\chi}$  equals the order of  $\chi$  and is therefore odd. As  $x^\alpha \neq z$  is by Step 1 an involution centralizing  $T$ , we have  $x^\alpha \notin T$  and thus  $(R\langle x^\alpha \rangle) \cap T = R$ . Moreover, clearly  $[R\langle x^\alpha \rangle, \hat{\chi}] \leq R$  and  $\hat{\chi}|_R = \chi$  is a morphism in  $\mathcal{C}$ . By [BLO03, Lemma 6.2], we have  $R \in C^c$ . So it follows from the definition of  $\mathcal{CP}$  in [Hen13] that  $\hat{\chi}$  is a morphism in  $\mathcal{CP}$ . Hence,  $\chi$  is a morphism in  $C_{\mathcal{CP}}(x^\alpha)$ . By Alperin's fusion theorem [AKO11, Theorem I.3.6],  $C_{\mathcal{C}}(z)$  is generated by  $\text{Inn}(T)$  and all the automorphism groups  $O^2(\text{Aut}_{C_{\mathcal{C}}(z)}(R))$  with  $R \in C_{\mathcal{C}}(z)^{fc}$ . As  $T \leq C_S(x^\alpha)$ , it follows that  $C_{\mathcal{C}}(z) \subseteq C_{\mathcal{CP}}(x^\alpha)$ .

*Step 3:* We show that  $P^\alpha \trianglelefteq C_{\mathcal{CP}}(x^\alpha)$  and  $P^\alpha \cap T \leq \langle z \rangle$ . As remarked above,  $\mathcal{F}_P(P)$  is tightly embedded in  $\mathcal{CP}$ . Hence, it follows from (T1) that  $P^\alpha \trianglelefteq N_{\mathcal{CP}}(\langle x^\alpha \rangle) = C_{\mathcal{CP}}(x^\alpha)$ . In particular, as  $C_{\mathcal{C}}(z) \subseteq C_{\mathcal{CP}}(x^\alpha)$  by Step 2, it follows that  $P^\alpha \cap T$  is strongly closed in  $C_{\mathcal{C}}(z)$ . As  $P^\alpha \cap T$  is abelian, [AKO11, Corollary I.4.7] gives that  $P^\alpha \cap T$  is normal in  $C_{\mathcal{C}}(z)$ . Since  $C_{\mathcal{C}}(z)/\langle z \rangle$  is simple, this implies  $P^\alpha \cap T \leq \langle z \rangle$  as required.

*Step 4:* We show that  $[T, P^\alpha] = 1$ . As  $C_{\mathcal{C}}(z) = O^p(C_{\mathcal{C}}(z))$ , we have

$$T = \text{hypp}(C_{\mathcal{C}}(z)) = \langle [Y, \beta] : Y \leq T, \beta \in \text{Aut}_{C_{\mathcal{C}}(z)}(Y) \text{ of odd order} \rangle.$$

Let  $Y \leq T$  and  $\beta \in \text{Aut}_{C_{\mathcal{C}}(z)}(Y)$  of odd order. We will show that  $[Y, \beta, P^\alpha] = 1$ , which is sufficient to complete Step 4. By Step 2,  $C_{\mathcal{C}}(z) \subseteq C_{\mathcal{CP}}(x^\alpha)$ . As  $P^\alpha \trianglelefteq C_{\mathcal{CP}}(x^\alpha)$  by Step 3, we can thus extend  $\beta$  to  $\hat{\beta} \in \text{Aut}_{\mathcal{CP}}(YP^\alpha)$  with  $(P^\alpha)^{\hat{\beta}} = P^\alpha$ . By the definition of  $\mathcal{CP}$  in [Hen13] and since  $P$  is abelian, we have  $[P^\alpha, \hat{\beta}] \leq P^\alpha \cap T \leq \langle z \rangle$ , where the last inclusion uses Step 3. In particular,  $[P^\alpha, \hat{\beta}, Y] = 1$ . As  $P^\alpha \trianglelefteq C_{\mathcal{CP}}(x^\alpha)$  and  $T$  centralizes  $x^\alpha$  by Step 1,  $T$  normalizes  $P^\alpha$ . Hence, again using Step 3, we conclude  $[Y, P^\alpha] \leq [T, P^\alpha] \leq T \cap P^\alpha \leq \langle z \rangle$  and so  $[Y, P^\alpha, \hat{\beta}] = 1$ . It follows now from the Three-Subgroup-Lemma that  $[Y, \beta, P^\alpha] = [\hat{\beta}, Y, P^\alpha] = 1$ . This finishes Step 4.

*Step 5:* We now derive the final contradiction. By Step 4, we have  $P^\alpha \leq C_S(T)$ . As we saw above,  $C_S(T) = Q\langle z \rangle$  and thus  $Q$  has index 2 in  $C_S(T)$ . Since  $|P^\alpha| = |Q| > 2$ , it follows  $P^\alpha \cap Q \neq 1$ . However, as  $Q \notin P^{CP}$ , the subgroup  $P^\alpha$  is an  $\mathcal{F}$ -conjugate of  $Q$  not equal to  $Q$ . Hence, by [Asc16, 3.1.8], we have  $P^\alpha \cap Q = 1$ . This contradiction completes the proof.  $\square$

Assuming Hypothesis 4.1, we are thus left with the case that  $Q$  has 2-rank 1, i.e. is either cyclic or quaternion. We end this section with a lemma which handles a residual situation occurring in

this context. It will be needed both in Section 5 to exclude the quaternion case and in Section 6 to handle the case that  $Q$  is cyclic.

**Lemma 4.4.** *Assume Hypothesis 4.1 with  $Q$  of 2-rank 1. Let  $t$  be the unique involution in  $Q$  and fix a subnormal subsystem  $\mathcal{F}_0$  of  $\mathcal{F}$  over  $S_0 \leq S$  such that  $t \in S_0$ . Then the following hold:*

- (a)  $\langle t \rangle$  is fully  $\mathcal{F}_0$ -normalized.
- (b) If  $[T, C_{S_0}(t)] \neq 1$ , then  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_0}(t)$ . Moreover,

$$\Omega_1(C_{S_0}(C_{S_0}(t))) = \Omega_1(Z(C_{S_0}(t))) = \langle t, z \rangle.$$

- (c) Assume that  $Q \leq S_0$  and  $\mathcal{C} \subseteq C_{\mathcal{F}_0}(t)$ . If  $\langle t \rangle \leq Z(S_0)$  is weakly  $\mathcal{F}_0$ -closed in  $Z(S_0)$ , then  $\langle t \rangle$  is weakly  $\mathcal{F}_0$ -closed.

*Proof.* As  $Q$  is tightly embedded, there is a fully  $\mathcal{F}$ -normalized  $\mathcal{F}$ -conjugate of  $\langle t \rangle$  in  $Q$  by [Asc16, 3.1.5]. It follows that  $\langle t \rangle$  is fully  $\mathcal{F}$ -normalized, since  $t$  is the unique involution in  $Q$ . So (a) follows from 2.4. In particular,  $C_{\mathcal{F}_0}(t)$  is saturated.

In the proof of (b) and (c), we will use that  $\mathcal{C}$  is normal in  $C_{\mathcal{F}}(t)$  by (S2). In particular  $\mathcal{C}$  is a component of  $C_{\mathcal{F}}(t)$ . In addition, we will use that  $C_S(T) = \langle z \rangle Q$  from Lemma 2.24 and Lemma 2.32.

For the proof of (b) assume that  $[T, C_{S_0}(t)] \neq 1$ . By (a) and [Asc11, 8.23.2],  $C_{\mathcal{F}_0}(t)$  is subnormal in  $C_{\mathcal{F}}(t)$ . So by [Asc11, 9.6],  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_0}(t)$  as  $[T, C_{S_0}(t)] \neq 1$ . In particular,  $T \leq C_{S_0}(t)$ . As  $\mathcal{C}$  is normal in  $C_{\mathcal{F}}(t)$ , we have  $T \trianglelefteq C_S(t)$  and in particular,  $z \leq Z(C_{S_0}(t))$ . As  $C_S(T) = \langle z \rangle Q$ , we obtain  $\langle t, z \rangle \leq \Omega_1(Z(C_{S_0}(t))) \leq \Omega_1(C_{S_0}(C_{S_0}(t))) \leq \Omega_1(C_S(T)) = \langle t, z \rangle$  and this implies that (b) holds.

For the proof of (c) assume now that  $Q \leq S_0$ ,  $\mathcal{C} \subseteq C_{\mathcal{F}_0}(t)$ , and  $\langle t \rangle \leq Z(S_0)$  is weakly  $\mathcal{F}_0$ -closed in  $Z(S_0)$ . Then in particular,  $S_0 = C_{S_0}(t)$  and  $C_{\mathcal{F}_0}(t)$  is saturated. As  $T \leq C_{S_0}(t)$  is non-abelian, (b) gives that  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_0}(t)$  and  $\Omega_1(Z(S_0)) = \langle t, z \rangle$ . As  $\mathcal{C}$  is normal in  $C_{\mathcal{F}}(t)$ , one easily checks that  $\mathcal{C}$  is  $C_{\mathcal{F}_0}(t)$ -invariant (using the equivalent definition of  $\mathcal{F}$ -invariant subsystems given in [AKO11, Proposition I.6.4(d)]). Hence, by a Theorem of Craven [Cra11],  $\mathcal{C} = O^{p'}(\mathcal{C})$  is normal in  $C_{\mathcal{F}_0}(t)$ .

Assume now there exists and  $\mathcal{F}_0$ -conjugate  $f$  of  $t$  with  $f \neq t$  and fix such  $f$ . We proceed in three steps to derive a contradiction.

*Step 1:* We show that  $f \notin QT$  and  $t$  is weakly  $\mathcal{F}_0$ -closed in  $QT$ . Assuming  $f \in QT$ , we would have  $f \leq \Omega_1(QT) \leq T\langle t \rangle$ . So  $f \in T$  or  $f = ut$  with  $u \in T$ . In the latter case, since  $t \leq Q \leq C_S(T)$  and  $f$  is an involution,  $u$  is an involution. By [LO02, Theorem 2.1(b)] all involutions in  $T$  are  $\mathcal{C}$ -conjugate. Moreover  $\mathcal{C} \subseteq C_{\mathcal{F}_0}(t)$ . So if  $f \in T$ , then  $f$  is  $\mathcal{F}_0$  conjugate to  $z$ , and if  $f = ut$  for some involution  $u \in T$ , then  $f$  is  $\mathcal{F}_0$ -conjugate to  $zt$ . In both cases we get a contradiction to the assumption that  $t$  is  $\mathcal{F}_0$ -closed in  $Z(S_0)$ . So  $f \notin QT$ . Because of the arbitrary choice of  $f$ , this completes Step 1.

We adopt Notation 2.30. In particular,  $E$  is the elementary abelian subgroup of order  $2^3$  specified there. As  $\mathcal{C}$  is normal in  $C_{\mathcal{F}_0}(t)$ , we can form the product system  $\mathcal{C}\langle f \rangle$  (as defined in [Hen13]) in  $C_{\mathcal{F}_0}(t)$  over the 2-group  $T\langle f \rangle$ .

*Step 2:* We show that  $f$  is  $\mathcal{C}\langle f \rangle$ -conjugate to every member of the coset  $fE$ .

Note first that  $F^*(\mathcal{C}\langle f \rangle) = \mathcal{C}$ . Thus, by [HL17, Theorem 4.3],  $\mathcal{C}\langle f \rangle$  is uniquely determined as the split extension of  $\mathcal{C}$  by a field automorphism of order 2. As all involutions in  $T\langle f \rangle - T$  are  $\mathcal{C}\langle f \rangle$ -conjugate by Lemma 2.35, after conjugating in  $\mathcal{C}\langle f \rangle$ , we may take  $f$  to be this field automorphism.

Appealing again to Lemma 2.35, we have  $C_{\mathcal{C}\langle f \rangle}(f) = \langle f \rangle \times \mathcal{C}_1$ , where  $\mathcal{C}_1 = O^2(C_{\mathcal{C}\langle f \rangle}(f))$  has Sylow group  $C_T(f)$  and is isomorphic to  $\mathcal{F}_{\text{Sol}}(q^{1/2})$ . Then  $T_{k-1} = \mathcal{U}^1(T_k)$  is the torus of  $\mathcal{C}_1$ . Moreover, there is an element of  $T_k$  that conjugates  $f$  to  $fz$  (for example, an element in  $T_k - T_{k-1}$  that powers to  $z$ ). Recall from Notation 2.30 that  $E = \Omega_1(T_k) = \Omega_1(T_{k-1})$ . Since  $\text{Aut}_{\mathcal{C}_1}(T_{k-1})$  acts transitively on  $E^\#$  by Lemma 2.31(b), we see that indeed  $f$  is  $\mathcal{C}\langle f \rangle$ -conjugate to every element of  $fE$ .

*Step 3:* We derive the final contradiction. Since  $\mathcal{C}$  is normal in  $C_{\mathcal{F}}(t)$  and  $S_0 \leq C_S(t)$ ,  $S_0$  induces automorphisms of  $\mathcal{C}$  by conjugation. As  $C_S(T) = Q\langle z \rangle$  and  $\text{Aut}(\mathcal{C})$  is cyclic by [HL17, Theorem 3.10], it follows that  $QT = TC_S(T)$  is normal in  $S_0$  and  $S_0/QT$  cyclic. Now let  $\alpha \in \mathfrak{A}_{\mathcal{F}_0}(f)$  with  $f^\alpha = t$ . Then  $\alpha$  is defined on  $\langle f \rangle E$  and, hence  $t$  is  $\mathcal{F}_0$ -conjugate to every member of the coset  $tE^\alpha$  by Step 2. Since  $E^\alpha$  is of 2-rank 3, while  $S_0/QT$  is cyclic, it follows that  $E^\alpha \cap QT \neq 1$ . For  $1 \neq e \in E^\alpha \cap QT$ ,  $t$  is conjugate to  $te \in QT$ . This contradicts Step 1.  $\square$

## 5. $Q$ QUATERNION

In this section, we show that  $Q$  is not quaternion using Aschbacher's classification of quaternion fusion packets [Asc17a]. When combined with Proposition 4.3, the results of this section reduce to the case in which  $Q$  is cyclic, which is handled in Section 6.

The Classical Involution Theorem identifies the finite simple groups which have a *classical involution*, that is, an involution whose centralizer has a component (or solvable component) isomorphic to  $SL_2(q)$  (or  $SL_2(3)$ ) [Asc77a, Asc77b]. With one exception ( $M_{11}$ ) the simple groups having a classical involution are exactly the groups of Lie type in odd characteristic other than  $L_2(q)$  or  ${}^2G_2(q)$ , where the  $SL_2(q)$  components in involution centralizers are fundamental subgroups generated by the center of a long root subgroup and its opposite.

In a group with a classical involution, the collection of these  $SL_2(q)$  subgroups satisfies special fusion theoretic properties that were identified and abstracted by Aschbacher in [Asc77a, Hypothesis  $\Omega$ ]. More recently, Aschbacher has formulated these conditions in fusion systems in the definition of a *quaternion fusion packet*, and his memoir [Asc17a] classifies all such packets.

**Definition 5.1.** A *quaternion fusion packet* is a pair  $\tau = (\mathcal{F}, \Omega)$ , where  $\mathcal{F}$  is a saturated fusion system on a finite 2-group  $S$ , and  $\Omega$  is an  $\mathcal{F}$ -invariant collection of subgroups of  $S$  such that

- (QFP1) There exists an integer  $m$  such that for all  $K \in \Omega$ ,  $K$  has a unique involution  $z(K)$  and is nonabelian of order  $m$ .
- (QFP2) For each pair of distinct  $K, J \in \Omega$ ,  $|K \cap J| \leq 2$ .
- (QFP3) If  $K, J \in \Omega$  and  $v \in J - Z(J)$ , then  $v^{\mathcal{F}} \cap C_S(z(K)) \subseteq N_S(K)$ .
- (QFP4) If  $K, J \in \Omega$  with  $z = z(K) = z(J)$ ,  $v \in K$ , and  $\varphi \in \text{Hom}_{C_{\mathcal{F}}(z)}(\langle v \rangle, S)$ , then either  $v^\varphi \in J$  or  $v^\varphi$  centralizes  $J$ .

We assume the following hypothesis until the last result in this section.

**Hypothesis 5.2.** Hypothesis 4.1 and its notation hold with  $Q$  quaternion. Let  $t$  be the unique involution in  $Q$ . Set  $\Omega = Q^{\mathcal{F}}$ , denote by  $\mathcal{F}^\circ$  the subnormal closure of  $Q$  in  $\mathcal{F}$  over the subgroup  $S^\circ \leq S$ , and set  $\Omega^\circ = Q^{\mathcal{F}^\circ}$ .

A tightly embedded subsystem with quaternion Sylow 2-subgroups, such as the centralizer system  $\mathcal{Q}$  in Hypothesis 5.2, always yields a quaternion fusion packet in a straightforward way.

**Lemma 5.3.**  $(\mathcal{F}, \Omega)$  is a quaternion fusion packet.

*Proof.* We go through the list of axioms. (QFP1) holds by definition of  $\Omega$ . Note that  $\Omega \subseteq \mathcal{P}^*$  in the sense of Definition 3.1.9 of [Asc16]. Hence, by [Asc16, 3.1.12.2],  $K \cap J = 1$  for each pair of distinct  $K, J \in \Omega$ . This shows that (QFP2) holds, and that any element of  $S$  centralizing  $z(K)$  must normalize  $K$ , so that (QFP3) also holds. Finally, under the hypotheses of (QFP4),  $K = J$  in the current situation. Fix  $1 \neq v \in K$  and  $\varphi \in \text{Hom}_{C_{\mathcal{F}}(z(K))}(\langle v \rangle, S)$ . Then  $z(K) \in \langle v \rangle$ , and  $z(K)^\varphi = z(K)$ . Also,  $\langle v \rangle \in \mathcal{P}$ , and  $K \in \mathcal{P}^*$ , in the sense of Definition 3.1.9 of [Asc16]. Since  $\langle v \rangle^\varphi \cap K > 1$ , we see from [Asc16, 3.1.14] (applied with  $\langle v \rangle$ ,  $\varphi$ , and  $K$  in the role of  $P$ ,  $\psi$ , and  $R$ ) that  $\langle v \rangle^\varphi \leq K$ . This shows that (QFP4) holds.  $\square$

**Lemma 5.4.** *Let  $\mathcal{F}_0$  be a subnormal subsystem of  $\mathcal{F}$  over the subgroup  $S_0 \leq S$ . Assume that  $Q \leq S_0$ , and that  $\mathcal{C} \leq C_{\mathcal{F}_0}(t)$ . Then  $Q^{\mathcal{F}_0} \neq \{Q\}$ .*

*Proof.* Suppose on the contrary that  $Q^{\mathcal{F}_0} = \{Q\}$ . Then  $Q$  is normal in  $S_0$ , and so  $t \in Z(S_0)$ . Let  $\alpha$  be a morphism in  $\mathcal{F}_0$  with  $t^\alpha \in Z(S_0)$ . By the extension axiom, we may assume that  $\alpha$  is defined on  $Q$ , and then  $Q^\alpha = Q$  by assumption, so that  $t^\alpha = t$ . This shows that  $\langle t \rangle$  is weakly  $\mathcal{F}_0$ -closed in  $Z(S_0)$ . Thus,  $\langle t \rangle$  is weakly  $\mathcal{F}_0$ -closed by Lemma 4.4(b).

Now as  $\langle t \rangle \leq Z(S_0)$  is weakly closed in  $\mathcal{F}_0$ , we have that  $\langle t \rangle \leq Z(\mathcal{F}_0)$  by Alperin's Fusion Theorem [AKO11, Theorem I.3.6]. Hence,  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_0}(t) = \mathcal{F}_0$  by Lemma 4.4(b). So  $\mathcal{C}$  is a component of  $\mathcal{F}$  contrary to Hypothesis 4.1.  $\square$

**Lemma 5.5.**  *$\mathcal{C}$  is a component of  $C_{\mathcal{F}^\circ}(t)$ . In particular,  $\mathcal{C}$  is contained in  $C_{\mathcal{F}^\circ}(t)$ .*

*Proof.* Define  $\text{sub}_0(\mathcal{F}, Q) = \mathcal{F}$ ,  $S_0 = S$ , and for each  $i \geq 0$ , define  $\text{sub}_{i+1}(\mathcal{F}, Q)$  to be the normal closure of  $Q$  in  $\text{sub}_i(\mathcal{F}, Q)$  with Sylow group  $S_{i+1}$ . Then  $\mathcal{F}_{i+1} \trianglelefteq \mathcal{F}_i$  for each  $i \geq 0$ , and  $\mathcal{F}^\circ$  is by definition the terminal member of this series. By Lemma 4.4(a),  $\langle t \rangle$  is fully normalized in  $\mathcal{F}_i$  for  $i \geq 0$ , so  $C_{\mathcal{F}_i}(t)$  is saturated for each  $i$ .

We argue by contradiction, and fix the least nonnegative integer  $i$  such that  $\mathcal{C}$  is not a component of  $C_{\mathcal{F}_{i+1}}(t)$ . Thus, as  $\mathcal{C}$  is normal in  $C_{\mathcal{F}}(t)$  by (S2), we have that  $i > 0$  and that  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_i}(t)$ . By Lemma 5.4, we have that  $Q^{\mathcal{F}_i} \neq \{Q\}$ . Fix  $Q' \in Q^{\mathcal{F}_i} - \{Q\}$ . As  $Q$  is tightly embedded in  $\mathcal{F}$ , we have  $Q \cap Q' = 1$  by [Asc16, 3.1.12.2], and we have

$$Q' \leq C_{S_i}(Q) \leq C_{S_i}(t)$$

by [Asc16, 3.3.5]. By definition of  $\mathcal{F}_{i+1}$ , we have  $Q' \leq S_{i+1}$  and thus  $Q' \leq C_{S_{i+1}}(t)$ . As  $C_S(T) = Q\langle z \rangle$  by Lemma 2.24 and Lemma 2.32, it follows  $[Q', T] \neq 1$ . and thus  $[T, C_{S_{i+1}}(t)] \neq 1$ . Hence,  $\mathcal{C}$  is a component of  $C_{\mathcal{F}_{i+1}}(t)$  by Lemma 4.4(b), and this contradicts our choice of  $i$ .  $\square$

**Lemma 5.6.** *The pair  $(\mathcal{F}^\circ, \Omega^\circ)$  is a quaternion fusion packet,  $\mathcal{F}^\circ$  is the normal closure of  $Q$  in  $\mathcal{F}^\circ$ , and  $\mathcal{F}^\circ$  is transitive on  $\Omega^\circ$ .*

*Proof.* Note that  $(\mathcal{F}^\circ, \Omega^\circ)$  is a quaternion fusion packet by Lemma 5.3 and [Asc17a, Lemma 6.4.2.1]. Recall that  $\mathcal{F}^\circ$  is the subnormal closure of  $Q$  in  $\mathcal{F}$ . So the second statement follows from the definition of subnormal closure, while the third holds by definition of  $\Omega^\circ$ .  $\square$

Let  $r$  be an odd prime. The class of finite groups denoted  $\text{Chev}(r)$  in [Asc17a, Chapter 0] is the same as the class denoted  $\mathcal{L}ie(r)$  in [GLS98, Definition 2.2.2]; see also [GLS98, Theorem 2.2.7]. (But this is not the same as the class “Chev( $r$ )” appearing in [GLS98, Theorem 2.2.8].) In the next lemma, we write  $\mathcal{L}ie(r)$  for the class of groups appearing in [GLS98, Definition 2.2.2].

**Lemma 5.7.** *Let  $r$  be an odd prime, let  $G \in \mathcal{L}ie(r)$  and let  $t \in G$  be an involution. Then for each component  $\bar{K}$  of  $C_G(t)/O(C_G(t))$ ,  $\bar{K}/Z(\bar{K})$  is a known finite simple group.*

*Proof.* Let  $C := C_G(t)$  and denote quotients by  $O(C)$  with bars. By [GLS98, Theorem 4.2.2],  $C$  has a normal subgroup  $L$  which is a central product of groups  $L_i \in \mathcal{L}ie(r)$ , and with  $C/L$  solvable. By [GLS98, Theorem 2.2.7], each central product factor  $L_i$  of  $L$  is either quasisimple or solvable, and hence each component of  $\bar{C}$  is the image of a component of  $C$  under the quotient map  $C \rightarrow \bar{C}$ . Hence, if  $\bar{K}$  is a component of  $\bar{C}$ , then there is a component  $K$  of  $C$  covering  $\bar{K}$ , and  $K \leq E(C) = E(L)$  as  $C/L$  is solvable. It follows that  $K = L_i$  for some  $i$ . Hence,  $\bar{K}/Z(\bar{K}) \cong L_i/Z(L_i)$  is known.  $\square$

Now remove the standing assumption that Hypothesis 5.2 holds.

**Proposition 5.8.** *Assume Hypothesis 4.1. Then  $Q$  is cyclic.*

*Proof.* We argue by contradiction, so that  $Q$  is quaternion by Proposition 4.3. Hence, Hypothesis 5.2 holds, and so we adopt the notation there. By Lemma 5.6, the pair  $(\mathcal{F}^\circ, \Omega^\circ)$  satisfies the hypotheses of Theorem 1 of [Asc17a]. Hence, by that theorem, one of the following holds: either

- (1)  $t \in Z(\mathcal{F}^\circ)$ , or
- (2)  $t \in O_2(\mathcal{F}^\circ) - Z(\mathcal{F}^\circ)$ , or
- (3) there is a finite group  $G$  with Sylow 2-subgroup  $S^\circ$  such that  $\mathcal{F}^\circ = \mathcal{F}_{S^\circ}(G)$ , and one of the following holds,
  - (a)  $S^\circ$  has 2-rank at most 3, or
  - (b)  $G \in \mathcal{L}ie(r)$  for some odd prime  $r$ , or
  - (c)  $G$  is quasisimple with  $Z(G)$  a 2-group, and  $G/Z(G) \cong Sp_6(2)$  or  $\Omega_8^+(2)$ .

Note that in all cases,

$$(5.9) \quad \mathcal{C} \text{ is a component of } C_{\mathcal{F}^\circ}(t),$$

by Lemma 5.4.

In Case (1),  $\mathcal{C}$  is a component of  $C_{\mathcal{F}^\circ}(t) = \mathcal{F}^\circ$ . Hence  $\mathcal{C}$  is a component of  $\mathcal{F}$  since  $\mathcal{F}^\circ$  is subnormal in  $\mathcal{F}$ , contrary to Hypothesis 4.1. In Case (2), the hypotheses of [Asc17a, Theorem 2] hold for  $(\mathcal{F}^\circ, \Omega^\circ)$ , and then by [Asc17a, Lemma 6.7.3], we have that  $\mathcal{F}^\circ$  is constrained. Thus,  $C_{\mathcal{F}^\circ}(t)$  is also constrained, and hence  $\mathcal{C} \leq E(C_{\mathcal{F}^\circ}(t)) = 1$ , a contradiction.

Case (3)(a) yields a contradiction, since  $QT \leq S^\circ$  is of 2-rank 5 by Lemma 2.31(d). In Case (3)(b), note that  $C_{\mathcal{F}^\circ}(t)$  is the fusion system of  $C_G(t)/O(C_G(t))$  by [AKO11, I.5.4]. So by (5.9) and Lemma 5.7, the hypothesis of Lemma 2.7 hold, and so there is a component  $K$  of  $C_G(t)/O(C_G(t))$  such that  $\mathcal{C}$  is the 2-fusion system of  $K$  by that lemma. This contradicts the fact that  $\mathcal{C}$  is exotic [LO02, Proposition 3.4].

In Case 3(c) we may assume that Case (2) does not hold, so that  $t \notin Z(\mathcal{F}^\circ)$ . Then  $t \notin Z(G)$ . As  $Sp_6(2)$  and  $\Omega_8^+(2)$  are of characteristic 2-type and as  $t \notin Z(G)$ , we have that  $C_G(t)$  is of characteristic 2. Hence  $C_{\mathcal{F}^\circ}(t)$  is constrained. We therefore obtain the same contradiction here as in Case (2).  $\square$

## 6. PROOF OF THEOREM 1.1

In this section, we finish the proof of Theorem 1.1 by handling the case in which  $Q$  is cyclic. We therefore assume the following hypothesis and notation for this section.

**Hypothesis 6.1.** Hypothesis 4.1 holds with  $Q$  cyclic and  $\mathcal{C}$  subintrinsic in  $\mathfrak{C}(\mathcal{F})$ . Write  $\Omega_1(Q) = \langle t \rangle$ ,  $S_t = C_S(t)$ , and  $\mathcal{F}_t = C_{\mathcal{F}}(t)$ .

**Lemma 6.2.** *The following hold.*

- (a)  $\langle t \rangle \in \mathcal{F}^f$ ,
- (b)  $C_S(T) = Q\langle z \rangle$ ,
- (c)  $\Omega_1(C_S(S_t)) = \Omega_1(Z(S_t)) = \langle t, z \rangle$ , and
- (d)  $t$  is not  $\mathcal{F}$ -conjugate to  $z$ .

*Proof.* Parts (a) and (c) follow from Proposition 4.4(a),(b) applied with  $\mathcal{F}_0 = \mathcal{F}$ , while (b) follows from Lemma 2.24 and Lemma 2.32.

It remains to prove (d). By (T1) in the definition of tight embedding (Definition 2.26), we have  $Q = \mathcal{Q} \trianglelefteq \mathcal{F}_t$ . Further,  $Q = C_{S_t}(\mathcal{C})$  by [Asc16, 9.1.6.3]. Write quotients by  $Q$  with bars. Note that  $C_{\bar{S}_t}(\bar{\mathcal{C}})$  is trivial by [Lyn15, Lemma 1.14], and  $\bar{\mathcal{C}} \cong \mathcal{C}$ . Thus,  $F^*(\bar{\mathcal{F}}_t) = \bar{\mathcal{C}}$  is isomorphic to a Benson-Solomon system. By [HL17, Theorem 4.3], this quotient is therefore a split extension of  $\bar{\mathcal{C}}$  by a 2-group of outer automorphisms, and in particular,  $O^2(\bar{\mathcal{F}}_t) = \bar{\mathcal{C}}$ . It follows that  $O^2(\mathcal{F}_t) \leq Q\mathcal{C}$ . Since  $O^2(Q\mathcal{C}) = \mathcal{C}$  and since  $O^2(O^2(\mathcal{F}_t)) = O^2(\mathcal{F}_t)$ , we have that  $O^2(\mathcal{F}_t) = \mathcal{C}$ . Hence,  $t$  is fully normalized and not in the hyperfocal subgroup of  $\mathcal{F}_t$ , while  $z^\alpha$  is contained in the hyperfocal subgroup of  $\mathcal{H}^\alpha \leq C_{\mathcal{F}}(z^\alpha)$  for every  $\alpha \in \mathfrak{A}(\langle z \rangle)$ . Thus,  $t$  and  $z$  are not  $\mathcal{F}$ -conjugate.  $\square$

The assumption that  $\mathcal{C}$  is subintrinsic in  $\mathfrak{C}(\mathcal{F})$  in Hypothesis 6.1 is important for the relative simplicity of the proof of the next lemma.

**Lemma 6.3.**  *$t \in Z(S)$ , and  $\langle t \rangle$  is weakly  $\mathcal{F}$ -closed in  $Z(S)$ .*

*Proof.* Assume first that  $t \notin Z(S)$ . Then  $S_t < S$ , so that  $S_t < N_S(S_t)$ . Fix  $a \in N_S(S_t) - S_t$ . Then  $t^a = tz$  and  $z^a = z$  by Lemma 6.2(c,d).

Since  $z^a = z$  and since  $\mathcal{H}$  is a component of  $C_{\mathcal{F}}(z)$  on  $T$  by Hypothesis 6.1, we see that  $\mathcal{H}^a$  is a component of  $C_{\mathcal{F}}(z)$  on  $T^a$ . However, if  $\mathcal{H}^a \neq \mathcal{H}$ , then since Sylow subgroups of distinct components commute, we would have  $T^a \leq C_{S_t}(T) \leq Q\langle z \rangle$  by Proposition 2.10, and we would be forced to conclude that  $T$  is abelian. Since this is not the case,  $a$  normalizes  $T$ . Thus, by (S4) in Definition 2.22, conjugation by  $a$  restricts to an automorphism of  $\mathcal{C}$ . Thus,  $a$  acts also on  $Q = C_{S_t}(\mathcal{C})$ , so that  $t^a = t$ . This contradicts the choice of  $a$ .

We have shown that  $t \in Z(S)$ . Then Lemma 6.2(c) yields  $\Omega_1(Z(S)) = \Omega_1(Z(S_t)) = \langle t, z \rangle$ , while Lemma 6.2(d) says that  $t$  is not  $\mathcal{F}$ -conjugate to  $z$ . Hence,  $\langle t \rangle$  is weakly  $\mathcal{F}$ -closed in  $Z(S)$  by Burnside's fusion lemma.  $\square$

**Lemma 6.4.**  *$\langle t \rangle$  is weakly  $\mathcal{F}$ -closed.*

*Proof.* This is a combination of Lemma 6.3 and Lemma 4.4(b).  $\square$

We now remove the standing assumption that Hypothesis 6.1 holds, and complete the proof of Theorem 1.1.

**Theorem 6.5.** *Assume Hypothesis 3.1 holds. Then  $\mathcal{C}$  is a component of  $\mathcal{F}$ .*

*Proof.* By Theorem 3.2, Hypothesis 4.1 holds. Then Proposition 5.8 yields that  $Q$  is cyclic, so that Hypothesis 6.1 holds. By Lemma 6.4,  $\langle t \rangle$  is weakly  $\mathcal{F}$ -closed. It follows that  $\langle t \rangle$  is preserved by each  $\mathcal{F}$ -automorphism of an  $\mathcal{F}$ -centric radical subgroup of  $S$ . Hence  $t \in Z(\mathcal{F})$  by Alperin's fusion theorem, so that  $\mathcal{C}$  is a component of  $C_{\mathcal{F}}(t) = \mathcal{F}$ .  $\square$

## REFERENCES

- [AC10] Michael Aschbacher and Andrew Chermak, *A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver*, Ann. of Math. (2) **171** (2010), no. 2, 881–978.
- [AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834
- [AO16] Michael Aschbacher and Bob Oliver, *Fusion systems*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 4, 555–615. MR 3544261
- [AOV12] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, *Reduced, tame and exotic fusion systems*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 1, 87–152. MR 2948790
- [Asc77a] Michael Aschbacher, *A characterization of Chevalley groups over fields of odd order*, Ann. of Math. (2) **106** (1977), no. 2, 353–398. MR 0498828 (58 #16865a)
- [Asc77b] ———, *A characterization of Chevalley groups over fields of odd order. II*, Ann. of Math. (2) **106** (1977), no. 3, 399–468. MR 0498829 (58 #16865b)
- [Asc08] ———, *Normal subsystems of fusion systems*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 1, 239–271. MR 2434097 (2009e:20044)
- [Asc10] ———, *Generation of fusion systems of characteristic 2-type*, Invent. Math. **180** (2010), no. 2, 225–299.
- [Asc11] ———, *The generalized Fitting subsystem of a fusion system*, Mem. Amer. Math. Soc. **209** (2011), no. 986, vi+110. MR 2752788
- [Asc15] ———, *Classifying finite simple groups and 2-fusion systems*, ICCM Not. **3** (2015), no. 1, 35–42. MR 3385504
- [Asc16] ———, *On fusion systems of component type*, Mem. Amer. Math. Soc. (to appear) (2016), 228pp.
- [Asc17a] ———, *Quaternion fusion packets*, preprint (2017), x+432pp.
- [Asc17b] ———, *Walter’s Theorem for fusion systems*, preprint (2017), 64pp.
- [BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).
- [BMO16] Carles Broto, Jesper M. Møller, and Bob Oliver, *Automorphisms of fusion systems of finite simple groups of lie type*, Mem. Amer. Math. Soc. (to appear) (2016), arXiv:1601.04566.
- [COS08] Andrew Chermak, Bob Oliver, and Sergey Shpectorov, *The linking systems of the Solomon 2-local finite groups are simply connected*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 1, 209–238. MR 2434096 (2009g:55018)
- [Cra11] David A. Craven, *Normal subsystems of fusion systems*, J. Lond. Math. Soc. (2) **84** (2011), no. 1, 137–158. MR 2819694
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple  $K$ -groups. MR 1490581 (98j:20011)
- [Hen13] Ellen Henke, *Products in fusion systems*, J. Algebra **376** (2013), 300–319. MR 3003728
- [HL17] Ellen Henke and Justin Lynd, *Extensions of the Benson-Solomon fusion systems*, preprint (2017), 14pp.
- [HS15] E. Henke and J. Semeraro, *Centralizers of normal subgroups and the  $Z^*$ -theorem*, J. Algebra **439** (2015), 511–514. MR 3373382
- [KS04] Hans Kurzweil and Bernd Stellmacher, *The theory of finite groups*, Universitext, Springer-Verlag, New York, 2004, An introduction, Translated from the 1998 German original. MR 2014408 (2004h:20001)
- [LO02] Ran Levi and Bob Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson*, Geom. Topol. **6** (2002), 917–990 (electronic).
- [LO05] ———, *Correction to: “Construction of 2-local finite groups of a type studied by Solomon and Benson” [Geom. Topol. **6** (2002), 917–990 (electronic); mr1943386]*, Geom. Topol. **9** (2005), 2395–2415 (electronic).
- [LS17] Justin Lynd and Jason Semararo, *Weights in a Benson-Solomon block*, arXiv:1712.02826 [math.GR] (2017), 20pp.
- [Lyn15] Justin Lynd, *A characterization of the 2-fusion system of  $L_4(q)$* , J. Algebra **428** (2015), 315–356. MR 3314296
- [Oli16] Bob Oliver, *Reductions to simple fusion systems*, Bull. Lond. Math. Soc. **48** (2016), no. 6, 923–934.

- [Pui00] Lluís Puig, *The hyperfocal subalgebra of a block*, Invent. Math. **141** (2000), no. 2, 365–397.
- [Sem15] Jason Semeraro, *Centralizers of subsystems of fusion systems*, J. Group Theory **18** (2015), no. 3, 393–405.  
MR 3341522

*E-mail address:* `ellen.henke@abdn.ac.uk`

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, FRASER NOBLE BUILDING, ABERDEEN AB24 3UE

*E-mail address:* `lynd@louisiana.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, MAXIM DOUCET HALL, LAFAYETTE,  
LA 70504