# MINIMAL PRIMAL IDEALS IN THE INNER CORONA ALGEBRA OF A $C_0(X)$-ALGEBRA

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ABSTRACT. Let $A = C(X) \otimes K(H)$, where $X$ is an infinite compact Hausdorff space and $K(H)$ is the algebra of compact operators on a separable, infinite-dimensional Hilbert space. Let $A^s$ be the norm-closed ideal of the multiplier algebra $M(A)$ consisting of all the strong*-continuous functions from $X$ to $K(H)$. Then $A^s/A$ is the *inner corona algebra of* $A$. We identify the space MinPrimal($A^s$) of minimal closed primal ideals in $A^s$. If $A$ is separable then MinPrimal($A^s$) is compact and extremally disconnected. Using ultrapowers, we exhibit a faithful family of irreducible representations of $A^s/A$ and hence show that if every point of $X$ lies in the boundary of a zero set (i.e. if $X$ has no P-points) then the minimal closed primal ideals of $A^s/A$ are precisely the images under the quotient map of the minimal closed primal ideals of $A^s$. The map between MinPrimal($A^s$) and MinPrimal($A^s/A$) need not be continuous, however, and MinPrimal($A^s/A$) is not weakly Lindelof. As an application, it is shown that if $X = \beta\mathbb{N}\backslash\mathbb{N}$ then the relation of inseparability on Prim($A^s/A$) is an equivalence relation but not an open equivalence relation.

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## 1. INTRODUCTION

Let $A = C_0(X) \otimes K(H) \cong C_0(X, K(H))$, where $C_0(X)$ is the C*-algebra of complex-valued functions vanishing at infinity on an infinite locally compact Hausdorff space $X$ and $K(H)$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space $H$. Then it is well known that $M(A)$, the multiplier algebra of $A$, is isomorphic to the algebra of bounded strong*-continuous functions from $X$ to $B(H)$, the algebra of bounded operators on $H$ [1]. In investigating the ideal structure of $M(A)$, and of the corona algebra $C(A) = M(A)/A$, a natural ideal of $M(A)$ to consider is $A^s$, the algebra of bounded strong*-continuous functions from $X$ to $K(H)$. Since $X$ is infinite, $A^s \neq A$ (see [12, Theorems 3.3 and 3.7] for more general results).

A general study of the ideal structure of $A^s/A$ in [12] exposed something of the complexity; and the purpose of this present paper is to continue this study by identifying the set of minimal closed primal ideals in $A^s/A$ in the case when $A$ is $\sigma$-unital (equivalently, when $X$ is $\sigma$-compact). We also say a little about the $\tau_w$-topology (see below) on this set of ideals, but this seems to be a difficult subject. The reason for trying to identify the minimal closed primal ideals is that they tend to form an accessible family in the ideal lattice of a C*-algebra, and they are closely related to the primitive ideals (in particular, the minimal closed primal ideals are the minimal elements in the $\tau_w$-closure of the set Prim($B$) of primitive ideals of a C*-algebra $B$ [3, Proposition 3.1]).

Recall that if $A$ is a ring and $I$ an ideal in $A$ then $I$ is *primal* if whenever $J_1, J_2, \ldots, J_n$ is a finite family of ideals of $A$ with $J_1 J_2 \ldots J_n = \{0\}$ then $J_i \subseteq I$ for at least one $i$. If $I$ is a closed ideal of a C*-algebra $A$ then $\mathrm{Prim}(A/I)$ can be canonically identified with the closed subset $\{P \in \mathrm{Prim}(A) : P \supseteq I\}$ of $\mathrm{Prim}(A)$ (with the hull-kernel topology) and it is well-known that $I$ is primal if and only if $\mathrm{Prim}(A/I)$ is contained in a limit set in $\mathrm{Prim}(A)$ (cf. [4, Proposition 3.2]). Every prime ideal (and hence every primitive ideal) of a C*-algebra $A$ is primal, and every ideal which contains a primal ideal is primal. A Zorn's Lemma argument shows that every closed primal ideal contains a minimal closed primal ideal. Let $\mathrm{MinPrimal}(A)$ denote the set of minimal closed primal ideals of $A$. The topology $\tau_w$ is defined on the set $\mathrm{Id}(A)$ of all closed ideals of $A$ by taking sets of the form $\{I \in \mathrm{Id}(A) : a \notin I\}$ $(a \in A)$ as sub-basic (see [3, p.525] where an equivalent definition is given). On $\mathrm{Prim}(A)$, $\tau_w$ coincides with the hull-kernel topology, and on $\mathrm{MinPrimal}(A)$, $\tau_w$ is a Hausdorff topology [3, Corollary 4.3].

Where possible we work in the context of a general $C_0(X)$-algebra $A$ (see the definition in Section 2), but for most of the results we have to impose some restrictions. The motivating example is $A = C_0(X) \otimes K(H)$, and we now describe the main results in this case.

In Section 3 we identify $\mathrm{MinPrimal}(A^s)$ in the case where $A$ is $\sigma$-unital (Theorem 3.3), and we show furthermore that if $A$ is separable then $\mathrm{MinPrimal}(A^s)$ is $\tau_w$-compact and extremally disconnected (Corollary 3.6). Our identification builds on that already established in [11] for $\mathrm{MinPrimal}(M(A))$.

In Section 4 we use ultrapowers to exhibit a faithful family of irreducible representations of $A^s/A$ in the case where $A$ is $\sigma$-unital (Theorem 4.7). This enables us to determine, in Section 5, the set of minimal closed primal ideals of $A^s/A$. We show that if $X$ has no P-points (recall that a *P-point* is one which does not lie in the boundary of any zero set) then the minimal closed primal ideals of $A^s/A$ are precisely the images of the minimal closed primal ideals of $A^s$ under the quotient map (Theorem 5.6).

In Section 6 we investigate the topology on $\mathrm{MinPrimal}(A^s/A)$, showing that while this can be described in some fairly simple cases it looks intractable in general. For example, if $A$ is separable and $X$ has no isolated points then $\mathrm{MinPrimal}(A^s/A)$ is not weakly Lindelof, and hence is certainly not homeomorphic to $\mathrm{MinPrimal}(A^s)$ (Theorem 6.5). Finally we study the case where $X$ is an F-space without isolated points, such as $\beta\mathbf{N}\setminus\mathbf{N}$ (recall that a completely regular topological space $Y$ is an *F-space* if disjoint cozero sets in $Y$ are contained in disjoint zero sets, see [17, 3.1, 14N4, 1.15 and 14.27]). We show, using $\mathrm{MinPrimal}(A^s/A)$ and its topology, that the relation $\sim$ of inseparability by disjoint open sets on $\mathrm{Prim}(A^s/A)$ is an equivalence relation on $\mathrm{Prim}(A^s/A)$ (Theorem 6.8) but is not an open equivalence relation (Theorem 6.9).

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## 2. Preliminaries

We begin by collecting some of the information that we need on $C_0(X)$-algebras. Recall that a C*-algebra $A$ is a $C_0(X)$-algebra if there is a continuous map $\phi$, called the base map, from $\mathrm{Prim}(A)$ to the locally compact Hausdorff space $X$ [32, Proposition C.5]. We will use $X_\phi$ to denote the image of $\phi$ in $X$. Then $X_\phi$ is completely regular; and if $A$ is $\sigma$-unital, $X_\phi$ is $\sigma$-compact and hence normal [8, Section 1].

For $x \in X_\phi$, set $J_x = \bigcap\{P \in \mathrm{Prim}(A) : \phi(P) = x\}$, and for $x \in X \setminus X_\phi$, set $J_x = A$. For $a \in A$, the function $x \to \|a + J_x\|$ $(x \in X)$ is upper semi-continuous [32, Proposition C.10]. The $C_0(X)$-algebra $A$ is said to be *continuous* if, for all $a \in A$, the norm function

$x \to \|a + J_x\|$ $(x \in X)$ is continuous. By Lee's theorem [32, Proposition C.10 and Theorem C.26], this happens if and only if the base map $\phi$ is open.

One case of special importance (through which all other cases factor) is when the base map $\phi$ is the complete regularization map

$$\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A) \subseteq \beta(\mathrm{Glimm}(A)$$

(see [6, Section 2] for the identification of the complete regularization of $\mathrm{Prim}(A)$ with the space $\mathrm{Glimm}(A)$ of Glimm ideals of $A$). If $\mathrm{Glimm}(A)$ is locally compact (for the complete regularization topology $\tau_{cr}$) then one may take $X = X_\phi = \mathrm{Glimm}(A)$; otherwise, one may take $X = \beta(\mathrm{Glimm}(A))$. In this setting, if $x \in X_\phi = \mathrm{Glimm}(A)$ then the ideal $J_x$ coincides with the Glimm ideal $x$.

Let $J$ be a proper, closed, two-sided ideal of a $C^*$-algebra $A$. The quotient map $q_J : A \to A/J$ has a canonical extension $\tilde{q_J} : M(A) \to M(A/J)$ such that $\tilde{q_J}(b)q_J(a) = q_J(ba)$ and $q_J(a)\tilde{q_J}(b) = q_J(ab)$ $(a \in A, b \in M(A))$. We define a proper, closed, two-sided ideal $\tilde{J}$ of $M(A)$ by

$$\tilde{J} = \ker \tilde{q_J} = \{b \in M(A) : ba, ab \in J \text{ for all } a \in A\}.$$

The following proposition was proved in [7, Proposition 1.1].

**Proposition 2.1.** *Let $J$ be a proper, closed, two-sided ideal of a $C^*$-algebra $A$. Then*

(i) *$\tilde{J}$ is the strict closure of $J$ in $M(A)$;*

(ii) *$\tilde{J} \cap A = J$;*

(iii) *if $P \in \mathrm{Prim}(A)$ then $\tilde{P}$ is primitive (and hence is the unique ideal in $\mathrm{Prim}(M(A))$ whose intersection with $A$ is $P$);*

(iv) *$\tilde{J} = \bigcap\{\tilde{P} : P \in \mathrm{Prim}(A)$ and $P \supseteq J\}$ and for all $b \in M(A)$*

$$\|b + \tilde{J}\| = \sup\{\|b + \tilde{P}\| : P \in \mathrm{Prim}(A) \text{ and } P \supseteq J\};$$

(v) *$(A + \tilde{J})/\tilde{J}$ is an essential ideal in $M(A)/\tilde{J}$.*

Furthermore, the map $P \mapsto \tilde{P}$ $(P \in \mathrm{Prim}(A))$ maps $\mathrm{Prim}(A)$ homeomorphically onto a dense, open subset of $\mathrm{Prim}(M(A))$ [26, 4.1.10].

The next proposition was proved in [7, Proposition 1.2].

**Proposition 2.2.** *Let $A$ be a $C_0(X)$-algebra with base map $\phi$. Then $\phi$ has a unique extension to a continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with base map $\overline{\phi}$ and $\mathrm{Im}(\overline{\phi}) = \mathrm{cl}_{\beta X}(X_\phi)$.*

Now let $A$ be a $C_0(X)$-algebra with base map $\phi$ and let $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ be as in Proposition 2.2. For $x \in \beta X$, we define $H_x = \bigcap\{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(Q) = x\}$, a closed two-sided ideal of $M(A)$. Thus $H_x$ is defined in relation to $(M(A), \beta X, \overline{\phi})$ in the same way that $J_x$ (for $x \in X$) is defined in relation to $(A, X, \phi)$. It follows that for each $b \in M(A)$, the function $x \to \|b + H_x\|$ $(x \in \beta X)$ is upper semi-continuous.

The next proposition was proved in [8, Proposition 1.3].

**Proposition 2.3.** *Let $A$ be a $C_0(X)$-algebra with base map $\phi$, and set $X_\phi = \mathrm{Im}(\phi)$.*

(i) *For all $x \in X$, $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $J_x = H_x \cap A$.*

(ii) *For all $x \in X$, $H_x$ is strictly closed if and only if $H_x = \tilde{J}_x$.*

(iii) *For all* $b \in M(A)$, $\|b\| = \sup\{\|b + \tilde{J}_x\| : x \in X_\phi\} = \sup\{\|b + H_x\| : x \in X_\phi\}$.

In the case where $A = C_0(X) \otimes K(H)$ for a locally compact Hausdorff space $X$, we shall assume that $\phi : \mathrm{Prim}(A) \to X$ is the canonical homeomorphism such that

$$\phi^{-1}(x) = \{f \in C_0(X) : f(x) = 0\} \otimes K(H) \quad (x \in X).$$

Then it follows from the definition of $\tilde{J}$ above that

$$\tilde{J}_x = \{f \in M(A) : f(x) = 0\},$$

On the other hand, by [7, Lemma 1.5(ii)],

$$H_x = \{f \in M(A) : \|f(x_\alpha)\| \to 0 \text{ as } x_\alpha \to x\}.$$

We now give the definition of the ideal $A^s$ of $M(A)$. Let $A$ be a $C_0(X)$-algebra with base map $\phi$, and for $x \in X_\phi$ set $N_x = A + \tilde{J}_x$. Define $A^s = \bigcap_{x \in X_\phi} N_x$. Then $A^s$ is a closed two-sided ideal in $M(A)$ and $A^s \supseteq A$. Clearly $A^s$ depends on the particular way in which $A$ is represented as a $C_0(X)$-algebra (there may be many continuous maps from $\mathrm{Prim}(A)$ to $X$ in general). If $A = C_0(X) \otimes K(H)$ for a locally compact Hausdorff space $X$, then since $\tilde{J}_x = \{f \in M(A) : f(x) = 0\}$, we have that $N_x = \{f \in M(A) : f(x) \in K(H)\}$. Hence $A^s$ is precisely the algebra of bounded strong*-continuous functions from $X$ to $K(H)$ referred to in the introduction. In this case, $A^s$ contains the algebra of bounded norm-continuous functions from $X$ to $K(H)$. A generalization of the latter algebra has been studied in [12, Section 3] and in [24].

The next lemma was proved in [12, Lemma 3.2].

**Lemma 2.4.** *Let $A$ be a $C_0(X)$-algebra with base map $\phi$ and let $b \in M(A)$. Then $b \in A$ if and only if*
(i) *for all $\epsilon > 0$ the set $\{x \in X_\phi : \|b + H_x\| \geq \epsilon\}$ is compact;*
(ii) *for all $x \in X_\phi$ there exists $a \in A$ such that $b - a \in H_x$.*

We now recall from [8, Lemma 5.6] a means of constructing elements which will be of considerable importance in this paper.

**Lemma 2.5.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Then for each zero set $Z$ in $X_\phi$ there exists a positive element $c^Z \in A^s$ such that*
(i) $\|c^Z + \tilde{J}_x\| = 0$ *for* $x \in Z$;
(ii) $\|c^Z + \tilde{J}_x\| = 1$ *for* $x \in X_\phi \setminus Z$;
(iii) *for all $x \in X_\phi \setminus Z$ there is a neighbourhood $V$ of $x$ in $X_\phi$ and an element $a \in A$ such that $c^Z - a \in H_y$ for all $y \in V$.*

It was not stated in [8, Lemma 5.6] that $c^Z$ could be chosen positive, but the proof shows that this is the case.

Now let $A$ be a $C_0(X)$-algebra with base map $\phi$. If $X = X_\phi$ (that is, $\phi$ is surjective) then we already have the canonical extension $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X_\phi$. If $X \neq X_\phi$ then we may replace $X$ by the compact Hausdorff space $\beta X_\phi$ (see the discussion in [12, p. 302] and so we have again $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X_\phi$. In either case, with the usual identifications, we may consider $\overline{\phi}|_{\mathrm{Prim}(A^s)}$ and $\overline{\phi}|_{\mathrm{Prim}(A^s/A)}$. Denoting the latter of these maps by $\psi$, we have that

$A^s/A$ is a $C(\beta X_\phi)$-algebra with base map $\psi$. The next result explains why the notion of a P-point crops up frequently in this paper. The assumption that $X_\phi$ is infinite ensures that $A^s$ strictly contains $A$ [12, p. 306]. Note that if the non-empty Hausdorff space $X_\phi$ contains no isolated points (as will be assumed in several results in Section 6) then it is automatically infinite.

**Theorem 2.6.** [12, Theorem 5.2] *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Then $A^s/A$ is a non-trivial $C(\beta X_\phi)$-algebra with base map $\psi$, and $X_\psi = \beta X_\phi \setminus W$ where $W$ is the set of P-points in $X_\phi$.*

## 3. Minimal primal ideals in $A^s$

In this section we determine the space of minimal closed primal ideals of $A^s$ where $A$ is a $\sigma$-unital, quasi-standard C*-algebra. We exploit the identification of MinPrimal($M(A)$) already established in [11].

Recall that a C*-algebra $A$ is *quasi-standard* if the relation $\sim$ of inseparability by disjoint open sets is an open equivalence relation on Prim($A$) [6]. This condition is a wide generalization of the special case when Prim($A$) is Hausdorff. If $A$ is quasi-standard then the complete regularization map $\phi_A$ is open [6, Theorem 3.3], so Glimm($A$) is locally compact and $A$ is a continuous $C_0(X)$-algebra with $X = X_{\phi_A} =$ Glimm($A$). Furthermore each Glimm ideal of $A$ is primal and the topological spaces Glimm($A$) and MinPrimal($A$) coincide [6, Theorem 3.3]. Examples of quasi-standard C*-algebras include von Neumann algebras, AW*-algebras, local multiplier algebras of C*-algebras [29], the group C*-algebras of amenable discrete groups (and many other groups) [19], [5], and algebras of the form $C_0(X) \otimes K(H)$ where $X$ is a locally compact Hausdorff space.

Let $X$ be a completely regular topological space [17, 3.1] and let $C_{\mathbb{R}}(X)$ denote the ring of continuous real-valued functions on $X$. For $f \in C_{\mathbb{R}}(X)$, let

$$Z(f) = \{x \in X : f(x) = 0\},$$

the *zero set* of $f$. Note that every zero set clearly arises as the zero set of a bounded continuous function. The set of all zero sets of $X$ is denoted $Z[X]$. A non-empty family $\mathcal{F}$ of zero sets of $X$ is called a *z-filter* if: (i) $\mathcal{F}$ is closed under finite intersections; (ii) $\emptyset \notin \mathcal{F}$; (iii) each zero set which contains a member of $\mathcal{F}$ belongs to $\mathcal{F}$. Each ideal $I \subseteq C_{\mathbb{R}}(X)$ yields a $z$-filter $Z[I] = \{Z(f) : f \in I\}$. An ideal $I$ is called a *z-ideal* if $Z(f) \in Z[I]$ implies $f \in I$; and if $\mathcal{F}$ is a $z$-filter on $X$ then the ideal $I(\mathcal{F})$ defined by

$$I(\mathcal{F}) = \{f \in C_{\mathbf{R}}(X) : Z(f) \in \mathcal{F}\}$$

is a $z$-ideal. There is a bijective correspondence between the set of $z$-ideals of $C_{\mathbb{R}}(X)$ and the set of $z$-filters on $X$, given by $I = I(Z[I]) \leftrightarrow Z[I]$ (see [17, Chapter 2]).

A $z$-filter $\mathcal{F}$ on a completely regular space $X$ is said to be *prime* if $Z_1 \cup Z_2 \in \mathcal{F}$ implies that either $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$, for zero sets $Z_1$ and $Z_2$. Let $PF(X)$ denote the set of prime $z$-filters, and let $PZ(X)$ be the set of prime $z$-ideals (recall that an ideal $P \subseteq C_{\mathbb{R}}(X)$ is prime if $fg \in P$ implies $f \in P$ or $g \in P$). The bijective correspondence between $z$-ideals and $z$-filters restricts to a bijective correspondence $j : PZ(X) \to PF(X)$ given by $j(P) = \{Z(f) : f \in P\}$ (see [17, Chapter 2]). Every $z$-ideal of $C_{\mathbb{R}}(X)$ is an intersection of prime $z$-ideals and the minimal prime ideals of $C_{\mathbb{R}}(X)$ are $z$-ideals [17, 2.8, 14.7]. The prime ideals containing a given prime ideal form a chain [17, 14.8].

Now let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$, and let $u \in A$ be a strictly positive element. For $a \in A$, set $Z(a) = \{x \in X_\phi : a \in J_x\}$. Unless norm functions of elements of $A$ are continuous on $X_\phi$, $Z(a)$ will not necessarily be a zero set of $X_\phi$. However, since $Z(u) = \emptyset$ and $A$ is closed under multiplication by $C^b(X_\phi)$, every zero set $Z(f)$ of $X_\phi$ arises as $Z(a)$ for the element $a = f \cdot u \in A$ ($f \in C^b_{\mathbb{R}}(X_\phi)$). For $b \in M(A)$, set $Z(b) = \{x \in X_\phi : b \in \tilde{J}_x\}$. Note that if $b \in A$ then this definition is consistent with the previous one because $\tilde{J}_x \cap A = J_x$ ($x \in X_\phi$). It is also useful to note that for $b \in M(A)$ and $x \in X_\phi$, $b \in \tilde{J}_x$ if and only if $bu \in \tilde{J}_x$ if and only if $bu \in J_x$. Hence $Z(b) = Z(bu)$, and this is a zero set in $X_\phi$ if $A$ is continuous.

For a $z$-filter $\mathcal{F}$ on $X_\phi$ define $L^{\mathrm{alg}}_{\mathcal{F}} = \{b \in M(A) : \exists Z \in \mathcal{F},\ Z(b) \supseteq Z\}$, and let $L_{\mathcal{F}}$ be the norm-closure of $L^{\mathrm{alg}}_{\mathcal{F}}$ in $M(A)$. Let $b \in L^{\mathrm{alg}}_{\mathcal{F}}$. Then for $a \in M(A)$, $Z(ab) \supseteq Z(b)$ and $Z(ba) \supseteq Z(b)$, while for $a \in L^{\mathrm{alg}}_{\mathcal{F}}$, $Z(a+b) \supseteq Z(a) \cap Z(b)$. Hence $L^{\mathrm{alg}}_{\mathcal{F}}$ is an ideal of $M(A)$, so $L_{\mathcal{F}}$ is a closed ideal of $M(A)$.

**Theorem 3.1.** [8, Theorem 3.2] *Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$. Suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $I$ and $J$ be $z$-ideals of $C_{\mathbb{R}}(X_\phi)$ and suppose that there exists a zero set $Z$ of $X_\phi$ such that $Z \in Z[I]$ but $Z \notin Z[J]$. Then $L_{Z[I]} \not\subseteq L_{Z[J]}$. Hence the assignment $I \to L_{Z[I]}$ defines an order-preserving injective map $L$ from the lattice of $z$-ideals of $C_{\mathbb{R}}(X_\phi)$ into the lattice of closed ideals of $M(A)$.*

For $x \in X_\phi$, let $M_x$ be the maximal ideal of $C_{\mathbb{R}}(X_\phi)$ given by $M_x = \{f \in C_{\mathbb{R}}(X_\phi) : f(x) = 0\}$, and let $O_x = \{f \in C_{\mathbb{R}}(X_\phi) : x \in \mathrm{int}(Z(f))\}$ where $\mathrm{int}(Z(f))$ denotes the interior of $Z(f)$ in $X_\phi$. Then $M_x$ and $O_x$ are $z$-ideals, and $O_x$ is the smallest ideal of $C_{\mathbb{R}}(X_\phi)$ which is not contained in any maximal ideal other than $M_x$. It is useful to extend the definitions just given as follows. Let $\mathrm{cl}_{\beta X} X_\phi$ denote the closure of $X_\phi$ in $\beta X$. For $p \in \mathrm{cl}_{\beta X} X_\phi$, let $M^p = \{f \in C_{\mathbb{R}}(X_\phi) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ and define $O^p$ to be the set of all $f \in C_{\mathbb{R}}(X_\phi)$ for which $\mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p$ in $\mathrm{cl}_{\beta X} X_\phi$. Then for $x \in X_\phi$, $M^x = M_x$ and $O^x = O_x$.

In the next result and in several subsequent results in this paper, we shall take $X$ to be the locally compact Hausdorff space Glimm($A$) associated with a $\sigma$-unital quasi-standard C*-algebra $A$. In this case, it should be understood that $\phi : \mathrm{Prim}(A) \to X$ is the complete regularization map $\phi_A$. Thus $X = X_\phi$, $\mathrm{cl}_{\beta X} X_\phi = \beta X$ and the sets $M^p$ and $O^p$ defined above coincide with those occurring in [17, 7.3 and 7.12]. Now suppose that $P \in PZ(X_\phi) = PZ(X)$. Then there exists $p \in \beta X$ such that $O^p \subseteq P \subseteq M^p$ [17, 7.15]. Hence, if $A/J_x$ is non-unital for all $x \in X$, it follows from Theorem 3.1 and [8, Theorem 4.3] that $H_p \subseteq L_{Z[P]}$ and, if $p \in X$, $L_{Z[P]} \subseteq \tilde{J}_p$.

For a ring $R$ let Min($R$) be the space of minimal (algebraic) primal ideals of $R$ with the *lower topology* generated by sub-basic sets of the form

$$\{P \in \mathrm{Min}(R) : a \notin P\}$$

as $a$ varies through elements of $R$. If $R$ is a commutative ring then an argument of Krull's shows that every minimal primal ideal of $R$ is prime, and Min($R$) is the usual space of minimal prime ideals of $R$ with the hull-kernel topology, see [28] and the references given there.

**Theorem 3.2.** [11, Theorem 3.4] *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra with $A/G$ non-unital for all $G \in \mathrm{Glimm}(A)$ and set $X = \mathrm{Glimm}(A)$. Then the assignment $P \mapsto L_{Z[P]}$ defines a homeomorphism from $\mathrm{Min}(C_{\mathbb{R}}(X))$ onto $\mathrm{MinPrimal}(M(A))$.*

For the next theorem, we need the following family of functions which is useful for relating $L_{\mathcal{F}}$ and $L_{\mathcal{F}}^{\mathrm{alg}}$. For $0 < \epsilon < 1/2$, define the continuous piecewise linear function $f_\epsilon : [0,\infty) \to [0,\infty)$ by: (i) $f_\epsilon(x) = 0$ $(0 \le x \le \epsilon)$; (ii) $f_\epsilon(x) = 2(x-\epsilon)$ $(\epsilon \le x \le 2\epsilon)$; (iii) $f_\epsilon(x) = x$ $(2\epsilon \le x)$. Note that for $b \in M(A)^+$, if $b \in L_{\mathcal{F}}$ then $f_\epsilon(b)$ belongs to the Pedersen ideal of $L_{\mathcal{F}}$ for all $\epsilon$ [26, 5.6.1], and hence $f_\epsilon(b) \in L_{\mathcal{F}}^{\mathrm{alg}}$. On the other hand, $\|b - f_\epsilon(b)\| \le \epsilon$. Thus we have that $b \in L_{\mathcal{F}}$ if and only if $f_\epsilon(b) \in L_{\mathcal{F}}^{\mathrm{alg}}$ for all $\epsilon \in (0, 1/2)$, which is Lemma 3.3 of [11].

We can now give the description of $\mathrm{MinPrimal}(A^s)$. For a C*-algebra $A$, let $\mathrm{Primal}(A)$ denote the set of closed primal ideals of $A$ and $\mathrm{Primal}'(A)$ the set of proper closed primal ideals of $A$.

**Theorem 3.3.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra and set $X = \mathrm{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$. Then the map $P \mapsto P \cap A^s$ is a homeomorphism from $\mathrm{MinPrimal}(M(A))$ onto $\mathrm{MinPrimal}(A^s)$.*

*Proof.* Let $P \in \mathrm{Primal}(M(A))$ and let $I_1, \dots, I_n \in \mathrm{Id}(A^s)$ with $I_1 \dots I_n = \{0\}$. Then $I_i \subseteq P$ for some $1 \le i \le n$, by the primality of $P$, so $I_i \subseteq P \cap A^s$. Hence $P \cap A^s \in \mathrm{Primal}(A^s)$. Now let $P, Q \in \mathrm{MinPrimal}(M(A))$ with $P \ne Q$. Then by [11, Theorem 2.4] there exist distinct minimal prime $z$-filters $\mathcal{F}$ and $\mathcal{G}$ on $X$ such that $P = L_{\mathcal{F}}$ and $Q = L_{\mathcal{G}}$. Let $Z \in Z[X]$ with $Z \in \mathcal{F} \setminus \mathcal{G}$ and let $c^Z \in A^s$ with the properties of Lemma 2.5. Note that $Z$ is the zero set of $c^Z$ and also the zero set of $f_\epsilon(c^Z)$ $(0 < \epsilon < 1/2)$. Hence $c^Z \in L_{\mathcal{F}}^{\mathrm{alg}} \subseteq L_{\mathcal{F}}$ but $f_\epsilon(c^Z) \notin L_{\mathcal{G}}^{\mathrm{alg}}$ for $0 < \epsilon < 1/2$, and thus $c^Z \notin L_{\mathcal{G}}$ by [11, Lemma 3.3]. Hence $P \cap A^s \not\subseteq Q \cap A^s$, and similarly $Q \cap A^s \not\subseteq P \cap A^s$.

Now let $R \in \mathrm{Primal}'(A^s)$ and define $\tilde{R} \in \mathrm{Primal}'(M(A))$ as follows. Let $W$ be the hull of $R$ in $\mathrm{Prim}(A^s)$ and let $i(W)$ be the image of $W$ in $\mathrm{Prim}(M(A))$ under the canonical injection $i$ from $\mathrm{Prim}(A^s)$ to $\mathrm{Prim}(M(A))$. Let $V$ be the closure of $i(W)$ in $\mathrm{Prim}(M(A))$ and set $\tilde{R} = \ker V$. Clearly $\tilde{R} \cap A^s = R$. Since $W$ is a limit set in $\mathrm{Prim}(A^s)$ it follows that $V$ is a limit set in $\mathrm{Prim}(M(A))$ so $\tilde{R}$ is primal. Hence there exists $P \in \mathrm{MinPrimal}(M(A))$ such that $\tilde{R} \supseteq P$, so $R = \tilde{R} \cap A^s \supseteq P \cap A^s$. Thus it follows from this paragraph and the previous one that

$$\mathrm{MinPrimal}(A^s) = \{P \cap A^s : P \in \mathrm{MinPrimal}(M(A))\}.$$

Next we show that the map $P \mapsto P \cap A^s$ is a homeomorphism. Sets of the form $\{P \in \mathrm{MinPrimal}(A^s) : a \notin P\}$ $(a \in A^s)$ are sub-basic for the $\tau_w$-topology on $\mathrm{MinPrimal}(A^s)$ and their inverse images are open in $\mathrm{MinPrimal}(M(A))$ since

$$\{P \in \mathrm{MinPrimal}(M(A)) : a \notin P \cap A^s\} = \{P \in \mathrm{MinPrimal}(M(A)) : a \notin P\}.$$

Thus the map is continuous.

Now let $b \in M(A)^+$ and set $V = \{P \in \mathrm{MinPrimal}(M(A)) : b \notin P\}$. We aim to show that $U := \{P \cap A^s : P \in V\}$ is $\tau_w$-open in $\mathrm{MinPrimal}(A^s)$. Since $\|f_{1/n}(b) - b\| \le 1/n$, we may write $V = \bigcup_{n \ge 2}\{P = L_{\mathcal{F}} \in \mathrm{MinPrimal}(M(A)) : f_{1/n}(b) \notin L_{\mathcal{F}}^{\mathrm{alg}}\}$. Let $n \ge 2$ and set $Z_n = Z(f_{1/n}(b))$. Let $c^{Z_n} \in A^s$ with the properties of Lemma 2.5. Then for a (minimal prime) $z$-filter $\mathcal{F}$, $c^{Z_n} \in L_{\mathcal{F}}^{\mathrm{alg}}$ if and only if $Z_n \in \mathcal{F}$ if and only if $f_{1/n}(b) \in L_{\mathcal{F}}^{\mathrm{alg}}$. But also $c^{Z_n} \in L_{\mathcal{F}}^{\text{alg}}$ if and only if $c^{Z_n} \in L_{\mathcal{F}}$ by [11, Lemma 3.3] (since $Z(f_\epsilon(c^{Z_n})) = Z_n$ for all $0 < \epsilon < 1/2$). Thus we have that $U = \bigcup_{n\geq 2}\{Q = P \cap A^s \in \mathrm{MinPrimal}(M(A)) : c^{Z_n} \notin Q\}$, a union of $\tau_w$-open sets. Thus $U$ is open and the map is a homeomorphism. □

Theorem 3.3 has various immediate consequences. The next four corollaries follow at once from Theorem 3.3 and [11, Corollaries 4.1, 4.2, 4.3, and 4.4] respectively. Recall that a topological space $Y$ is *countably compact* if every countable open cover of $Y$ has a finite subcover. If $Y$ is a $T_1$-space then $Y$ is countably compact if and only if every infinite subset of $Y$ has a limit point in $Y$ [25, p. 181]. If $Y$ is countably compact then $Y$ is *pseudocompact*, that is, $Y$ does not admit any unbounded continuous real-valued functions [17, 1.4].

**Corollary 3.4.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra with $A/G$ non-unital for all $G \in \mathrm{Glimm}(A)$. Then*

(i) *the Hausdorff space* $\mathrm{MinPrimal}(A^s)$ *is totally disconnected and countably compact;*

(ii) *if* $\mathrm{MinPrimal}(A^s)$ *is locally compact then it is basically disconnected.*

**Corollary 3.5.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra and suppose that $A/G$ is non-unital for all $G \in \mathrm{Glimm}(A)$. Then the following are equivalent:*

(i) $\mathrm{MinPrimal}(A^s)$ *is compact;*

(ii) $\mathrm{Glimm}(A)$ *is cozero-complemented; that is, for every cozero set $U$ in* $\mathrm{Glimm}(A)$ *there exists a cozero set $V$ in* $\mathrm{Glimm}(A)$ *such that $U \cap V = \emptyset$ and $U \cup V$ is dense in* $\mathrm{Glimm}(A)$.

Let $D$ be an infinite discrete space with one-point compactification $\alpha D = D \cup \{p\}$. For $f \in C(\alpha D)$, if $p \in Z(f)$ then $Z(f)$ is co-countable, and if $p \notin Z(f)$ then $Z(f)$ is finite. It follows from the 'cozero-complemented' criterion used in Corollary 3.5 that $\mathrm{Min}(C_{\mathbb{R}}(\alpha D))$ is compact if and only if $D$ is countable. Applying this to $\mathrm{MinPrimal}(A^s)$ where $A$ is as in Corollary 3.5 with $\mathrm{Glimm}(A)$ homeomorphic to $\alpha D$, we have that $\mathrm{MinPrimal}(A^s)$ is compact if and only if $D$ is countable.

If $A$ in Corollary 3.5 is separable then (ii) holds so $\mathrm{MinPrimal}(A^s)$ is compact, but much more can be said. Recall that a *regular closed* set is one that is the closure of its interior. If $A$ is separable then $\mathrm{Glimm}(A)$ is perfectly normal [9, Lemma 3.9] (i.e. every closed subset of $\mathrm{Glimm}(A)$ is a zero set) so $A$ certainly satisfies condition (ii) of the next corollary.

**Corollary 3.6.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra. Suppose that $A/G$ is non-unital for $G \in \mathrm{Glimm}(A)$. Then the following are equivalent:*

(i) $\mathrm{MinPrimal}(A^s)$ *is compact and extremally disconnected;*

(ii) *every regular closed set in* $\mathrm{Glimm}(A)$ *is the closure of a cozero set.*

*In particular, if $A$ is separable then $A$ satisfies these equivalent conditions.*

**Corollary 3.7.** *Set $A = C(\beta\mathbb{N} \setminus \mathbb{N}) \otimes K(H)$. Then* $\mathrm{MinPrimal}(A^s)$ *is nowhere locally compact. If Martin's Axiom holds then* $\mathrm{MinPrimal}(A^s)$ *is not an F-space.*

We conclude this section by describing the space of Glimm ideals of $A^s$ in the case where $A$ is a $\sigma$-unital, quasi-standard C*-algebra with $A/G$ non-unital for all $G \in \mathrm{Glimm}(A)$. Set $X = \mathrm{Glimm}(A)$ and let $\phi = \phi_A : \mathrm{Prim}(A) \to X$ be the complete regularization map. Let $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ be the canonical extension of $\phi$. Then the Glimm ideals of $M(A)$ are the ideals $H_x$ ($x \in \beta X$) and the assignment $x \to H_x$ defines a homeomorphism of $\beta X$ onto

Glimm($M(A)$) (see the comment after [10, Proposition 4.4]). Since $A \subseteq A^s \subseteq M(A)$, it follows that $M(A)$ is the multiplier algebra of $A^s$, and hence that the ring of bounded continuous functions on Prim($A^s$) is isomorphic to the ring of bounded continuous functions on Prim($A$) by the restriction map (regarding Prim($A$) as an open subset of Prim($A^s$) in the usual way). Thus Glimm($A^s$) is homeomorphic to $\overline{\phi}(\mathrm{Prim}(A^s))$. Furthermore, $\overline{\phi}(\mathrm{Prim}(A^s)) = \beta X$. To see this, note first of all that if $X$ is finite then $A^s = A$ and $\beta X = X$. On the other hand, if $X$ is infinite then we may apply Theorem 2.6 to the continuous $C_0(X)$-algebra $A$ to obtain that $\overline{\phi}(\mathrm{Prim}(A^s/A)) = \beta X \setminus W$, where $W$ is the set of P-points of $X$. But

$$W \subseteq X = \phi(\mathrm{Prim}(A)) \subseteq \overline{\phi}(\mathrm{Prim}(A^s))$$

and hence $\overline{\phi}(\mathrm{Prim}(A^s)) = \beta X$, as required. Finally, the Glimm ideals of $A^s$ have the form $H_x \cap A^s$ $(x \in \beta X)$, where $H_x$ is as above.

One consequence of this is worth recording.

**Corollary 3.8.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra and suppose that $A/G$ is non-unital for all $G \in$ Glimm($A$). Then the following are equivalent:*

(i) *$A^s$ is quasi-standard;*

(ii) *$M(A)$ is quasi-standard;*

(iii) Glimm($A$) *is basically disconnected.*

*Proof.* The equivalence of (ii) and (iii) was established in [7, Corollary 4.9].

(ii)⇒(i). This follows from the general fact that closed ideals of quasi-standard C*-algebras are quasi-standard [6, p. 356].

(i)⇒(iii). If $A^s$ is quasi-standard then Glimm($A^s$) and MinPrimal($A^s$) coincide as sets and topological spaces [6, Theorem 3.3]. We have seen above that Glimm($A^s$) is compact and thus MinPrimal($A^s$) is also compact, and hence locally compact. Thus it follows from Corollary 3.4 that MinPrimal($A^s$) is basically disconnected, and hence that Glimm($A^s$) is basically disconnected. But we saw above that Glimm($A^s$) is homeomorphic to the Stone-Cech compactification of Glimm($A$), and thus Glimm($A$) is basically disconnected by [17, 6M]. □

## 4. A faithful family of irreducible representations of $A^s/A$

In this section we exhibit a family of irreducible representations of the corona algebra $M(A)/A$ where $A$ is either a stable, separable, quasi-standard C*-algebra or an algebra of the form $C_0(X) \otimes K(H)$ with $X$ $\sigma$-compact. The representations are constructed as irreducible representations of $M(A)$ with kernels containing $A$. We show that these irreducible representations form a faithful family for $A^s/A$. The method is based on the ultraproduct construction [16]. We begin with irreducible representations 'at infinity'.

**Theorem 4.1.** *Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$. Let $W$ be a countably infinite, closed, relatively discrete subset of $X_\phi$ and suppose that $J_x$ is a primitive ideal of $A$ for each $x \in W$. Let $\mathcal{F}^\sharp$ be a free ultrafilter on $W$ and let $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}^\sharp\}$, a z-filter on $X_\phi$. Then $L_\mathcal{F} \in$ Prim($M(A)$) and $L_\mathcal{F} \supseteq A$.*

*Proof.* Set $Y = \phi^{-1}(W)$. Then $Y$ is a closed subset of Prim($A$). Set $I = \ker Y$ and $B = A/I$. Then $B$ is isomorphic to the $c_0$-direct sum $\sum_{x \in W} A/J_x$, so $M(B)$ is isomorphic to the $l_\infty$-direct sum $\prod_{x \in W} M(A)/\tilde{J}_x$ [2, Lemma 1.2.21] (recall that, since $A$ is $\sigma$-unital, $M(A/J_x) \cong$

$M(A)/\tilde{J}_x$ for $x \in X_\phi$, by definition of $\tilde{J}_x$). Let $\pi : A \to B$ be the quotient map. Since $A$ is $\sigma$-unital, $\pi$ extends to a surjective homomorphism $\tilde{\pi} : M(A) \to M(B)$.

Let $C = \prod^{\mathcal{F}^\sharp} M(A)/\tilde{J}_x$ be the ultraproduct, and let $\rho : M(B) \to C$ denote the quotient map. Then $\ker\rho = \{c \in M(B) : \lim_{\mathcal{F}^\sharp} \|c_x\| = 0\}$. Since each $M(A)/\tilde{J}_x$ is primitive, $C$ is a primitive C*-algebra by [16, Theorem 5.4]. Hence $C$ is a primitive quotient of $M(A)$, and we now show that $J := \ker\rho \circ \tilde{\pi}$ is equal to $L_{\mathcal{F}}$.

Suppose that $b \in L_{\mathcal{F}}^{\mathrm{alg}}$. Then $Z(b) \supseteq Z$ for some $Z \in \mathcal{F}$ and hence $Z(b) \cap W \supseteq Z \cap W \in \mathcal{F}^\sharp$. Thus $\tilde{\pi}(b) \in \ker\rho$, so $b \in J$. Hence $L_{\mathcal{F}} \subseteq J$. Conversely, let $b \in J$ with $0 \le b \le 1$. Then $\tilde{\pi}(b) \in \ker\rho$, and hence for $1/2 > \epsilon > 0$ there exists $Z' \in \mathcal{F}^\sharp$ such that $f_\epsilon(b) + \tilde{J}_x = 0$ for all $x \in Z'$ (where $f_\epsilon$ is as defined before Theorem 3.3). Thus $Z' \subseteq Z(f_\epsilon(b)) \cap W$. Since $Z'$ is closed in $X_\phi$ it follows from [8, Lemma 4.1] that there is a zero set $Z$ in $X_\phi$ with $Z' \subseteq Z \subseteq Z(f_\epsilon(b))$. Hence $Z \cap W \in \mathcal{F}^\sharp$, so $Z \in \mathcal{F}$. Since $Z \subseteq Z(f_\epsilon(b))$, it follows that $f_\epsilon(b) \in L_{\mathcal{F}}^{\mathrm{alg}}$, and hence $b \in L_{\mathcal{F}}$ by [11, Lemma 3.3]. Thus $J \subseteq L_{\mathcal{F}}$, so $J = L_{\mathcal{F}}$.

Finally, for $a \in A$ with $0 \le a \le 1$ and $0 < \epsilon < 1/2$, the set $\{x \in X_\phi : \|a + J_x\| \ge \epsilon\}$ is compact (it is the image under $\phi$ of the compact subset $\{P \in \mathrm{Prim}(A) : \|a + P\| \ge \epsilon\}$ of $\mathrm{Prim}(A)$) and hence has finite intersection with $W$. Arguing as in the previous paragraph, we see that there exists $Z \in \mathcal{F}$ such that $\|f_\epsilon(a) + J_x\| = 0$ for all $x \in Z$, and hence $a \in L_{\mathcal{F}}$. □

If $A$ is a $C_0(X)$-algebra with base map $\phi$ and $Z$ is a non-empty closed subset of $X_\phi$ then we define $J_Z = \bigcap_{x \in Z} J_x$ (cf. [9, p. 5]). It is straightforward to check that $J_Z = \{b \in A : Z(b) \supseteq Z\}$ and that $\tilde{J}_Z = \{b \in M(A) : Z(b) \supseteq Z\}$.

**Lemma 4.2.** *Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$ and let $Z$ be a non-empty closed subset of $X_\phi$. Then $J_Z$ is $\sigma$-unital if and only if $Z$ is a $G_\delta$.*

*Proof.* Suppose that $Z$ is a $G_\delta$ subset of $X_\phi$. Then since $X_\phi$ is normal there is a continuous function $f : X_\phi \to [0, 1]$ such that $Z(f) = Z$. Set $z = \theta_A(f \circ \phi)$ where $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ is the Dauns-Hofmann isomorphism. Let $v = zu$ where $u$ is a strictly positive element for $A$. Then $v$ is a strictly positive element for the C*-algebra $J_Z$, so $J_Z$ is $\sigma$-unital.

Conversely, suppose that $J_Z$ is $\sigma$-unital. Then the set $W = \{P \in \mathrm{Prim}(A) : P \not\supseteq J_Z\}$ is $\sigma$-compact, so $\phi(W) = X_\phi \setminus Z$ is also $\sigma$-compact. Hence $Z$ is a $G_\delta$. □

In particular, if $A$ in Lemma 4.2 is separable, then every ideal $J_Z$ is $\sigma$-unital so every non-empty closed subset $Z$ of $X_\phi$ is a $G_\delta$ (cf. [9, Lemma 3.9]).

**Lemma 4.3.** *Let $A$ be a $C_0(X)$-algebra with base map $\phi$ and let $Z$ be a non-empty closed subset of $X_\phi$. Then $\tilde{J}_Z$ is an hereditary C*-subalgebra of $M(J_Z)$.*

*Proof.* We work in $A^{**}$, identifying $A$ with its canonical image in $A^{**}$. Hence we may identify $M(A)$ with the idealizer of $A$ in $A^{**}$ and $M(J_Z)$ with the idealizer of $J_Z$ in $J_Z^{**}$ where the latter is canonically embedded in $A^{**}$ [26, Proposition 3.12.3]. First note that if $b \in \tilde{J}_Z$ and $c \in M(J_Z)$ then $bcb \in \tilde{J}_Z$. To see this observe that $\tilde{J}_Z = \{b \in M(A) : bA + Ab \subseteq J_Z\}$. Hence for $a \in A$, $bcba \in bcJ_Z \subseteq J_Z$. Similarly $abcb \in J_Zcb \subseteq J_Z$. Thus $bcb$ belongs firstly to $M(A)$ and secondly to $\tilde{J}_Z$.

Now suppose that $b \in \tilde{J}_Z$ and $c \in M(J_Z)$ with $0 \le c \le b$. Then $c \in \overline{bM(J_Z)b}$, the hereditary C*-subalgebra of $M(J_Z)$ generated by $b$ [23, 1.5.9]. But $\overline{bM(J_Z)b} \subseteq \tilde{J}_Z$ by the previous paragraph. Hence $c \in \tilde{J}_Z$ as required. □

We want now to apply Theorem 4.1 in the context of ideals of the form $J_Z$. Suppose that $A$ is a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$, so that the C*-algebra $J_Z$ is $\sigma$-unital by Lemma 4.2. Let $\psi = \phi|_{\mathrm{Prim}(J_Z)}$. Then $J_Z$ is a $C_0(X)$-algebra with base map $\psi$ and $X_\psi = X_\phi \setminus Z$, where $X_\psi = \mathrm{Im}(\psi)$. Hence $M(J_Z)$ is a $C(\beta X)$-algebra with base map $\overline{\psi}$. Let $W$ be a countably infinite, relatively closed, relatively discrete subset of $X_\psi$ and suppose that $J_x \in \mathrm{Prim}(A)$ for each $x \in W$ so that $J_x \cap J_Z$ is a primitive ideal of $J_Z$. Let $\mathcal{F}^\sharp$ be a free $z$-ultrafilter on $W$ and let $\mathcal{F}' = \{Z \in Z[X_\psi] : Z \cap W \in \mathcal{F}^\sharp\}$. Using the superscript $M(J_Z)$ to indicate that we are applying Theorem 4.1 with $J_Z$ in place of $A$, we obtain an ideal $L_{\mathcal{F}'}^{M(J_Z)}$ in $\mathrm{Prim}(M(J_Z))$ such that $L_{\mathcal{F}'}^{M(J_Z)} \supseteq J_Z$.

The next lemma compares $L_{\mathcal{F}'}^{M(J_Z)}$ (an ideal of $M(J_Z)$) with $L_{\mathcal{F}}^{M(A)} := L_{\mathcal{F}}$ (an ideal of $M(A)$), where $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}^\sharp\}$. From now on, it will be convenient for various reasons to assume that $A$ is a $\sigma$-unital, continuous $C_0(X)$-algebra. In particular, this will imply that $Z(b)$ is a zero set in $X_\phi$ for every $b \in M(A)$ (see Section 3).

**Lemma 4.4.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$ and let $Y$ be a countably infinite, relatively closed, relatively discrete subset of $X_\phi \setminus Z$ such that $J_x \in \mathrm{Prim}(A)$ for $x \in Y$. Then with the notation above, $L_{\mathcal{F}'}^{M(J_Z)} \cap M(A) \subseteq L_{\mathcal{F}}^{M(A)}$. If $J_Z$ is an essential ideal of $A$ then $L_{\mathcal{F}'}^{M(J_Z)} \cap M(A) = L_{\mathcal{F}}^{M(A)}$.*

*Proof.* Let $b \in L_{\mathcal{F}'}^{M(J_Z)} \cap M(A)$ with $b \geq 0$, and let $0 < \epsilon < 1/2$. Then $f_\epsilon(b) \in (L_{\mathcal{F}'}^{\mathrm{alg}})^{M(J_Z)}$. Since $b \in M(A)$, $Z(f_\epsilon(b)) \cap X_\psi$ is a zero set of $X_\psi$ and hence belongs to $\mathcal{F}'$. It follows that $Z(f_\epsilon(b)) \cap Y \in \mathcal{F}^\sharp$ and hence that $b \in L_{\mathcal{F}}^{M(A)}$.

Now suppose that $J_Z$ is an essential ideal of $A$, that $0 \leq b \in L_{\mathcal{F}}^{M(A)}$, and that $0 < \epsilon < 1/2$. Then $f_\epsilon(b) \in (L_{\mathcal{F}}^{\mathrm{alg}})^{M(A)}$ so $Z(f_\epsilon(b)) \in \mathcal{F}$. Hence $Z(f_\epsilon(b)) \cap Y \in \mathcal{F}^\sharp$. Since $J_Z$ is essential, $b \in M(J_Z)$, and $(Z(f_\epsilon(b)) \cap X_\psi) \cap Y \in \mathcal{F}^\sharp$. Thus $b \in L_{\mathcal{F}'}^{M(J_Z)}$. □

Note that, under the hypotheses of Lemma 4.4, the base map $\phi$ is open, so the ideal $J_Z$ is essential in $A$ if and only if the zero set $Z$ has empty interior in $X_\phi$.

**Theorem 4.5.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $Z$ be a zero set of $X_\phi$ and let $Y$ be a countably infinite, relatively closed, relatively discrete subset of $X_\phi \setminus Z$ such that $J_x \in \mathrm{Prim}(A)$ for $x \in Y$. Let $\mathcal{F}^\sharp$ be a free $z$-ultrafilter on $Y$ and set $\mathcal{F} = \{Z' \in Z[X_\phi] : Z' \cap Y \in \mathcal{F}^\sharp\}$. Then there exists an irreducible representation $\pi_{\mathcal{F}}$ of $M(A)$ such that*

(i) $\ker \pi_{\mathcal{F}} \supseteq A$;
(ii) $\ker \pi_{\mathcal{F}} \not\supseteq A^s$;
(iii) $\ker \pi_{\mathcal{F}} = \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L_{\mathcal{F}'}^{M(J_Z)} \cap M(A)\}$;
(iv) $\ker \pi_{\mathcal{F}} \cap \tilde{J}_Z \subseteq L_{\mathcal{F}}^{M(A)}$.

*Proof.* Since $Z$ is a zero set of $X_\phi$, the C*-algebra $J_Z$ is $\sigma$-unital by Lemma 4.2. Hence it follows from Theorem 4.1 applied to $J_Z$, as detailed above, that there is an irreducible representation $\rho$ of $M(J_Z)$ on a Hilbert space $K$ with $\ker \rho = L_{\mathcal{F}'}^{M(J_Z)}$ In particular, $L_{\mathcal{F}'}^{M(J_Z)} \supseteq J_Z$.

Let $c^Z \in A^s$ be as in Lemma 2.5 ([8, Lemma 5.6]) with $c^Z \in \tilde{J}_x$ for $x \in Z$ and $\|c^Z + \tilde{J}_x\| = 1$ for $x \in X_\phi \setminus Z$. Then $c^Z \in \tilde{J}_Z$, but by the construction of $\rho$, $c^Z \notin \ker\rho$, and hence $\tilde{J}_Z \not\subseteq \ker\rho$. Since $\tilde{J}_Z$ is an hereditary subalgebra of $M(J_Z)$ by Lemma 4.3, it follows that the representation $\pi = \rho|_{\tilde{J}_Z}$ is an irreducible representation of $\tilde{J}_Z$ on the Hilbert space $H = \rho(\tilde{J}_Z)K$ [26, 4.1.5] with kernel $\tilde{J}_Z \cap L_{\mathcal{F}'}^{M(J_Z)}$. Let $\pi_{\mathcal{F}}$ be the canonical extension of $\pi$ to an irreducible representation of $M(A)$ on $H$. Then $c^Z \notin \ker\pi_{\mathcal{F}}$ so $\pi_{\mathcal{F}}$ does not annihilate $A^s$, establishing (ii).

By the construction of $\pi_{\mathcal{F}}$ we have that

$$\begin{aligned}\ker\pi_{\mathcal{F}} &= \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq \ker\pi = L_{\mathcal{F}'}^{M(J_Z)} \cap \tilde{J}_Z\}\\ &= \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L_{\mathcal{F}'}^{M(J_Z)}\}\\ &= \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L_{\mathcal{F}'}^{M(J_Z)} \cap M(A)\},\end{aligned}$$

establishing (iii). Furthermore, for $a \in A$,

$$a\tilde{J}_Z + \tilde{J}_Z a \subseteq J_Z \subseteq L_{\mathcal{F}'}^{M(J_Z)} \cap M(A),$$

so $A \subseteq \ker\pi_{\mathcal{F}}$, establishing (i).

Finally, suppose that $b \geq 0$ and that $b \in \ker\pi_{\mathcal{F}} \cap \tilde{J}_Z$. Then $2b^2 \in L_{\mathcal{F}'}^{M(J_Z)} \cap M(A)$ by (iii), and hence $2b^2 \in L_{\mathcal{F}}^{M(A)}$ by Lemma 4.4. Thus $b \in L_{\mathcal{F}}^{M(A)}$, establishing (iv). □

**Corollary 4.6.** *In the context of Theorem 4.5, if $J_Z$ is an essential ideal of $A$ then (iii) and (iv) may be replaced by*

*(iii)′* $\ker\pi_{\mathcal{F}} = \{b \in M(A) : b\tilde{J}_Z + \tilde{J}_Z b \subseteq L_{\mathcal{F}}^{M(A)}\}$*;*

*(iv)′* $\ker\pi_{\mathcal{F}} \cap \tilde{J}_Z = L_{\mathcal{F}}^{M(A)} \cap \tilde{J}_Z$*.*

*If $Z = \{x\}$ is a singleton with $x$ non-isolated in $X_\phi$ then* $\ker\pi_{\mathcal{F}}|_{A^s} = (L_{\mathcal{F}}^{M(A)} \cap A^s) + A$*, so* $(L_{\mathcal{F}}^{M(A)} \cap A^s) + A \in \mathrm{Prim}(A^s)$*.*

*Proof.* If $J_Z$ is an essential ideal of $A$ then (iii)′ follows from Theorem 4.5(iii) and Lemma 4.4. This then implies that $L_{\mathcal{F}}^{M(A)} \subseteq \ker\pi_{\mathcal{F}}$, and thus (iv)′ follows from Theorem 4.5(iv).

Now suppose that $Z = \{x\}$ is a singleton. Then $J_x = J_Z$ is an essential ideal in $A$ since $x$ is non-isolated in $X_\phi$, so we know from the previous paragraph and Theorem 4.5(i) that $\ker\pi_{\mathcal{F}}|_{A^s} \supseteq (L_{\mathcal{F}}^{M(A)} \cap A^s) + A$. Suppose that $b \in \ker\pi_{\mathcal{F}}|_{A^s}$. Let $a \in A$ such that $b - a \in \tilde{J}_x$. Then $b - a \in \ker\pi_{\mathcal{F}}|_{A^s}$, so $b - a \in L_{\mathcal{F}}^{M(A)}$ by (iv)′ above. Hence $b \in (L_{\mathcal{F}}^{M(A)} \cap A^s) + A$ as required. □

We now define $\mathcal{S}$ to be the family of irreducible representations of $M(A)$ obtainable by the methods of Theorem 4.1 and Theorem 4.5. Obviously the size of $\mathcal{S}$ depends on the size of the set $\{x \in X_\phi : J_x \in \mathrm{Prim}(A)\}$.

**Theorem 4.7.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$ such that $X_\phi$ is infinite. Suppose that $A/J_x$ is non-unital for all $x \in X_\phi$ and that the set $\{x \in X_\phi : J_x \in \mathrm{Prim}(A)\}$ is dense in $X_\phi$. Then the family $\mathcal{S}$ of irreducible representations is faithful on $A^s/A$.*

*Proof.* Since $X_\phi$ is infinite, there exists $b \in A^s \setminus A$ [12, p. 306]. Then either (i) or (ii) of Lemma 2.4 fails.

Suppose first that (i) fails. Then there exists $\epsilon > 0$ such that the set

$$Y = \{x \in X_\phi : \|b + H_x\| \geq \epsilon\}$$

is non-compact. Since $Y$ is closed in $X_\phi$ by the upper semi-continuity of norm functions, it follows that $X_\phi$ is non-compact. Hence, since $A$ is continuous and $\sigma$-unital and $X_\phi$ is locally compact and $\sigma$-compact, we may write $X_\phi = \bigcup_{i=1}^\infty X_n$ where each $X_n$ is compact and is strictly contained in the interior of $X_{n+1}$. The non-compactness of $Y$ implies that for each $n \geq 1$, $Y \cap (X_\phi \setminus X_n)$ is non-empty so we may choose a sequence $(y_n)_{n\geq 1}$ with $y_n \in Y \cap (X_\phi \setminus X_n)$. Temporarily fix $n \geq 1$. Then $\|b + H_{y_n}\| \geq \epsilon$, so by [7, Lemma 1.5] there exists $x_n$ in the open neighbourhood $X_\phi \setminus X_n$ of $y_n$ such that $\|b + \tilde{J}_{x_n}\| > \epsilon/2$. By the lower semi-continuity of norm functions [12, Lemma 6.2(i)], and the density of the set $\{x \in X_\phi : J_x \in \mathrm{Prim}(A)\}$ in $X_\phi$, it follows that there exists $w_n \in X_\phi \setminus X_n$ with $J_{w_n} \in \mathrm{Prim}(A)$ and $\|b + \tilde{J}_{w_n}\| > \epsilon/2$. Set $W = \{w_n : n \geq 1\}$. Then $W$ is countably infinite, and at most finitely many elements of $W$ belong to each $X_n$, so $W$ is a closed, relatively discrete set in $X_\phi$. Let $\mathcal{F}^\sharp$ be a free $z$-ultrafilter on $W$ and let $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}^\sharp\}$. Then it follows from Theorem 4.1 that there is an irreducible representation $\pi_\mathcal{F} \in \mathcal{S}$ with kernel $L_\mathcal{F}$. But $f_\delta(b) \notin L_\mathcal{F}^{\mathrm{alg}}$ for all $\delta \in (0, \epsilon/2)$, so $\pi_\mathcal{F}(b) \neq 0$.

Now suppose instead that (ii) fails. Then there exists $x \in X_\phi$ such that $b - a \notin H_x$ for all $a \in A$. Let $a \in A$ such that $c := b - a \in \tilde{J}_x$, and set $Z = Z(c)$. Then $Z$ is a zero set in $X_\phi$ and $x \in Z$. Since $c \notin H_x$ we see that $x$ is in the boundary of $Z$ by [7, Lemma 1.5]. We seek an irreducible representation $\pi_\mathcal{F} \in \mathcal{S}$ such that $\pi_\mathcal{F}(c) \neq 0$; and replacing $c$ by $c^*c$, we may assume that $c \geq 0$. Since $c \notin H_x$, there exists $\epsilon > 0$ such that $f_\epsilon(c) \notin L^{\mathrm{alg}}_{Z[O_x]}$ [8, Theorem 4.3(ii)]. Thus for every open set $U$ containing $x$ we may find $y \in U \setminus Z$ such that $\|c + H_y\| > \epsilon$.

Since $X_\phi$ is locally compact and $\sigma$-compact, the cozero set $X_\phi \setminus Z$ is also locally compact and $\sigma$-compact; and as $x$ is a boundary point in $Z$, $X_\phi \setminus Z$ is non-compact. Thus we may write $X_\phi \setminus Z$ as a countable, strictly increasing union of compact sets $Y_n$ ($n \geq 1$) where each $Y_n$ is contained in the interior of $Y_{n+1}$.

For each $n \geq 1$, set $U_n = X_\phi \setminus Y_n$. Then $x \in U_n$. Temporarily fix $n$ and choose $y_n \in U_n \setminus Z$ such that $\|c + H_{y_n}\| > \epsilon$. Then by [7, Lemma 1.5] there exists $x_n \in U_n \setminus Z$ such that $\|c + \tilde{J}_{x_n}\| > \epsilon$. By the lower semi-continuity of norm functions [12, Lemma 6.2(i)], and the density of the set $\{x \in X_\phi : J_x \in \mathrm{Prim}(A)\}$ in $X_\phi$, it follows that there exists $w_n \in U_n \setminus Z$ with $J_{w_n} \in \mathrm{Prim}(A)$ and $\|c + \tilde{J}_{w_n}\| > \epsilon$.

Set $W = \{w_n : n \geq 1\}$. The sets $U_n \setminus Z$ are decreasing and $\bigcap_{n=1}^\infty (U_n \setminus Z) = \emptyset$, so $W$ is countably infinite. Let $y \in X_\phi$ be an accumulation point of $W$. Since $W$ has finite intersection with each $Y_{n+1}$, $y$ cannot be in the interior of $Y_{n+1}$, so in particular $y \notin Y_n$. Thus $y \in Z$. It follows that $W$ is relatively closed and relatively discrete in $X_\phi \setminus Z$. Let $\mathcal{F}^\sharp$ be a free $z$-ultrafilter on $W$ and set $\mathcal{F} = \{Z \in Z[X_\phi] : Z \cap W \in \mathcal{F}^\sharp\}$. Then by Theorem 4.5(iv) there exists an irreducible representation $\pi_\mathcal{F} \in \mathcal{S}$ such that

$$\ker \pi_\mathcal{F} \cap \tilde{J}_Z \subseteq L_\mathcal{F}^{M(A)}.$$

Since $c \in \tilde{J}_Z \setminus L_\mathcal{F}^{M(A)}$ we have that $c \notin \ker \pi_\mathcal{F}$. □

The question of whether this family $\mathcal{S}$ of irreducible representations is faithful on the whole of $C(A) = M(A)/A$ is equivalent to the question whether $A^s/A$ is an essential ideal in $C(A)$, i.e. whether for each $b \in M(A) \setminus A$ there exists $a \in A^s$ such that either $ba \notin A$ or $ab \notin A$.

It was shown in [12, Theorem 4.1] that if $A = C_0(X) \otimes K(H)$ where $X$ is a locally compact Hausdorff space then $A^s/A$ is an essential ideal in $C(A)$ if and only if $X$ has no isolated points.

## 5. Primal ideals in $A^s/A$

In this section we use the irreducible representations just constructed to identify the set $\mathrm{MinPrimal}(A^s/A)$ of minimal closed primal ideals of $A^s/A$ where $A$ is a $\sigma$-unital, quasi-standard C*-algebra with non-unital Glimm quotients such that $\mathrm{Glimm}(A)$ is infinite and $\mathrm{Prim}(A) \cap \mathrm{Glimm}(A)$ is dense in $\mathrm{Glimm}(A)$ (this latter condition is automatic if $A$ is separable and quasi-standard [6, Corollary 3.5]).

We have already mentioned that if $A$ is a C*-algebra and $I$ a closed ideal in $A$ then $I$ is primal if $\mathrm{Prim}(A/I)$ is contained in a limit set in $\mathrm{Prim}(A)$. Let $J$ be a closed ideal in $A$ and $\pi : A \to A/J$ the quotient map. If $K$ is a primal ideal in $A/J$ then $\pi^{-1}(K)$ is evidently primal in $A$. On the other hand there is no general reason why the image $\pi(I)$ of a primal ideal $I$ in $A$ should be primal in $A/J$. For example, let $A$ be the Kaplansky example of the C*-algebra of sequences $x = (x_n)_{n \geq 1}$ of $2 \times 2$ complex matrices converging at infinity to a diagonal matrix $\mathrm{diag}(\lambda(x), \mu(x))$. Then the closed ideal $J = \ker \lambda \cap \ker \mu$ is primal in $A$, but $A/J \cong \mathbf{C} \oplus \mathbf{C}$ so the image of $J$ in the quotient (namely $\{0\}$) is not primal in $A/J$. It is therefore somewhat surprising to find that the primal ideals in $A^s/A$ in the context above are precisely the images of the primal ideals in $A^s$ (see Theorem 5.5).

The first point to clarify is when the image of a certain kind of ideal in $A^s$ consists of the whole of $A^s/A$. We shall need the following lemma.

**Lemma 5.1.** *Let $\mathcal{F}$ be a z-filter on a completely regular space $X$ and suppose that there is a non-empty compact subset $Z'$ of $X$ such that $Z' \in \mathcal{F}$. Then*
(i) *$Y := \bigcap \{Z : Z \in \mathcal{F}\}$ is compact and non-empty;*
(ii) *if $W$ is a zero-set neighbourhood of $Y$ in $X$ then $W \in \mathcal{F}$.*

*Proof.* (i) This follows from the finite intersection property.

(ii) Let $U$ be the interior of $W$. Then $Y \subseteq U$ so there exists $Z \in \mathcal{F}$ such that $Z' \cap Z$ does not meet $X \setminus U$ (for otherwise $Y$ would meet the compact set $Z' \cap (X \setminus U)$). Hence $W \supseteq U \supseteq Z' \cap Z \in \mathcal{F}$ and so $W \in \mathcal{F}$. □

Now let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. For a $z$-filter $\mathcal{F}$ on $X_\phi$, we define

$$D_{\mathcal{F}} = ((L_{\mathcal{F}} \cap A^s) + A)/A.$$

**Proposition 5.2.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $\mathcal{F}$ be a z-filter on $X_\phi$. Then the following are equivalent:*
(i) *$D_{\mathcal{F}}$ is a proper closed ideal of $A^s/A$;*
(ii) *there is no non-empty finite subset $W$ of $X_\phi$ such that $\mathcal{F} = \{Z \in Z[X_\phi] : Z \supseteq W\}$.*

*Proof.* First suppose that $\mathcal{F}$ does not satisfy (ii). Then there exists a non-empty finite subset $W$ of $X_\phi$ such that $\mathcal{F} = \{Z \in Z[X_\phi] : Z \supseteq W\}$. Let $b \in A^s$. Then since $W$ is finite it is easy to find $a \in A$ such that $b - a \in \tilde{J}_x$ for all $x \in W$. Hence $Z(b-a) \supseteq W$, so $b - a \in L_\mathcal{F} \cap A^s$. Thus $b \in (L_\mathcal{F} \cap A^s) + A$ so $(L_\mathcal{F} \cap A^s) + A = A^s$, and hence $D_\mathcal{F} = A^s/A$. Thus (i) fails.

Now suppose that $\mathcal{F}$ satisfies (ii). We deal first with the case when every zero set in $\mathcal{F}$ is non-compact. Let $c^Z \in A^s$ be an element with the properties of Lemma 2.5 in the case when $Z$ is the empty set. Let $V \in \mathcal{F}$. Then $\|c^Z + \tilde{J}_x\| = 1$ for all $x \in V$. For $a \in A$ and $\epsilon > 0$, the set $\{x \in X_\phi : \|a + J_x\| \geq \epsilon\}$ is compact (being the image under $\phi$ of the compact subset $\{P \in \mathrm{Prim}(A) : \|a + P\| \geq \epsilon\}$ of $\mathrm{Prim}(A)$). Therefore, since $V$ is closed and non-compact, $\inf_{x\in V} \|a + J_x\| = 0$ and so $\sup_{x\in V} \|(c^Z - a) + \tilde{J}_x\| \geq 1$. Hence there does not exist $a \in A$ such that $\|(c^Z - a) + L_\mathcal{F}\| < 1$. Thus $c^Z \notin L_\mathcal{F} + A$, so $c^Z + A \notin D_\mathcal{F}$.

Next suppose that $\mathcal{F}$ satisfies (ii) and contains a compact zero set. Then $Y := \bigcap\{Z : Z \in \mathcal{F}\}$ is compact and is non-empty by Lemma 5.1(i). First we deal with the case when $Y$ is infinite. If $Y$ is infinite then $Y$ contains a non-P-point by [17, 4K.1], so there is a zero set $Z'$ in $Y$ with non-empty boundary in $Y$. The normality of $X_\phi$ implies that there is a zero set $Z$ of $X_\phi$ such that $Z \cap Y = Z'$ (in other words, $Y$ is '$z$-embedded' in $X_\phi$). Let $c^Z \in A^s$ be an element with the properties of Lemma 2.5. Then $c^Z \in \tilde{J}_x$ for $x \in Z$ but $\|c^Z + \tilde{J}_x\| = 1$ for $x \in X_\phi \setminus Z$. Let $b \in L_\mathcal{F}^{\mathrm{alg}}$ and let $a \in A$. Then $Z(b) \in \mathcal{F}$ so $Z(b) \supseteq Y$. Since $Z'$ has non-empty boundary in $Y$, there is a net $(x_\alpha)$ in $Y \setminus Z'$ converging to some $x \in Z'$. Then $\|(b+a) + \tilde{J}_{x_\alpha}\| \to \|(b+a) + \tilde{J}_x\|$, while $\|c^Z + \tilde{J}_{x_\alpha}\| = 1$ and $\|c^Z + \tilde{J}_x\| = 0$. Hence $\|(b+a) - c^Z\| \geq 1/2$, so $c^Z \notin L_\mathcal{F} + A$. Thus $c^Z + A \notin D_\mathcal{F}$.

Now suppose that $Y$ is finite. Then since $\mathcal{F}$ satisfies (ii), there is a zero set $Z$ containing $Y$ such that $Z \notin \mathcal{F}$. Let $V \in \mathcal{F}$. We claim that $Z \cap V$ is not clopen in $V$. Suppose to the contrary. Since $V \cap Z$ and $V \setminus (V \cap Z)$ are disjoint closed sets in the normal space $X_\phi$, they can be separated by disjoint closed neighbourhoods and hence there is a zero set neighbourhood $U$ of $Y$ containing $Z \cap V$ with $U$ disjoint from $V \setminus (Z \cap V)$. But $U \in \mathcal{F}$ by Lemma 5.1(ii) and $U \cap V = Z \cap V$, so $Z \cap V \in \mathcal{F}$. Hence $Z \in \mathcal{F}$, a contradiction.

Let $c^Z \in A^s$ be an element with the properties of Lemma 2.5. Then $c^Z \in \tilde{J}_x$ for $x \in Z$ but $\|c^Z + \tilde{J}_x\| = 1$ for $x \in X_\phi \setminus Z$. Let $b \in L_\mathcal{F}^{\mathrm{alg}}$ and let $a \in A$. Then $Z(b) \in \mathcal{F}$ so $Z \cap Z(b)$ is not clopen in $Z(b)$. Hence there is a net $(x_\alpha)$ in $Z(b) \setminus Z$ converging to some $x \in Z \cap Z(b)$. Then $\|(b+a) + \tilde{J}_{x_\alpha}\| \to \|(b+a) + \tilde{J}_x\|$, while $\|c^Z + \tilde{J}_{x_\alpha}\| = 1$ and $\|c^Z + \tilde{J}_x\| = 0$. Hence $\|(b+a) - c^Z\| \geq 1/2$, so $c^Z \notin L_\mathcal{F} + A$. Thus $c^Z + A \notin D_\mathcal{F}$. □

Suppose that $A$ is a $C_0(X)$-algebra with base map $\phi$, that $\mathcal{F}$ is a $z$-filter on $X_\phi$ and that $p \in \mathrm{cl}_{\beta X} X_\phi$. Then $\mathcal{F}$ is said to *converge* to $p$ if every neighbourhood (in $\mathrm{cl}_{\beta X} X_\phi$) of $p$ contains a member of $\mathcal{F}$ [17, 6.2]. In this case, $\mathcal{F} \supseteq Z[O^p]$. Indeed, if $f \in O^p$ then $\mathrm{cl}_{\beta X} Z(f)$ contains some member $Z$ of $\mathcal{F}$ and hence the closed subset $Z(f)$ of $X_\phi$ contains $Z$. Hence, in the context of Lemma 5.3 below, it follows from Theorem 3.1 and [8, Theorem 4.3] that $H_p \subseteq L_\mathcal{F}$ and, if $p \in X_\phi$, $L_\mathcal{F} \subseteq \tilde{J}_p$.

**Lemma 5.3.** *Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$ and suppose that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $\mathcal{F}$ be a $z$-filter on $X_\phi$ that converges to some $y \in X_\phi$ and let $b \in M(A)$. Then $b \in L_\mathcal{F} + A$ if and only if*

(a) *$b \in A + \tilde{J}_y = N_y$ and*

(b) *for all $a \in A$ such that $b - a \in \tilde{J}_y$, $b - a \in L_\mathcal{F}$.*

*Proof.* If $b \in L_{\mathcal{F}} + A$ then there exist $c \in L_{\mathcal{F}}$ and $a \in A$ such that $b = c + a$. Since $L_{\mathcal{F}} \subseteq \tilde{J}_y$, we have that $b + \tilde{J}_y = a + \tilde{J}_y \in A + \tilde{J}_y$. Hence (a) holds. For (b), notice that if $a_1 \in A$ with $b - a_1 \in \tilde{J}_y$ then $a - a_1 \in J_y \subseteq H_y \subseteq L_{\mathcal{F}}$. Hence $b - a_1 = c + (a - a_1) \in L_{\mathcal{F}}$.

Conversely, if (a) holds then there exists $a \in A$ such that $b - a \in \tilde{J}_y$, and then (b) implies that $b - a \in L_{\mathcal{F}}$. □

**Proposition 5.4.** *Let $A$ be a $\sigma$-unital, continuous $C_0(X)$-algebra with base map $\phi$. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be z-filters on $X_\phi$ which converge to points in $\mathrm{cl}_{\beta X} X_\phi$ and suppose that neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ has the form $Z[M_y]$ ($y \in X_\phi$). If $\mathcal{F}_1 \not\subseteq \mathcal{F}_2$ then $D_{\mathcal{F}_1} \not\subseteq D_{\mathcal{F}_2}$. In particular, if $\mathcal{F}_1 \neq \mathcal{F}_2$ then $D_{\mathcal{F}_1} \neq D_{\mathcal{F}_2}$.*

*Proof.* Suppose that $\mathcal{F}_1 \not\subseteq \mathcal{F}_2$. By assumption, there is a zero zet $Z$ with $Z \in \mathcal{F}_1 \setminus \mathcal{F}_2$. Let $c^Z \in A^s$ with the properties of Lemma 2.5. Then it follows from the proof of [8, Theorem 5.7] that $c^Z \in (A^s \cap L^{\mathrm{alg}}_{\mathcal{F}_1}) \setminus L_{\mathcal{F}_2}$.

Suppose first that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same limit $x \in X_\phi$. Since $x \in Z$, $c^Z \in \tilde{J}_x$. Thus Lemma 5.3 implies that $c^Z \notin L_{\mathcal{F}_2} + A$. Hence $c^Z + A \in D_{\mathcal{F}_1} \setminus D_{\mathcal{F}_2}$.

Suppose next that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same limit $p \in \mathrm{cl}_{\beta X} X_\phi \setminus X_\phi$. Towards a contradiction, suppose that $c^Z + a \in L_{\mathcal{F}_2}$ for some $a \in A$. Then there exists $b \in M(A)$ such that $Z(b) \in \mathcal{F}_2$ and $\|c^Z + a - b\| < 1/4$. Let $C := \{t \in X_\phi : \|a + J_t\| \geq 1/4\}$, a compact subset of $X_\phi$. For $t \in X_\phi \setminus (C \cup Z)$, $\|(c^Z + a) + \tilde{J}_t\| \geq 3/4$ and hence $\|b + \tilde{J}_t\| \geq 1/2$. In particular, $Z(b) \subseteq C \cup Z$. Since $\mathcal{F}_2$ converges to $p$ and $\mathrm{cl}_{\beta X} X_\phi \setminus C$ is a neighbourhood of $p$ in $\mathrm{cl}_{\beta X} X_\phi$, there exists $Z' \in \mathcal{F}_2$ such that $Z' \cap C = \emptyset$. It follows that $Z$ contains $Z(b) \cap Z'$ and so $Z \in \mathcal{F}_2$, a contradiction. Hence $c^Z \notin L_{\mathcal{F}_2} + A$ and so $c^Z + A \in D_{\mathcal{F}_1} \setminus D_{\mathcal{F}_2}$.

Finally, suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ have different limits $p_1$ and $p_2$ (respectively) in $\mathrm{cl}_{\beta X} X_\phi$. Then there exists $f_1 \in C(\beta X)$ with $0 \leq f_1 \leq 1$ such that $f_1(p_1) = 0$ and $f_1(p_2) = 1$. Set $f_2 = 1 - f_1$, and let $\overline{\mu} : C(\beta X) \to ZM(A)$ be the extension of the structure map $\mu$ of the $C_0(X)$-algebra $A$ [7, Proposition 1.2]. Then for $b \in A^s$, $b = \overline{\mu}(f_1)b + \overline{\mu}(f_2)b$. Now

$$\overline{\mu}(f_i)b \in H_{p_i} \cap A^s \subseteq L_{\mathcal{F}_i} \cap A^s \quad (i = 1, 2)$$

(see [7, p. 77] for the first inclusion and the remarks preceding Lemma 5.3 for the second). Thus $A^s = (L_{\mathcal{F}_1} \cap A^s) + (L_{\mathcal{F}_2} \cap A^s)$ and so $D_{\mathcal{F}_1} + D_{\mathcal{F}_2} = A^s/A$. Suppose that $D_{\mathcal{F}_1} = A^s/A$. By Proposition 5.2, there is a non-empty finite subset $W$ of $X_\phi$ such that $\mathcal{F}_1 = \{Z \in Z[X_\phi] : Z \supseteq W\}$. Since $\mathcal{F}_1$ converges to $p_1$, no point of $X_\phi \setminus \{p_1\}$ belongs to $W$. Thus $W = \{p_1\}$ and $\mathcal{F}_1 = Z[M_{p_1}]$, a contradiction. Thus $D_{\mathcal{F}_1}$ is a proper ideal of $A^s/A$ and a similar argument applies to $D_{\mathcal{F}_2}$. Since their sum is $A^s/A$, neither can contain the other. □

**Theorem 5.5.** *Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra. Suppose that $X := \mathrm{Glimm}(A)$ is infinite, that $X \cap \mathrm{Prim}(A)$ is dense in $X$ and that $A/J_x$ is non-unital for all $x \in X$. Let $P \in \mathrm{MinPrimal}(A^s)$. Then $(P + A)/A$ is a primal ideal of $A^s/A$.*

*Proof.* Set $P' = P + A$. By [11, Theorem 3.4] and Theorem 3.3, there is a minimal prime z-filter $\mathcal{F}$ on $X$ such that $P = L_{\mathcal{F}} \cap A^s$. Suppose first that there exists $x \in X$ such that $Z[O_x] \subseteq \mathcal{F} \subseteq Z[M_x]$ (see [17, Theorem 7.15]) and hence such that $H_x \subseteq L_{\mathcal{F}} \subseteq \tilde{J}_x$, [8, Theorem 4.3]. If $x$ is a P-point of $X$ then $H_x = \tilde{J}_x$, so $P = \tilde{J}_x \cap A^s$. Hence $P' = (\tilde{J}_x \cap A^s) + A = A^s$, so $P'/A$ is trivially primal in $A^s/A$. Thus we may assume that $x$ is a non-P-point of $X$. This implies that $L_{\mathcal{F}}$ is strictly contained in $\tilde{J}_x$ [17, 14.12] and hence that $P$ is strictly contained in $\tilde{J}_x \cap A^s$ [8, Theorem 5.7]. Thus $P'$ is strictly contained in $A^s$ by Lemma 5.3 (let $b \in (\tilde{J}_x \cap A^s) \setminus P$: if $b \in P + A$ then we get a contradiction in Lemma 5.3(b) by taking $a = 0$).

Let $b_i \in A^s$ $(1 \leq i \leq n)$ with $\|b_i\| = \|b_i + P'\| = 1$. For the primality of $P'/A$ it is enough to show that $b_1 A^s b_2 \ldots A^s b_n \not\subseteq A$. For $1 \leq i \leq n$, let $a_i \in A$ such that $b_i - a_i \in \tilde{J}_x \cap A^s$ and set $c_i = b_i - a_i$. Then it will be enough to show that $c_1 A^s c_2 \ldots A^s c_n \not\subseteq A$. We have that $c_i \in \tilde{J}_x \cap A^s$, but $\|c_i + P\| \geq 1$ (for if there exists $p \in P$ such that $\|c_i - p\| < 1$ then $\|b_i - (p + a_i)\| = \|c_i - p\| < 1$, and $p + a_i \in P + A = P'$). Set $Z = \bigcap_{i=1}^n Z(c_i)$, so that $c_i \in \tilde{J}_Z$ for $1 \leq i \leq n$. Then $Z$ is a zero set in $X$ and $x \in Z(c_i)$ for each $i$ so $x \in Z$.

Since $c_i \notin P$, for each $1 \leq i \leq n$, $c_i c_i^* \notin P$, so there exists $0 < \epsilon_i < 1/2$ such that $f_{\epsilon_i}(c_i c_i^*) \notin P$ (for otherwise $c_i c_i^* = \lim_{\epsilon \to 0} f_\epsilon(c_i c_i^*) \in P$). Hence $Z(f_{\epsilon_i}(c_i c_i^*)) \notin \mathcal{F}$, for otherwise $f_{\epsilon_i}(c_i c_i^*) \in L_{\mathcal{F}}^{\text{alg}} \cap A^s \subseteq L_{\mathcal{F}} \cap A^s = P$. Set $W = \bigcup_{i=1}^n Z(f_{\epsilon_i}(c_i c_i^*))$ and note that $W$ is a zero set in $X$ and that $Z(c_i) = Z(c_i c_i^*) \subseteq Z(f_{\epsilon_i}(c_i c_i^*))$ for each $i$, so $Z \subseteq W$. Hence, in particular, $x \in W$. Since $\mathcal{F}$ is a prime $z$-filter, $W \notin \mathcal{F}$. This implies that $W \neq X$, and also that $x$ does not lie in the interior of $W$, for otherwise $W \in Z[O_x] \subseteq \mathcal{F}$. Set $U = X \setminus W$. Then $U$ is a non-empty cozero set in $X$ and $x$ lies in the closure of $U$.

Let $f \in C_{\mathbb{R}}(X)$ with $0 \leq f \leq 1$ such that $Z(f) = Z$. Then $f(x) = 0$, and $x$ lies in the boundary of $U$ so we may inductively construct a sequence $(x_n)_{n \geq 1}$ in $U \cap \text{Prim}(A)$ such that $f(x_1) < 1$ and $f(x_n) < \min\{f(x_{n-1}), 1/n\}$ for all $n \geq 2$. Thus $f(x_n) \to 0$ as $n \to \infty$. Hence the set $Y = \{x_n : n \geq 1\}$ is contained in $U$ and is relatively closed and relatively discrete in $X \setminus Z$.

Let $\mathcal{G}^\sharp$ be a free ultrafilter on $Y$ and set $\mathcal{G} = \{Z' \in Z[X] : Z' \cap Y \in \mathcal{G}^\sharp\}$. With $Z$ and $Y$ as above, let $\pi_{\mathcal{G}}$ be an irreducible representation of $M(A)$ constructed as in the proof of Theorem 4.5. Note that for $y \in Y$, $f_{\epsilon_i}(c_i c_i^*) + \tilde{J}_y \neq 0$ and hence $\|c_i + \tilde{J}_y\|^2 > \epsilon_i$. Thus $c_i \notin L_{\mathcal{G}'}^{M(J_Z)}$ (where $\mathcal{G}' = \{Z' \in Z[X \setminus Z] : Z' \cap Y \in \mathcal{G}^\sharp\}$). On the other hand, $c_i \in \tilde{J}_Z$; so $c_i \notin \ker \pi_{\mathcal{G}}$ using Theorem 4.5(iii). It follows that $\pi_{\mathcal{G}}(c_1 A^s c_2 \ldots A^s c_n) \neq \{0\}$ since $\pi_{\mathcal{G}}$ is an irreducible representation. Thus $c_1 A^s c_2 \ldots A^s c_n \not\subseteq A$, by Theorem 4.5(i), so $b_1 A^s b_2 \ldots A^s b_n \not\subseteq A$, as required.

Now suppose that there exists $x \in \beta X \setminus X$ such that $Z[O^x] \subseteq \mathcal{F} \subseteq Z[M^x]$ (see [17, Theorem 7.15] again). Let $a$ belong to the Pedersen ideal of $A$ [26, 5.6.1]. Then the set $\{P \in \text{Prim}(A) : a \notin P\}$ is contained in a compact subset $C$ of $\text{Prim}(A)$, so the complement of $Z(a) = \{y \in X : a \in J_y\}$ is contained in the compact set $\phi_A(C)$. Thus $\text{cl}_{\beta X} Z(a) \supseteq \beta X \setminus \phi_A(C)$ which is an open subset of $\beta X$ containing $x$. Hence $Z(a) \in Z[O^x] \subseteq \mathcal{F}$, and thus $L_{\mathcal{F}}^{\text{alg}}$ contains the Pedersen ideal of $A$. Hence $P \supseteq A$, so $P' = P$ and $P'$ is a proper subset of $A^s$. Again let $b_i \in A^s$ $(1 \leq i \leq n)$ with $\|b_i\| = \|b_i + P\| = 1$. We argue as in the previous case, but the argument is now slightly simpler. Since $b_i \notin P$, for each $1 \leq i \leq n$ there exists $0 < \epsilon_i < 1/2$ such that $f_{\epsilon_i}(b_i b_i^*) \notin P$ (for otherwise $b_i b_i^* = \lim_{\epsilon \to 0} f_\epsilon(b_i b_i^*) \in P$). Hence $Z(f_{\epsilon_i}(b_i b_i^*)) \notin \mathcal{F}$, for otherwise $f_{\epsilon_i}(b_i b_i^*) \in L_{\mathcal{F}}^{\text{alg}} \cap A^s \subseteq L_{\mathcal{F}} \cap A^s = P$. Set $W = \bigcup_{i=1}^n Z(f_{\epsilon_i}(b_i b_i^*))$ and note that $W$ is a zero set in $X$. Since $\mathcal{F}$ is a prime $z$-filter, $W \notin \mathcal{F}$. This implies that $W \neq X$, and also that $x$ does not lie in the interior in $\beta X$ of $\text{cl}_{\beta X} W$, for otherwise $W \in Z[O^x] \subseteq \mathcal{F}$. Set $U = X \setminus W$. Then $U$ is a non-empty cozero set in $X$ and $x$ lies in $\text{cl}_{\beta X} U$. Let $y \in U$. Then $f_{\epsilon_i}(b_i b_i^*) + \tilde{J}_y \neq 0$ and hence $\|b_i + \tilde{J}_y\|^2 > \epsilon_i$.

Since $A$ is $\sigma$-unital and quasi-standard, $X$ is a $\sigma$-compact open subset of $\beta X$, and hence is a cozero set in $\beta X$. Let $f \in C_{\mathbb{R}}(\beta X)$ with $0 \leq f \leq 1$ and $Z(f) = \beta X \setminus X$. Then $f(x) = 0$, and $x$ lies in the closure of $U$ in $\beta X$ so we may inductively construct a sequence $(x_n)_{n\geq 1}$ in $U \cap \mathrm{Prim}(A)$ such that $f(x_1) < 1$ and $f(x_n) < \min\{f(x_{n-1}), 1/n\}$ for all $n \geq 2$. Thus $f(x_n) \to 0$ as $n \to \infty$. Hence the set $Y = \{x_n : n \geq 1\}$ is a closed, relatively discrete subset of $X$. Let $\pi$ be an irreducible representation of $M(A)$ obtained from $Y$ as in the proof of Theorem 4.1. Then $A \subseteq \ker \pi$ but $\|\pi(b_i)\|^2 \geq \epsilon_i > 0$ for $1 \leq i \leq n$. Hence $b_1 A^s b_2 \dots A^s b_n \not\subseteq A$, as required. □

**Theorem 5.6.** *Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra. Suppose that $X := \mathrm{Glimm}(A)$ is infinite, that $X \cap \mathrm{Prim}(A)$ is dense in $X$ and that $A/J_x$ is non-unital for all $x \in X$. Let $Y$ be the set of P-points of $X$ and set $S = \{\tilde{J}_y \cap A^s : y \in Y\}$. Let $\sigma : A^s \to A^s/A$ be the quotient map. Then*

$$\mathrm{MinPrimal}(A^s/A) = \{\sigma(P) : P \in \mathrm{MinPrimal}(A^s) \setminus S\}.$$

*Furthermore, the map induced by $\sigma$ from* $\mathrm{MinPrimal}(A^s) \setminus S$ *onto* $\mathrm{MinPrimal}(A^s/A)$ *is bijective.*

*Proof.* Let $P \in \mathrm{MinPrimal}(A^s)$. Then by Theorem 3.3, $P = Q \cap A^s$ for some $Q \in \mathrm{MinPrimal}(M(A))$; and by [11, Theorem 2.4], $Q = L_{\mathcal{F}}$ for some minimal prime $z$-filter $\mathcal{F}$ on $X$. Hence $\sigma(P) = D_{\mathcal{F}}$. Suppose that $\sigma(P) = A^s/A$. Then, by Proposition 5.2, there is a non-empty finite subset $W$ of $X$ such that $\mathcal{F} = \{Z \in Z[X] : Z \supseteq W\}$. Since $\mathcal{F}$ is prime, $W = \{y\}$ for some $y \in X$ and so $\mathcal{F} = Z[M_y]$. Then $M_y$ is a minimal prime $z$-ideal and hence $M_y = O_y$ [17, 14.12] and so $y$ is a P-point of $X$ [17, 4L]. We have $L_{\mathcal{F}} = L_{Z[O_y]} = H_y = \tilde{J}_y$ [8, Theorem 4.5] and so $P = \tilde{J}_y \cap A^s$.

Conversely, if $y \in Y$ then, again, $\tilde{J}_y = H_y$. Since $H_y$ is a Glimm ideal and $\tilde{J}_y$ is primal [7, Lemma 4.5], it follows that $\tilde{J}_y$ is a minimal closed primal ideal of $M(A)$, so $\tilde{J}_y \cap A^s$ is a minimal closed primal ideal of $A^s$ by Theorem 3.3. But $(\tilde{J}_y \cap A^s) + A = A^s$, so we see that $\sigma(\tilde{J}_y \cap A^s)$ is not a proper ideal in $A^s/A$ and hence is not a minimal closed primal ideal in $A^s/A$.

Let $R$ be any proper closed primal ideal of $A^s/A$. Then $T := \sigma^{-1}(R)$ is a proper closed primal ideal of $A^s$, so there exists $P' \in \mathrm{MinPrimal}(A^s)$ such that $T \supseteq P'$. Hence $\sigma(T) = R \supseteq \sigma(P')$, which is primal by Theorem 5.5. This shows that $P' \notin S$, and also that every minimal closed primal ideal of $A^s/A$ has the form $\sigma(P')$ for some $P' \in \mathrm{MinPrimal}(A^s) \setminus S$.

Let $P \in \mathrm{MinPrimal}(A^s) \setminus S$. By Theorem 5.5, $\sigma(P)$ is a closed primal ideal of $A^s/A$ and hence contains a minimal closed primal ideal of $A^s/A$. As seen above, the latter has the form $\sigma(P')$ for some $P' \in \mathrm{MinPrimal}(A^s) \setminus S$. Since $\sigma(P) \supseteq \sigma(P')$, $P + A \supseteq P' + A$. We have $P = L_{\mathcal{F}} \cap A^s$ and $P' = L_{\mathcal{F}'} \cap A^s$ where $\mathcal{F}$ and $\mathcal{F}'$ are minimal prime $z$-filters on $X$. Note that $\mathcal{F}$ and $\mathcal{F}'$ are convergent to points of $\beta X$ by [17, 7.15 and 10I1]. Suppose that $\mathcal{F} = Z[M_y]$ for some $y \in X$. Then $M_y$ is a minimal prime $z$-ideal and hence $M_y = O_y$ [17, 14.12] and so $y \in Y$ [17, 4L], contradicting the fact that $P \notin S$. Similarly, $\mathcal{F}'$ does not have the form $Z[M_y]$ $(y \in X)$. We have that $D_{\mathcal{F}} \supseteq D_{\mathcal{F}'}$ and hence $\mathcal{F} \supseteq \mathcal{F}'$ by Proposition 5.4. By minimality, $\mathcal{F} = \mathcal{F}'$ and so $\sigma(P) = \sigma(P')$ and hence $\sigma(P)$ is a minimal closed primal ideal of $A^s/A$ as required.

Finally, the injectivity of the map induced by $\sigma$ follows from Proposition 5.4. □

Note that Theorem 5.6 shows that the minimal closed primal ideals of $A^s/A$ are those proper ideals of $A^s/A$ which are obtained by intersecting with $A^s/A$ those ideals of the corona algebra

$C(A)$ which lie in the image of $\mathrm{Min}(C_{\mathbf{R}}(X))$ under the second embedding map of [8, Theorem 3.3].

## 6. Some examples and applications

In this section we investigate some examples, and study the topology on $\mathrm{MinPrimal}(A^s/A)$. We show that if $X$ is locally compact, $\sigma$-compact and without isolated points and $A = C_0(X) \otimes K(H)$ then $\mathrm{MinPrimal}(A^s/A)$ is not weakly Lindelof (recall that a topological space $Y$ is *weakly Lindelof* if for any open cover $\mathcal{U}$ of $Y$, there is a countable $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in $Y$). It follows that if $X$ is also second countable then the induced bijective map in Theorem 5.6 is not continuous (Corollary 6.6). On the other hand, if $X$ is an infinite F-space without isolated points, such as $\beta\mathbb{N} \setminus \mathbb{N}$, then we are able to show that $\mathrm{MinPrimal}(A^s/A)$ and $\mathrm{Glimm}(A^s/A)$ coincide as sets (Theorem 6.8) but not as topological spaces (Theorem 6.9).

As well as the topology $\tau_w$, we shall use another topology $\tau_s$, defined on the set $\mathrm{Id}(A)$ of closed ideals of a C*-algebra $A$ as the weakest topology with regard to which all the norm functions $I \mapsto \|a + I\|$ $(I \in \mathrm{Id}(A), a \in A)$ are continuous [15], [3]. It is known that $\tau_s$ coincides with $\tau_w$ when restricted to $\mathrm{MinPrimal}(A)$ [3, Corollary 4.3]. We shall also make use of the following lemma in Example 6.3, Theorem 6.8 and Theorem 6.9. Recall the definition of $D_{\mathcal{F}}$ that was given before Proposition 5.2.

**Lemma 6.1.** *Let $A$ be a $\sigma$-unital $C_0(X)$-algebra. Suppose that $X_\phi$ is infinite and that $A/J_x$ is non-unital for all $x \in X_\phi$. Let $\sigma : A^s \to A^s/A$ be the quotient map. Let $\mathcal{F}$ be a $z$-filter on $X_\phi$ that converges to some $y \in X_\phi$ and let $b \in A^s \cap \tilde{J}_y$. Then*

$$\|\sigma(b) + D_{\mathcal{F}}\| = \|b + L_{\mathcal{F}}\|.$$

*Proof.* Let $\nu : A^s \cap \tilde{J}_y \to (A^s/A)/D_{\mathcal{F}}$ be the *-homomorphism given by $\nu(c) = \sigma(c) + D_{\mathcal{F}}$. By Lemma 5.3, $\ker \nu = (A^s \cap \tilde{J}_y) \cap L_{\mathcal{F}} = A^s \cap L_{\mathcal{F}}$ (recall that $L_{\mathcal{F}} \subseteq \tilde{J}_y$). Thus

$$\|\sigma(b) + D_{\mathcal{F}}\| = \|b + \ker \nu\| = \|b + L_{\mathcal{F}}\|$$

(by two standard isomorphisms). □

**Example 6.2.** Let $X := \mathbb{N}$ and set $A = C_0(X) \otimes K(H)$. Then the (minimal) prime $z$-filters on $X$ are precisely the fixed and the free ultrafilters. The minimal prime $z$-filters corresponding to the points $p \in \mathbb{N}$ have the form $Z[O_p] = \{Z \in Z[X] : p \in Z\}$ while those corresponding to the points of $\beta\mathbb{N} \setminus \mathbb{N}$ have the form $Z[O^p] = \{Z \in Z[X] : Z \in \mathcal{F}_p\}$ where $\mathcal{F}_p$ is the free ultrafilter on $\mathbb{N}$ associated with $p \in \beta\mathbb{N} \setminus \mathbb{N}$. It follows from Theorem 3.3 and [11, Theorem 3.4] that $\mathrm{MinPrimal}(A^s)$ is homeomorphic to $\beta\mathbb{N}$. Furthermore, the ideals $L_{\mathcal{F}_p} \cap A^s$ are primitive by Theorem 4.1, and contain $A$. Thus Theorem 5.6 implies that $\mathrm{MinPrimal}(A^s/A) = \{D_{\mathcal{F}_p} : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$. Since each $L_{\mathcal{F}_p} \cap A^s$ contains $A$, the bijective map of Theorem 5.6 from $\mathrm{MinPrimal}(A^s) \setminus S$ onto $\mathrm{MinPrimal}(A^s/A)$ is bi-continuous for the $\tau_w$ topologies and so $\mathrm{MinPrimal}(A^s/A)$ is canonically homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$. Hence $\mathrm{MinPrimal}(A^s/A)$ is a compact F-space and every minimal closed primal ideal of $A^s/A$ is primitive.

Since the minimal closed primal ideals of $A^s$ coincide with the Glimm ideals of $A^s$ (see the discussion preceding Corollary 3.8 and [8, Theorem 4.3(ii)]) and since continuous functions on $\mathrm{Prim}(A^s)$ restrict to continuous functions on $\mathrm{Prim}(A^s/A)$, the minimal closed primal ideals of $A^s/A$ are not only primitive ideals but also Glimm ideals. Using [6, Lemma 3.1(iii)], we see that the identity map from the compact space $(\mathrm{MinPrimal}(A^s/A), \tau_w)$ to the Hausdorff space $(\mathrm{Glimm}(A^s/A), \tau_{cr})$ is continuous and hence is a homeomorphism. Thus $A^s/A$ is quasi-standard (as are $A^s$ and $M(A)$ by Corollary 3.8).

**Example 6.3.** Let $X := \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and set $A = C(X) \otimes K(H)$. Then it was shown by Kohls that $\mathrm{Min}(C_{\mathbb{R}}(X))$ is homeomorphic to $\beta\mathbb{N}$ [21], [18, p. 110], [17, 14G]. Again the minimal prime $z$-filters corresponding to the points $p \in \mathbb{N}$ have the form $Z[O_p] = \{Z \in Z[X] : p \in Z\}$ while those corresponding to $p \in \beta\mathbb{N} \setminus \mathbb{N}$ have the form $\mathcal{F}_p = \{Z \in Z[X] : Z \setminus \{\infty\} \in \mathcal{F}_p^\sharp\}$ where $\mathcal{F}_p^\sharp$ is the free ultrafilter on $\mathbb{N}$ associated with $p$. Once again it follows from Theorem 3.3 and [11, Theorem 3.4] that $\mathrm{MinPrimal}(A^s)$ is homeomorphic to $\beta\mathbb{N}$, and the ideals $(L_{\mathcal{F}_p} \cap A^s) + A$ $(p \in \beta\mathbb{N} \setminus \mathbb{N})$ are primitive ideals of $A^s$ by Corollary 4.6 (with $Z = \{\infty\}$ and $Y = \mathbb{N}$). Thus again Theorem 5.6 implies that $\mathrm{MinPrimal}(A^s/A) = \{D_{\mathcal{F}_p} : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$.

For the topology on $\mathrm{MinPrimal}(A^s/A)$, let $b \in A^s$ and let $\sigma : A^s \to A^s/A$ denote the quotient map. Let $a \in A$ such that $c := b - a \in \tilde{J}_\infty$. For $p \in \beta\mathbb{N} \setminus \mathbb{N}$, $\mathcal{F}_p$ is convergent in $X$ to the point $\infty$ and so by Lemma 6.1

$$\|\sigma(b) + D_{\mathcal{F}_p}\| = \|\sigma(c) + D_{\mathcal{F}_p}\| = \|c + L_{\mathcal{F}_p}\| = \|c + (L_{\mathcal{F}_p} \cap A^s)\|.$$

It follows that the bijective map of Theorem 5.6 from the compact space $\mathrm{MinPrimal}(A^s) \setminus S$ onto the Hausdorff space $\mathrm{MinPrimal}(A^s/A)$ is continuous for the $\tau_s$ topologies and therefore is a homeomorphism. Hence again $\mathrm{MinPrimal}(A^s/A)$ is a compact F-space and every minimal closed primal ideal of $A^s/A$ is primitive.

Example 6.2 and Example 6.3 are unusual in the prevalence of isolated points in $X$, and we shall now see that if $A$ is separable and $X$ has no isolated points then the topologies on $\mathrm{Min}(C_{\mathbb{R}}(X))$ and $\mathrm{MinPrimal}(A^s/A)$ are very different from each other.

**Proposition 6.4.** *Let $B$ be a C*-algebra such that* MinPrimal(B) *is weakly Lindelof. Then* $\mathrm{Prim}(B)$ *is weakly Lindelof.*

*Proof.* Let $\{U_\alpha\}$ be an open cover of $\mathrm{Prim}(B)$. Then for each $\alpha$ there is a closed ideal $I_\alpha$ of $B$ such $U_\alpha = \{P \in \mathrm{Prim}(B) : P \not\supseteq I_\alpha\}$. Set $V_\alpha = \{R \in \mathrm{Primal}'(B) : R \not\supseteq I_\alpha\}$. Since each proper primal ideal of $B$ is contained in a primitive ideal it follows that $\{V_\alpha\}$ is an open cover of $\mathrm{Primal}'(B)$. By assumption there is a countable family $\{V_i\}_{i\ge 1}$ of the sets $V_\alpha$ such that $W := \mathrm{MinPrimal}(B) \cap \bigcup_{i\ge 1} V_i$ is dense in $\mathrm{MinPrimal}(B)$. We claim that $Y := \bigcup_{i\ge 1} U_i$ is dense in $\mathrm{Prim}(B)$. Let $U = \{P \in \mathrm{Prim}(B) : P \not\supseteq I\}$ be any non-empty open subset of $\mathrm{Prim}(B)$ and set $V = \{R \in \mathrm{Primal}'(B) : R \not\supseteq I\}$. Then $W \cap V$ is a non-empty open subset of MinPrimal(B) so there exists $S \in W \cap V$. Then $S \in V_i$ for some $i \ge 1$, so $V \cap V_i$ is an open neighbourhood of $S$ in $\mathrm{Primal}'(B)$. Hence by [3, Proposition 3.1] there exists $P \in \mathrm{Prim}(B)$ such that $P \in V_i \cap V$. But $V_i \cap V \cap \mathrm{Prim}(B) = U_i \cap U$, so $Y \cap U$ is non-empty as required. □

**Theorem 6.5.** *Let $A$ be a continuous $C_0(X)$-algebra with base map $\phi$. Suppose either that $A$ is separable or that $A = C_0(X) \otimes K(H)$ where $X$ is a $\sigma$-compact, locally compact Hausdorff space. If $A/J_x$ is non-unital for all $x \in X_\phi$ and if $X_\phi$ has no isolated points then* $\mathrm{MinPrimal}(A^s/A)$ *is not weakly Lindelof.*

*Proof.* This follows from [12, Theorem 6.7] and Proposition 6.4. □

**Corollary 6.6.** *Let $A$ be a separable quasi-standard $C^*$-algebra and set $X = \mathrm{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$ and that $X$ has no isolated points. Then the bijective map from* $\mathrm{MinPrimal}(A^s)$ *to* $\mathrm{MinPrimal}(A^s/A)$ *described in Theorem 5.6 is not continuous.*

*Proof.* Since $A$ is separable and quasi-standard, the set $X \cap \mathrm{Prim}(A)$ is dense in $X$ and so Theorem 5.6 applies. Since every singleton subset of $X$ is a zero set [9, Lemma 3.9], every P-point in $X$ is an isolated point, and thus the sets $Y$ and $S$ of Theorem 5.6 are empty. The map $P \mapsto \sigma(P)$ ($P \in \mathrm{MinPrimal}(A^s)$) is therefore a bijection between $\mathrm{MinPrimal}(A^s)$ and $\mathrm{MinPrimal}(A^s/A)$. But $\mathrm{MinPrimal}(A^s)$ is compact and extremally disconnected by Corollary 3.6, whereas $\mathrm{MinPrimal}(A^s/A)$ is not weakly Lindelof by Theorem 6.5. Thus the map $P \mapsto \sigma(P)$ is not continuous. □

We are far from having a complete description of the topology on $\mathrm{MinPrimal}(A^s/A)$ but the next theorem sheds further light on the failure of the weak Lindelof property.

**Theorem 6.7.** *Let $A$ be a $\sigma$-unital quasi-standard $C^*$-algebra and set $X = \mathrm{Glimm}(A)$. Suppose that $A/J_x$ is non-unital for all $x \in X$ and that $X$ is first countable. Let $p$ be a non-isolated point in $X$. Then the set of minimal closed primal ideals of $A^s/A$ which contain $D_{Z[O_p]}$ is a non-empty clopen subset of* $\mathrm{MinPrimal}(A^s/A)$.

*Proof.* Set $Z = \{p\}$. Then $Z$ is a zero set in $X$ by the first countability of $X$. Let $c^Z \in A^s$ with the properties of Lemma 2.5. In particular, $c^Z \in \tilde{J}_p$. Since $X$ is locally compact, there exists a continuous function $f : X \to [0, 1]$ such that $f(V) = \{1\}$ for some neighbourhood $V$ of $p$ in $X$ and $C := \mathrm{supp}(f)$ is compact. Furthermore, $V$ contains a zero set neighbourhood $N$ of $p$. Set $z := \mu(f) \in ZM(A)$, where $\mu : C_0(X) \to ZM(A)$ is the structure map. Let $\mathcal{F}$ be a minimal prime $z$-filter on $X$.

Suppose first of all that there exists $F \in \mathcal{F}$ such that $C \cap F = \emptyset$. Then

$$F \subseteq X \setminus C \subseteq Z(z) \subseteq Z(zc^Z)$$

and so $Z(zc^Z) \in \mathcal{F}$, $zc^Z \in L_{\mathcal{F}} \cap A^s$ and $\sigma(zc^Z) \in D_{\mathcal{F}}$.

On the other hand, suppose that $C \cap F$ is non-empty for all $F \in \mathcal{F}$. Since $C$ is compact, there exists $q \in C$ such that, for all $F \in \mathcal{F}$, $q \in C \cap F$. Thus $I(\mathcal{F}) \subseteq M_q$ and so, since $I(\mathcal{F})$ is a prime $z$-ideal, $O_q \subseteq I(\mathcal{F})$ [17, 4I]. Hence $Z[O_q] \subseteq \mathcal{F}$, that is, $\mathcal{F}$ is convergent to $q$. Suppose that $q \neq p$. Then by Lemma 2.5(iii) there exists $a \in A$ such that $c^Z - a \in H_q = L_{Z[O_q]} \subseteq L_{\mathcal{F}}$. Thus $c^Z - a \in L_{\mathcal{F}} \cap A^s$ and again $\sigma(zc^Z) \in D_{\mathcal{F}}$.

Finally, suppose that $q = p$ and note that this case will occur since $Z[O_p]$ is the intersection of the minimal prime $z$-filters on $X$ that contain it [17, 14.12]. Let $b \in L_{\mathcal{F}}^{\mathrm{alg}}$. Then $Z(b) \in \mathcal{F}$ and so $Z(b) \cap N \in \mathcal{F}$. Since $p$ is not an isolated point in $X$, $Z(b) \cap N \neq \{p\}$ and so the non-empty set $Z(b) \cap N$ contains some point $x \in V \setminus \{p\}$. Then

$$\|zc^Z - b\| \geq \|(zc^Z - b) + \tilde{J}_x\| = \|zc^Z + \tilde{J}_x\| = f(x)\|c^Z + \tilde{J}_x\| = 1.$$

Since $zc^Z \in \tilde{J}_p$, it follows from Lemma 6.1 that

$$1 \geq \|\sigma(zc^Z) + D_{\mathcal{F}}\| = \|zc^Z + L_{\mathcal{F}}\| \geq 1$$

and hence $\|\sigma(zc^Z) + D_{\mathcal{F}}\| = 1$.

It now follows that the set of minimal closed primal ideals of $A^s/A$ which contain $D_{Z[O_p]}$ is a non-empty $\tau_s$-clopen subset of $\mathrm{MinPrimal}(A^s/A)$. □

Theorem 6.7 can be used to give an alternative proof of Theorem 6.5 in the case $A = C_0(X) \otimes K(H)$ where the locally compact Hausdorff space $X$ is $\sigma$-compact, first countable, and without isolated points. Theorem 6.7 shows that in this case $\mathrm{MinPrimal}(A^s/A)$ can be covered by an uncountable family of disjoint clopen subsets, and hence is not weakly Lindelof.

We conclude with an application to a very different class of C*-algebras.

**Theorem 6.8.** *Let $A$ be a $\sigma$-unital, quasi-standard C*-algebra. Set $X = \mathrm{Glimm}(A)$ and suppose that*
(i) *$X \cap \mathrm{Prim}(A)$ is dense in $X$ and $A/J_x$ is non-unital for all $x \in X$,*
(ii) *$X$ is an infinite F-space.*
*Then every Glimm ideal of $A^s/A$ is primal. Hence the relation $\sim$ of inseparability by disjoint open sets is an equivalence relation on $\mathrm{Prim}(A^s/A)$.*

*Proof.* Since $X$ is an F-space, the minimal prime $z$-filters on $X$ are of the form $Z[O^x]$ for $x \in \beta X$ [17, 7.15, 14.25]. Hence the minimal closed primal ideals of $M(A)$ are of the form $H_x = L_{Z[O^x]}$ $(x \in \beta X)$ by [11, Proposition 2.5 and Theorem 3.4]. Thus by Theorem 5.6, the minimal closed primal ideals of $A^s/A$ have the form $D_{Z[O^x]} = ((H_x \cap A^s) + A)/A$ for $x \in \beta X \setminus Y$ (where $Y$ is the set of P-points in $X$).

For distinct $x, y \in \beta X \setminus Y$, let $h \in C(\beta X)$ with $0 = h(x) \neq h(y)$. Set $f := h \circ \overline{\phi_A} \in C(\mathrm{Prim}(M(A)))$ and let $g = f|_{\mathrm{Prim}(A^s/A)}$. Then $g \in C^b(\mathrm{Prim}(A^s/A))$ and $g(R) = 0$ for $R \in \mathrm{hull}(\sigma(H_x \cap A^s))$ but $g(R) \neq 0$ for $R \in \mathrm{hull}(\sigma(H_y \cap A^s))$. Thus distinct minimal closed primal ideals of $A^s/A$ contain distinct Glimm ideals. Every closed primal ideal in a C*-algebra $B$ contains a unique Glimm ideal [6, Lemma 2.2], from which it follows that every Glimm ideal of $B$ is the intersection of the minimal closed primal ideals containing it. Thus we have shown that every Glimm ideal of $A^s/A$ is a minimal closed primal ideal; and from this it follows that the relation $\sim$ is an equivalence relation on $\mathrm{Prim}(A^s/A)$ [6, Lemma 3.1(i)]. □

Although the sets $\mathrm{MinPrimal}(A^s/A)$ and $\mathrm{Glimm}(A^s/A)$ coincide in the context of Theorem 6.8, the topologies on the two spaces are problematic. Various set-theoretic considerations come into play, such as the presence or absence of P-points in $X$. Although we may not be able to identify the topologies $\tau_s$ and $\tau_{cr}$, we do have enough information about them to show that if $X$ has no isolated points then they are not equal.

In preparation for this, we assume that $A$ satisfies the hypotheses of Theorem 6.8. Firstly, it follows from [6, Lemma 3.1(iii)] that the $\tau_s$-topology on $\mathrm{MinPrimal}(A^s/A)$ is finer than the complete regularization topology $\tau_{cr}$ on $\mathrm{Glimm}(A^s/A)$. Secondly, set $Y^c := \beta X \setminus Y$ where $Y$ is the set of P-points in $X$. It follows from [12, Theorem 5.2] that $Y^c = \psi(\mathrm{Prim}(A^s/A))$, where $\psi$ is the restriction to $\mathrm{Prim}(A^s/A)$ of the continuous map $\overline{\phi_A} : \mathrm{Prim}(M(A)) \to \beta X$. On the other hand, it follows from the proof of Theorem 6.8 that there is a bijection $\rho : Y^c \to \mathrm{Glimm}(A^s/A)$, given by

$$\rho(x) = D_{Z[O^x]} = \frac{(H_x \cap A^s) + A}{A} \qquad (x \in Y^c),$$

such that $\rho \circ \psi = \phi_{(A^s/A)}$, the complete regularisation map for the $C^*$-algebra $A^s/A$. As a subspace of the compact Hausdorff space $\beta X$, $Y^c$ is completely regular. It follows from the universal property for the complete regularisation that $\rho^{-1}$ is $\tau_{cr}$-continuous. Thirdly, the bijection $\rho$ induces from the topologies $\tau_s$ and $\tau_{cr}$ on $\mathrm{Glimm}(A^s/A)$ two topologies on $Y^c$ which we denote again by $\tau_s$ and $\tau_{cr}$ respectively. Thus, on $Y^c$, $\tau_s$ is finer than $\tau_{cr}$ which in turn is finer than the relative topology on $Y^c$ from $\beta X$.

**Theorem 6.9.** *Let $A$ be a $\sigma$-unital, quasi-standard $C^*$-algebra. Set $X = \mathrm{Glimm}(A)$ and suppose that*
(i) $X \cap \mathrm{Prim}(A)$ *is dense in $X$ and $A/J_x$ is non-unital for all $x \in X$,*
(ii) $X$ *is an F-space without isolated points.*
*Then $A^s/A$ is not quasi-standard.*

*Proof.* Let $V := X \setminus Y$ be the set of non-P-points of the locally compact space $X$. Thus $V$ is the union of the boundaries of the zero sets in $X$ and $V$ is contained in the subset $Y^c$ of $\beta X$ discussed above. Since a compact P-space is finite and $X$ has no isolated points, $V$ is dense in $X$. Let $U$ be an arbitrary non-empty, proper cozero set in X, let $Z := X \setminus U$ and let $c^Z \in A^s$ with the properties of Lemma 2.5. Then $\|c^Z\| = 1$ and, for $x \in Z$, $c^Z \in \tilde{J}_x$ and so by Lemma 6.1

$$\|\sigma(c^Z) + D_{Z[O_x]}\| = \|c^Z + H_x\|.$$

On the other hand, if $y \in U$ then

$$1 \geq \|c^Z + H_y\| \geq \|c^Z + \tilde{J}_y\| = 1,$$

so that $\|c^Z + H_y\| = 1$ and hence, by upper semi-continuity, $\|c^Z + H_x\| = 1$ for all $x$ in the boundary (in $X$) of Z. Thus, for $x$ in the boundary of $Z$, $\|\sigma(c^Z) + D_{Z[O_x]}\| = 1$.

Suppose next that $x \in \mathrm{int}(Z) \cap V$. Then $\|c^Z + \tilde{J}_y\| = 0$ for all $y$ in a neighbourhood of $x$ and hence $\|c^Z + H_x\| = 0$ by [7, Lemma 1.5]. Thus $c^Z \in H_x \cap A^s$ and hence $\sigma(c^Z) \in D_{Z[O_x]}$. Now suppose that $x \in U \cap V$. By Lemma 2.5(iii) there exists $a \in A$ such that $c^Z - a \in H_x \cap A^s$ and hence $\sigma(c^Z) \in D_{Z[O_x]}$. Bearing in mind that the $\tau_s$-topology on $\mathrm{MinPrimal}(A^s/A)$ is finer than the relative topology on $V$ from $X$, we see that the three sets consisting of $Z \cap V$, the boundary of $Z$, and $U \cap V$ are all $\tau_s$-clopen in $V$.

Now suppose that we can find subsets $U$ and $W$ of $X$ with the following properties: $U$ is a non-empty cozero set of $X$ such that $U$ is non-closed in $X$ but has compact closure, and $W$ is a zero set of $U$ such that the boundary (in $U$) of $W$ is non-compact. Set $T = (X \setminus U) \cup W$. Then $T$ is easily seen to be a zero set of $X$. Let $c^T \in A^s$ with the properties of Lemma 2.5. Since $X$ is locally compact, $X$ is open in $\beta X$. We have $U \cap V = U \cap Y^c$, which is relatively open in $Y^c$ (for the topology from $\beta X$) and hence $\tau_{cr}$-open in $Y^c$. On the other hand, since the closure of $U$ in $X$ is compact and hence closed in $\beta X$, it follows that the $\tau_{cr}$-closure of $U \cap V$ in $Y^c$ is contained in $V$. We now suppose, for a contradiction, that $U \cap V$ is $\tau_{cr}$-closed in $V$. Then $U \cap V$ is clopen in the $\tau_{cr}$ topology on $Y^c$. The characteristic function $\chi$ of the $\tau_{cr}$-clopen subset $\rho(U \cap V)$ of $\mathrm{Glimm}(A^s/A)$ induces a central projection $z \in M(A^s/A)$ such that

$$zd + D_{Z[O^x]} = \chi(\rho(x))(d + D_{Z[O^x]}) \qquad (d \in A^s/A,\ x \in Y^c).$$

Set $b := z\sigma(c^T) \in A^s/A$. Then the function $x \mapsto \|b + D_{Z[O_x]}\|$ $(x \in V)$ is upper semi-continuous for $\tau_{cr}$ on $V$. If $x$ is in the boundary of $W$ in $U$, then $x$ is in the boundary of $T$ in $X$ and hence $\|c^T + D_{Z[O_x]}\| = 1$ (as for $c^Z$ in the first part of the proof). Since

$x \in U \cap V$, $\|b + D_{Z[O_x]}\| = 1$. On the other hand, $\|b + D_{Z[O_x]}\| = 0$ for $x \in V \setminus U$. As the boundary of $W$ in $U$ is non-compact, its closure in $X$ meets $V \setminus U$, contradicting the $\tau_{cr}$ upper semi-continuity of the norm function. Thus if such $U$ and $W$ exist, we can conclude that $U \cap V$ is not closed in the $\tau_{cr}$ topology on $V$, whereas $U \cap V$ *is* closed in the $\tau_s$ topology on $V$ by the first part of the proof.

To complete the proof it remains to show that such sets $U$ and $W$ can be found. Let $x$ be any point in $V$. There exists a continuous function $f_1 : X \to [0,1]$ such that $x$ lies in the boundary of $Z(f_1)$. Let $f_2 : X \to [0,1]$ be a continuous function with compact support which is identically 1 on some neighbourhood of $x$. Set $f = f_1 f_2$ and $U = X \setminus Z(f)$. Then $\overline{U}$ is compact but $x \in \overline{U} \setminus U$. Since $U$ is open and $V$ is dense in $X$, there is a net $(v_\alpha)$ in $U \cap V$ with limit $x$. Then $f(v_\alpha) \to f(x) = 0$ and so there exists a sequence $(x_i)_{i \geq 1}$ of distinct points of $U \cap V$ such that $f(x_i) \to 0$. Let $C = \{x_i : i \geq 1\}$, a relatively discrete subset of $U$.

For each $i \geq 1$, let $U_i$ be a neighbourhood of $x_i$ in $U$ disjoint from $C \setminus \{x_i\}$ and let $g_i : U \to [0,1]$ be a continuous function supported in $U_i$ such that $x_i$ lies in the boundary (in $U$) of the zero set of $g_i$. [Since $x_i \in V$ there is a continuous function $f_i : X \to [0,1]$ such that $x_i$ lies in the boundary of $Z(f_i)$. Let $h_i = f_i|_U$. Since $U$ is open, $x_i$ lies in the boundary in $U$ of $Z(h_i)$. Then we may obtain $g_i$ by multiplying $h_i$ by a continuous function from $U$ to $[0,1]$ which is identically 1 on a neighbourhood of $x_i$ and supported in $U_i$.] Set $g = \sum_{i=1}^{\infty} g_i/2^i$. Then $g \in C_{\mathbb{R}}(U)$ and $C$ lies in the boundary in $U$ of the zero set of $g$. Suppose, for a contradiction, that the boundary in $U$ of $Z(g)$ is compact. Then there is a subnet of the sequence $(x_i)$ convergent to some point $u \in U$. Since $f$ is continuous, $f(u) = 0$, contradicting the fact that $u \in U$. Hence taking $W = Z(g)$ gives the required set. It follows that $U \cap V$ is not closed in the $\tau_{cr}$ topology on $V$, so the $\tau_s$ and the $\tau_{cr}$ topologies do not coincide on $V$, and $A^s/A$ is not quasi-standard [6, Theorem 3.3]. □

**Corollary 6.10.** *Under the hypothesis of Theorem 6.9, the corona algebra $M(A)/A$ is not quasi-standard and also $M(A^s/A)$ is not quasi-standard.*

*Proof.* If $B$ is a quasi-standard C*-algebra then every closed ideal of $B$ is quasi-standard [6, p. 356]. Hence it follows from Theorem 6.9 that neither $M(A)/A$ nor $M(A^s/A)$ is quasi-standard. □

It was shown in [12, Theorem 5.5] that if $A = C(\beta\mathbb{N}\setminus\mathbb{N}) \otimes K(H)$ then, under the Continuum Hypothesis, $M(A)/A$ is canonically isomorphic to a proper C*-subalgebra of $M(A^s/A)$.

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