DIRECTED TOPOLOGICAL COMPLEXITY OF SPHERES

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Abstract. We show that the directed topological complexity (as defined by E. Goubault [4]) of the directed n-sphere is 2, for all $n \geq 1$.

1. Introduction

Topological complexity is a numerical homotopy invariant, defined by Michael Farber [2, 3] as part of his topological study of the motion planning problem from robotics. Given a path-connected space X, let PX denote the space of all paths in X endowed with the compact open topology, and let $\pi : PX \to X \times X$ denote the endpoint fibration given by $\pi(\gamma) = (\gamma(0), \gamma(1))$. Viewing X as the configuration space of some mechanical system, one defines a *motion planner* on a subset $A \subseteq X \times X$ to be a local section of π on A, that is, a continuous map $\sigma : A \to PX$ such that $\pi \circ \sigma$ equals the inclusion of A into $X \times X$. Assuming X is an Euclidean Neighbourhood Retract (ENR), the topological complexity of X, denoted $\mathsf{TC}(X)$, is defined to be the smallest natural number k such that $X \times X$ admits a partition into k disjoint ENRs, each of which admits a motion planner.

Many basic properties of this invariant were established in the papers [2, 3], which continue to inspire a great deal of research by homotopy theorists (a snapshot of the current state-of-the-art can be found in the conference proceedings volume [6]). Here we simply mention that the topological complexity of spheres was calculated in [2]; it is given by

$$\mathsf{TC}(S^n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

In the recent preprint [4], Eric Goubault defined a variant of topological complexity for directed spaces. Recall that a directed space, or d-space, is a space X together with a distinguished class of paths in X called directed paths, satisfying certain axioms (full definitions will be given in Section 2). Partially ordered spaces give examples of d-spaces. The directed paths of a d-space form a subspace $\overrightarrow{P}X$ of PX. The endpoint fibration restricts to a map $\chi : \overrightarrow{P}X \to X \times X$, which is not surjective in general. Its image, denoted $\Gamma_X \subseteq X \times X$, is the set of $(x, y) \in X \times X$ such that there exists a directed path from x to y. A directed motion planner on a subset $A \subseteq \Gamma_X$ is defined to be a local section of χ on A. The directed topological complexity of the d-space X, denoted $\overrightarrow{TC}(X)$, is the smallest natural number k such that Γ_X admits a partition into k disjoint ENRs, each of which admits a directed motion planner.

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As remarked in the introduction to [4], the directed topological complexity seems more suited to studying the motion planning problem in the presence of control constraints on the movements of the various parts of the system. It was shown in [4] to be invariant under a suitable notion of directed homotopy equivalence, and a few simple examples were discussed. It remains to find useful upper and lower bounds for this invariant, and to compute its value for familiar d-spaces.

The contribution of this short note is to compute the directed topological complexity of directed spheres. For each $n \geq 1$ the directed sphere $\overrightarrow{S^n}$ is the directed space whose underlying topological space is the boundary ∂I^{n+1} of the $(n+1)$-dimensional unit cube, and whose directed paths are those paths which are non-decreasing in every coordinate.

Theorem. *The directed topological complexity of directed spheres is given by*

$$\overrightarrow{TC}(\overrightarrow{S^n}) = 2 \textit{ for all } n \geq 1.$$

This theorem will be proved in Section 3 below by exhibiting a partition of $\Gamma_{\overrightarrow{S^n}}$ into 2 disjoint ENRs with explicit motion planners.

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2. Preliminaries

Definition 2.1 (M. Grandis, [5])**.** A *directed space* or *d-space* is a pair $(X, \overrightarrow{P}X)$ consisting of a topological space X and a subspace $\overrightarrow{P}X \subseteq PX$ of the path space of X satisfying the following axioms:

- constant paths are in $\overrightarrow{P}X$;
- $\overrightarrow{P}X$ is closed under pre-composition with non-decreasing continuous maps $r : [0,1] \to [0,1]$;
- $\overrightarrow{P}X$ is closed under concatenation.

The paths in $\overrightarrow{P}X$ are called *directed paths* or *dipaths*, and the space $\overrightarrow{P}X$ is called the *dipath space*.

Examples of d-spaces include partially ordered spaces (where dipaths in $\overrightarrow{P}X$ consist of continuous order-preserving maps $\gamma : ([0,1], \leq) \to (X, \leq))$ and cubical sets. We can also view any topological space X as a d-space by taking $\overrightarrow{P}X = PX$. The dipath space is usually omitted from the notation for a d-space.

Definition 2.2 ([4])**.** Given a d-space X, let

$$\Gamma_X = \{(x,y) \in X \times X \mid \exists \gamma \in \overrightarrow{P}X \text{ such that } \gamma(0) = x, \gamma(1) = y\} \subseteq X \times X.$$

The *dipath space map* is given by

$$\chi : \overrightarrow{P}X \to \Gamma_X, \qquad \chi(\gamma) = \big(\gamma(0), \gamma(1)\big).$$

That is, the dipath space map is obtained from the classical endpoint fibration $\pi : PX \to X \times X$ by restriction of domain and codomain.

Definition 2.3 ([4])**.** Given a d-space X, its *directed topological complexity*, denoted $\overrightarrow{\mathsf{TC}}(X)$, is defined to be the smallest natural number k such that there exists a partition $\Gamma_X = A_1 \sqcup \cdots \sqcup A_k$ into disjoint ENRs, each of which admits a continuous map $\sigma_i : A_i \to \overrightarrow{P}X$ such that $\chi \circ \sigma_i = \mathrm{incl} : A_i \hookrightarrow \Gamma_X$.

Remarks 2.4. The dipath space map is not a fibration, in general. One can easily imagine directed spaces X for which the homotopy type of the fibre $\overrightarrow{P}X(x,y)$ is not constant on the path components of Γ_X. Related to this is the fact that, unlike in the classical case of $\mathsf{TC}(X)$, the above definition does not coincide with the alternative definition using open (or closed) covers. Both of these remarks are due to E. Goubault.

Note that we are using the unreduced version of $\overrightarrow{\mathsf{TC}}$, as in the article [4].

A notion of *dihomotopy equivalence* was defined in [4, Definition 3], and it was shown in [4, Lemma 6] that if X and Y are dihomotopy equivalent d-spaces then $\overrightarrow{\mathsf{TC}}(X) = \overrightarrow{\mathsf{TC}}(Y)$. Furthermore, a notion of *dicontractibility* for d-spaces was outlined in [4, Definition 4], and [4, Theorem 1] asserts that a d-space X that is contractible in the classical sense has $\overrightarrow{\mathsf{TC}}(X) = 1$ if and only if X is dicontractible. Here we will only require the following weaker assertion.

Lemma 2.5. *Let X be a d-space for which $\overrightarrow{\mathsf{TC}}(X) = 1$. Then for all $(x,y) \in \Gamma_X$, the corresponding fibre $\overrightarrow{P}X(x,y)$ of the dipath space map is contractible.*

Proof. We reproduce the relevant part of the proof of [4, Theorem 1]. Suppose $\overrightarrow{\mathsf{TC}}(X) = 1$, and let $\sigma : \Gamma_X \to \overrightarrow{P}X$ be a global section of $\chi : \overrightarrow{P}X \to \Gamma_X$. Given $(x,y) \in \Gamma_X$, let $f : \{\sigma(x,y)\} \to \overrightarrow{P}X(x,y)$ and $g : \overrightarrow{P}X(x,y) \to \{\sigma(x,y)\}$ denote the inclusion and constant maps, respectively. Clearly $g \circ f = \mathrm{Id}_{\{\sigma(x,y)\}}$, so to prove the lemma it suffices to give a homotopy $H : \overrightarrow{P}X(x,y) \times I \to \overrightarrow{P}X(x,y)$ from $f \circ g$ to $\mathrm{Id}_{\overrightarrow{P}X(x,y)}$. Such a homotopy is defined explicitly by setting

$$H(\gamma,t)(s) = \begin{cases} \gamma(s) & \text{for } 0 \le s \le \frac{t}{2} \\ \sigma\left(\gamma\left(\frac{t}{2}\right), \gamma\left(1-\frac{t}{2}\right)\right)\left(\frac{s-\frac{t}{2}}{1-t}\right) & \text{for } \frac{t}{2} \le s \le 1-\frac{t}{2} \\ \gamma(s) & \text{for } 1-\frac{t}{2} \le s \le 1 \end{cases}$$

for all $t, s \in [0,1]$. $\square$

3. Directed topological complexity of directed spheres

Definition 3.1. Let $n \ge 1$ be a natural number. The *directed n-sphere*, denoted $\overrightarrow{S^n}$, is the d-space whose underlying space is the boundary ∂I^{n+1} of the unit cube $I^{n+1} = [0,1]^{n+1} \subseteq \mathbb{R}^{n+1}$, and whose dipaths are those paths which are non-decreasing in each coordinate.

We now fix some notation, most of which is borrowed from [1, Section 6]. A point $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be denoted $\mathbf{x} = x_0 \cdots x_n$ for brevity. We use $-$ to denote an arbitrary element of $(0,1)$. Therefore we may indicate an arbitrary point in ∂I^{n+1} by a string $x_0 \cdots x_n$ where each $x_i \in \{0, -, 1\}$, and at least one $x_i \in \{0,1\}$.

For example, $--0$ denotes an arbitrary point in the interior of the bottom $(z=0)$ face of ∂I^3, while -11 denotes a point on the interior of the top $(z=1)$, back $(y=1)$ edge. The point $---$ is not in ∂I^3.

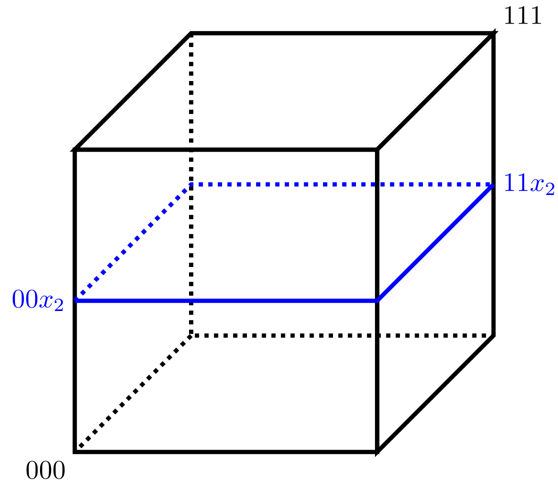

Figure 1. Directed paths in $\overrightarrow{S^2}$ from $00x_2$ to $11x_2$ must remain in the blue square, illustrating the homeomorphism $\overrightarrow{P}\overrightarrow{S^2}(00x_2, 11x_2) \cong \overrightarrow{P}\overrightarrow{S^1}(00, 11)$.

With these notations, if $(x_0 \cdots x_n, y_0 \cdots y_n) \in \Gamma_{\overrightarrow{S^n}}$ then $x_i \leq y_i$ for $i = 0, \ldots, n$, but the converse does not hold. For example, any pair of the form $(0 - -, 1 - -)$ is not in $\Gamma_{\overrightarrow{S^2}}$.

We are now ready to prove our main result, restated here for convenience.

Theorem. *The directed topological complexity of directed spheres is given by*
$$\overrightarrow{TC}(\overrightarrow{S^n}) = 2 \textit{ for all } n \geq 1.$$

Proof. To see that $\overrightarrow{TC}(\overrightarrow{S^n}) > 1$, it suffices by Lemma 2.5 to find $(\mathbf{x}, \mathbf{y}) \in \Gamma_{\overrightarrow{S^n}}$ such that $\overrightarrow{P}\overrightarrow{S^n}(\mathbf{x}, \mathbf{y})$ is not contractible. Fix $x_2, \ldots, x_n \in (0, 1)$. It is clear that
$$\overrightarrow{P}\overrightarrow{S^n}(00x_2 \cdots x_n, 11x_2 \cdots x_n) \cong \overrightarrow{P}\overrightarrow{S^1}(00, 11),$$
and that the latter space is disconnected (since not all dipaths from 00 to 11 are dihomotopic), see Figure 1. In particular, $\overrightarrow{P}\overrightarrow{S^n}(00x_2 \cdots x_n, 11x_2 \cdots x_n)$ is not contractible, hence $\overrightarrow{\mathsf{TC}}(\overrightarrow{S^n}) > 1$.

To prove that $\overrightarrow{\mathsf{TC}}(\overrightarrow{S^n}) \leq 2$, we will exhibit a partition $\Gamma_{\overrightarrow{S^n}} = A_1 \sqcup A_2$ into two disjoint ENRs, each equipped with a continuous directed motion planner $\sigma_i : A_i \to \overrightarrow{P}(\overrightarrow{S^n})$.

Consider the d-space $\overrightarrow{\mathbb{R}^{n+1}}$, where the dipaths are non-decreasing in each coordinate. Here we have $\overrightarrow{\mathsf{TC}}(\overrightarrow{\mathbb{R}^{n+1}}) = 1$, for we can describe a directed motion planner $\widetilde{\sigma_1}$ on
$$\Gamma_{\overrightarrow{\mathbb{R}^{n+1}}} = \{(x_0 \cdots x_n, y_0 \cdots y_n) \mid x_i \leq y_i \text{ for all } i = 0, \ldots, n\}$$
by first increasing x_0 to y_0, then increasing x_1 to y_1, and so on, finally increasing x_n to y_n. It is not difficult to write a formula for $\widetilde{\sigma_1}$, and check that is it continuous. Similarly, we can define a second motion planner $\widetilde{\sigma_2}$ which first increases x_n to y_n, then increases x_{n-1} to y_{n-1}, and so on, finally increasing x_0 to y_0.

For $i = 1, 2$, let B_i be the set of pairs $(\mathbf{x}, \mathbf{y})$ in $\Gamma_{\overrightarrow{S^n}} \subseteq \Gamma_{\overrightarrow{\mathbb{R}^{n+1}}}$ such that the path $\widetilde{\sigma_i}(\mathbf{x}, \mathbf{y})$ has image contained in ∂I^{n+1}. The restriction $\widetilde{\sigma_i}|_{B_i} : B_i \to \overrightarrow{P}(\overrightarrow{S^n})$ is clearly continuous, and is a directed motion planner on B_i.

We will show that $B_1 \cup B_2 = \Gamma_{\overrightarrow{S^n}}$, and that both B_1 and its complement $U_1 := \Gamma_{\overrightarrow{S^n}} \setminus B_1$ are ENRs. Hence we may set $A_1 = B_1$ and $A_2 = U_1 \subseteq B_2$ to obtain a cover by disjoint ENRs with motion planners $\sigma_i = \widetilde{\sigma_i}|_{A_i}$, as required.

The sets B_1 and B_2 are best understood in terms of their complements U_1 and $U_2 := \Gamma_{\overrightarrow{S^n}} \setminus B_2$, and in fact we will show that $U_1 \cap U_2 = \varnothing$.

Observe that U_1 is the set of pairs $(\mathbf{x}, \mathbf{y}) \in \Gamma_{\overrightarrow{S^n}}$ such that $\widetilde{\sigma_1}(\mathbf{x}, \mathbf{y})$ enters the interior of the cube, and this can happen upon increasing any of the $n+1$ coordinates. Thus an element $(x_0 \cdots x_n, y_0 \cdots y_n) \in U_1$ falls into one of the following cases:

(1) $x_0 < y_0$ and $x_1 = \cdots = x_n = -$;
(2) For some $j \in \{1, \ldots, n-1\}$ we have
$x_j < y_j$ and $y_0 = \cdots = y_{j-1} = x_{j+1} = \cdots = x_n = -$;
(3) $y_0 = \cdots = y_{n-1} = -$ and $x_n < y_n$.

Similarly, an element $(x_0 \cdots x_n, y_0 \cdots y_n) \in U_2$ falls into one of the following cases:

(A) $x_0 = \cdots = x_{n-1} = -$ and $x_n < y_n$;
(B) For some $k \in \{1, \ldots, n-1\}$ we have
$x_k < y_k$ and $x_0 = \cdots = x_{k-1} = y_{k+1} = \cdots = y_n = -$;
(C) $x_0 < y_0$ and $y_1 = \cdots = y_n = -$.

A case-by-case analysis will show that an element $(\mathbf{x}, \mathbf{y}) = (x_0 \cdots x_n, y_0 \cdots y_n) \in U_1$ cannot be in U_2.

Suppose first that $(\mathbf{x}, \mathbf{y})$ is in case (1). Since $x_0 \cdots x_n \in \partial I^{n+1}$ and $x_0 < y_0$, we must have $x_0 = 0$, and therefore we cannot be in either case (A) or case (B). If we are in case (C), then since $y_0 \cdots y_n \in \partial I^{n+1}$ it follows that $y_0 = 1$, and that in fact $(\mathbf{x}, \mathbf{y}) = (0 - \cdots -, 1 - \cdots -)$. This element is not in $\Gamma_{\overrightarrow{S^n}}$ (since there are no directed paths from the interior of a face to the interior of the opposite face), a contradiction.

Next suppose $(\mathbf{x}, \mathbf{y}) \in U_1$ is in case (2). Then $y_0 = x_n = -$, and so we cannot be in either case (A) (which would give $\mathbf{x} = - \cdots -$) or case (C) (which would give $\mathbf{y} = - \cdots -$). If we are in case (B), then there exist $j, k \in \{1, \ldots, n-1\}$ such that

$$\begin{aligned}(\mathbf{x}, \mathbf{y}) &= (x_0 \cdots x_j - \cdots -, - \cdots - y_j \cdots y_n) \\ &= (- \cdots - x_k \cdots x_n, y_0 \cdots y_k - \cdots -).\end{aligned}$$

We observe that if $j < k$ then $\mathbf{x} = - \cdots -$, while if $j > k$ then $\mathbf{y} = - \cdots -$. Hence $j = k$, and $(\mathbf{x}, \mathbf{y}) = (- \cdots - x_j \cdots -, - \cdots - y_j - \cdots -)$ where $x_j < y_j$. Since $\mathbf{x}$ and $\mathbf{y}$ are in ∂I^{n+1}, we must have $x_j = 0 < 1 = y_j$. This gives $(\mathbf{x}, \mathbf{y}) = (- \cdots - 0 - \cdots -, - \cdots - 1 - \cdots -)$ which is not in $\Gamma_{\overrightarrow{S^n}}$, a contradiction.

Now suppose $(\mathbf{x}, \mathbf{y}) \in U_1$ is in case (3). Since $y_0 \cdots y_n \in \partial I^{n+1}$ and $x_n < y_n$, we conclude that $y_n = 1$. Hence we are not in cases (B) or (C). In case (A), since $x_0 \cdots x_n \in \partial I^{n+1}$ it follows that $x_n = 0$, and so $(\mathbf{x}, \mathbf{y}) = (- \cdots - 0, - \cdots - 1)$ which is not in $\Gamma_{\overrightarrow{S^n}}$, a contradiction.

Thus $U_1 \cap U_2 = \varnothing$, and $B_1 \cup B_2 = \Gamma_{\overrightarrow{S^n}}$. It only remains to observe that both B_1 and U_1 are semi-algebraic subsets of $\mathbb{R}^{2n+2}$ (they are the solution sets of a finite number of linear equalities and inequalities) hence are ENRs. $\square$

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