A HYPERFINITE FACTOR WHICH IS NOT AN INJECTIVE C*-ALGEBRA

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Abstract

We exhibit a wild monotone complete C*-algebra which is a hyperfinite factor but is not an injective C*-algebra.

1 INTRODUCTION

A von Neumann factor is hyperfinite if it is generated by an increasing sequence of finite dimensional matrix algebras. It is straightforward to show that such a factor is an injective C^*-algebra. See [14], Chapter XVI Corollary 1.8 or [8].

A much harder result, due to Connes [2], is the converse. If a (small) von Neumann factor is an injective C^*-algebra then it is hyperfinite.

Let M be a C*-algebra with self-adjoint part M_{sa}. If each upper bounded, upward directed subset of M_{sa} has a least upper bound then M is said to be *monotone complete.* (For a detailed account of monotone complete C*-algebras, see [11]). Every von Neumann algebra is monotone complete but the converse is false. However it seemed plausible to conjecture that if a monotone complete factor is hyperfinite, in a suitable sense, then it is injective. If this could be established for a particular monotone complete factor ([13] and [4]) then this would imply a positive solution of the Marzewski problem [18]. (See also [15, 1]). More precisely, Banach-Tarski paradoxical decompositions can be achieved using pieces which are measurable with respect to the Baire Property. In an impressive tour de force, Dougherty and Foreman [3] obtained this conclusion by completely different methods. This made the injectivity of the Takenouchi-Dyer factor seem even more plausible. But this is false.

We exhibit a monotone complete C*-algebra which is a hyperfinite factor but is not injective. See Theorem 10.

This can be deduced by applying some delicate and intricate arguments of Hjorth and Kechris [7]. Here we use a different approach which can be applied more generally. We give an argument which we hope readers will find transparent; key tools used include Theorem 3.4 [12] and our Lemma 1, (on properties of the Free Group, $\mathbb{F}_2$, and equivariant linear maps).

2 PRELIMINARIES AND BACKGROUND

A commutative unital C*-algebra $C(E)$ is monotone complete if and only if the compact Hausdorff space E is extremally disconnected. In other words, the closure of each open subset of E is clopen.

Let T be an Hausdorff topological space. Then T is said to be *perfect* if it has no isolated points. Furthermore T is a *Polish space* if it is homeomorphic to a complete separable metric space. Let $B(T)$ be the (commutative) algebra of bounded complex-valued Borel functions on a perfect Polish space T. Let $M(T)$ be the ideal of those f in $B(T)$ for which $\{t \in T : f(t) \neq 0\}$ is meagre. Then $B(T)/M(T)$ is isomorphic to $B(\mathbb{R})/M(\mathbb{R})$. This quotient algebra is known as the *Dixmier algebra*. It is monotone complete; we denote it by $C(S)$. The compact extremally disconnected space S has no isolated points but has a countable dense subset. $C(S)$ is not a von Neumann algebra because it has no normal states [11].

Let B be any commutative unital C*-algebra. Then $B \cong C(K)$, where K is compact Hausdorff. Let θ be a homeomorphism of K onto itself. Define h_θ on $C(K)$ by $h_\theta(f) = f \circ \theta$. Then $\theta \to h_\theta$ is a group anti-isomorphism from $Homeo(K)$, (the group of all homeomorphisms of K onto K), onto $Auto(C(K))$, (the group of all $*$-automomorphisms of $C(K)$). It follows that $\theta \to h_\theta^{-1}$ is an isomorphism of $Homeo(K)$ onto $Auto(C(K))$.

Throughout this paper, G will be a countable group. Let X be a perfect Polish space or a dense G_δ subset of S. (So the Baire category theorem is valid for X.) Then an action of G on X is a homomorphism α from G into the group of homeomorphisms of X onto X. Much of what follows is valid for more general extremally disconnected spaces than S but we shall focus on the Dixmier algebra, $C(S)$.

Let Z be a G-invariant subset of X. Then the action α is said to be *topologically free* over Z, if whenever $g \in G$ and g is not the identity, then α_g has no fxed points in Z.

The action α is said to be *(generically) free*, if whenever $g \in G$ and g is not the identity, then the set of fixed points of α_g is a closed nowhere dense subset of X. When this occurs, there is a, G-invariant, dense G_δ subset $Z \subset X$, such that the action α is topologically free over Z.

The action α is said to be *(generically) ergodic* if, for some $x_0 \in X$, the orbit $\{\alpha_g(x_0) : g \in G\}$ is dense in X. When this occurs, it follows from Lemma 1.1 [12] that there is a, G-invariant, dense G_δ subset $Y \subset X$ such that for every $y \in Y$, the orbit $\{\alpha_g(y) : g \in G\}$ is dense in Y.

Let A be any commutative monotone complete C*-algebra. Let Γ be a countable group of $*$-automorphisms of A. So $A \cong C(E)$, where E is compact Hausdorff and extremally disconected.

We recall that a $*$-automorphism, h of A is said to be *properly outer* if there does not exist a nonzero projection e in A such that the restriction of h to eA is the identity map. The action of Γ on A is said to be *free* if every element of Γ, other than the identity, is a properly outer automorphism. The action of Γ on A is said to be *ergodic* if, given a projection p, $g(p) = p$ for all g implies that $p = 0$ or $p = 1$.

Let Γ be a countable infinite group of $*$-automorphisms of $C(S)$ which acts freely and ergodically. There is a corresponding monotone cross-product C*-algebra, $M(C(S), \Gamma)$. (For monotone cross-products see [9, 10, 5, 11, 12, 13]). This algebra is a $TypeIII$ factor which contains the Dixmier algebra as a maximal abelian $*$-subalgebra and hence is not a von Neumann algebra [9, 10]. See also [16, 17]. By Theorem 3.4 [12] this algebra is unique. Every free, ergodic action of an infinite countable group G on $C(S)$ gives rise to the same factor. By [12] this algebra is hyperfinite. It corresponds, modulo meagre sets, to a canonical hyperfinite Borel equivalence relation on a perfect Polish space. We shall show that it is not injective.

The strategy is as follows. Let $A = C(S)$ and let B be a commutative monotone complete C*-algebra with A a subalgebra of B. Let G be a (countable) group of $*$-automorphisms of B which are also automorphisms of A. Then, by Proposition 2.3 [18], if $M(A, G)$ is injective there exists a G-equivariant, positive linear projection Φ of B onto A. On putting $B = \ell^\infty(G)$ and taking G to be the free group on two generators, it will follow from our technical lemma, Lemma 1, that Φ vanishes on B. But this is impossible since $\Phi(1) = 1$. So this contradiction shows that the hyperfinite factor, $M(C(S), G)$ is not injective.

3 EQUIVARIANT LINEAR MAPS OVER THE FREE GROUP

Let G be a countable group. Let $\ell^\infty(G)$ be the commutative C^*-algebra of all complex valued, bounded sequences on G. Let γ^g be the automorphism of $\ell^\infty(G)$ defined by

$$(\gamma^g f)(h) = f(g^{-1}h) \text{ for each } g,\ h \in G \text{ and } f \in \ell^\infty(G).$$

Let $\mathbb{F}_2$ be the free group on two generators a and b. Let $< a >$ be the subgroup of $\mathbb{F}_2$ generated by a (so $< a > \cong \mathbb{Z}$). Let e be the empty word. For each reduced word x, let $W(x)$ be the set of reduced words beginning with x.

There are many ways known of obtaining paradoxical decompositions of $\mathbb{F}_2$. Wagon [15] gives a lucid and elegant exposition.See in particular his Theorem 4.2 and the related discussions. For our purposes, it is convenient to use the following decomposition.

For each $i \in \mathbb{Z}$ we put

$$H_i = \bigcup \{W(a^i b^{i_1}) : i_1 \neq 0\}.$$

Then we have $H_i \cap H_j = \varnothing$ if $i \neq j$. Moreover, we have

$$\mathbb{F}_2 = < a > \cup \bigcup_{i \in \mathbb{Z}} H_i$$

where each $H_i \cap < a > = \varnothing$.

Observe that the map $\lambda_a : x \mapsto ax$ ($x \in \mathbb{F}_2$) satisfies

$$\lambda_a H_i = H_{i+1} \text{ for each } i \in \mathbb{Z}.$$

Also the map $\lambda_b : y \mapsto by$ ($y \in \mathbb{F}_2$) satisfies $\lambda_b H_i \subset H_0$ ($i \neq 0$) and $\lambda_b < a > \subset H_0$.

Observe that, for each $E \subset \mathbb{F}_2$, $\chi_E \in \ell^\infty(\mathbb{F}_2)$ and for each $g \in \mathbb{F}_2$,

$$(\gamma^g \chi_E)(h) = \chi_E(g^{-1}h) = \chi_{\lambda_g E}(h) \text{ for each } h.$$

Let T be a Hausdorff topological space with no isolated points. Let $g \to \alpha_g$ be a group homomorphism from a countable group G into $Homeo(T)$, the group of homeomorphisms of T. Let t_0 be a point such that $\{\alpha_g(t_0) : g \in G\}$ is a dense orbit. Also suppose the orbit is free, that is, the map

$$g \to \alpha_g(t_0)$$

is a bijection from G onto the orbit.

Let $\alpha^g(f)(t) = f(\alpha_{g^{-1}}t)$. Then $g \to \alpha^g$ is a group isomorphism of G into $Aut(C_B(T))$, the group of all automorphisms of $C_B(T)$, the C^*-algebra of all bounded, complex-valued continuous functions on T.

There is a natural embedding J of $C_B(T)$ into $\ell^\infty(G)$ given by

$$Jf = (f(\alpha_h(t_0)))_{h \in G}.$$

Then γ^g restricted to $J[C_B(T)]$ coincides with the automorphism of $J[C_B(T)]$ induced by α^g.

Indeed, for $f \in C_B(T)$ and $g, h \in G$, we have

$$(\gamma^g Jf)(h) = (Jf)(g^{-1}h) = f(\alpha_{g^{-1}h}t_0) = f(\alpha_{g^{-1}}\alpha_h t_0) = (\alpha^g(f))(\alpha_h t_0) = J\alpha^g(f)(h),$$

which implies $\gamma^g J = J\alpha^g$ for all $g \in G$.

Now specialise by putting $G = \mathbb{F}_2$ and require T to be compact and extremally disconnected. Then $C_B(T) = C(T)$. Extremal disconnectedness implies that whenever D is a dense subset of T then the Stone-Čech compactification of D can be identified with T. (See Theorem 6.2.7 in [11]) Each element of $J[C_B(T)]$ restricts to a bounded continuous function defined on the orbit $(\alpha_h(t_0))_{h \in G}$, equipped with the relative topology induced by T. Conversely, by Stone-Čech, each such function is the restriction of a unique continuous function in $C(T)$.

Lemma 1 *Let ϕ be a positive linear map from $\ell^\infty(F_2)$ to $J[C(T)]$ such that*

$$\gamma^g \circ \phi = \phi \circ \gamma^g \text{ for each } g \in \mathbb{F}_2.$$

Further suppose that the sub orbit $\{\alpha_{a^n} t_0 : n \in \mathbb{Z}\} =< a > [t_0]$ is dense in T. We recall that T has no isolated points. Then the linear map ϕ is identically zero.

Proof. We have

$$\gamma^a \phi(\chi_{H_i}) = \phi(\gamma^a \chi_{H_i}) = \phi(\chi_{\lambda_a H_i}) = \phi(\chi_{H_{i+1}}) \text{ for each } i \in \mathbb{Z}.$$

This implies that

$$(\gamma^a)^n \phi(\chi_{H_i}) = \phi(\chi_{H_{i+n}}) \text{ for all } i, n \in \mathbb{Z}.$$

We have

$$\phi(\chi_{\mathbb{F}_2})(t_0) \geq \phi\left(\sum_{i=-m}^{m} (\chi_{H_i})\right)(t_0) = \sum_{i=-m}^{m} \phi(\chi_{H_i})(t_0)$$

$$\geq \sum_{i=-m}^{m} (\gamma^a)^i (\phi(\chi_{H_0}))(t_0) \text{ for each } m \in \mathbb{N}.$$

So $(\gamma^a)^n(\phi(\chi_{H_0}))(t_0) \to 0$ as $|n| \to \infty$. But

$$(\gamma^a)^n(\phi(\chi_{H_0}))(t_0) = \phi(\chi_{H_0})(a^{-n}t_0) \text{ for each } n \in \mathbb{Z}.$$

So $\phi(\chi_{H_0})(a^{-n}t_0) \to 0$ as $|n| \to \infty$. So, for any given positive number ϵ, there is an $m_0 \in \mathbb{N}$ such that

$$|(\gamma^a)^n(\phi(\chi_{H_0}))(t_0)| = |\phi(\chi_{H_0})(a^{-n}t_0)| < \epsilon \text{ for all } n \text{ with } |n| \geq m_0.$$

Since T has no isolated points, $\{a^n(t_0) : |n| \geq m_0\}$ is dense in T. It follows that $|\phi(\chi_{H_0})(a^n(t_0))| \leq \epsilon$ for all n. By applying the Stone-Čech compactification theorem, $\phi(\chi_{H_0})$ has a unique extension to a continuous function on T which is norm bounded by ϵ for all $\epsilon > 0$. So $\phi(\chi_{H_0}) = 0$. By positivity, if $S \subset H_0$ then $\phi(\chi_S) = 0$. Also $0 = \gamma^g \phi(\chi_S) = \phi(\gamma^g(\chi_S)) = \phi(\chi_{\lambda_g S})$. We now recall $\lambda_b H_i \subset H_0$ $(i \neq 0)$. So

$$\lambda_b \left(\bigcup_{i \neq 0} H_i\right) \subset H_0.$$

So ϕ vanishes on the characteristic function $\chi_{\bigcup_{i\neq 0} H_i}$. Similarly, $\lambda_b < a > \subset H_0$ implies that ϕ vanishes on the characteristic function of $< a >= \{a^n : n \in \mathbb{Z}\}$. Hence ϕ is identically zero. ■

4 CONSTRUCTING FREE GROUP ACTIONS WITH DENSE SUB-ORBITS

In this section we construct actions of the free group, $\mathbb{F}_2$, that we shall need when applying Lemma 1.

First we shall consider the Cantor space over $\mathbb{F}_2$ with the shift action induced by $\mathbb{F}_2$. See [6]. We equip, $2^{\mathbb{F}_2}$ with the product topology. Then it is a compact Hausdorff space which is a perfect Polish space, that is, it is homeomorphic to a complete seperable metric space with no isolated points. (In fact it is homeomorphic to the Cantor set.) The action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is defined, for each $g \in \mathbb{F}_2$ by

$$(\alpha_g(x))(h) = x(g^{-1}h) \text{ for } x \in 2^{\mathbb{F}_2} \text{ and } h \in \mathbb{F}_2.$$

When there is no risk of ambiguity, we put $g \cdot x = \alpha_g(x)$. So

$$(g \cdot x)(h) = x(g^{-1}h) \text{ for } x \in 2^{\mathbb{F}_2} \text{ and } h \in \mathbb{F}_2.$$

Let us recall that for each $g \in \mathbb{F}_2$, this map $2^{\mathbb{F}_2} \ni x \mapsto g \cdot x \in 2^{\mathbb{F}_2}$ is a *homeomorphism* from $2^{\mathbb{F}_2}$ onto itself and the action $\mathbb{F}_2 \ni g \mapsto \alpha_g \in Homeo(2^{\mathbb{F}_2})$ is a group homomorphism such that for $g_1, g_2 \in \mathbb{F}_2$ and $x \in 2^{\mathbb{F}_2}$,

$$((g_1 g_2) \cdot x)(h) = x(g_2^{-1} g_1^{-1} h) = (g_2 \cdot x)(g_1^{-1} h) = [g_1 \cdot (g_2 \cdot x)](h) \text{ for each } h \in \mathbb{F}_2.$$

Take $g \in \mathbb{F}_2 \setminus \{e\}$ and let

$$F(g) = \{x \in 2^{\mathbb{F}_2} : g \cdot x = x\}.$$

Since α_g is a homeomorphism, $F(g)$ is closed. So its compliment

$$O(g) = \{x \in 2^{\mathbb{F}_2} : g \cdot x \neq x\}$$

is open. Since $\mathbb{F}_2$ acts on $2^{\mathbb{F}_2}$ as the shift action, $O(g) \neq \emptyset$.

Moreover, we have $y \in F(hgh^{-1})$ iff $hgh^{-1} \cdot y = y$ iff $g \cdot (h^{-1} \cdot y) = h^{-1} \cdot y$ iff $h^{-1} \cdot y \in F(g)$ iff $y \in h \cdot F(g)$. So

$$h \cdot O(g) = O(hgh^{-1}) \text{ for each } h \in \mathbb{F}_2.$$

We claim $O(g)$ is dense. To do this, take a non-empty open subset O. We may assume that O is a *basic open subset* and so we may assume

$$O = \{x \in 2^{\mathbb{F}_2} : x(h) = \iota_h \text{ if } h \in F\}$$

for some *non-empty finite subset* F of $\mathbb{F}_2$ and $\iota_h \in \{0,1\}$ for $h \in F$. Since $F \cup gF$ is *finite*, there is $h_0 \in \mathbb{F}_2$ such that $h_0 \notin F$ and $g^{-1}h_0 \notin F$. Let

$$x(h) = \begin{cases} 0 & \text{if } h = h_0 \\ 1 & \text{if } h = g^{-1}h_0 \\ \iota_h & \text{if } h \in F \\ 0 & \text{if } h \in \mathbb{F}_2 \setminus (F \cup \{h_0, g^{-1}h_0\}). \end{cases}$$

Clearly $x \in O$. On the other hand, we have

$$(g \cdot x)(h_0) = x(g^{-1}h_0) = 1 \text{ but } x(h_0) = 0$$

and so $g \cdot x \neq x$, that is, $x \in O(g)$. Hence it follows that $O \cap O(g) \neq \emptyset$ for all non-empty open subsets O. So, $O(g)$ is dense in $2^{\mathbb{F}_2}$. By the Baire category theorem, $\bigcap_{g \in \mathbb{F}_2 \setminus \{e\}} O(g)$ is also dense in $2^{\mathbb{F}_2}$.

Since for each $g, h \in \mathbb{F}_2 \setminus \{e\}$, $h \cdot O(g) = O(hgh^{-1})$, $\bigcap_{g \in \mathbb{F}_2 \setminus \{e\}} O(g)$ is $\mathbb{F}_2$-*invariant.*

Let $X = \bigcap_{g \in \mathbb{F}_2 \setminus \{e\}} O(g)$. Then X is a dense G_δ subset of the perfect Polish space $2^{\mathbb{F}_2}$. So X is a perfect Polish space. Also X is an $\mathbb{F}_2$-invariant subset of $2^{\mathbb{F}_2}$ such that for any $x \in X$, $g \cdot x \neq x$ for every $g \in \mathbb{F}_2 \setminus \{e\}$. In other words, the action of $\mathbb{F}_2$ is *topologically free* on X.

Lemma 2 *There exists $x_0 \in 2^{\mathbb{F}_2}$ such that the orbit $\{\alpha_g(x_0) : g \in \mathbb{F}_2\}$ is free and the suborbit $\{a^n \cdot x_0 : n \in \mathbb{Z}\}$ is dense in $2^{\mathbb{F}_2}$.*

Proof. Since the action on X is topologically free, for any $x_1 \in X$, the orbit $\{g \cdot x_1 : g \in \mathbb{F}_2\}$is free.

Take any pair of basic clopen subsets of $2^{\mathbb{F}_2}$. Call them W_1 and W_2 with

$$W_1 = \{x \in 2^{\mathbb{F}_2} : x(g) = i_g \text{ for } g \in S_1\} \text{ and } W_2 = \{y \in 2^{\mathbb{F}_2} : y(g) = j_g \text{ for } g \in S_2\}$$

where S_1 and S_2 are non-empty finite subsets of $\mathbb{F}_2$ and $i_g \in \{0,1\}$ for $g \in S_1$ and $j_g \in \{0,1\}$ for $g \in S_2$. Since $S_2 S_1^{-1}$ is a finite set and $< a >= \{a^m : m \in \mathbb{Z}\}$ (subgroup of $\mathbb{F}_2$) is *infinite*, there exists $k_0 \in \mathbb{N}$ such that $a^n \notin S_2 S_1^{-1}$ for $|n| > k_0$, that is, $a^n S_1 \cap S_2 = \emptyset$.

Let $y \in 2^{\mathbb{F}_2}$ be defined by

$$y(h) = \begin{cases} i_{a^{-n}h} & \text{when } h \in a^n S_1 \\ j_h & \text{when } h \in S_2 \\ 1 & \text{otherwise} \end{cases}.$$

Then $y \in W_2$. Suppose $h \in S_1$. Then $a^n h \in a^n S_1$. Since

$$(a^{-n} \cdot y)(h) = y(a^n h) = i_{a^{-n}a^n h} = i_h,$$

it follows that $a^{-n} \cdot y \in W_1$, which implies that $y \in a^n \cdot W_1$.

Hence $(a^n \cdot W_1) \cap W_2$ is a *non-empty clopen subset* of $2^{\mathbb{F}_2}$ for $|n| > k_0$.

This definition of y makes sense because $a^n S_1 \cap S_2 = \emptyset$. Since X is a dense subset of $2^{\mathbb{F}_2}$ it follows that

$$(a^n \cdot (X \cap W_1)) \cap (X \cap W_2) = X \cap ((a^n \cdot W_1) \cap W_2) \neq \emptyset.$$

Let $\mathcal{W}$ be the colllection of all sets of the form W_1. Then $\mathcal{W}$ is a countable base for the topology of $2^{\mathbb{F}_2}$. We remark that each $W \in \mathcal{W}$ is clopen.

Then $\mathcal{V} = \{W \cap X : W \in \mathcal{W}\}$ is a base for the topology of X. Then $\mathcal{V}$ is a countable base, of non-empty clopen sets. Let $(V_1, V_2, \cdots)$ be an enumeration of $\mathcal{V}$.

Since X is Polish it may be regarded as a separable metric space (with complete metric d). Let D be a countable dense subset of X. Let $\mathcal{B}$ be the collection of all open balls with rational radius and centred on points of D. Then $\mathcal{B}$ is a countable base for the topology of X.

Given any $\varepsilon > 0$ and any $V \in \mathcal{V}$ there is $B \in \mathcal{B}$ such that $B \subset V$ and the diameter of B is less than ε. Since $\mathcal{V}$ is also a base for the topology there is $U \in \mathcal{V}$ with $U \subset B$. So diameter U is less than ϵ and $U \subset V$.

We shall now obtain a sequence (U_n) in $\mathcal{V}$ such that

(i) for each n, $a^{-m(n)} \cdot U_n \subset V_n$ for some $m(n) \in \mathbb{N}$,

(ii) $diameter(U_n) < \frac{1}{2^n}$ and

(iii) $U_n \subset U_{n-1}$ for $n > 1$.

Take V_1. Then we can find $B \subset V_1$, where B is an open ball of diameter less than $\frac{1}{2}$. Now let $U_1 \in \mathcal{V}$ be a subset of B. Clearly (ii) is satisfied. So also is (i) on putting $m(1) = 0$.

Now suppose that $(U_1, U_2, ..., U_n)$ have been constructed.

Now take V_{n+1}. Arguing as above, there exists a (clopen) $Z \in \mathcal{V}$ such that $Z \subset U_n$ and its diameter is less than $\frac{1}{2^{n+1}}$. Then one has $m \in \mathbb{N}$ such that

$$(a^m \cdot V_{n+1}) \cap Z \neq \emptyset.$$

Since $(a^m \cdot V_{n+1}) \cap Z$ is a non-empty open subset, one has a non-empty clopen set U_{n+1} in $\mathcal{V}$ such that

$$U_{n+1} \subset (a^m \cdot V_{n+1}) \cap Z.$$

Since $U_{n+1} \subset Z \subset U_n$, the diameter of U_{n+1} is less than that of Z which is less than $\frac{1}{2^{n+1}}$ and $U_{n+1} \subset U_n$. Also we have $U_{n+1} \subset a^m \cdot V_{n+1}$, that is, $a^{-m} \cdot U_{n+1} \subset V_{n+1}$.

We now obtain a sequence in X by choosing x_n from each U_n. Clearly (x_n) is a Cauchy sequence. Since the metric space is complete, the sequence converges to a point x_0.

We claim $\{a^m \cdot x_0 : m \in \mathbb{Z}\}$ is dense in X.

Take any non-empty open subset O of X, one has $V_n \in \mathcal{V}$ such that $V_n \subset O$. By (i), there exists $m \in \mathbb{Z}$ with $m \geq 0$ such that

$$a^m \cdot U_n \subset V_n \subset O.$$

Since $x_0 \in U_n$, $a^m \cdot x_0 \in O$, which means that $< a > x_0$ is dense in X. Since X is dense in $2^{\mathbb{F}_2}$so also is $< a > x_0$. ■

Corollary 3 *(i)There exists Y, a dense G_δ subset of X, such that, for every $y \in Y$, the sub-orbit $< a > y$ is dense in X and hence dense in $2^{\mathbb{F}_2}$.*

(ii) There exists Z, a dense G_δ subset of $2^{\mathbb{F}_2}$, such that, for every $z \in Z$, the sub-orbit $< a > z$ is dense in $2^{\mathbb{F}_2}$. Furthermore we may suppose that Z is invariant under the action of $\mathbb{F}_2$.

Proof. (i) This follows from Lemma 1.1 [12]. Alternatively, apply Proposition 6.5.5 and Lemma 6.4.7 [11].

(ii) By the Baire category theorem, the intersection of countably many dense G_δ subsets of $2^{\mathbb{F}_2}$, is a dense G_δ subset. Put $Z = \cap\{gY : g \in \mathbb{F}_2\}$. ■

We wish to obtain similar results for actions of the free group on the structure space, S, of the Dixmier algebra. We shall do this by applying results on induced actions given in Section 6.4 of [11]. Although this can be done more generally, we shall focus on the Dixmier algebra.

Let K be a perfect Polish space which is compact, for example $2^{\mathbb{F}_2}$. Then there is a natural injective $*$-homomorphism j from $C(K)$ into $B(K)/M(K)\cong C(S)$, the Dixmier algebra. Using the familiar duality between commutative (unital) C^*-algebras and compact Hausdorff spaces, there is a continuous surjection $\rho : S \to K$ such that $j(f) = f \circ \rho$.

Let θ be a homeomorphism of K onto K. As in the beginning of Section 2, let h_θ be the corresponding $*$-automorphism of $C(K)$. Also $f \longmapsto f \circ \theta$ induces an automorphism $\widehat{h}_\theta$ of $B(K)/M(K)$. Since $B(K)/M(K)$ can be identified with $C(S)$, there exists $\widehat{\theta}$ in $Homeo(S)$ corresponding to $\widehat{h}_\theta$. Clearly, $\widehat{h}_\theta$ restricts to the automorphism, h_θ, of $C(K)$. It can be shown that $\widehat{h}_\theta$ is the unique extension of h_θ to a $*$-automorphism of $C(S)$. Then $\theta \to \widehat{\theta}$ is an injective group homomorphism from $Homeo(K)$ into $Homeo(S)$.

Lemma 4 *For each $s \in S$, $\theta(\rho s) = \rho(\widehat{\theta}\ s)$.*

Proof. See Corollary 6.4.3 [11]. ∎

To apply this lemma we now put $K = 2^{\mathbb{F}_2}$ and recall the action, $g \to \alpha_g$, of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. Here this action is an injective group homomorphism of $\mathbb{F}_2$ into $Homeo(2^{\mathbb{F}_2})$. It follows that $g \to \widehat{\alpha_g}$, is an injective group homomorphism of $\mathbb{F}_2$ into $Homeo(S)$. So we can define an induced action $\widehat{\alpha}$ on S by $\widehat{\alpha}_g = \widehat{\alpha_g}$ for $g \in \mathbb{F}_2$.

Proposition 5 *There exists $s_0 \in S$ such that the orbit $\{\widehat{\alpha}_g(s_0) : g \in \mathbb{F}_2\}$ is free and the suborbit $\{a^n \cdot s_0 : n \in \mathbb{Z}\}$ is dense in S. Here we define $a \cdot s = \widehat{\alpha}_a s$.*

Proof. Let x_0 be as in 2. There exists s_0 in S such that $\rho s_0 = x_0$ because ρ is a surjective map from S onto $2^{\mathbb{F}_2}$. By Lemma 4 $\theta(x_0) = \rho(\widehat{\theta}_{s_0})$, whenever $\theta = \alpha_g$. It now follows from Proposition 6.4.4 [11] that the sub-orbit $\{a^n \cdot s_0 : n \in \mathbb{Z}\}$ is dense in S.

Also, by Proposition 6.4.5 [11], the orbit $\{\widehat{\alpha}_g(s_0) : g \in \mathbb{F}_2\}$ is free. ∎

Corollary 6 *There exists a $\mathbb{F}_2$-invariant Y, which is a dense G_δ subset of S, such that, for $g \in \mathbb{F}_2$ and $g \neq e$, $\widehat{\alpha}_g$ has no fixed point in Y. Also $s_0 \in Y$.*

Proof. This follows from Lemma 6.4.7 [11]. ∎

By applying Lemma 1.1 (3) [12] (see also Proposition 6.5.5 [11]) we obtain:

Corollary 7 *There exists a $\mathbb{F}_2$-invariant Z, which is a dense $G_\delta$subset of S, such that, for $g \in \mathbb{F}_2$ and $g \neq e$, $\widehat{\alpha}_g$ has no fixed point in Z. Also for $y \in Z$, the sub-orbit $\{a^n \cdot y : n \in \mathbb{Z}\}$ is dense in Z.*

5 CONCLUSIONS

In Lemma 1 we replace T by S and observe that Proposition 5 implies that we can suppose the dense sub-orbit hypothesis is satisfied. This gives:

Proposition 8 *Let ϕ be a positive linear map from $\ell^\infty(\mathbb{F}_2)$ to $J[C(S)]$ such that*

$$\gamma^g \circ \phi = \phi \circ \gamma^g$$

for each $g \in \mathbb{F}_2$. Then the linear map ϕ is identically zero.

In the above, the action $g \to \gamma^g$ of $\mathbb{F}_2$ on $J[C(S)]$ is free and ergodic. To see this we can argue as follows.

Take $g \in G$ with $g \neq e$, the neutral element of G. If γ^g were not properly outer, one would have a G-invariant non-empty clopen subset E of S such that $\gamma^g \big|_{C(S)\chi_E}$ is the identity. Then we would have $\gamma^g J\chi_E Jf = J\chi_E Jf$ for all $f \in C(S)$ which means that

$$(\chi_E f)(\alpha_{g^{-1}}\alpha_h s_0) = \chi_E f(\alpha_h s_0) \text{ for all } f \in C(S) \text{ and } h \in G.$$

Since Gs_0 is dense and E is non-empty clopen, one has $h \in G$ such that $hs_0 \in E$. Since $\alpha_{g^{-1}}\alpha_h s_0 \neq \alpha_h s_0$, one has $f \in C(S)$ such that $f(\alpha_{g^{-1}}\alpha_h s_0) \neq f(\alpha_h s_0)$. Since $\alpha_h s_0 \in E$, $\alpha_{g^{-1}}\alpha_h s_0 \in E$ and so we have $f(\alpha_{g^{-1}}\alpha_h s_0) = f(\alpha_h s_0)$. But this would be a contradiction. So, the action $g \mapsto \gamma^g$ is free on $J[C(S)]$. Next take any $f \in C(S)$ and suppose $\gamma^g(Jf) = Jf$ for all $g \in G$. Then we have $f(\alpha_{g^{-1}}\alpha_h s_0) = f(\alpha_h s_0)$ for all $g, h \in G$ and so in particular, $f(\alpha_h s_0) = f(s_0)$ for all $h \in G$. Since Gs_0 is dense in S and f is continuous, it follows that $f(s) = f(s_0)$ for all $s \in S$ and so $f = f(s_0)1$. So, the action γ is ergodic.

Let G be a countably infinite group which acts freely and ergodically as $*$-automorphisms of $C(S)$. Then there exists a corresponding monotone cross-product C*-algebra $M(C(S), G)$ which is a wild factor. Then Theorem 3.4 [12] tells us that $M(C(S), G)$ is isomorphic to the Takenouchi-Dyer factor, for every choice of G, provided the action is free and ergodic. It is also isomorphic to $M(C(S), \oplus\mathbb{Z}_2)$. In particular $M(C(S), \mathbb{F}_2) \cong M(C(S), \oplus\mathbb{Z}_2)$.Each of these monotone cross-products corresponds to a hyperfinite Borel equivalence relation on a perfect Polish space (see [12]). See Example 6.1.4 [11]. For more details and generalisations, see Chapter 7 [11]. We have that the factor $M(C(S), \mathbb{F}_2)$ is hyperfinite. We shall show that it is not an injective C*-algebra. (For monotone cross-products see [9, 10, 5, 11, 12, 13]).

Lemma 9 *Let A and B be commutative monotone complete C*-algebras. Let A be embedded as a C*-subalgebra of B. Let G be a countable group of $*$-automorphisms of B, each of which maps A into A. Let $M(A, G)$ be injective. Then there exists a positive linear projection (conditional expectation) Φ from B onto A with the following properties. For each $g \in G$, and $x \in B$,*

$$\Phi g(x) = g\Phi(x).$$

Proof. This follows from Proposition 2.3 [18]. ■

Theorem 10 *The hyperfiniteTakenouchi-Dyer factor is not injective.*

Proof. $\{sketch\}$ In Lemma 9 put $B = \ell^\infty(\mathbb{F}_2)$ and put $A = J[C(S)]$.

Asssume that $M(C(S), \mathbb{F}_2)$ is an injective C*-algebra. Then, by Lemma 9, there exists an equivariant positive linear projection from B onto A. Then, by Proposition 8, Φ vanishes on B. But this is impossible since $\Phi 1 = 1$. ■

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