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COMMUTATIVE ν -ALGEBRA
AND
SUPERTROPICAL ALGEBRAIC GEOMETRY

ZUR IZHAKIAN [†]

ABSTRACT. This paper lays out a foundation for a theory of supertropical algebraic geometry, relying on commutative ν -algebra. To this end, the paper introduces \mathfrak{q} -congruences, carried over ν -semirings, whose distinguished ghost and tangible clusters allow both quotienting and localization. Utilizing these clusters, \mathfrak{g} -prime, \mathfrak{g} -radical, and maximal \mathfrak{q} -congruences are naturally defined, satisfying the classical relations among analogous ideals. Thus, a foundation of systematic theory of commutative ν -algebra is laid. In this framework, the underlying spaces for a theoretic construction of schemes are spectra of \mathfrak{g} -prime congruences, over which the correspondences between \mathfrak{q} -congruences and varieties emerge directly. Thereby, scheme theory within supertropical algebraic geometry follows the Grothendieck approach, and is applicable to polyhedral geometry.

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1. INTRODUCTION

The evolution of supertropical mathematics having been initiated in [21] by introducing a semiring structure whose arithmetic intrinsically formulates combinatorial properties that address the lack of subtraction in semirings. This mathematics is carried over ν -semirings whose structure permits a systematic development of algebraic theory, analogous to the theory over rings [32], in which fundamental notions can be interpreted combinatorially [33, 34, 35, 37].

The introduction of supertropical theory was motivated by the aim of capturing tropical varieties in a purely algebraic sense by extending the max-plus semiring (cf. Example 3.26). The ultimate goal has been to establish a profound theory of polyhedral algebraic geometry in the spirit of A. Grothendieck, whose foundations are built upon commutative algebra. The present paper introduces an algebraic framework for such a theory, evolving further supertropical mathematics.

The aspiration of this theory has been to provide an intuitive algebraic language, clean and closer as possible to classical theory, but at the same time abstract enough to frame objects having a discrete nature. In this theory, the mathematical formalism involves no complicated combinatorial formulas and enables a direct implementation of familiar algebro-geometric approaches. The conceptual ideas and main principles of the theory are summarized below.

1.1. Supertropical structures.

The underlying additive structure of a ν -algebra is a ν -**monoid** $(\mathcal{M}, \mathcal{G}, \nu)$ – a monoid \mathcal{M} with a distinguished (partially ordered) **ghost submonoid** \mathcal{G} and a projection $\nu : \mathcal{M} \rightarrow \mathcal{G}$ – which satisfies the key

property

$$a + b = \nu(a) \quad \text{whenever} \quad \nu(a) = \nu(b).$$

In particular, $a + a = \nu(a)$ for any $a \in \mathcal{M}$. Thus, an element $\nu(a)$ of \mathcal{G} can be thought of as carrying an “additive multiplicity” higher than 1, i.e., a sum $a + \cdots + a$ with a repeated at least twice. When the monoid operation $+$ is maximum, this arithmetic intrinsically formulates the common tropical phrase “the maximum is attained at least twice”, replacing the “vanishing condition” in classical theory.

Equipping an additive ν -monoid with a multiplicative operation that respects the relevant axioms, a ν -**semiring** $(R, \mathcal{T}, \mathcal{G}, \nu)$ is obtained. In ν -semirings the ghost submonoid \mathcal{G} becomes an ideal – a “ghost absorbing” submonoid – which plays the customary role of zero element in commutative algebra. In this view, a traditional equation $\diamond \cdots \diamond = 0$ is reformulated as $\diamond \cdots \diamond \in \mathcal{G}$, replacing the familiar “vanishing condition” (which is often meaningless for semiring structures) by possessing ghost. On the other edge, to obtain a suitable multiplicative substructure, the subset \mathcal{T} of **tangible elements** in the complement of \mathcal{G} is distinguished, satisfying the conditions: if $a + b \in \mathcal{T}$ then $a, b \notin \mathcal{G}$. As well, \mathcal{T} contains the well behaved subset \mathcal{T}° of **t-persistent elements**, closed for taking powers.

Together, the ghost ideal \mathcal{G} and the tangible set \mathcal{T} of a ν -semiring enable a reformulation of basic algebraic notions. For instance, a **q-homomorphism** is a homomorphism of ν -semirings that preserves the components \mathcal{T} and \mathcal{G} . The **tangible core** and the **ghost kernel** are, respectively, the preimages of \mathcal{T} and \mathcal{G} which characterize q-homomorphisms. With these objects in place, the category of ν -semirings whose morphisms are q-homomorphisms is established.

ν -semirings generalize the max-plus semiring $(\mathbb{R}, \max, +)$ and the boolean algebra $(\mathbb{B}, \vee, \wedge)$, enriching them with extra algebraic properties, as detailed in Examples 3.26 and 3.27. Furthermore, any ordered monoid (\mathcal{M}, \cdot) gives rise to a semiring structure by setting its addition to be maximum, and therefore, supertropical theory is carried over transparently to ordered monoids. Former works have frequently assumed multiplicative cancellativity (i.e., $ca = cb \Rightarrow a = b$ for any a, b, c), to compensate the lack of inverses in semirings. This condition is too restrictive for ν -semirings (see Example 3.22). Therefore, we avoid any cancellativity conditions. As a consequent, pathological elements, as ghost divisors and ghostpotents, emerge in this setting and are treated by the use of congruences.

1.2. Congruences versus ideals.

Quotienting and localization are central notions in algebra. In commutative ring theory and in classical algebraic geometry these notions are delivered by ideals. A ring ideal defines an equivalence relation, and thus a quotient, while the complement of a prime ideal is a multiplicative system, used for localizing. Since subtraction is absent in semirings, ideals do not determine equivalence relations and, therefore, are not applicable for quotienting. Consequently, one has to work directly with congruences, i.e., equivalence relations that respect the semiring operations. However, by itself, this approach does not address localization. But, with extra properties, quotienting and localization are accessible via congruences on ν -semirings.

To introduce **clustering** on congruences – a coarser decomposition of classes – we enhance the classification of elements as tangibles or ghosts to equivalence classes. In particular, an equivalence class $[a]$ is tangible if it consists only of tangibles, $[a]$ is ghost if it contains some ghost, and $[a]$ is neither tangible nor ghost otherwise. Thus, a congruence \mathfrak{A} on a ν -semiring R is endowed with two disjoint clusters, consisting of equivalence classes:

✂ **tangible cluster** whose classes are preserved as tangibles in R/\mathfrak{A} ,

✂ **ghost cluster** whose classes are identified as ghosts in R/\mathfrak{A} .

The former serves for localization and the latter serves for quotienting. These clusters are not necessarily the complement of each other; so one has to cope with an extra degree of freedom. This divergency is addressed by the tangible and ghost projections of \mathfrak{A} on R , which are respectively determined by the diagonals of classes within the clusters of \mathfrak{A} .

A **q-congruence** \mathfrak{A} on a ν -semiring R is a congruence whose tangible projection contains the group of units of R , and thus a submonoid of t-persistent (tangible) elements. In the special case, when the tangible projection is a monoid by itself, \mathfrak{A} is an **l-congruence**. In our theory, q-congruences are elementary entity, providing the building stones for commutative algebra. Quotienting by a q-congruence is done in the standard way, while the monoid structure of tangible projections of l-congruences allows for executing localization. The tangible projection and the ghost cluster enable the utilization of familiar methods

in commutative algebra. Moreover, \mathfrak{q} -congruences preserve the ν -semiring structure in the transition to quotient structures and, at the same time, perfectly coincide with \mathfrak{q} -homomorphisms. Therefore, concerning ν -semirings, \mathfrak{q} -congruences play the traditional role of ideals.

\mathfrak{q} -congruences support executing “**ghostification**” of elements – a redeclaration of elements as ghosts in quotient structures. This redeclaration is the supertropical analogy to quotienting by an ideal, whose elements are “identified” with zero. Formally, an element $a \in R$ is ghostified by a congruence \mathfrak{A} , if \mathfrak{A} includes the equivalence $a \equiv \nu(a)$. This means that, despite the absence of additive inverses, we are capable to quotient out a ν -semiring by its substructure (even by a subset) in a meaningful sense. Hence, cokernels of \mathfrak{q} -homomorphisms can be defined naturally.

1.3. Congruences and spectra.

Classical theory employs spectra of prime ideals, endowed with Zariski topology, as underlying topological spaces. In supertropical theory, ideals are replaced by \mathfrak{q} -congruences, whose special structure permits the key definitions:

- ✦ a \mathfrak{q} -congruence is **g-radical** if $a^k \equiv_{\mathfrak{r}} \nu(a^k)$ implies $a \equiv_{\mathfrak{r}} \nu(a)$,
- ✦ an ℓ -congruence is **g-prime** if $ab \equiv_{\mathfrak{p}} \nu(ab)$ implies $a \equiv_{\mathfrak{p}} \nu(a)$ or $b \equiv_{\mathfrak{p}} \nu(b)$.

Note that the conditions in these definitions are determined solely by equivalence to ghosts, while localization by **g-prime** congruences is performed through their tangible projections. This setup enables to formulate many tropical relations, analogous to well known relations among radical, prime, and maximal ideals. With these relations, manifest over \mathfrak{q} -congruences, the notions of **noetherian ν -semirings** and **Krull dimension** are obtained.

To cope simultaneously with congruences and “ghost absorbing” subsets of ν -semirings, we study several types of radicals, which are shown to coincide. Moreover, they provide a version of abstract Nullstellensatz (Theorem 4.69). In comparison to ring ideals, the hierarchy of \mathfrak{q} -congruences contains unique types of congruences, including deterministic ℓ -congruence and interweaving congruences. However, maximal congruences are not applicable for defining locality, since they do not necessarily coincide with maximality of clusters. Instead, **t-minimal ℓ -congruences**, determined by maximality of non-tangible projections, are used to define locality. A ν -semiring R is **local**, if all **t-minimal ℓ -congruences** on R share the same tangible projection.

The well behaved interplay between localization and various types of \mathfrak{q} -congruences allows for introducing the **spectrum of g-prime congruences**, endowed with a Zariski type topology. With this setting, our study follows the standard methodology of exploring the correspondences among closed sets, open sets, \mathfrak{q} -congruences, and ν -semiring structures, resulting eventually in a construction of sheaves.

1.4. Sheaves and schemes.

Upon the spectrum $X = \text{Spec}(A)$ of all **g-prime** congruences on a ν -semiring A , realized as a topological space, a closed set $\mathcal{V}(f)$ in X is the set of all **g-prime** congruences \mathfrak{P} that ghostify a given $f \in A$. That is, f belongs to the ghost projection of \mathfrak{P} , i.e., $f \equiv_{\mathfrak{p}} \nu(f)$ in \mathfrak{P} . Therefore, since $f \equiv_{\mathfrak{p}} \nu(f)$ for any ghost f , we focus on non-ghost elements $f \in A$ to avoid triviality. Yet, this focusing does not imply that a non-ghost element f belongs to the tangible projection of all **g-prime** congruences composing the complement $\mathcal{D}(f)$ of $\mathcal{V}(f)$ in X . Thus, f does not necessarily possess tangible values over the entire open set $\mathcal{D}(f)$, in particular when f is not tangible. This bearing requires a special care in the construction of sheaves, as explained below.

A **variety** in X is a closed set which can be determined either as the set of **g-prime** congruences containing a \mathfrak{q} -congruence \mathfrak{A} on A , or as the set of **g-prime** congruences that ghostify a subset E of A . These equivalent definitions establish the correspondences between properties of \mathfrak{q} -congruences on A and subsets of X . For example, the one-to-one correspondence between **g-prime** congruences and **irreducible varieties** (Theorem 6.24). In this setting, closed immersions $\text{Spec}(A/\mathfrak{A}) \xrightarrow{\sim} \mathcal{V}(\mathfrak{A})$ of spectra appear naturally (Corollary 6.25). Open immersions are more subtle and require a further adaptation, that is, a restriction $\mathcal{D}(C, f)$ of $\mathcal{D}(f)$ to **g-prime** congruences whose tangible projection contains a given tangible monoid C of A . This restriction yields the bijection $\text{Spec}(A_C) \xrightarrow{\sim} \bigcap_{f \in C} \mathcal{D}(C, f)$.

To preserve the ν -semiring structure, localization is feasible only by **t-persistent** elements. Nevertheless, to construct a sheaf, each open set $\mathcal{D}(f)$ has to be assigned with a ν -semiring, allocated by the means of localization, even when f is not tangible. To overcome this drawback, we consider the submonoid

$S(f)$ of particular \mathfrak{t} -persistent elements h such that $\mathcal{D}(f) \subseteq \mathcal{D}(h)$, which delivers the well defined map $\mathcal{D}(f) \mapsto A_{S(f)}$ for every $f \in A$. To ensure that this map coincides with the map $\mathfrak{P} \mapsto A_{\mathfrak{P}}$, sending a point \mathfrak{P} in $\mathcal{D}(f)$ to the localization of A by \mathfrak{P} , sections over $\mathcal{D}(f)$ are customized to \mathfrak{g} -prime congruences in $\mathcal{D}(f)$ whose tangible projection contains the submonoid $S(f)$. The subset of these \mathfrak{g} -prime congruences assembles the **focal zone** $\check{\mathcal{D}}(f)$ of the open set $\mathcal{D}(f)$. Accordingly, ν -stalks are determined as inductive limits taken with respect to focal zones determined by \mathfrak{t} -persistent elements, whereas the naive definition of morphism applies and respects focal zones (Lemma 7.16). The building of structure ν -sheaves and locally ν -semiringed spaces is then standard, admitting functoriality as well. This construction provides a scheme structure (X, \mathcal{O}_X) of ν -semirings, called ν -**scheme**.

While most of our theory follows classical scheme theory, a ν -semiring $\mathcal{O}_X(\mathcal{D}(f))$ of sections in an affine ν -scheme (X, \mathcal{O}_X) , with $X = \text{Spec}(A)$, might not be isomorphic to the localized ν -semiring $A_{S(f)}$, since sections are specialized to focal zones. However, this excludes **strict elements** $f \in A$ (and in particular units) for which the equality $\check{\mathcal{D}}(f) = \mathcal{D}(f)$ holds. Therefore, in the theory of ν -schemes, global sections $\Gamma(X, \mathcal{O}_X)$ are isomorphic to the underlying ν -semiring A (Theorem 7.20). Moreover, there is a one-to-one correspondence between \mathfrak{q} -homomorphisms of ν -semirings and morphisms of ν -schemes (Theorem 7.24). Fiber products of ν -schemes also exist (Proposition 8.18).

The advantage of this framework appears in the analysis of the local ν -semiring at a point, which includes the notions of **tangent space**, **local dimension**, and **singularity**. These notions are not so evident in standard tropical geometry, since ideals are not well applicable in semirings.

1.5. Varieties towards polyhedral geometry.

Traditional tropical geometry is a geometry over the max-plus semiring $(\mathbb{R}, \max, +)$. Features of this geometry are balanced polyhedral complexes of pure dimension [20, 52, 54], called tropical varieties. These varieties are determined as the so-called “corner loci” of tropical polynomials and are obtained as valuation images of toric varieties, linking tropical geometry to classical theory. Although $(\mathbb{R}, \max, +)$ suffices for describing tropical varieties, from the perspective of polyhedral geometry this family is rather restrictive; for example, it does not include ploytopes.

Supertropical algebraic sets are defined directly as ghost loci of polynomial equations [32, 37, 38]; that is, $\mathcal{Z}(f) = \{\mathbf{x} \in A^{(n)} \mid f(\mathbf{x}) \in \mathcal{G}\}$. Thereby, tropical varieties are captured as algebraic sets of tangible polynomials over the supertropical extension of $(\mathbb{R}, \max, +)$, cf. Example 3.26. This approach does not rely on the so-called “tropicalization” of toric varieties and dismisses the use of the balancing condition [39]. Furthermore, it yields formulation of additional polyhedral features, previously inaccessible by tropical geometry, such as polytopes, and more generally subvarieties of the same dimension as their ambient variety (Example 3.55).

Supertropical structures provide a sufficiently general framework to deal with finite and infinite underlying semirings, as well as with bounded semirings. Our theory allows for approaching convex geometry and discrete geometry, utilizing similar principles as in this paper. We leave the study of these geometries for future work.

1.6. A brief overview of related theories.

Tropical semirings $(R, \max, +)$, where $R = \mathbb{B}, \mathbb{N}, \mathbb{Z}$, are linked to number theory [6, 7, 8] and arithmetic geometry via the Banach semifield theory of characteristic one [44]. Features of traditional tropical geometry are received as the Euclidean closures of “tropicalization” of subvarieties of a torus $(\mathbb{K}^\times)^n$, where \mathbb{K} is a non-archimedean algebraically closed valued field, complete with respect to the valuation [14, 20, 54]. A generalization to subvarieties of a toric variety is given by Payne [56], using stratification by torus orbits. With this geometry, a translation of algebro-geometric questions into combinatorial problems is obtained, where varieties are replaced by polyhedral complexes, helping to solve problems in enumerative geometry [52]. The translations of classical problems into combinatorial framework have been motivating the development of profound algebraic foundations for geometry over semirings.

Over the past decade, many works have dealt with scheme theory over semirings and monoids, aiming mainly to develop a characteristic one arithmetic geometry over the field \mathbb{F}_1 with one element, e.g., Connes-Consani [6], Deitmar [10], Durov [12], Toën-Vaquié [59]. See [48] for survey. Berkovich uses abstract skeletons [3], while Lorscheid [49, 50] uses blueprints. Giansiracusa-Giansiracusa [15] and Maclagan-Rincón [53] specialize \mathbb{F}_1 to propose tropical schemes, subject to “bend relations”. The point of departure of these works is an underlying spectrum whose atoms are prime ideals. Alternatively,

other works employ congruences as atoms of spectra. However, the use of congruences raises the issue of defining primeness that exhibits the desired attributes. This approach is taken by Joó-Mincheva [41] and Rowen [58], who use twisted products, by Bertram-Easton [4], and by Lescot [45, 46, 47]. Primeness in [45, 46, 47] depends only on equivalence to zero, ignoring other relations that are determined by a congruence. In [9], primes are defined in terms of cancellative quotient monoids.

Although the above approaches address the needs of other theories, for polyhedral algebraic geometry they seem to have their own deficiencies. Some approaches are too restrictive to capture the wide range of polyhedral objects, while others are much too abstract and difficult to be implemented. The concept in the current paper is different in the sense that it uses congruences, enriched by supertropical attributes, as atoms of spectra that fulfill the required correspondences between topological spaces and algebras. The algebraic theory then follows the classical commutative algebra, allowing for the development of scheme theory within algebraic geometry along the track of Grothendieck. This concept provides a clear-cut algebraic framework for a straightforward study of polyhedral varieties and schemes, in particular tropical varieties, without the need to constantly referring back to their classical valuation preimages.

1.7. Paper outline.

For the reader's convenience, the paper is designed as stand-alone. We include all the relevant definitions, detailed proofs, and basic examples. We follow standard structural theory of scheme within algebraic geometry, involving a categorical viewpoint, e.g., [5, 13, 18, 19]. The paper consists of two parts. The first part (Sections 2–5) is devoted to commutative ν -algebra and the second part (Sections 6–8) develops scheme theory over ν -semirings.

Section 2 opens by recalling the setup of known algebraic structures, to be used in the paper. Section 3 provides the definitions of various ν -structures, as well as their connections to familiar semiring structures. Section 4 intensively studies congruences on ν -semirings, including the introduction of \mathfrak{q} -congruences and ℓ -congruences. Section 5 introduces ν -modules and tensor products. Section 6 discusses varieties through the correspondences of their characteristic properties to \mathfrak{q} -congruences. Section 7 constructs ν -sheaves, combining categorical and algebraic perspectives. Finally, Section 8 develops the notions of ν -schemes and locally ν -semiringed spaces.

Part I: Commutative ν -Algebra

2. ALGEBRAIC STRUCTURES

In this section we recall definitions of traditional algebraic structures and their morphisms, to be used in this paper. As customary, \mathbb{N} denotes the positive natural numbers, while \mathbb{N}_0 stands for $\mathbb{N} \cup \{0\}$; \mathbb{Q} and \mathbb{R} denote respectively the rational and real numbers. We write $A^{(n)}$ for the cartesian product $A \times \cdots \times A$ of a structure A , with A repeated n times.

In this paper addition and multiplication are always assumed to be associative operations.

2.1. Congruences in universal algebra.

A **congruence** \mathfrak{A} on an algebra A – a carrier algebra – is an equivalence relation \equiv that preserves all the relevant operations and relations of A . That is, if $a_i \equiv b_i$, $i = 1, 2$, then

- (i) $a_1 + a_2 \equiv b_1 + b_2$,
- (ii) $a_1 a_2 \equiv b_1 b_2$.

Note that to prove (ii) it is enough to show that $a_1 a \equiv b_1 a$ and $aa_1 \equiv ab_1$ for all $a \in A$, since then

$$a_1 b_1 \equiv a_2 b_1 \equiv a_2 b_2.$$

We call \equiv the **underlying equivalence** of \mathfrak{A} and write $[a]$ for the equivalence class of an element $a \in A$. (These are called the **homomorphic equivalences** in [23].) We write A/\mathfrak{A} for the factor algebra, whose elements are equivalence classes $[a]$ determined by \mathfrak{A} , with operations

$$[a] \cdot [b] = [ab], \quad [a] + [b] = [a + b], \tag{2.1}$$

for $a, b \in A$.

A congruence \mathfrak{A} on algebra A may be viewed either as a subalgebra of $A \times A$ containing the diagonal and satisfying two additional conditions corresponding to symmetry and transitivity (in which case \mathfrak{A} is described as the appropriate set of ordered pairs), or otherwise \mathfrak{A} may be viewed as an equivalence relation \equiv satisfying certain algebraic conditions. To make the exposition clearer we utilize both views, relying on the context.

We denote the set of all congruences on an algebra A by

$$\text{Cong}(A) := \{ \mathfrak{A} \text{ is a congruence on } A \},$$

which is closed for intersection. We write $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ if $a \equiv_1 b$ implies $a \equiv_2 b$, in other terms $(a, b) \in \mathfrak{A}_1$ implies $(a, b) \in \mathfrak{A}_2$. Thus, $\text{Cong}(A)$ is endowed with a partial order \leq induced from \subseteq , i.e., $\mathfrak{A}_1 \leq \mathfrak{A}_2$ if and only if $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$.

The **diagonal** $\Delta(A)$ of $A \times A$ is a congruence by itself, contained in any congruence \mathfrak{A} on A , and is minimal with respect to inclusion. It provides the bijection

$$\iota : A \xrightarrow{\sim} \Delta(A) \subseteq \mathfrak{A}, \quad a \mapsto (a, a), \quad (2.2)$$

with the inverse map $\Delta^{-1}((a, a)) = a$ for any $(a, a) \in \Delta(A)$. On the other hand, $\Delta(A)$ is also obtained as

$$\Delta(A) = \bigcap_{\mathfrak{A} \in \text{Cong}(A)} \mathfrak{A}, \quad (2.3)$$

since an intersection of congruences is again a congruence.

Remark 2.1. For any collection $\{(a_i, b_i)\}_{i \in I}$ of pairs $(a_i, b_i) \in A \times A$, there is a unique minimal congruence in which $a_i \equiv b_i$ for every $i \in I$. It is termed the congruence **determined** by the pairs (a_i, b_i) . This congruence can be obtained in two equivalent ways, either by taking the transitive closure of all the equivalences $a_i \equiv b_i$ with respect to the operations and relations of A , or by considering the intersection of all congruences on A that include the equivalences $a_i \equiv b_i$ for all $i \in I$.

In this paper we follow the latter approach, as it better fits our purpose, allowing a direct restriction to congruences subject to particular properties.

Definition 2.2. A congruence \mathfrak{A} on an algebra A is said to be

- (i) **proper**, if it has more than one equivalence class;
- (ii) **trivial**, if each of its equivalence classes is a singleton (i.e., $\mathfrak{A} = \Delta(A)$);
- (iii) **maximal**, if there is no other congruence properly containing \mathfrak{A} ;
- (iv) **irreducible**, if it cannot be written as the intersection of two congruences properly containing \mathfrak{A} ;
- (v) **cancellative**, if $ca \equiv cb$ implies $a \equiv b$.

An element $a \in A$ is called **isolated** with respect to \mathfrak{A} if it is congruent only to itself.

In this context, an algebra A is called **simple**, if the trivial congruence is the only proper congruence on A . One may also consider a specialization of the above congruences to a restricted family of congruences on A , determined by particular attributes. For example, maximality with respect to a given property.

Remark 2.3. We recall some key results from [40, §2].

- (i) Given a congruence \mathfrak{A} on an algebra A , one can endow the set

$$A/\mathfrak{A} := \{[a] \mid a \in A\}$$

of equivalence classes with the same (well-defined) algebraic structure via (2.1). The **canonical surjective homomorphism**

$$\pi_{\mathfrak{A}} : A \longrightarrow A/\mathfrak{A}, \quad a \mapsto [a], \quad (2.4)$$

is defined trivially.

- (ii) In the opposite direction, for any homomorphism $\varphi : A_1 \longrightarrow A_2$ one can obtain a congruence \mathfrak{A}_φ on A_1 , defined by

$$a \equiv_\varphi b \quad \text{iff} \quad \varphi(a) = \varphi(b).$$

\mathfrak{A}_φ is termed the **congruence-kernel** (written **c-kernel**) of φ , and is also denoted by $\ker_c(\varphi)$. Then, φ induces a one-to-one homomorphism $\tilde{\varphi} : A_1/\mathfrak{A}_\varphi \longrightarrow A_2$, via $\tilde{\varphi}([a]) = \varphi(a)$, where φ factors through

$$A_1 \longrightarrow A_1/\mathfrak{A}_\varphi \xrightarrow{\sim} A_2,$$

as indicated in [40, p. 62]. Therefore, the homomorphic images of A_1 correspond to the congruences defined on A_1 .

- (iii) A homomorphism $\varphi : A_1 \longrightarrow A_2$ induces a map of congruences $\varphi^\vee : \mathfrak{A}_2 \longmapsto \mathfrak{A}_1$, sending a congruence \mathfrak{A}_2 on A_2 to the congruence $\mathfrak{A}_1 = \mathfrak{A}_{\pi_2 \circ \varphi}$ on A_1 , where $\pi_2 : A_2 \longrightarrow A_2/\mathfrak{A}_2$. That is

$$a \equiv_1 b \quad \text{iff} \quad \varphi(a) \equiv_2 \varphi(b).$$

When $\varphi = \pi_1 : A_1 \longrightarrow A_2 = A_1/\mathfrak{A}_1$ is the canonical surjection (2.4), we denote the map π_2^{-1} also by π_2^\vee .

Given a homomorphism $\varphi : A_1 \longrightarrow A_2$, there is the induced map

$$\widehat{\varphi} : A_1 \times A_1 \longrightarrow A_2 \times A_2, \quad (a, b) \longmapsto (\varphi(a), \varphi(b))$$

that restricts to a map of congruences $\mathfrak{A}_1 \subset A_1 \times A_1$. Note that $\widehat{\varphi}(\mathfrak{A}_1)$ need not be a congruence on A_2 , as transitivity and reflexivity may fail, but it induces a congruence-inclusion relation:

$$\widehat{\varphi}(\mathfrak{A}_1) \subset \mathfrak{A}_2 \quad \text{iff} \quad (\varphi(a), \varphi(b)) \in \mathfrak{A}_2 \quad \text{for all} \quad (a, b) \in \mathfrak{A}_1. \quad (2.5)$$

To approach restrictive relations on subsets of A we use the following terminology.

Definition 2.4. The *restriction* of a congruence \mathfrak{A} on A to a subset B of A is

$$\mathfrak{A}|_B := \{(a, b) \in \mathfrak{A} \mid a, b \in B\}.$$

A *partial congruence* on A is a congruence on a subalgebra B of A .

Set theoretically, a congruence \mathfrak{A}' on a subalgebra $B \subset A$ is a subset of $B \times B \subset A \times A$ containing the diagonal of B , but it is not a subalgebra of $A \times A$ that contains the diagonal of A . For this reason we call it ‘‘partial’’. However, any partial congruence extends naturally to a congruence on the whole A by joining the diagonal of $A \setminus B$, subject to the transitive closure of \mathfrak{A}' over A . A restriction $\mathfrak{A}|_B$ of \mathfrak{A} is a congruence on B , if B is a subalgebra of A .

Definition 2.5. Let \mathfrak{A}_1 and \mathfrak{A}_2 be congruences on algebras A_1 and A_2 , respectively. A *homomorphism of congruences* $\Psi : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_2$ is a homomorphism $\Psi = (\psi, \psi)$, where $\psi : A_1 \longrightarrow A_2$ is a homomorphism for which

$$a \equiv_1 b \quad \Rightarrow \quad \psi(a) \equiv_2 \psi(b),$$

i.e., $\Psi((a, b)) = (\psi(a), \psi(b)) \in \mathfrak{A}_2$ for all $(a, b) \in \mathfrak{A}_1$.

We recall some standard relations on congruences. The product of two congruences $\mathfrak{A}_1, \mathfrak{A}_2 \in \text{Cong}(A)$ is defined as

$$\mathfrak{A}_1 \mathfrak{A}_2 := \{(ac, bd) \mid (a, b) \in \mathfrak{A}_1, (c, d) \in \mathfrak{A}_2\},$$

for which the relation $\mathfrak{A}_1 \mathfrak{A}_2 \subseteq \mathfrak{A}_1 \cap \mathfrak{A}_2$ holds. Then, for a given congruence \mathfrak{A} on A , we have the chain

$$\mathfrak{A} \supseteq \mathfrak{A}^2 \supseteq \dots \supseteq \mathfrak{A}^n \supseteq \dots, \quad (2.6)$$

which induces quotients on \mathfrak{A} .

Definition 2.6. The *quotient* $\mathfrak{A}_1/\mathfrak{A}_2$ of congruences $\mathfrak{A}_1 \subset \mathfrak{A}_2$ on A is defined as

$$\mathfrak{A}_1/\mathfrak{A}_2 := \{([a]_2, [b]_2) \mid (a, b) \in \mathfrak{A}_1\},$$

for which there exists the homomorphism

$$\Psi : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_1/\mathfrak{A}_2$$

of congruences.

Composing this definition with (2.6), one obtains a sequence of congruence homomorphisms

$$\mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{A}^2 \longrightarrow \dots \longrightarrow \mathfrak{A}^{n-1}/\mathfrak{A}^n \longrightarrow \dots. \quad (2.7)$$

Later, we mainly refer to the initial step of this sequence.

Remark 2.7. If $\mathfrak{A}_1 \subset \mathfrak{A}_2$ are congruences on A , then $\mathfrak{A}_1/\mathfrak{A}_2$ is a congruence on A/\mathfrak{A}_1 , for which $(A/\mathfrak{A}_1)/(\mathfrak{A}_1/\mathfrak{A}_2) \cong A/\mathfrak{A}_2$. This also gives a factorization of the homomorphism $A \longrightarrow A/\mathfrak{A}_2$ as $A \longrightarrow A/\mathfrak{A}_1 \longrightarrow A/\mathfrak{A}_2$.

We defined the **congruence closure** of $\mathfrak{A}_1 \cup \mathfrak{A}_2$ and $\mathfrak{A}_1 + \mathfrak{A}_2$, for $\mathfrak{A}_1, \mathfrak{A}_2 \in \text{Cong}(A)$, to be respectively the intersections, when are nonempty ¹,

$$\overline{\mathfrak{A}_1 \cup \mathfrak{A}_2} := \bigcap_{\substack{\mathfrak{A} \in \text{Cong}(A) \\ \mathfrak{A}_1 \cup \mathfrak{A}_2 \subset \mathfrak{A}}} \mathfrak{A}, \quad \overline{\mathfrak{A}_1 + \mathfrak{A}_2} := \bigcap_{\substack{\mathfrak{A} \in \text{Cong}(A) \\ \mathfrak{A}_1 + \mathfrak{A}_2 \subset \mathfrak{A}}} \mathfrak{A}, \quad (2.8)$$

where $\mathfrak{A}_1 \cup \mathfrak{A}_2$ is the set theoretic union of \mathfrak{A}_1 and \mathfrak{A}_2 and $\mathfrak{A}_1 + \mathfrak{A}_2$ is induced by the addition of $A \times A$.

2.2. Semigroups, monoids, and semirings.

A (multiplicative) **semigroup** $\mathcal{S} := (\mathcal{S}, \cdot)$ is a set of elements \mathcal{S} , closed with respect to an associative binary operation (\cdot) . A **monoid** $\mathcal{M} := (\mathcal{M}, \cdot)$ is a semigroup with an identity element $\mathbb{1}_{\mathcal{M}}$. Formally, any semigroup \mathcal{S} can be attached with identity element $\mathbb{1}_{\mathcal{S}}$ by declaring that $\mathbb{1}_{\mathcal{S}} \cdot a = a \cdot \mathbb{1}_{\mathcal{S}} = a$ for all $a \in \mathcal{S}$. So, when dealing with multiplication, we work with monoids. As usual, when (\cdot) is clear from the context, $a \cdot b$ is written as ab . An **abelian** monoid is a commutative monoid, i.e., $ab = ba$ for all $a, b \in \mathcal{M}$. Analogously, we use additive notation for monoids, written $\mathcal{M} := (\mathcal{M}, +)$, whose identity is denoted by $\mathbb{0}_{\mathcal{M}}$.

Remark 2.8. *The intersection $\mathcal{A} \cap \mathcal{B}$ of two submonoids $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ is again a submonoid. Indeed, if $a, b \in \mathcal{A} \cap \mathcal{B}$, then $ab \in \mathcal{A}$ and $ab \in \mathcal{B}$, and hence $ab \in \mathcal{A} \cap \mathcal{B}$.*

An abelian monoid $\mathcal{M} := (\mathcal{M}, \cdot)$ is **cancellative** with respect to a subset $T \subseteq \mathcal{M}$, if $ac = bc$ implies $a = b$ whenever $a, b \in \mathcal{M}$ and $c \in T$. In this case, we say that T is a **cancellative subset** of \mathcal{M} . Clearly, when T is cancellative, the monoid generated by T in \mathcal{M} also is cancellative, so one can assume that T is a submonoid. A monoid \mathcal{M} is **strictly cancellative**, if \mathcal{M} is cancellative with respect to itself.

An element a of \mathcal{M} is **absorbing**, if $ab = ba = a$ for all $b \in \mathcal{M}$. Usually, it is identified as $\mathbb{0}_{\mathcal{M}}$. A monoid \mathcal{M} is called **pointed monoid** if it has an absorbing element $\mathbb{0}_{\mathcal{M}}$. An element $a \in \mathcal{M}$ is a **unit** (or **invertible**), if there exists $b \in \mathcal{M}$ such that $ab = ba = \mathbb{1}_{\mathcal{M}}$. The subgroup of all units in \mathcal{M} is denoted by \mathcal{M}^\times .

Definition 2.9. *A **partially ordered monoid** is a monoid \mathcal{M} with a partial order \leq that respects the monoid operation:*

$$a \leq b \quad \text{implies} \quad ca \leq cb, \quad ac \leq bc, \quad (2.9)$$

for all $a, b, c \in \mathcal{M}$. A monoid \mathcal{M} is **ordered** if the ordering \leq is a total order.

When working with pointed ordered monoid we usually assume that $\mathbb{0}_{\mathcal{M}}$ is a minimal (or maximal) element in \mathcal{M} .

Definition 2.10. *A **homomorphism** of monoids is a map $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ that respects the monoid operation:*

- (i) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$;
- (ii) $\varphi(\mathbb{1}_{\mathcal{M}}) = \mathbb{1}_{\mathcal{N}}$.

φ is called **local monoid homomorphism**, if $\varphi^{-1}(\mathcal{N}^\times) = \mathcal{M}^\times$ (every φ satisfies “ \supset ”).

A standard general reference for structural theory of semirings is [16].

Definition 2.11. *A (unital ²) **semiring** $R := (R, +, \cdot)$ is a set R equipped with two (associative) binary operations $(+)$ and (\cdot) , addition and multiplication respectively, such that:*

- (i) $(R, +)$ is an abelian monoid with identity $\mathbb{0}_R$;
- (ii) (R, \cdot) is a monoid with element $\mathbb{1}_R$;
- (iii) $\mathbb{0}_R$ is an absorbing element, i.e., $a \cdot \mathbb{0}_R = \mathbb{0}_R \cdot a = \mathbb{0}_R$ for every $a \in R$;
- (iv) multiplication distributes over addition.

R is a **commutative semiring**, if $ab = ba$ for all $a, b \in R$.

¹The intersection of all congruences containing $\mathfrak{A}_1 * \mathfrak{A}_2$ for a given operation $*$, possibly subject to certain properties, implicitly produces the minimal transitive closure of $\mathfrak{A}_1 * \mathfrak{A}_2$.

²The given definition is for a unital semiring. But, as in this paper deals only with unital semirings, we call it semiring, for short.

A semiring R is said to be **idempotent semiring**, if $a + a = a$ for every $a \in R$. It is called **bipotent** (sometimes called **selective**) if $a + b \in \{a, b\}$ for any a, b . For example, the max-plus semiring $\mathbb{R}_{-\infty} := (\mathbb{R} \cup \{-\infty\}, \max, +)$ is a bipotent semiring with $\mathbb{1}_{\mathbb{R}_{-\infty}} = 0$ and $\mathbb{0}_{\mathbb{R}_{-\infty}} = -\infty$, see e.g. [57].

Remark 2.12. Any (totally) ordered monoid (\mathcal{M}, \cdot) gives rise to a bipotent semiring. We formally add the zero element $\mathbb{0}$ as the smallest element, employed also as absorbing element, and define the idempotent addition $a + b$ to be $\max\{a, b\}$. Indeed, this is a semiring. Associativity is clear, and distributivity follows from (2.9).

To designate meaningful sums in semirings, we exclude inessential terms in the following sense.

Definition 2.13. A sum $\sum_i a_i$ of elements is said to be **redundant** if $\sum_i a_i = \sum_{j \neq i} a_j$ for some i ; in this case a_i is said to be **inessential**. Otherwise, $\sum_i a_i$ is called **reduced**, in which each a_i is **essential**.

For example, in an idempotent semiring every sum of the form $a + a$ is a redundant sum, while $\mathbb{0}$ is always inessential.

Definition 2.14. An **ideal** \mathfrak{a} of a semiring $R := (R, +, \cdot)$, written $\mathfrak{a} \triangleleft R$, is an additive submonoid of $(R, +)$ such that $ab \in \mathfrak{a}$ and $ba \in \mathfrak{a}$ for all $a \in \mathfrak{a}$ and $b \in R$.

An ideal \mathfrak{a} of R determines a ‘‘Rees type’’ congruence $\mathfrak{A}_{\text{Rees}}(\mathfrak{a})$ on R by letting $(a, b) \in \mathfrak{A}_{\text{Rees}}(\mathfrak{a})$ iff $a, b \in \mathfrak{a}$. Then, every element of $R \setminus \mathfrak{a}$ is isolated with respect to $\mathfrak{A}_{\text{Rees}}(\mathfrak{a})$, so $\mathfrak{A}_{\text{Rees}}(\mathfrak{a})$ includes no nontrivial relations on $R \setminus \mathfrak{a}$, which makes it very limited.

Semiring ideals do not have the extensive role as ideals have in ring theory, since, due to the lack of negation, they do not determine congruences naturally. Furthermore, their correspondence to kernels of semiring homomorphisms is not so obvious. Yet, they are useful for classifying special substructures of semirings, and we employ them only for this purpose. The passage to factor semirings is done by semiring congruences, which are a particular case of congruences on algebras, cf. §2.1.

Definition 2.15. The **equaliser** of two elements a, b in a semiring R is the set

$$\text{Eq}(a, b) = \{s \in R \mid sa = sb, as = bs\},$$

which might be empty.

When R is a commutative semiring, the equaliser $\text{Eq}(a, b)$ is a semiring ideal of R , often playing the role of annihilators in ring theory.

Definition 2.16. A **homomorphism** of semirings is a map $\varphi : R \rightarrow S$ that preserves addition and multiplication. To wit, φ satisfies the following properties for all $a, b \in R$:

- (i) $\varphi(a + b) = \varphi(a) + \varphi(b)$;
- (ii) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$;
- (iii) $\varphi(\mathbb{0}_R) = \mathbb{0}_S$;

A **unital** semiring homomorphism is a semiring homomorphism that preserves the multiplicative identity, i.e., $\varphi(\mathbb{1}_R) = \mathbb{1}_S$.

In the sequel, unless otherwise is specified, semiring homomorphisms are all assumed to be unital.

2.3. Monoid localization.

Recall the well-known construction of the **localization** $C^{-1}\mathcal{M}$ of an abelian monoid $\mathcal{M} := (\mathcal{M}, \cdot)$ by a cancellative submonoid C , cf. Bourbaki [1]. The elements of $C^{-1}\mathcal{M}$ are the fractions $\frac{a}{c}$ with $a \in \mathcal{M}$ and $c \in C$, where

$$\frac{a}{c} = \frac{a'}{c'} \quad \text{iff} \quad ac' = a'c,$$

and multiplication given by

$$\frac{a_1}{c_1} \frac{a_2}{c_2} = \frac{a_1 a_2}{c_1 c_2}.$$

(Although many texts treat localization for algebras, precisely the same constructions and proofs work for monoids.) This construction is easily generalized to non-cancellative submonoids, where now

$$\frac{a}{c} = \frac{a'}{c'} \quad \text{iff} \quad ac'c'' = a'cc'' \quad \text{for some } c'' \in C. \quad (2.10)$$

(In the case of a pointed monoid \mathcal{M} , we assume that $0_{\mathcal{M}} \notin C$.) If the monoid \mathcal{M} is ordered, then $C^{-1}\mathcal{M}$ is also ordered, by letting $\frac{a}{c} \leq \frac{a'}{c'}$ iff $ac'c'' \leq a'cc''$ for some $c'' \in C$.

Any monoid homomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$, for which $\varphi(c)$ is invertible for every $c \in C$, extends naturally to a unique monoid homomorphism $\widehat{\varphi} : C^{-1}\mathcal{M} \rightarrow \mathcal{N}$, given by

$$\widehat{\varphi}\left(\frac{a}{c}\right) = \varphi(a)\varphi(c)^{-1}.$$

If \mathcal{M} is endowed also with addition (+), then (+) extends to $C^{-1}\mathcal{M}$ via the rule:

$$\frac{a_1}{c_1} + \frac{a_2}{c_2} = \frac{c_2a_1 + c_1a_2}{c_1c_2}.$$

Each of the basic properties (associativity of (+) and distributivity) can be extended straightforwardly from \mathcal{M} to $C^{-1}\mathcal{M}$.

For a strictly cancellative monoid \mathcal{M} (and thus without $0_{\mathcal{M}}$), or when $C = \mathcal{M}$, we write $Q(\mathcal{M})$ for the localization $C^{-1}\mathcal{M}$. Therefore, $Q(\mathcal{M})$ is an abelian group, since $(\frac{a}{c})^{-1} = \frac{c}{a}$.

2.4. Modules over semirings.

Modules (called also semi-module in the literature) over semirings are defined similarly to modules over rings [42].

Definition 2.17. A (left) *R-module* over a semiring R is an abelian monoid $M := (M, +)$ with an operation $R \times M \rightarrow M$ – a left action of R – which is associative and distributive over addition, and which satisfies $a0_M = 0_M = 0_Rv$, $1_Rv = v$ for all a in R , $v \in M$.

An *R-module homomorphism* is a map $\varphi : M \rightarrow N$ that satisfies the conditions:

- (i) $\varphi(v + w) = \varphi(v) + \varphi(w)$ for all $v, w \in M$;
- (ii) $\varphi(av) = a\varphi(v)$ for all $a \in R$ and $v \in M$.

$\text{Hom}_R(M, N)$ denotes the set of R -module homomorphisms from M to N .

Right modules are defined dually. If R and S are semirings and M is a left R -module and a right S -module, then M is called (R, S) -**bimodule**, if $(av)b = a(vb)$ for all $a \in R$, $v \in M$, $b \in S$.

Example 2.18. Let R be a semiring.

- (i) A semiring ideal $\mathfrak{a} \triangleleft R$ is an R -module, cf. Definition 2.14.
- (ii) The direct sum of R -modules is clearly an R -module. In particular, we define $R^{(n)}$ to be the direct sum of n copies of R .

Congruences of monoids, cf. §2.1, extend directory to modules.

Definition 2.19. An *R-module congruence* is a monoid congruence on M that satisfies the additional property that, if $v \equiv w$, then $av \equiv aw$ for all $v, w \in M$ and $a \in R$.

3. SUPERTROPICAL STRUCTURES

To develop a solid algebraic theory, we introduce a new monoid structure which leads to existing definitions of supertropical structures [21, 22, 32, 39], these are later generalized by applying additional modifications.

3.1. Additive ν -monoids.

We start with our basic underlying additive structure.

Definition 3.1. An *additive ν -monoid* is a triplet $\mathcal{M} := (\mathcal{M}, \mathcal{G}, \nu)$, where $\mathcal{M} = (\mathcal{M}, +)$ is an additive abelian monoid³, \mathcal{G} is a distinguished partially ordered submonoid of \mathcal{M} with $0_{\mathcal{M}} < a$ for all $a \in \mathcal{G}$, and $\nu : \mathcal{M} \rightarrow \mathcal{G}$ is an idempotent monoid homomorphism (i.e., $\nu^2 = \nu$)—a projection on \mathcal{G} —satisfying for every $a, b \in \mathcal{M}$ the conditions:

- NM1: $a + b = a$ whenever $\nu(a) > \nu(b)$,
- NM2: $a + b = \nu(a)$ whenever $\nu(a) = \nu(b)$,

³Supertropical monoids in [24] regard with a multiplicative monoid structure, which does not involve the monoid ordering, and are of a different nature.

NM3: If $a + b \notin \mathcal{G}$ and $a + \nu(b) \in \mathcal{G}$, then $a + b = \nu(a) + b$.

The element $0_{\mathcal{M}}$ is the monoid identity: $0_{\mathcal{M}} + a = a + 0_{\mathcal{M}} = a$ for every $a \in \mathcal{M}$.

By definition $\nu(a) = a$ for every $a \in \mathcal{G}$, and $0_{\mathcal{M}} \in \mathcal{G}$, since \mathcal{G} is an (additive) submonoid of \mathcal{M} . As ν is a monoid homomorphism, we have $\nu(a + b) = \nu(a) + \nu(b)$ for any $a, b \in \mathcal{M}$. We often term \mathcal{M} a ν -**monoid**, for short.

In the extreme case that the partial ordering of \mathcal{G} is degenerate, i.e., only trivial relations $a = a$ occur, Axiom NM1 is dismissed. Replacing Axiom NM2 by the (weaker) axiom

NM2': $a + a = \nu(a) + a = \nu(a)$ for all $a \in \mathcal{M}$,

Axiom NM3 implies NM2. Indeed, suppose $\nu(a) = \nu(b)$ but $a + b \notin \mathcal{G}$, then $a + \nu(b) = a + \nu(a) \in \mathcal{G}$, and hence $a + b = \nu(a) + b = \nu(b) + b \in \mathcal{G}$ – a contradiction. Literally, Axiom NM3 gives a condition whether the contribution of a to a sum $a + b$ is “idempotent”, namely $a + b = a + a + b = \nu(a) + b$. This includes that case that $a + b$ is redundant (Definition 2.13), i.e., $a + b = \nu(a) + b = b$.

The elements of \mathcal{G} are termed **ghost elements** and \mathcal{G} is called the **ghost submonoid** of \mathcal{M} . An element $a \notin \mathcal{G}$ is said to be **non-ghost**. Although, $0_{\mathcal{M}} \in \mathcal{G}$, as an exceptional case, it is sometimes considered as a non-ghost element. The projection map $\nu : \mathcal{M} \rightarrow \mathcal{G}$ is called the **ghost map** of \mathcal{M} . We write a^ν for the image $\nu(a)$ of a in \mathcal{G} . (\mathcal{M} might have elements $a \in \mathcal{G}$ without ν -preimage in $\mathcal{M} \setminus \mathcal{G}$.) A ν -monoid \mathcal{M} is said to be **ghost monoid**, if $\mathcal{M} = \mathcal{G}$, i.e., if all its elements are ghost. In this case, the ghost $\nu = \text{id}_{\mathcal{M}}$ is the identity map.

We say that elements a and b in \mathcal{M} are ν -**equivalent**, written $a \cong_\nu b$, if $a^\nu = b^\nu$. That is

$$a \cong_\nu b \iff \nu(a) = \nu(b). \quad (3.1)$$

The ν -**fiber** of an element $a \in \mathcal{M}$ is the set

$$\text{fib}_\nu(a) := \{b \in \mathcal{M} \mid b \cong_\nu a\}, \quad (3.2)$$

usually considered for $a \in \mathcal{G}$. Accordingly, for any $b, b' \in \text{fib}_\nu(a)$ we have $b + b' = a^\nu$, while $\text{fib}_\nu(a) \cap \mathcal{G} = \{a^\nu\}$ for every $a \in \mathcal{M}$.

Remark 3.2. One can easily verify the following properties, obtained from Axioms NM1, NM2 and the ordering of \mathcal{G} .

- (a) $a + b = 0_{\mathcal{M}} \Rightarrow a = b = 0_{\mathcal{M}}$.⁴
- (b) $\nu(a) = 0_{\mathcal{M}} \Rightarrow a = 0_{\mathcal{M}}$.
- (c) $a + a = a + a^\nu = a^\nu$ for every $a \in \mathcal{M}$.

We define the **ghost surpassing** relation:

$$a \models b \quad \text{if } a = b + c \text{ for some } c \in \mathcal{G}. \quad (3.3)$$

Accordingly, since \mathcal{G} is a submonoid, if $b \in \mathcal{G}$ and $a \models b$, then $a \in \mathcal{G}$. Also, we always have $a^\nu \models a$, while $a \models a^\nu$ only when $a \in \mathcal{G}$.

Given a ν -monoid $\mathcal{M} := (\mathcal{M}, \mathcal{G}, \nu)$, one can recover the partial ordering of \mathcal{G} from the additive structure of \mathcal{M} . That is, $a > b$ iff $a + b = a$, for $a, b \in \mathcal{G}$. The ordering of $\mathcal{G} \subseteq \mathcal{M}$ induces a (partial) ν -ordering \geq_ν on the whole \mathcal{M} via the ghost map, defined as

$$\begin{aligned} a \geq_\nu b &\iff a^\nu \geq b^\nu, \\ a >_\nu b &\iff a^\nu > b^\nu, \end{aligned} \quad (3.4)$$

in which $a \geq_\nu 0_{\mathcal{M}}$ for every $a \in \mathcal{M}$.

Example 3.3. In the degenerate case that \mathcal{M} is a ν -monoid with $\mathcal{G} = \{0_{\mathcal{M}}, a\}$ and ghost map $\nu : b \mapsto a$ for all $b \neq 0_{\mathcal{M}}$ in \mathcal{M} , by Axiom NM2, $b + c = a$ for all $b, c \neq 0_{\mathcal{M}}$. When $\mathcal{G} = \{0_{\mathcal{M}}\}$, we have $\nu : b \mapsto 0_{\mathcal{M}}$ for all b , implying that $b = b + 0_{\mathcal{M}} = \nu(b) = 0_{\mathcal{M}}$, and thus $\mathcal{M} = \{0_{\mathcal{M}}\}$ is trivial.

For reduced sums $x = \sum_{i \in I} a_i$ and $y = \sum_{j \in J} a_j$ with nonempty finite sets of indices I, J , we define the relation $<$ as

$$x < y \iff I \subset J, \quad (3.5)$$

which introduces an additional partial ordering on \mathcal{M} .

⁴This property is called “lack zero sums” in [28].

Definition 3.4. A *homomorphism* of additive ν -monoids is a monoid homomorphism

$$\varphi : (\mathcal{M}, \mathcal{G}, \nu) \longrightarrow (\mathcal{M}', \mathcal{G}', \nu') \quad (3.6)$$

of ν -monoids, i.e., $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in \mathcal{M}$, where $\varphi(0_{\mathcal{M}}) = 0_{\mathcal{M}'}$. A **gs-morphism** (a short for ghost surpassing morphism) is a map $\phi : \mathcal{M} \longrightarrow \mathcal{M}'$ that satisfies $\phi(a) + \phi(b) \models \phi(a + b)$ for all $a, b \in \mathcal{M}$.

The **image** of φ , denoted by $\text{im}(\varphi)$, is the set $\varphi(\mathcal{M}) \subset \mathcal{M}'$. The **ghost-kernel** of φ , abbreviated **g-kernel**, is defined as

$$\text{gker}(\varphi) := \{a \in \mathcal{M} \mid \varphi(a) \in \mathcal{G}'\} \subset \mathcal{M}.$$

We say that φ is **ghost injective**, if $\text{gker}(\varphi) = \mathcal{G}$.⁵ A homomorphism φ is a **ghost homomorphism**, if $\text{gker}(\varphi) = \mathcal{M}$.

Clearly any gs-morphism is a homomorphism. It easy to verify that $\text{gker}(\varphi)$ is a ν -submonoid of \mathcal{M} containing the ghost submonoid \mathcal{G} . A ν -monoid homomorphism (3.6) respects the ghost map $\nu : \mathcal{M} \longrightarrow \mathcal{G}$, as well as the induced ν -ordering (3.4):

Lemma 3.5. Any homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ of additive ν -monoids satisfies the following properties for all $a, b \in \mathcal{M}$:

- (i) $\varphi(a^\nu) = \varphi(a)^{\nu'}$;
- (ii) $\varphi(\mathcal{G}) \subseteq \mathcal{G}'$, and thus $\mathcal{G} \subseteq \text{gker}(\varphi)$;
- (iii) If $a >_\nu b$, then $\varphi(a) >_{\nu'} \varphi(b)$;
- (iv) If $a \cong_\nu b$, then $\varphi(a) \cong_{\nu'} \varphi(b)$.

Proof. (i): Write $\varphi(a^\nu) = \varphi(a + a) = \varphi(a) + \varphi(a) = \varphi(a)^{\nu'}$.

(ii): Immediate from (i), since any $a \in \mathcal{G}$ satisfies $\varphi(a) = \varphi(a^\nu) = \varphi(a)^{\nu'} \in \mathcal{G}'$.

(iii): If $a >_\nu b$, then $a + b = a$, implying $\varphi(a) = \varphi(a + b) = \varphi(a) + \varphi(b)$, i.e., $\varphi(a) >_{\nu'} \varphi(b)$ by (3.4).

(iv): If $a \cong_\nu b$, then $a + b = a^\nu = b^\nu$, implying $\varphi(a)^{\nu'} = \varphi(a + b) = \varphi(b)^{\nu'}$, i.e., $\varphi(a) \cong_{\nu'} \varphi(b)$. \square

Lemma 3.6. Let $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ be a homomorphism of additive ν -monoids.

- (i) If φ is injective, then $\text{gker}(\varphi) = \mathcal{G}$.
- (ii) If $\text{gker}(\varphi) = \mathcal{G}$, then $a + b \in \mathcal{G}$ for any a, b with $\varphi(a) = \varphi(b)$.

Proof. (i): $\mathcal{G} \subseteq \text{gker}(\varphi)$ by Lemma 3.5.(ii). If $a \in \text{gker}(\varphi) \setminus \mathcal{G}$, then $a \neq a^\nu$, but $\varphi(a)^{\nu'} = \varphi(a^\nu) = \varphi(a)$ by Lemma 3.5.(i), since $a \in \text{gker}(\varphi)$, which contradicts the injectivity of φ .

(ii): $\varphi(a) = \varphi(b) \Rightarrow \varphi(a) + \varphi(b) \in \mathcal{G}' \Rightarrow \varphi(a + b) \in \mathcal{G}' \Rightarrow a + b \in \text{gker}(\varphi) = \mathcal{G}$. \square

Given additive ν -monoids \mathcal{M}' and \mathcal{M} , as customarily, $\text{Hom}(\mathcal{M}, \mathcal{M}')$ denotes the set of all homomorphisms $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$. $\text{Hom}(\mathcal{M}, \mathcal{M}')$ is equipped with a partial ordering, determined for $\varphi, \psi \in \text{Hom}(\mathcal{M}, \mathcal{M}')$ by

$$\varphi >_{\nu_{\text{Hom}}} \psi \iff \varphi(a) >_{\nu'} \psi(a) \text{ for all } a \in \mathcal{M},$$

and further with the ghost map $\nu_{\text{Hom}} : \text{Hom}(\mathcal{M}, \mathcal{M}') \longrightarrow \text{Hom}(\mathcal{M}, \mathcal{G}')$ defined by $\varphi \longmapsto \nu' \circ \varphi$.

Proposition 3.7. $\text{Hom}(\mathcal{M}, \mathcal{M}')$ is a ν -monoid (Definition 3.1).

Proof. The operation $\varphi + \psi$ is well defined, as φ and ψ are homomorphism. It is associative and commutative, since \mathcal{M}' is an abelian monoid. The function $0_{\text{Hom}} : a \longmapsto 0_{\mathcal{M}'}$ is the neutral element for $(+)$. So $\text{Hom}(\mathcal{M}, \mathcal{M}')$ is an abelian monoid. The ghost map ν' of \mathcal{M}' is idempotent, therefore $\nu' \circ \nu' \circ \varphi = \nu' \circ \varphi$, and hence $\nu_{\text{Hom}} = \nu_{\text{Hom}} \circ \nu_{\text{Hom}}$ is also idempotent. The ghost submonoid of $\text{Hom}(\mathcal{M}, \mathcal{M}')$ is the image of ν_{Hom} . Axioms NM1–NM3 are obtained by point-wise verification, provided that they hold in \mathcal{M}' . \square

⁵Note that ghost injectivity does not imply that the restriction $\varphi|_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}'$ is an injective map.

3.2. Colimits, pullbacks, and pushouts.

We denote the category of additive ν -monoids by $\nu\mathbf{Mon}$, whose objects are ν -monoids and its morphisms are ν -monoid homomorphisms.

Proposition 3.8. *Colimits exist for additive ν -monoids.*

Proof. Let I be a directed index set, and let $(\mathcal{M}_i, \varphi_{ij})_{i,j \in I}$ be a direct system of ν -monoids, where $\varphi_{ij} : \mathcal{M}_i \rightarrow \mathcal{M}_j$ are homomorphisms of ν -monoids. Take \sim to be the equivalence determined by $x \sim y$ if $\varphi_{ik}(x) = \varphi_{jk}(y)$ for some $k \geq i, j$ in I , where $x \in \mathcal{M}_i$ and $y \in \mathcal{M}_j$. The direct limit $\widehat{\mathcal{M}} := \varinjlim_{i \in I} \mathcal{M}_i = (\coprod_{i \in I} \mathcal{M}_i) / \sim$ exists in the category of monoids, together with the associated maps $\varphi_i : \mathcal{M}_i \rightarrow \widehat{\mathcal{M}}$. Similarly, $\widehat{\mathcal{G}} := \varinjlim_{i \in I} \mathcal{G}_i = (\coprod_{i \in I} \mathcal{G}_i) / \sim$ is the direct limit of the system $(\mathcal{G}_i, \psi_{ij})_{i,j \in I}$ of ghost submonoids $\mathcal{G}_i \subset \mathcal{M}_i$ such that $\psi_{ij} = \varphi_{ij}|_{\mathcal{G}_i}$. The ghost maps $\nu_i : \mathcal{M}_i \rightarrow \mathcal{G}_i$ satisfy $\psi_{ij} = \nu_j \circ \varphi_{ij}$, and are preserved under φ_{ij} , by Lemma 3.5. Therefore, $(\widehat{\mathcal{M}}, \widehat{\mathcal{G}}, \widehat{\nu})$ is a ν -monoid with ghost map $\widehat{\nu} : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{G}}$. \square

Definition 3.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \nu\mathbf{Mon}$ be ν -monoids with morphisms $\phi_1 : \mathcal{A} \rightarrow \mathcal{C}$ and $\phi_2 : \mathcal{B} \rightarrow \mathcal{C}$. A ν -monoid $\mathcal{P} \in \nu\mathbf{Mon}$, with morphisms $\pi_1 : \mathcal{P} \rightarrow \mathcal{A}$ and $\pi_2 : \mathcal{P} \rightarrow \mathcal{B}$, is a **pullback** along ϕ_1 and ϕ_2 if it renders the following diagram commute and universal,

$$\begin{array}{ccc}
 \mathcal{U} & & \\
 \psi_2 \swarrow & & \searrow \psi_1 \\
 \mathcal{P} & \xrightarrow{\pi_2} & \mathcal{B} \\
 \downarrow \pi_1 & & \downarrow \phi_2 \\
 \mathcal{A} & \xrightarrow{\phi_1} & \mathcal{C}
 \end{array}$$

i.e., for every other ν -monoid \mathcal{U} with morphisms $\psi_1 : \mathcal{U} \rightarrow \mathcal{A}$ and $\psi_2 : \mathcal{U} \rightarrow \mathcal{B}$ there is a unique morphism $\xi : \mathcal{U} \rightarrow \mathcal{P}$ which makes the diagram commute.

In the standard way we define the ν -monoid \mathcal{P} as the subset of the direct product $\mathcal{A} \times \mathcal{B}$:

$$\mathcal{A} \times_{\mathcal{C}} \mathcal{B} := \{(a, b) \in \mathcal{A} \times \mathcal{B} \mid \phi_1(a) = \phi_2(b)\},$$

endowed with component-wise addition (+) and identity element $(0_{\mathcal{A}}, 0_{\mathcal{B}})$; its ghost map is induced from $\mathcal{A} \times \mathcal{B}$.

Proposition 3.10. $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a pullback and it is universal.

Proof. First, to see that $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a ν -monoid, notice that $\phi_1(0_{\mathcal{A}}) = 0_{\mathcal{C}} = \phi_2(0_{\mathcal{B}})$, since ϕ_1 and ϕ_2 are ν -monoid homomorphisms, thus $(0_{\mathcal{A}}, 0_{\mathcal{B}}) \in \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$. Clearly, $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is closed for addition in $\mathcal{A} \times \mathcal{B}$, and is also closed under ν . Indeed, $\phi_1(a) = \phi_2(b) \Rightarrow \phi_1(a^\nu) = \phi_1(a)^\nu = \phi_2(b)^\nu = \phi_2(b^\nu)$.

Observe that $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is set-theoretic pullback. Given $\mathcal{U} \in \nu\mathbf{Mon}$ with $\psi_1 : \mathcal{U} \rightarrow \mathcal{A}$, $\psi_2 : \mathcal{U} \rightarrow \mathcal{B}$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, we have

$$\xi : \mathcal{U} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}, \quad u \longmapsto (\psi_1(u), \psi_2(u)),$$

hence $\phi_1 \circ \pi_1 \circ \xi = \phi_2 \circ \pi_2 \circ \xi$. Then, a routine check shows that there is a one-to-one correspondence between pairs of homomorphisms $(\psi_1, \psi_2) : \mathcal{U} \rightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$ and homomorphisms $\xi : \mathcal{U} \rightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$. \square

Pushout of ν -monoids as usual is the dual notion of a pullback, determined by the diagram

$$\begin{array}{ccc}
 \mathcal{V} & & \\
 \psi_1 \swarrow & & \searrow \psi_2 \\
 \mathcal{P} & \xleftarrow{\pi_2} & \mathcal{B} \\
 \uparrow \pi_1 & & \uparrow \phi_2 \\
 \mathcal{A} & \xleftarrow{\phi_1} & \mathcal{C}
 \end{array}$$

It is universal, where the proof follows similar arguments as in the proof of Proposition 3.10.

3.3. ν -semirings.

Equipping an additive ν -monoid (Definition 3.1) by a multiplicative operation with additional traits, we obtain the following semiring structure, which is the central structure of the study in this paper.

Definition 3.11. A ν -**semiring** is a quadruple $R := (R, \mathcal{T}, \mathcal{G}, \nu)$, where R is a semiring, (R, \mathcal{G}, ν) is an additive ν -monoid, and $\mathcal{T} \subseteq R \setminus \mathcal{G}$ is a distinguished subset, containing a designated subset \mathcal{T}° with $R^\times \subseteq \mathcal{T}^\circ$ which satisfies the conditions:

NS1: $a \in \mathcal{T}^\circ$ implies $a^n \in \mathcal{T}^\circ$ for any $n \in \mathbb{N}$,

NS2: $a + b \in \mathcal{T}$ implies $a + b^\nu \notin \mathcal{G}$, unless $a + b$ is redundant.

The additive identity 0_R is an absorbing element, i.e., $0_R a = a 0_R = 0_R$ for every $a \in R$.

A ν -semiring R is said to be

- **faithful** if $\nu|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$ is injective;
- **commutative** if $ab = ba$ for all $a, b \in R$;
- **definite** if $\mathcal{T} = R \setminus \mathcal{G}$;
- **tame** if any $a \in R \setminus (\mathcal{T} \cup \mathcal{G})$ can be written as $a = c + d^\nu$, where $c, d \in \mathcal{T}$;
- **persistent full** if $\mathcal{T} = \mathcal{T}^\circ$;
- **persistent closed** if \mathcal{T}° is a (multiplicative) monoid;
- **\mathcal{T} -closed** if \mathcal{T} is a monoid.

We set $e_R := \mathbb{1}_R + \mathbb{1}_R = (\mathbb{1}_R)^\nu$, hence $e = e^2 = e + e = e^\nu$ for $e = e_R$. In particular $e \in \mathcal{G}$, and $a^\nu = a + a = a(\mathbb{1}_R + \mathbb{1}_R) = (\mathbb{1}_R + \mathbb{1}_R)a$; thus $a^\nu = ea = ae$ for every $a \in R$. Therefore, in ν -semirings the ghost submonoid \mathcal{G} becomes a semiring ideal (Definition 2.14), called the **ghost ideal** (which can be thought of as a “ghost absorbing” subset). Indeed,

$$ab^\nu = aeb = a(\mathbb{1}_R + \mathbb{1}_R)b = a(b + b) = ab + ab = (ab)^\nu \in \mathcal{G},$$

furthermore $a^\nu b^\nu = eab = e(ab) = (ab)^\nu$. Note also that, for any a_1, \dots, a_n in R ,

$$\left(\sum a_i\right)^\nu = e\left(\sum a_i\right) = \sum ea_i = \sum a_i^\nu, \quad (3.7)$$

which implies that if x is a ghost, then all reduced sums $y < x$ are also ghosts, cf. (3.5). The ν -**equivalence** \cong_ν on R is induced from the ν -equivalence (3.1) of its additive ν -monoid structure (cf. §3.1), respecting multiplication as well, i.e., $a \cong_\nu b$ implies $ca \cong_\nu cb$ for every $c \in R$. On ν -semirings \cong_ν can be stated as $a \cong_\nu b$ iff $ea = eb$.

The elements of the distinguished subset \mathcal{T} are called **tangible elements** and \mathcal{T} is termed the **tangible set** of R .⁶ When R is \mathcal{T} -closed, \mathcal{T} is called the **tangible monoid** of R and we say that R is **tangibly closed**, for emphasis. We write \mathcal{T}_0 for $\mathcal{T} \cup \{0_R\}$, which is a pointed monoid when R is a tangibly closed ν -semiring. The element 0_R can be considered either as ghost or tangible. An element $a \notin \mathcal{T}$ is termed **non-tangible**.

A tangible element $a \in \mathcal{T}^\circ$ is said to be **tangibly persistent**, written **t-persistent**, i.e., if $a^n \in \mathcal{T}^\circ \subseteq \mathcal{T}$ for any $n \in \mathbb{N}$. In particular, every tangible idempotent is t-persistent. The designated subset $\mathcal{T}^\circ \subseteq \mathcal{T}$ is called the **t-persistent set** of R , and is never empty, since $\mathbb{1}_R \in R^\times \subseteq \mathcal{T}^\circ$. \mathcal{T}° may contain a larger monoid than $\{\mathbb{1}_R\}$, which we denote by \mathcal{T}^\bullet and call it a **t-persistent monoid**, or a **tangible monoid**, in which any product of t-persistent elements is t-persistent. Accordingly, for a persistent closed ν -semiring we have $\mathcal{T}^\bullet = \mathcal{T}^\circ$, while $\mathcal{T}^\bullet = \mathcal{T}^\circ = \mathcal{T}$ in a tangibly closed ν -semiring.

Example 3.12. A tangible element $a = b + c$ that can be written as a reduced sum of tangibles $b, c \in \mathcal{T}$ in a commutative ν -semiring is not t-persistent. Indeed, $a^2 = (b + c)^2 = b^2 + ebc + b^2$, which is not tangible, unless the term ebc is inessential.

Literally, Axiom NS1 means that any power of a t-persistent element is t-persistent, but a product of t-persistent elements need not be t-persistent. On the other hand, $ab \in \mathcal{T}$ (reps. $ab \in \mathcal{T}^\circ$) does not imply that $a, b \in \mathcal{T}$ (resp. $a, b \in \mathcal{T}^\circ$).

⁶Intuitively, the tangible elements in a ν -semiring correspond to the original max-plus algebra in Example 3.26 below, although now $a + a = a^\nu$ instead of $a + a = a$. Therefore, as $a + \dots + a = a^\nu$, this formulation encodes an additive multiplicity > 1 , or equivalently the phrase “the sum attends by at least two different terms”. Taking the sum to be maximum, this is the tropical analogy of the vanishing condition in ring theory.

Axiom NS2 strengthens Axiom NM3 of ν -monoids (Definition 3.1) to tangible sums in ν -semirings. Practically, it follows from Axiom NS2 that a reduced tangible sum cannot involve ghosts:

$$a + b \in \mathcal{T} \Rightarrow a, b \notin \mathcal{G}. \quad (3.8)$$

Indeed, otherwise, if $a + b \in \mathcal{T}$, say with $b \in \mathcal{G}$, then both a^ν and b are ghost and hence $a^\nu + b \in \mathcal{G}$, since \mathcal{G} is an ideal, which contradicts Axiom NS2. This implies that if $a \models b$ where $a \in \mathcal{T}$, cf. (3.3), then $a = b$.

Remark 3.13. *In a tame ν -semiring $ab \notin \mathcal{T}$ for any $b \notin \mathcal{T}$, since $b = c + d^\nu$ and $ab = ac + (cd)^\nu \notin \mathcal{T}$ by (3.8).*

Recall that R^\times denotes the subgroup of units in R , where an element $a \in R$ is a **unit**, and said to be **invertible**, if there exists $b \in R$ such that $ab = ba = \mathbb{1}_R$. In this case, b is called the **inverse of a** and is denoted by a^{-1} . A unit in ν -semirings must be \mathfrak{t} -persistent, to stress this we sometimes write \mathcal{T}^\times for R^\times .

Remark 3.14. *In a commutative ν -semiring R , $ab \in R^\times$ implies $a, b \in R^\times$, since $c(ab) = \mathbb{1}_R = (ca)b$ for some $c \in \mathcal{T}$, and hence $b \in R^\times$. Commutativity implies that also $a \in R^\times$.*

To summarize our setting, for a ν -semiring R we have the (multiplicative) components

$$\mathbb{1}_R \in \mathcal{T}^\times = \underset{\text{group}}{R^\times} \subseteq \underset{\text{set}}{\mathcal{T}^\circ} \subseteq \underset{\text{set}}{\mathcal{T}} \subseteq \underset{\text{set}}{R \setminus \mathcal{G}}, \quad e_R \in \underset{\text{ideal}}{\mathcal{G}} \triangleleft R,$$

each of which is nonempty.

We say that two non-invertible elements $a, b \notin R^\times$ are **associates** if $a = bc$ for some $c \in R^\times$. An element $a \notin R^\times$ is **irreducible**, if $a = bc$ implies $b \in R^\times$ or $c \in R^\times$. Accordingly, a ghost element $a \in \mathcal{G}$ is never irreducible, as $a = a^\nu = ea$, where e is not a unit.

Since R is a semiring, where multiplication distributes over addition, and (R, \mathcal{G}, ν) is a partially ν -ordered monoid (3.4), then

$$a >_\nu b \Rightarrow ca >_\nu cb, ac >_\nu bc \quad \text{for any } a, b, c \in R, c \neq \mathbb{0}_R. \quad (3.9)$$

So, the multiplication of R respects the partial ν -ordering of R , and in this sense all nonzero elements may be realized as ‘‘positive’’ elements.

When R is tangibly closed, we have the strong property

$$a \in \mathcal{T} \text{ and } b \in \mathcal{T} \Rightarrow ab \in \mathcal{T},$$

which provides \mathcal{T} as a monoid.

Remark 3.15. *The following structural relations hold:*

- (a) *definite \Rightarrow tame;*
- (b) *tangibly closed \Rightarrow persistent full and persistent closed.*

One can construct a ν -semiring (more precisely a supertropical semiring, as defined below in §3.4) from any ordered monoid.

Construction 3.16. *Given an ordered monoid $\mathcal{M} := (\mathcal{M}, \cdot)$, cf. Definition 2.9, we duplicate \mathcal{M} to have a second copy \mathcal{M}^ν of \mathcal{M} and create the set $\text{STR}(\mathcal{M}) := \mathcal{M} \cup \{\mathbb{0}\} \cup \mathcal{M}^\nu$, where $\mathbb{0}$ is formally added as the smallest element. $\text{STR}(\mathcal{M})$ becomes an ordered set by declaring that*

$$a < a^\nu < b < b^\nu$$

for any $a < b$ in \mathcal{M} . We define the addition of $x, y \in \text{STR}(\mathcal{M})$ as

$$x + y := \max\{x, y\},$$

and take the multiplication induced by the monoid operation, i.e., $ab = a \cdot b$ for $a, b \in \mathcal{M}$ and $ab^\nu = a^\nu b = a^\nu b^\nu = (a \cdot b)^\nu$, where $a^\nu, b^\nu \in \mathcal{M}^\nu$. Accordingly, \mathcal{M} is assigned as the tangible monoid \mathcal{T} , while $\mathcal{M}^\nu \cup \{\mathbb{0}\}$ is allocated as the ghost ideal \mathcal{G} . Then, $\text{STR}(\mathcal{M})$ is endowed with the structure of tangibly closed ν -semiring, which is also faithful and definite, with ghost map $\nu|_{\mathcal{T}} = \text{id}$, in which $\mathbb{0}a = \mathbb{0}a = \mathbb{0}$ is an absorbing element.

The elements of the Cartesian product $R^{(n)} = R \times \cdots \times R$ are n -tuples $\mathbf{a} = (a_1, \dots, a_n)$ with $a_i \in R$. A point $\mathbf{a} \in R^{(n)}$ is **tangible** if $a_i \in \mathcal{T}_0$, for all i , and is ghost if $a_i \in \mathcal{G}$ for every i . We write $(\mathbb{0}_R)$ for the point $(\mathbb{0}_R, \dots, \mathbb{0}_R)$, which belongs to both $\mathcal{T}_0^{(n)}$ and $\mathcal{G}^{(n)}$. Given a subset $E \subset R^{(n)}$, we write $E|_{\text{tng}}^\circ$

for $E \cap (\mathcal{T}^\circ)^{(n)}$ – called the **tangibly persistent part** of E . We write $E|_{\text{tng}}$ for $E \cap \mathcal{T}_0^{(n)}$ – called the **tangible part** of E , $E|_{\text{gh}} := E \cap \mathcal{G}^{(n)}$ is called the **ghost part** of E . The latter parts, $E|_{\text{tng}}$ and $E|_{\text{gh}}$, need not be the complement of each other.

Example 3.17. *Suppose R is a commutative, tame, tangibly closed ν -semiring.*

- (i) *The semiring $R[\lambda_1, \dots, \lambda_n]$ of polynomials over R is a commutative ν -semiring; its tangible set consists of the polynomials with tangible coefficients. This ν -semiring is tame but not persistent full, as the tangible polynomials do not form a monoid. As explained below in §3.9, $R[\lambda_1, \dots, \lambda_n]$ is canonically associated to the persistent full ν -semiring of polynomial functions on $R^{(n)}$.*
- (ii) *The semiring of $n \times n$ matrices over R is a noncommutative ν -semiring, which is tame but not tangibly closed. It contains the subgroup of invertible matrices (i.e., the group of generalized permutation matrices) and the subgroup of diagonal tangible matrices, which give rise to (tame) tangibly closed ν -subsemiring structures.*
- (iii) *The set $R^{(n)}$ of all n -tuples over R , with entry-wise addition and multiplication, is a commutative tangibly closed ν -semiring. The identity element $\mathbb{1}_{R^{(n)}}$ of this ν -semiring can be generated additively and multiplicatively by other elements of $R^{(n)}$.*

Parts (ii) and (iii) provide examples of ν -semiring which are tame, but this property can be dependent on the way that their tangible elements are defined, e.g., if zero entries are allowed in tangible elements.

The present paper deals with commutative structures arising from commutative ν -semirings, for example, from polynomial functions that establish a persistent full ν -semiring, as described below in §3.9. Therefore, in the sequel, **our underlying ν -semirings are always assumed to be commutative**. (A similar but more involved theory can be developed for noncommutative ν -semirings, to cope also with noncommutative structures, e.g., with matrices.)

Although semiring ideals (Definition 2.14) in the present paper are mostly employed to classify special subsets of ν -semirings, rather than for factoring out substructures, for a matter of completeness and for future use, we introduce their special types.

Definition 3.18. *An ideal $\mathfrak{a} \triangleleft R$ of a ν -semiring R is called:*

- (i) **ghost radical**, *if for any $a^n \in \mathfrak{a}$, with $n \in \mathbb{N}$, the following condition holds*

$$a^n \in \mathfrak{a}|_{\text{gh}} \Rightarrow a \in \mathfrak{a}|_{\text{gh}},$$
- (ii) **ghost prime**, *if for any $ab \in \mathfrak{a}$ the following condition holds*

$$ab \in \mathfrak{a}|_{\text{gh}} \Rightarrow a \in \mathfrak{a}|_{\text{gh}} \text{ or } b \in \mathfrak{a}|_{\text{gh}};$$
- (iii) **ghost primary**, *if for any $ab \in \mathfrak{a}$ the following condition holds*

$$ab \in \mathfrak{a}|_{\text{gh}} \Rightarrow a \in \mathfrak{a}|_{\text{gh}} \text{ or } b^n \in \mathfrak{a}|_{\text{gh}} \text{ for some } n \in \mathbb{N};$$
- (iv) **maximal**, *if \mathfrak{a} is proper and maximal with respect to inclusion.*

As one sees, the structural condition on these ideals applies only to ghost products; this is the curtail merit of these ideals.⁷ As will be seen later, this is a conceptual idea in our theory.

3.4. ν -domains and ν -semifields.

To avoid pathological cases, e.g., as in Example 3.17, one can turn to more rigid ν -structures which manifest a better behavior. Henceforth, when it is clear from the context, we write $\mathbb{1}$, e , $\mathbb{0}$, for $\mathbb{1}_R$, e_R , $\mathbb{0}_R$, respectively.

Definition 3.19. *An element $a \notin \mathcal{G}$ in a ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ is said to be a **ghost divisor** if there exists an element $b \notin \mathcal{G}$ such that $ab \in \mathcal{G}$ or $ba \in \mathcal{G}$. It is a **zero divisor** if $ab = ba = \mathbb{0}$. We denote the set of all ghost divisors in R by $\text{gdiv}(R)$.*

*A tangibly closed ν -semiring R is called **ν -domain**, if it contains no ghost divisors.⁸ A commutative ν -domain is called **integral ν -domain**. If furthermore \mathcal{T} is an abelian group, then R is called **ν -semifield**.*

⁷These types of ideals here are different from the ideals studied in [37, 38].

⁸This definition changes the definition provided in [39], which is based on cancellativity of multiplication. The latter definition does not suite here, as seen in Examples 3.22 below.

In the sequel, we restrict to commutative structures, and write ν -domain for integral ν -domain, for short.

Clearly any zero divisor is a ghost divisor. Suppose that a is a ghost divisor with $ab \in \mathcal{G}$, where $b \notin \mathcal{G}$, then any product ca with $ca \notin \mathcal{G}$ is also a ghostdivisor. This shows that $\text{gdiv}(R) \cup \mathcal{G}$ is a monoid (with R commutative). Moreover, a cannot be a unit, since otherwise $a^{-1}ab = b \in \mathcal{G}$ – a contradiction. If a is \mathfrak{t} -persistent in a \mathfrak{t} -persistent monoid \mathcal{T}^\bullet , then $b \notin \mathcal{T}^\bullet$, as \mathcal{T}^\bullet is a tangible monoid.

Remark 3.20. *Suppose that $a, b \notin \text{gdiv}(R)$, then $ab \notin \text{gdiv}(R)$. Indeed, otherwise $(ab)c \in \mathcal{G}$ with $c \notin \mathcal{G}$ implies $a(bc) \in \mathcal{G}$, where $bc \notin \mathcal{G}$ since $b \notin \text{gdiv}(R)$, and thus $a \in \text{gdiv}(R)$ – a contradiction.*

We have the following trivial example.

Example 3.21. *A definite tangibly closed ν -semiring has no ghost divisors, cf. Remark 3.15, and therefore it is a ν -domain.*

In comparison to zero divisors in rings, ghost divisors in ν -semirings appear much often, both in commutative and noncommutative ν -semirings.

Example 3.22. *Let $a = b + ec$ be a reduced sum, where $b, c \in \mathcal{T}$ are tangibles and R is commutative. Then*

$$(b + ec)(eb + c) = eb^2 + bc + ebc + ec^2 = eb^2 + ebc + ec^2.$$

and thus a is ghost divisor. Furthermore, writing

$$(b + ec)(eb + c) = (eb + ec)(eb + c) = (b + ec)(eb + ec)$$

shows that R is not cancellative with respect to multiplication, and also that unique factorization fails.

Similarly, even when $a = b + c$ is tangible we have $a^2 = (b + c)^2 = b^2 + ebc + c^2$, where

$$(b^2 + ebc + c^2)(eb^2 + bc + ec^2) = e(b^4 + b^3c + b^2c^2 + bc^3 + c^4).$$

Thus a^2 is ghost divisor (unless ebc is inessential), implying that a is a ghost divisor.

Lemma 3.23. *Let R be a commutative tame ν -semiring.*

- (i) *Every $a \in R \setminus (\mathcal{T}^\circ \cup \mathcal{G})$ is a ghost divisor.*
- (ii) *$\mathcal{T}^\circ \setminus \text{gdiv}(R)$ is a multiplicative monoid.*
- (iii) *If a product ab is \mathfrak{t} -persistent, then a and b are both \mathfrak{t} -persistent.*

Proof. (i): Assume first that $a \in R \setminus (\mathcal{T} \cup \mathcal{G})$. Since R is tame, we can write $a = c + ed$, where $c, d \in \mathcal{T}$, which is a ghost divisor, by Example 3.22. If $a \in R \setminus (\mathcal{T}^\circ \cup \mathcal{G})$, then for some m either $a^m \in \mathcal{G}$ or $a^m \in R \setminus (\mathcal{T} \cup \mathcal{G})$, and hence a is a ghost divisor. Indeed, for the latter take the minimal m so that a^m is a ghost divisor and there is $b \notin \mathcal{G}$ such that $a^mb \in \mathcal{G}$. Write $a^mb = a(a^{m-1}b)$, where $a^{m-1}b \notin \mathcal{G}$, since otherwise we would contradict to the minimality of m .

(ii): Let $a, b \in \mathcal{T}^\circ \setminus \text{gdiv}(R)$, and assume that $ab \notin \mathcal{T}^\circ$. The product ab is not ghost since $a, b \in \mathcal{T}^\circ \setminus \text{gdiv}(R)$. Then, also bab is not ghost, since $b \notin \text{gdiv}(R)$ and $ab \notin \mathcal{G}$, implying that $(ab)^2 \notin \mathcal{G}$, and iteratively that $(ab)^n \notin \mathcal{G}$. So, $(ab)^n \in R \setminus (\mathcal{T} \cup \mathcal{G})$ for some n , and thus $(ab)^n = c + ed$, with $c, d \in \mathcal{T}$. But then $(ab)^n$ is a ghost divisor by part (i), and there is $q \notin \mathcal{G}$ such that $(ab)^nq \in \mathcal{G}$. Now $bq \notin \mathcal{G}$ since $b \notin \text{gdiv}(R)$, so if $a(bq) \in \mathcal{G}$, then $a \in \text{gdiv}(R)$ – a contradiction. Iteratively, we get that $(ab)^nq \in \mathcal{G}$, where $b(ab)^{n+1} \notin \mathcal{G}$, contradictively implying $a \in \text{gdiv}(R)$.

(iii): Suppose $a \notin \mathcal{T}^\circ$, then there exists m such that $a^m \notin \mathcal{T}$, where $a^m \notin \mathcal{G}$, since otherwise $(ab)^m \in \mathcal{G}$. Hence $a^m = c + ed$, with $c, d \in \mathcal{T}$. But, then $(ab)^m = a^mb^m = (c + ed)b^m = cb^m + edb^m$ – contradicting (3.8). \square

3.5. Supertropical semirings.

So far all our ν -structures have been considered to have an (additive) ghost submonoid which is partially ordered (Definition 3.11). The next supertropical structures strengthen this property to totally ordered submonoids [32].⁹

⁹The present paper develops the theory in the more abstract setting of ν -semirings and supertropical structures are brought as particular examples. Supertropical semirings were employed for an extensive development of linear and matrix algebra [25, 26, 33, 34, 35].

Definition 3.24. A *supertropical semiring* $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ is a ν -semiring whose ghost ideal is totally ordered. A *supertropical (integral) domain* is a (commutative) definite tangibly closed ν -semiring R . i.e., $\mathcal{T}^\circ = \mathcal{T} = R \setminus \mathcal{G}$ is a abelian monoid, in which the restriction $\nu|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$ is onto. A *supertropical semifield* is a supertropical integral domain whose tangible monoid is an abelian group, i.e., $R^\times = \mathcal{T}$.

In a supertropical domain R the tangible set $\mathcal{T}^\circ = \mathcal{T} = R \setminus \mathcal{G}$ is a monoid, which directly implies that R has no ghost divisors (Example 3.21), and thus it is a ν -domain. Furthermore, R is tame, as it is definite (cf. Remark 3.15). In addition, the ν -fiber (3.2) of each ghost $b \in \mathcal{G}$ contains a tangible element $a \in \mathcal{T}$, not necessarily unique.

Example 3.25. Letting the ordered monoid \mathcal{M} in Construction 3.16 be an ordered abelian group, the ν -semiring $\text{STR}(\mathcal{M})$ is then a supertropical semifield.

Particular examples for $\text{STR}(\mathcal{M})$ are obtained by taking \mathcal{M} to be $(\mathbb{N}_0, +)$, $(\mathbb{Z}, +)$, and $(\mathbb{Q}, +)$, where $+$ stands for the standard summation. The ν -semirings $\text{STR}(\mathbb{Z})$ and $\text{STR}(\mathbb{Q})$ are supertropical semifields, while $\text{STR}(\mathbb{N}_0)$ is not a supertropical semifield, since $\text{STR}(\mathbb{N}_0)|_{\text{tng}} = \mathbb{N}_0$ where $\mathbb{N}_0^\times = 0$. But, it is a supertropical domain (Definition 3.24).

The next example establishes the natural extension of the familiar (tropical) max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and its connection to standard tropical geometry [20].

Example 3.26. Our main supertropical example is the *extended tropical semifield* [21], that is $\mathbb{T} := \text{STR}(\mathcal{M})$ with $\mathcal{M} = (\mathbb{R}, +)$ ordered traditionally, cf. Construction 3.16. Explicitly,

$$\mathbb{T} := \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu,$$

with $\mathcal{T} = \mathbb{R}$, $\mathcal{G} = \mathbb{R}^\nu \cup \{-\infty\}$, where the restriction of the ghost map $\nu|_{\mathcal{T}} : \mathbb{R} \rightarrow \mathbb{R}^\nu$ is the identity map and addition and multiplication are induced respectively by the maximum and standard summation of the real numbers [21]. The supertropical semifield \mathbb{T} extends the familiar max-plus semifield, which is the underlying structure of tropical geometry. It serves as a main numerical examples, which we traditionally call *logarithmic notation* (in particular $\mathbb{1} = 0$ and $0 = -\infty$).

The following example presents the smallest finite supertropical semifield, extending the well-known boolean algebra.

Example 3.27. The *superboolean semifield* $\mathbb{B} := (\{\mathbb{1}, 0, \mathbb{1}^\nu\}, \{\mathbb{1}\}, \{0, \mathbb{1}^\nu\}, \nu)$ is a three element supertropical semifield, extending the boolean semiring $(\{0, \mathbb{1}\}, \vee, \wedge)$, endowed with the following addition and multiplication:

$+$	0	$\mathbb{1}$	$\mathbb{1}^\nu$	\cdot	0	$\mathbb{1}$	$\mathbb{1}^\nu$
0	0	$\mathbb{1}$	$\mathbb{1}^\nu$	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}^\nu$	$\mathbb{1}^\nu$	$\mathbb{1}$	0	$\mathbb{1}$	$\mathbb{1}^\nu$
$\mathbb{1}^\nu$	$\mathbb{1}^\nu$	$\mathbb{1}^\nu$	$\mathbb{1}^\nu$	$\mathbb{1}^\nu$	0	$\mathbb{1}^\nu$	$\mathbb{1}^\nu$

\mathbb{B} is totally ordered as $\mathbb{1}^\nu > \mathbb{1} > 0$. The tangible element of \mathbb{B} is $\mathbb{1}$, while $e = \mathbb{1} + \mathbb{1} = \mathbb{1}^\nu$ is its ghost element; $\mathcal{G} := \{0, \mathbb{1}^\nu\}$ is the ghost ideal of \mathbb{B} with the obvious ghost map $\nu : \mathbb{1} \rightarrow e$.

In other words, by Construction 3.16, the superboolean semifield \mathbb{B} is $\text{STR}(\mathcal{M})$ with \mathcal{M} being the trivial group. This semifield suffices for realizations of matroids, and more generally of finite abstract simplicial complexes, as semimodules [30, 31].

Proposition 3.28 (Frobenius Property [32, Proposition 3.9]). *If R is a supertropical semiring, then*

$$(a + b)^n = a^n + b^n, \quad n \in \mathbb{N},$$

for any $a, b \in R$.

3.6. ν -topology.

Let $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ be a ν -semiring. Given a nonempty subset $U \subseteq R$, using the ν -fibers (3.2) of its members we define the subset

$$\text{fib}_\nu(U) := \{b \in \text{fib}_\nu(a) \mid a \in U\}. \tag{3.10}$$

In particular, if $U \subseteq \mathcal{G}$, then $U \subseteq \text{fib}_\nu(U)$. Employing these subsets, a given topology Ω on the ghost ideal \mathcal{G} induces a topology $\tilde{\Omega}$ on R whose open sets are

$$V \subseteq \text{fib}_\nu(U) \text{ such that } \nu(V) = U, \quad U \text{ is open in } \Omega.$$

Clearly, $\tilde{\Omega}$ admits continuity of semiring operations on V , whenever Ω preserves continuity of operations on U . We call this topology the ν -**topology** induced by Ω . A ν -topology $\tilde{\Omega}$ on R is extended to $R^{(n)}$ by taking the product ν -topology of $\tilde{\Omega}$.

Example 3.29. *In the case of the extended tropical semifield \mathbb{T} in Example 3.26, the Euclidian topology on \mathbb{R}^ν induces a ν -topology on \mathbb{T} which admits continuity of addition. However, this induced ν -topology is not Hausdorff, since a and a^ν cannot be separated by neighbourhoods.*

In general, when \mathcal{G} is totally ordered, we can define a ν -topology directly.

Example 3.30. *A supertropical semiring R , i.e., \mathcal{G} is totally ordered, is endowed with a ν -topology on R having the intervals*

$$V_{a,b} = \{x \in R \mid a <_\nu x <_\nu b\}, \quad a, b \in \mathcal{G},$$

as a base of its open sets.

3.7. Homomorphisms of ν -semirings.

Recall from Lemma 3.5 that $\varphi(\mathcal{G}) \subseteq \mathcal{G}'$ for any homomorphism of ν -monoids $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ (Definition 3.4).

Definition 3.31. *A **homomorphism** of ν -semirings is a semiring homomorphism (Definition 2.16)*

$$\varphi : (R, \mathcal{T}, \mathcal{G}, \nu) \longrightarrow (R', \mathcal{T}', \mathcal{G}', \nu') \quad (3.11)$$

which is also a ν -monoid homomorphism. φ is **unital**, if $\varphi(\mathbb{1}_R) = \mathbb{1}_{R'}$.

A homomorphism φ is a **q-homomorphism**, abbreviation for quotient homomorphism, if $\varphi(a) \in \mathcal{T}'$ implies $a \in \mathcal{T}$ for all $\varphi(a) \in \mathcal{T}'$, that is $\varphi^{-1}(\mathcal{T}') \subseteq \mathcal{T}$.

The **tangible core** and the **persistent tangible core** of a q-homomorphism φ are respectively the subsets

$$\text{tcor}(\varphi) := \{a \in \mathcal{T} \mid \varphi(a) \in \mathcal{T}'\}, \quad \text{tcor}^\circ(\varphi) := \{a \in \mathcal{T} \mid \varphi(a) \in (\mathcal{T}')^\circ\}.$$

We say that φ is **tangibly injective**, if $\varphi(\mathcal{T}) \subseteq \mathcal{T}'$. We call φ a **tangibly local homomorphism**, if $\varphi^{-1}(\mathcal{T}') = \mathcal{T}$, namely if $\text{tcor}(\varphi) = \mathcal{T}$.

In other words, a q-homomorphism $\varphi : R \longrightarrow R'$ maps only tangible elements of R to the tangible set \mathcal{T}' of R' .¹⁰ When φ is unital, it cannot be a ghost homomorphism (Definition 3.4) and $\varphi(e_R) = \varphi(\mathbb{1}_R) + \varphi(\mathbb{1}_R) = \mathbb{1}_{R'} + \mathbb{1}_{R'} = e_{R'}$, which shows again that $\varphi(\mathcal{G}) \subseteq \mathcal{G}'$, since $\varphi(a^\nu) = \varphi(e_{Ra}) = \varphi(e_R)\varphi(a) = e_{R'}\varphi(a)$. In addition, $\varphi(R^\times) \subseteq (R')^\times$, since $\varphi(\mathbb{1}_R) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}) = \mathbb{1}_{R'}$.

By definition, we see that $\text{tcor}^\circ(\varphi) \subseteq \text{tcor}(\varphi)$, where $\text{tcor}^\circ(\varphi)$ of any q-homomorphism φ is nonempty, since φ is unital and thus $\mathbb{1}_R \in \text{tcor}^\circ(\varphi)$.

Lemma 3.32. $\text{tcor}^\circ(\varphi) \subset \mathcal{T}^\circ$ for any q-homomorphism $\varphi : R \longrightarrow R'$.

Proof. Assume $a \in \text{tcor}^\circ(\varphi)$ is not t-persistent, then $a^n \notin \mathcal{T}$ for some n , and thus $a^n \notin \varphi^{-1}(\mathcal{T}')$, as φ is a q-homomorphism. On the other hand, $\varphi(a^n) = \varphi(a)^n \in (\mathcal{T}')^\circ$, and therefore $a^n \in \varphi^{-1}(\mathcal{T}') - a$ contradiction. \square

Proposition 3.33. *Let $\varphi : R \longrightarrow R'$ be a q-homomorphism. Then, $\varphi^{-1}((\mathcal{T}')^\bullet) \subseteq \mathcal{T}^\circ$ is a monoid for any tangible monoid $(\mathcal{T}')^\bullet \subset (\mathcal{T}')^\circ$.*

Proof. $\varphi^{-1}((\mathcal{T}')^\circ) \subseteq \mathcal{T}^\circ$ by the Lemma 3.32. In particular, $\mathbb{1}_R \in \text{tcor}^\circ(\varphi)$, since φ is a q-homomorphism. Assume that $a, b \in \varphi^{-1}((\mathcal{T}')^\bullet)$, i.e., $\varphi(a), \varphi(b) \in (\mathcal{T}')^\bullet$. Then, $\varphi(ab) = \varphi(a)\varphi(b) \in (\mathcal{T}')^\bullet$, hence $ab \in \varphi^{-1}((\mathcal{T}')^\bullet)$, implying that $\varphi^{-1}((\mathcal{T}')^\bullet)$ is a monoid. \square

Corollary 3.34. *If $\varphi : R \longrightarrow R'$ is a q-homomorphism, where R' is a tangibly closed ν -semiring, then $\text{tcor}(\varphi) = \text{tcor}^\circ(\varphi)$ is a monoid.*

Example 3.35. *The superboolean semifield \mathbb{B} (Example 3.27) embeds naturally in any ν -semiring R via*

$$\iota : \mathbb{B} \longrightarrow R, \quad \mathbb{1} \longmapsto \mathbb{1}_R, \quad \mathbb{0} \longmapsto \mathbb{0}_R, \quad \mathbb{1}^\nu \longmapsto e_R.$$

Note that the surjective map $\psi : R \longrightarrow \mathbb{B}$, given by $a \longmapsto \mathbb{1}$ for any $a \in \mathcal{T}$, $\mathbb{0}_R \longmapsto \mathbb{0}$, and $a \longmapsto \mathbb{1}^\nu$ for every $a \in R \setminus \mathcal{T}_0$, is not a homomorphism of ν -semirings.

¹⁰This condition prevents a mapping of non-tangibles of R which are not ghosts to tangibles in R' .

Example 3.36. If $\varphi : R \rightarrow R'$ is a ghost injective homomorphism (Definition 3.4) of supertropical domains, then $\varphi|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}'$ is injective. Indeed, if $a \neq b$, say with tangibles $a >_{\nu} b$, such that $\varphi(a) = \varphi(b)$, then

$$\varphi(a^{\nu}) = \varphi(a)^{\nu'} = \varphi(a) + \varphi(b) = \varphi(a + b) = \varphi(a),$$

so $\varphi(a) \in \mathcal{G}'$ – a contradiction.

It is easy to verify that the ghost kernel $\text{gker}(\varphi)$ (Definition 3.4) of a ν -semiring homomorphism $\varphi : R \rightarrow R'$ is a semiring ideal containing the ghost ideal \mathcal{G} of R . Clearly, $\text{gker}(\varphi) \cap \text{tcor}(\varphi) = \emptyset$ for any \mathfrak{q} -homomorphism φ of ν -semirings.

Definition 3.37. The category $\nu\mathbf{Smr}$ of ν -semirings, is defined to be the category whose objects are ν -semirings (Definition 3.11) and whose morphisms are \mathfrak{q} -homomorphisms (Definition 3.31).

In what follows, all our objects are taken from the category $\nu\mathbf{Smr}$.

3.8. Localization of ν -semirings.

Let $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ be a commutative ν -semiring, and let $C \subset R$ be a multiplicative submonoid with $\mathbb{1}_R \in C$. We always assume that C is not pointed, i.e., that $\mathbb{0}_R \notin C$. When it is clear from the context, we write $\mathbb{1}$ and $\mathbb{0}$ for $\mathbb{1}_R$ and $\mathbb{0}_R$, for short.

We define the **localization** $C^{-1}R$ of R by C as the monoid localization by a non-cancellative submonoid, as described in §2.3. To wit, this localization is determined by the equivalence \sim_C on $R \times C$ given as

$$(a, c) \sim_C (a', c') \quad \text{iff} \quad ac'c'' = a'cc'' \quad \text{for some } c'' \in C, \quad (3.12)$$

written $\frac{a}{c} = \frac{a'}{c'}$. The addition and multiplication of $C^{-1}R$ are defined respectively via

$$\frac{a_1}{c_1} + \frac{a_2}{c_2} = \frac{c_2a_1 + c_1a_2}{c_1c_2}, \quad \frac{a_1}{c_1} \frac{a_2}{c_2} = \frac{a_1a_2}{c_1c_2},$$

for $a_1, a_2 \in R$, $c_1, c_2 \in C$. Then, $C^{-1}R$ could become a ν -semiring by defining $\frac{a_i}{c_j}$ to be tangible if and only if both a_i and c_j are tangibles in R , and letting $\frac{a_i}{c_j}$ be ghost if a_i or c_j is ghost in R . But then, the elements $\frac{a}{c}$ are not necessarily invertible, as $c \in C$ could be non-tangible. Moreover, taking an arbitrary monoid C does not suit here, as seen by the next remark.

Remark 3.38. The idea of localization by non-tangible elements has a major defect. For example, suppose that $a = p + eq$ is a non-tangible element in C , where $p, q \in \mathcal{T}$. In this case we would have

$$\mathbb{1} = \frac{a}{a} = \frac{p + eq}{p + eq} = \frac{p + eq + eq}{p + eq} = \frac{p + eq}{p + eq} + \frac{eq}{p + eq} = \mathbb{1} + e \left(\frac{q}{p + eq} \right),$$

which for a large q contradicts NS2, cf. (3.8). (In addition, $\mathbb{1}$ is then a \mathfrak{g} -divisor by Example 3.22, implying contradictively that $b = \mathbb{1}b = \text{ghost}$ for some non-ghost $b \in R \setminus \mathcal{G}$.) Similarly, for $a = eq$ we would have $\mathbb{1} = \frac{a}{a} = \frac{eq}{eq} = \frac{eeq}{eq} = e \frac{eq}{eq} = e$, introducing a contradiction again.

With this nature, to ensure that a localized ν -semiring is well defined, where the elements of C become units in the localization R_C , initially, all the members of C must be tangibles. More precisely, they must be \mathfrak{t} -persistents, i.e., $C \subseteq \mathcal{T}^{\circ}$, since C should be a tangible monoid.

Definition 3.39. When $C \subseteq \mathcal{T}$ is a tangible submonoid, we say that $C^{-1}R$ is a **tangible localization** of R . If $C = \mathcal{T}$ (and hence $\mathcal{T} = \mathcal{T}^{\bullet}$), then $C^{-1}R$ is called the **ν -semiring of fractions** of R and is denoted by $Q(R)$. When R is a ν -domain, $C^{-1}R$ is called the **ν -semifield of fractions** of R .

Note that C may contain \mathfrak{g} -divisors, as far as it is a tangible monoid. Henceforth, we assume that $C \subseteq \mathcal{T}^{\circ}$ is a tangible monoid.

The canonical \mathfrak{q} -homomorphism (Definition 3.31) is given by

$$\tau_C : R \longrightarrow C^{-1}R, \quad a \longmapsto \frac{a}{\mathbb{1}}, \quad (3.13)$$

and is an injection. We identify $C^{-1}R$ with the ν -semiring $(C^{-1}R, C^{-1}\mathcal{T}, C^{-1}\mathcal{G}, \nu')$, whose ghost map is given by $\nu' : \frac{a}{c} \mapsto \frac{a^{\nu}}{c}$, and write

$$R_C := C^{-1}R$$

for the localization of R by C . When the multiplicative submonoid C is generated by a single element $a \in R$, i.e., $C = \{\mathbb{1}, a, a^2, a^3, \dots\}$, we sometimes write R_a for R_C .

Remark 3.40. Let $\tau_C : R \longrightarrow R_C$ be the (canonical) injective \mathfrak{q} -homomorphism (3.13).

(i) The ghost kernel of τ_C is determined as

$$\text{gker}(\tau_C) = \{a \in R \mid ac \in \mathcal{G} \text{ for some } c \in C\}.$$

Indeed, $\frac{a}{\mathbb{1}} = \frac{b^\nu}{\mathbb{1}}$ is ghost in R_C iff $ac = b^\nu c$ for some $c \in C$, but $b^\nu c = (bc)^\nu \in \mathcal{G}$ is ghost, implying that $ac \in \mathcal{G}$.

(ii) For all $c \in C$, $\tau_C(c)$ is a tangible unit in R_C , i.e., $\tau_C(c) \in (R_C)^\times$.

(iii) τ_C is bijective, if C consists of tangible units in R .

For a \mathfrak{q} -homomorphism $\varphi : R \longrightarrow R'$ of ν -semirings, where R' is a tangibly closed ν -semiring, we define the tangible localization R_φ of R by φ to be

$$R_\varphi := R_{\text{tcor}^\circ(\varphi)} = (\text{tcor}^\circ(\varphi))^{-1}R. \quad (3.14)$$

It is well defined, since $\text{tcor}^\circ(\varphi) \subset R$ is a multiplicative tangible submonoid by Corollary 3.33.

Proposition 3.41 (Universal property of tangible localization). *Let R be a ν -semiring, and let R_C be its tangible localization by a (multiplicative) submonoid $C \subseteq \mathcal{T}$. The canonical \mathfrak{q} -homomorphism $\tau_C : R \longrightarrow R_C$ satisfies $\tau_C(C) \subseteq (R_C)^\times$ and it is universal: For any ν -semiring \mathfrak{q} -homomorphism $\varphi : R \longrightarrow S$ with $\varphi(C) \subseteq S^\times$ there is a unique ν -semiring \mathfrak{q} -homomorphism $\hat{\varphi} : R_C \longrightarrow S$ such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\tau_C} & R_C \\ & \searrow \varphi & \downarrow \hat{\varphi} \\ & & S \end{array}$$

commutes. Furthermore, if $\varphi : R \longrightarrow S$ satisfies the same universal property as τ_C does, then the \mathfrak{q} -homomorphism $\hat{\varphi} : R_C \longrightarrow S$ is an isomorphism.

Proof. Uniqueness: For $a \in R$ and $c \in C$, we have $\varphi(a) = \hat{\varphi}\left(\frac{a}{\mathbb{1}}\right) = \hat{\varphi}\left(\frac{a \cdot c}{c \cdot \mathbb{1}}\right) = \hat{\varphi}\left(\frac{a}{c}\right)\varphi(c)$, hence $\hat{\varphi}\left(\frac{a}{c}\right) = \varphi(a)\varphi^{-1}(c)$ since $\varphi(C) \subseteq S^\times$, implying that $\hat{\varphi}$ is uniquely determined by φ .

Existence of a \mathfrak{q} -homomorphism $\hat{\varphi} : R_C \longrightarrow S$: Set $\hat{\varphi}\left(\frac{a}{c}\right) = \varphi(a)\varphi(c)^{-1}$, with $a \in R$ and $c \in C$. Suppose $\frac{a}{c} = \frac{a'}{c'}$, that is $ac'c'' = a'cc''$ for some $c'' \in C$. Then $\varphi(a)\varphi(c')\varphi(c'') = \varphi(a')\varphi(c)\varphi(c'')$, which implies $\varphi(a)\varphi(c') = \varphi(a')\varphi(c)$, since $\varphi(c'')$ is a unit in S , and this is equivalent to $\varphi(a)\varphi(c)^{-1} = \varphi(a')\varphi(c')^{-1}$. Hence, $\hat{\varphi} : R_C \longrightarrow S$ is well-defined, and it is easily checked that $\hat{\varphi}$ satisfies $\varphi = \hat{\varphi} \circ \tau_C$.

Assume that both τ_C and φ are universal in the above sense. Then, $\hat{\varphi} : R_C \longrightarrow S$ satisfies $\varphi = \hat{\varphi} \circ \tau_C$, and there is a \mathfrak{q} -homomorphism $\phi : S \longrightarrow R_C$ such that $\tau_C = \phi \circ \varphi$. Applying the uniqueness part to

$$\text{id}_S \circ \varphi = \hat{\varphi} \circ \tau_C = (\hat{\varphi} \circ \phi) \circ \varphi, \quad \text{id}_{R_C} \circ \tau_C = \phi \circ \varphi = (\phi \circ \hat{\varphi}) \circ \tau_C,$$

we conclude that $\hat{\varphi} \circ \phi = \text{id}_S$ and $\phi \circ \hat{\varphi} = \text{id}_{R_C}$. Consequently, $\hat{\varphi} : R_C \longrightarrow S$ is an isomorphism. \square

Corollary 3.42. *Suppose $C_2 \subset C_1$ are multiplicative submonoids of R , then R_{C_1} is isomorphic to $(R_{C_2})_{\tau_{C_2}(C_1)}$.*

Proof. Once we have the universal property of tangible localization in Proposition 3.41, the proof is similar to the case of rings, e.g., [40, Proposition 7.4]. \square

Lemma 3.43. *Suppose $a \in \mathcal{T}^\circ \setminus \text{gdiv}(R)$ is irreducible \mathfrak{t} -persistent, where R is a tame ν -semiring. If $a^n = bc$, then $b = ua^s$ and $c = va^t$ with u, v units.*

Proof. As a is irreducible, we may assume that a cannot be written as a product of two elements, otherwise we can divide by one terms which is a unit. Since a is \mathfrak{t} -persistent and R is tame, b and c are also \mathfrak{t} -persistent by Lemma 3.23.(iii). Localizing by b , we have $\frac{a^n}{b} = \frac{c}{\mathbb{1}}$, implying that $\frac{a^s}{b} = \frac{c}{\mathbb{1}}$ for some s , since a is irreducible; hence $a^s = b$. \square

3.9. Functions and polynomials.

We repeat some basic definitions from [39], for the reader's convenience. Given a semiring R , in the usual way via point-wise addition and multiplication, we define the semiring $\text{Fun}(X, R)$ of set-theoretic functions from a set X to R . As customarily, we write $f|_Y$ for the restriction of a function f to a nonempty subset Y of X .

For a ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$, $\text{Fun}(X, R)$ is a ν -semiring whose ghost elements are functions defined as $f^\nu(x) = (f(x))^\nu$, for all $x \in X$. Defining tangible functions is more subtle, and includes several possibilities [39]. We elaborate this issue later.

Definition 3.44. *The **ghost locus** of a function $f \in \text{Fun}(X, R)$, with R a ν -semiring, is defined as*

$$\mathcal{Z}(f) := \{x \in X \mid f(x) \in \mathcal{G}\}.$$

*When f is determined by a polynomial, $\mathcal{Z}(f)$ is called an **algebraic set**.*

An equivalent way to define the ghost locus of f is by

$$\mathcal{Z}(f) := \{x \in X \mid f(x) = f^\nu(x)\}. \tag{3.15}$$

By this definition we see that $X = \mathcal{Z}(f^\nu)$ for every ghost function f^ν over any set X . When $X \subset R^{(n)}$, the distinguishing of tangible subsets and ghost subsets in X can be made by a point-wise classification.

Polynomials in n indeterminates $\Lambda := \{\lambda_1, \dots, \lambda_n\}$ over a ν -semiring R are defined as customarily by formulas

$$\sum_{\mathbf{i} \in I} \alpha_{\mathbf{i}} \Lambda^{\mathbf{i}}, \quad \alpha_{\mathbf{i}} \in R,$$

where $I \subset \mathbb{N}^n$ is a finite subset and $\mathbf{i} = (i_1, \dots, i_n)$ is a multi-index. $R[\Lambda] := R[\lambda_1, \dots, \lambda_n]$ denotes the ν -semiring of all polynomials over R , whose addition and multiplication are induced from R in the standard way. A polynomial $f \in R[\Lambda]$ is a **tangible polynomial**, if $\alpha_{\mathbf{i}} \in \mathcal{T}$ for all $\mathbf{i} \in I$. f is a **ghost polynomial**, if $\alpha_{\mathbf{i}} \in \mathcal{G}$ for all $\mathbf{i} \in I$. (Note that, as a function, a tangible polynomial does not necessarily take tangible values everywhere).

Remark 3.45. *Clearly, if R is a tame ν -semiring, then $R[\Lambda]$ is also tame, cf. Example 3.17.(i).*

The polynomial ν -semiring $R[\Lambda]$ is not a tangibly closed ν -semiring (resp. definite ν -semiring), even if R is tangibly closed (resp. definite), as a product (or even powers) of tangible polynomials can be non-tangible (e.g. $(\lambda + a)^2 = \lambda^2 + a^\nu \lambda + a^2$, and $\lambda + a$ is not \mathbf{t} -persistent). Therefore, tangible polynomials do not constitute a monoid. To resolve this drawback, we view polynomials as functions under the natural map

$$\phi : R[\Lambda] \longrightarrow \text{Fun}(X, R),$$

defined by sending a polynomial f to the function $\tilde{f} : \mathbf{a} \longmapsto f(\mathbf{a})$, where $\mathbf{a} = (a_1, \dots, a_n) \in X \subset R^{(n)}$.

We denote the image of $R[\Lambda]$ in $\text{Fun}(X, R)$ by $\text{Pol}(X, R)$, which is a ν -subsemiring of $\text{Fun}(X, R)$. The map ϕ induces a natural congruence \mathfrak{F}_X on X , whose underlying equivalence \cong_X is determined by

$$f \cong_X g \Leftrightarrow \tilde{f}|_X = \tilde{g}|_X.$$

Accordingly, the ν -semiring $\text{Pol}(X, R)$ is isomorphic to $R[\Lambda]/\mathfrak{F}_X$, whose elements are termed **polynomial functions**.¹¹

In this setting, an element $[f] \in R[\Lambda]/\mathfrak{F}_X$ is

- a **tangible polynomial function** if and only if $f \cong_X h$ only to tangible polynomials $h \in R[\Lambda]$,
- a **ghost polynomial function** if and only if $f \cong_X g$ for some ghost polynomial $g \in R[\Lambda]$,
- a **zero function** if and only if $f \cong_X 0$.

We write $f|_X \cong_\nu g|_X$, if $f(\mathbf{a}) \cong_\nu g(\mathbf{a})$ for all $\mathbf{a} \in X$.

As a consequence of these definitions, if R is a tangibly closed ν -semiring, then $R[\Lambda]/\mathfrak{F}_X$ is also tangibly closed. In particular, $F[\Lambda]/\mathfrak{F}_X$ is tangibly closed for any ν -semifield F .

Remark 3.46. *Let $f \in R[\Lambda]/\mathfrak{F}_X$ be a polynomial function.*

¹¹Polynomial functions have been studied in [32], and yielded a supertropical version of Hilbert Nullstellensatz. Their restriction to an algebraic set forms a coordinate semiring [39]. This view provides an algebraic formulation, given in terms of congruences, for the balancing condition used in tropical geometry [20].

- (i) The property of being tangible for f depends on the domain X . A polynomial function f can be tangible over X but non-tangible (or even ghost) over a subset $Y \subset X$, for example take $Y = \mathcal{Z}(f)$. In particular, a tangible polynomial can be a ghost polynomial function over some subsets $X \subset R^{(n)}$.
- (ii) If f is a ghost function over X , then it is ghost over any subset $Y \subseteq X$.
- (iii) If a product fg is ghost, where f is tangible, then g is ghost. However, we may have a ghost product of two non-ghost functions, i.e., a ghost divisor as in Example 3.22.
- (iv) $f \cong_X 0$ iff $f = 0$, independently on the subset X , as long as $X \neq \{0\}$.

When $X = R^{(n)}$, we denote by $\tilde{R}[\Lambda]$ the ν -semiring $\text{Pol}(X, R) = R[\Lambda]/\mathfrak{F}_X$ of polynomial functions. Similarly, we write $\tilde{F}[\Lambda]$ for the tangibly closed ν -semiring $F[\Lambda]/\mathfrak{F}_X$ with F a ν -semifield. (In this case, every tangible polynomial is assigned with a tangible polynomial function.) The restriction of $\tilde{F}[\Lambda]$ to a subset $Y \subset X$, denoted as $\tilde{F}[Y]$, is considered as the **coordinate ν -semiring** of Y , cf. [39].

Remark 3.47. Given a subset $Y \subseteq X$ and a point $\mathbf{a} \in Y$, we define the “evaluation mapping”

$$\epsilon_{\mathbf{a}} : \tilde{R}[Y] \longrightarrow R$$

by $\epsilon_{\mathbf{a}}(f) = f(\mathbf{a})$. If $\epsilon_{\mathbf{a}} = \epsilon_{\mathbf{b}}$ for $\mathbf{a}, \mathbf{b} \in X$, then $\mathbf{a} = \mathbf{b}$. (Indeed, if $\epsilon_{\mathbf{a}} = \epsilon_{\mathbf{b}}$, then $f(\mathbf{a}) = f(\mathbf{b})$ for every $f \in \tilde{R}[X]$, and in particular for $f(\mathbf{x}) = \mathbf{x}$.) The map $\epsilon_{\mathbf{a}}$ is a \mathfrak{q} -homomorphism of ν -semirings, for which an element $\mathbf{a} \in \mathcal{Z}(f)$ iff $f \in \text{gker}(\epsilon_{\mathbf{a}})$.

Any \mathfrak{q} -homomorphism $\varphi : \tilde{R}[Y] \longrightarrow \tilde{R}[Z]$ defines a morphism $\phi : Z \longrightarrow Y$. Indeed, the composition $\epsilon_{\mathbf{a}} \circ \varphi$ is clearly a function $\tilde{R}[Y] \longrightarrow R$, and thus equals $\epsilon_{\mathbf{b}}$ for some $\mathbf{b} \in Y$. Then, the identification ϕ sending $\mathbf{a} \longmapsto \mathbf{b}$ defines a morphism.

The view of polynomial functions over supertropical semirings provides the **Frobenius Property**, written as

$$f^m = \left(\sum_{\mathbf{i}} a_{\mathbf{i}} \Lambda^{\mathbf{i}} \right)^m = \sum_{\mathbf{i}} (a_{\mathbf{i}} \Lambda^{\mathbf{i}})^m, \quad (3.16)$$

which is a direct consequence of the point-wise computation in Proposition 3.28.

Theorem 3.48 ([32, Theorem 8.35]). *Let F be divisibly closed supertropical semifield. Then, any polynomial function $f \in \tilde{F}[\lambda]$ factorizes uniquely to a product of linear and quadratic terms of the form*

$$\lambda + a, \quad e\lambda + a, \quad \lambda + a^\nu, \quad \lambda^2 + b^\nu\lambda + a, \quad \text{with } b > \sqrt{a}.$$

These terms are called primitive elements.

The extended tropical semiring \mathbb{T} in Example 3.26 is a supertropical semifield that admits the conditions of the theorem. However, in general, unique factorization fails for ν -semirings (as seen in Example 3.22), especially for supertropical polynomial functions, even for tangible functions.

Example 3.49. Consider the ν -semiring of polynomial functions $\tilde{F}[\Lambda]$ over a ν -semifield F .

- (i) The tangible polynomial function ¹²

$$\begin{aligned} f = g_1 g_2 &= (\lambda_1 + \lambda_2 + \mathbb{1})(\lambda_1 + \lambda_2 + \lambda_1 \lambda_2) \\ &= \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 + \lambda_1^2 + \lambda_2^2 + e\lambda_1 \lambda_2 + \lambda_1 + \lambda_2 \\ &= (\lambda_1 + \mathbb{1})(\lambda_2 + \mathbb{1})(\lambda_1 + \lambda_2) = h_1 h_2 h_3 \end{aligned}$$

has two different factorizations $g_1 g_2$ and $h_1 h_2 h_3$ [39, Example 5.22]. Note that $h_i < g_1$ for $i = 1, 2, 3$, cf. (3.5), but $\mathcal{Z}(h_i) \not\subseteq \mathcal{Z}(g_1)$.

- (ii) The square of the linear function $f = e\lambda_1 + e\lambda_2 + \mathbb{1}$ can be written as

$$(e\lambda_1 + e\lambda_2 + \mathbb{1})^2 = (e\lambda_1 + \lambda_2 + \mathbb{1})(\lambda_1 + e\lambda_2 + \mathbb{1}) = g_1 g_2,$$

although f is irreducible. But still $f \cong_\nu g_i$, for $i = 1, 2$.

By Lemma 3.43, for powers of tangible polynomial functions in $\tilde{F}[\Lambda]$, with \tilde{F} a tame ν -semifield, we have the following property.

¹²This example holds also for polynomials over the standard tropical semiring.

Remark 3.50. Assume that $f \in \tilde{F}[\Lambda]$ is a tangible irreducible which is not a ghost divisor; thus f is \mathfrak{t} -persistent by Lemma 3.23. If $f^m = gh$, then $g = af^s$ and $h = bf^t$ for some units $a, b \in R^\times$, cf. Lemma 3.43. More generally, this holds for $f \in \tilde{R}[\Lambda]$, where R is a tame tangibly closed ν -semiring.

Considering the ghost locus $\mathcal{Z}(f)$ in Definition 3.44 as the root set of a polynomial function $f \in \tilde{R}[\lambda]$, a very easy analog of the *Fundamental Theorem of Algebra* is obtained.

Proposition 3.51. Over a ν -semiring R , every $f \in \tilde{R}[\lambda]$ which is not a tangible constant has a root.

Proof. As a function, $f(a) \in \mathcal{G}$ for a large enough $a \in \mathcal{G}$, unless f is a tangible constant. \square

Restricting to tangible roots, we recall Proposition 5.8 from [32]:

Proposition 3.52. Over a divisibly closed supertropical semifield F , every $f \in \tilde{F}[\lambda]$ which is not a tangible monomial has a tangible root.

The present paper develops the theory in the general framework of ν -semirings and the above results serve later as particular examples.

3.10. Hyperfields and valuations.

The link of classical theory to ν -semirings is established by valuations. Recall that a **valuation** of a valued field \mathbb{K} is a map

$$\text{val} : \mathbb{K} \longrightarrow \mathcal{T}_0 := \mathcal{T} \cup \{0\}, \quad (0 := -\infty),$$

where $\mathcal{T} := (\mathcal{T}, +)$ is a totally ordered abelian group, that satisfies

- (a) $\text{val}(f \cdot g) = \text{val}(f) + \text{val}(g)$;
- (b) $\text{val}(f + g) \leq \max\{\text{val}(f), \text{val}(g)\}$ with equality if $\text{val}(f) \neq \text{val}(g)$.

By Example 3.25, the ordered group \mathcal{T} extends to the supertropical semifield $\text{STR}(\mathcal{T}) = (\mathcal{T}_0 \cup \mathcal{T}^\nu, \mathcal{T}, \mathcal{T}_0^\nu, \nu)$, where $\text{val} : \mathbb{K} \longrightarrow \mathcal{T}_0$ gives the supervaluation $\text{sval} : \mathbb{K} \longrightarrow \text{STR}(\mathcal{T})$, sending \mathbb{K} to the tangible submonoid of $\text{STR}(\mathcal{T})$, as studied in [22].

A valuation in general is not a homomorphism, as it does not preserve associativity. Yet, one wants to realize this map at least as a ‘‘homomorphic relation’’. To receive such realization, we view $\text{STR}(\mathcal{T})$, usually with $\mathcal{T} = \mathbb{R}$, as a hyperfield [2, 60]. This is done by assigning every tangible $a \in \mathcal{T}_0$ with the singleton $P_a := \{a\} \subset \mathcal{T}_0$, while each ghost $a^\nu \in \mathcal{T}^\nu$ is associated to the subset $P_{a^\nu} := \{b \in \mathcal{T}_0 \mid b \leq a\} \subset \mathcal{T}_0$. The hyperfield operations are induced from the operations of $\text{STR}(\mathcal{T})$:

$$P_x + P_y := P_{x+y}, \quad P_x \cdot P_y := P_{xy}.$$

For $x \neq y$, this construction provides the inclusions

$$P_x \subseteq P_y \quad \text{iff} \quad x \leq_\nu y, \quad \text{for all } x \in \text{STR}(\mathcal{T}), y \in \mathcal{T}^\nu, \quad (3.17)$$

while the (non-unique) inclusion $\text{val}(f) \in P_x$ gives the binary relation

$$\text{val}(f) \in P_x \quad \text{or} \quad \text{val}(f) \notin P_x, \quad (3.18)$$

for every $x \in \text{STR}(\mathcal{T})$ and $f \in \mathbb{K}$.

Proposition 3.53. The relation (3.18) is homomorphic in the sense that

$$\text{val}(fg) \in P_{xy} \quad \text{and} \quad \text{val}(f + g) \in P_{x+y},$$

for $\text{val}(f) \in P_x, \text{val}(g) \in P_y$.

Proof. Let $a = \text{val}(f), b = \text{val}(g)$. Then, $\text{val}(fg) = \text{val}(f) + \text{val}(g) = ab \in P_{ab}$, and

$$\text{val}(f + g) \leq \max\{a, b\} = \begin{cases} a \in P_a = P_{a+b}, & a > b, \\ a \in P_a \subset P_{a+a} = P_{a^\nu}, & a = b, \\ b \in P_b = P_{a+b}, & b > a. \end{cases} \quad \square$$

Example 3.54. Consider the supertropical semifield $\mathbb{T} = \text{STR}((\mathbb{R}, +))$ in Example 3.26. To realize \mathbb{T} as a hyperfield over \mathbb{R} , each $a \in \mathbb{R} \cup \{-\infty\}$ is assigned with the one-element set $\{a\} \subset \mathbb{R} \cup \{-\infty\}$, while $b \in \mathbb{R}^\nu$ is assigned with the closed ray $[-\infty, a] \subset \mathbb{R} \cup \{-\infty\}$.

3.11. A view to polyhedral geometry.

Traditional tropical varieties may be obtained by taking different viewpoints, as outlined below, see e.g. [14]. We write $\mathbb{R}_{-\infty}$ for the max-plus semiring $\mathbb{R}_{-\infty} := \{\mathbb{R} \cup \{-\infty\}, \max, +\}$, where $\mathbb{R}_{-\infty}^{(n)}$ stands for the cartesian product of n copies of $\mathbb{R}_{-\infty}$.

The **amoeba** of a complex affine variety $Y = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{C}\}$ is defined as

$$\mathcal{A}_t(Y) = \{(\log_t |z_1|, \dots, \log_t |z_n|) \mid (z_1, \dots, z_n) \in Y\} \subset \mathbb{R}_{-\infty}^{(n)},$$

which by taking limit $t \rightarrow 0$ degenerates to a **non-Archimedean amoeba** \mathcal{A}_0 in the n -space over the max-plus semiring $\mathbb{R}_{-\infty}$, cf. [14]. \mathcal{A}_0 is a finite polyhedral complex of pure dimension, i.e., all its maximal faces (termed facets) have the same dimension. This symplectic viewpoint leads to the topological definition [20]: a **tropical variety** $X \subset \mathbb{R}^{(n)}$ is a finite rational polyhedral complex of pure dimension whose weighted facets δ carry positive integral values $m(\delta)$ such that for each face σ of codimension 1 in X the following **balancing condition** holds

$$\sum_{\sigma \subset \delta} m(\delta) n_{\sigma}(\delta) = 0, \quad (3.19)$$

where δ runs over all facets of X containing σ , and $n_{\sigma}(\delta)$ is the primitive unit vector normal to σ lying in the cone centered at σ and directed by δ . Thereby, a tropical hypersurface must have (topological) dimension $n - 1$.

The “combinatorial–algebraic” approach to tropical varieties starts with a polynomial $f_{\mathbb{R}}$ over the max-plus semiring $\mathbb{R}_{-\infty}$, which determines a piecewise linear convex function $f_{\mathbb{R}} : \mathbb{R}^{(n)} \rightarrow \mathbb{R}$. Its domain of non-differentiability $\text{Cor}(f_{\mathbb{R}})$, called **corner locus**, defines a tropical hypersurface. In combinatorial sense, $\text{Cor}(f_{\mathbb{R}})$ is the set of points in $\mathbb{R}^{(n)}$ on which the evaluation of $f_{\mathbb{R}}$ is attained by at least two of its monomials. Yet, this formalism is not purely algebraic.

Valuations as described in §3.10 give a direct passage from classical algebraic varieties to tropical varieties [54]. For example, take $\mathcal{T} = \mathbb{R}$ to be the valued group of the field \mathbb{K} of Puiseux series $p(t) = \sum_{q \in Q} c_q t^q$, with $c_q \in \mathbb{C}$ and $Q \subseteq \mathbb{Q}$ bounded from below, where $\text{val} : \mathbb{K} \rightarrow \mathbb{R}_{-\infty}$ is given by

$$\text{val}(p(t)) := \begin{cases} -\min\{q \in Q \mid c_q \neq 0\}, & p(t) \in \mathbb{K}^{\times}, \\ -\infty, & p(t) = 0. \end{cases}$$

A tropical variety is now defined as the closure $\overline{\text{val}(Y)}$ of a subvariety Y of the torus $(\mathbb{K}^{\times})^{(n)}$, where val is applied coordinate-wise to Y . A parallel way is to tropicalize the generating elements of the ideal that determines Y and then to consider the intersection of their corner loci.

Supertropical theory provides a purely algebraic way to capture tropical varieties as ghost loci of systems of polynomials (Definition 3.44). In this setting, standard tropical varieties are a subclass of ghost loci obtained as the tangible domains of tangible polynomial functions [32]. Furthermore, ghost loci allow to frame a larger family of polyhedral objects, including objects whose dimension equals to that of their ambient space.

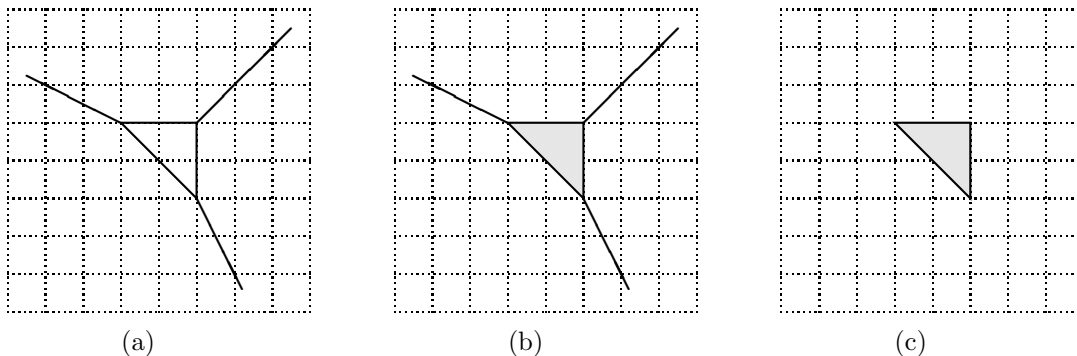


FIGURE 1. Tangible parts of supertropical algebraic sets.

Example 3.55. Let $f = \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 + \alpha \lambda_1 \lambda_2 + 0$ be a polynomial in $\mathbb{T}[\lambda_1, \lambda_2]$, where \mathbb{T} is the extended tropical semiring of Example 3.26. Assume $\alpha <_\nu 0$, and let $X = \mathcal{Z}(f)$ be the ghost locus of f .

- (i) For a tangible $\alpha \in \mathcal{T} = \mathbb{R}$, the restriction $X|_{\text{tng}}$ is a standard tropical elliptic curve, as described in Figure 1.(a).
- (ii) When $\alpha \in \mathcal{G} = \mathbb{R}^\nu$ is a ghost, we obtain the supertropical curve $X|_{\text{tng}}$ illustrated in Figure 1.(b). This type of objects is not accessible by traditional tropical geometry.
- (iii) Consider the set of polynomials $E = \{\alpha^\nu + \lambda_1, \alpha^\nu + \lambda_2, 0 + (-\alpha)^\nu \lambda_1 \lambda_2, \} \subset \mathbb{T}[\lambda_1, \lambda_2]$ with $\alpha > 0$, and let $Y = \mathcal{Z}(E) = \bigcap_{f \in E} \mathcal{Z}(f)$ be ghost locus of E . Then, restricting to tangibles, $Y|_{\text{tng}}$ is the filled triangle given in Figure 1.(c).

In fact, by the same way as in (iii), any convex polytope with m facets having rational slopes can be described as the tangible restriction of the ghost locus of a set of m binomials over \mathbb{T} . More specifically, to capture the geometric features with tropical geometry one can simply use the extension \mathbb{T} of the max-plus semiring $\mathbb{R}_{-\infty}$, as demonstrated Example 3.55.(i).

4. CONGRUENCES ON SUPERTROPICAL STRUCTURES

In this section we employ congruences on ν -semirings, starting with their underlying additive ν -monoid structure, and later concern their multiplicative structure as well. To enable a meaningful passage to quotient structures, congruences in our theory play the customary role of ideals in ring theory. In this view, we study main types of congruences, analogous to types of ideals, and explore their meaning in supertropical theory. Types of these congruences are of different nature, due to the special structure of ν -semirings.

4.1. Congruences on additive ν -monoids.

We denote by $\text{Cong}(\mathcal{M})$ the set of all congruences on a given additive ν -monoid $\mathcal{M} := (\mathcal{M}, \mathcal{G}, \nu)$, cf. Definition 3.1. A congruence $\mathfrak{A} \in \text{Cong}(\mathcal{M})$ is an equivalence relation that respects the ν -monoid operation and all relevant relations (especially associativity), cf. §2.1. Its underlying equivalence is denoted by \equiv , unless otherwise is specified. Recall from (2.4) that for any congruence \mathfrak{A} on \mathcal{M} there exists the canonical surjection $\pi_{\mathfrak{A}} : \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathfrak{A}$, see Remark 2.3.

Given a congruence $\mathfrak{A} \in \text{Cong}(\mathcal{M})$ with underlying equivalence \equiv , we write

$$a \equiv \text{ghost} \quad \text{if} \quad a \equiv b \text{ for some } b \in \mathcal{G}.$$

(This notation includes the case that $a \equiv 0$, in which also $a^\nu \equiv 0$.)

Lemma 4.1. *Suppose $a \equiv \text{ghost}$ in a congruence $\mathfrak{A} \in \text{Cong}(\mathcal{M})$, then $a \equiv a^\nu$.*

Proof. By assumption $a \equiv b$ for some $b \in \mathcal{G}$. As \mathfrak{A} respects the ν -monoid operation $+$, we have $a + a \equiv b + b = b$, since $b \in \mathcal{G}$. Hence $a^\nu \equiv b$, and the transitivity of \mathfrak{A} implies $a \equiv a^\nu$. \square

We have the obvious characterization of ghosts in terms of congruences:

Corollary 4.2. *An element $a \in \mathcal{M}$ is ghost if and only if $a \equiv a^\nu$ in all congruences \mathfrak{A} on \mathcal{M} .*

Viewing a congruence \mathfrak{A} on \mathcal{M} as subalgebra of $\mathcal{M} \times \mathcal{M}$, we define its **ghost cluster** of \mathfrak{A} to be

$$\mathbf{G}_{\text{cls}}(\mathfrak{A}) := \{(a, b) \in \mathfrak{A} \mid a \equiv a^\nu\} \subseteq \mathcal{M} \times \mathcal{M}. \quad (4.1)$$

This ghost cluster provides a coarse classification of the ghost equivalence classes of \mathfrak{A} . In particular, for every \mathfrak{A} we always have $\mathcal{G} \times \mathcal{G} \subseteq \mathbf{G}_{\text{cls}}(\mathfrak{A})$. Moreover, by Lemma 4.1, an inclusion $(a, b) \in \mathbf{G}_{\text{cls}}(\mathfrak{A})$ implies that $(a, a) \in \mathbf{G}_{\text{cls}}(\mathfrak{A})$ and $(b, b) \in \mathbf{G}_{\text{cls}}(\mathfrak{A})$, whereas $a \equiv a^\nu$ and $b \equiv b^\nu$.

The set-theoretic complement of $\mathbf{G}_{\text{cls}}(\mathfrak{A})$ in \mathfrak{A} is denoted by

$$\mathbf{G}_{\text{cls}}^c(\mathfrak{A}) := \mathfrak{A} \setminus \mathbf{G}_{\text{cls}}(\mathfrak{A}).$$

A congruence \mathfrak{A} on \mathcal{M} is said to be a **ghost congruence**, if $\mathfrak{A} = \mathbf{G}_{\text{cls}}(\mathfrak{A})$, namely $\mathbf{G}_{\text{cls}}^c(\mathfrak{A}) = \emptyset$.

Remark 4.3. *If the ghost cluster $\mathbf{G}_{\text{cls}}(\mathfrak{A})$ of $\mathfrak{A} \in \text{Cong}(\mathcal{M})$ consists of elements from a single (ghost) equivalence class, then \mathfrak{A} is not a proper congruence (Definition 2.2). Indeed, in this case $a^\nu \equiv 0$ for every $a^\nu \in \mathcal{G}$, and then*

$$a = a + 0 \equiv a + a^\nu = a + (a + a + a) = (a + a) + (a + a) = a^\nu + a^\nu \equiv 0 + 0 = 0.$$

Therefore, each $a \in \mathcal{M}$ is congruent to $\mathbb{0}$.

Formally, in special cases, we use the **zero congruence** \mathfrak{D} , whose underlying equivalence \equiv_{\circ} is given by

$$a \equiv_{\circ} \mathbb{0} \quad \text{for all } a \in \mathcal{M}. \quad (4.2)$$

\mathfrak{D} consists of the single equivalence class $[\mathbb{0}]$.

We define the **ghost projection** of the ghost cluster $\mathbf{G}_{\text{cls}}(\mathfrak{A})$ of a congruence \mathfrak{A} on \mathcal{M} to be the subset

$$\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A}) := \{a \in \mathcal{M} \mid a \equiv a^{\nu}\} \subseteq \mathcal{M}. \quad (4.3)$$

In other words, $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ is the preimage of the diagonal of $\mathbf{G}_{\text{cls}}(\mathfrak{A})$ under the map $\iota : \mathcal{M} \rightarrow \Delta(\mathfrak{A})$, cf. (2.2). In this setup, $\mathbf{G}_{\text{cls}}(\mathfrak{A})$ is the restriction of \mathfrak{A} to $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A}) \subseteq \mathcal{M}$, and by itself is a congruence on $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$. Note that an element $a \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ need not be a ghost belonging to \mathcal{G} , and that $a \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ if and only if $(a, a) \in \mathbf{G}_{\text{cls}}(\mathfrak{A})$. Clearly, the inclusion $\mathcal{G} \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ holds for any congruence \mathfrak{A} on \mathcal{M} . For short, we write

$$\mathbf{G}_{\text{cls}}^{\text{c}}(\mathfrak{A}) := (\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A}))^{\text{c}} \quad (4.4)$$

for the set-theoretic complement of $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ in \mathcal{M} .

The quotient of a ν -monoid $\mathcal{M} := (\mathcal{M}, \mathcal{G}, \nu)$ by a congruence \mathfrak{A} is defined as

$$\mathcal{M}/\mathfrak{A} := (\mathcal{M}/\mathfrak{A}, \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})/\mathbf{G}_{\text{cls}}(\mathfrak{A}), [\nu]), \quad (4.5)$$

where the ghost map $\nu : \mathcal{M} \rightarrow \mathcal{G}$ of \mathcal{M} induces the ghost map $[\nu]$ of \mathcal{M}/\mathfrak{A} via $[a]^{[\nu]} = [a^{\nu}]$. A class $[a]$ of \mathcal{M}/\mathfrak{A} is a ghost class, if it contains a ghost element of \mathcal{M} , where Lemma 4.1 implies that $a \equiv a^{\nu}$, and hence also $a^{\nu} \in [a]$. The partial ordering of the ghost submonoid of \mathcal{M}/\mathfrak{A} is induced by addition, i.e., $[a] > [b]$ if $[a] + [b] = [a]$, and thus $[a] + [b] = [a]^{[\nu]}$ whenever $[a]^{[\nu]} = [b]^{[\nu]}$. Hence, \mathcal{M}/\mathfrak{A} is an additive ν -monoid (Definition 3.1).

We see that in fact the ghost projection $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ is the preimage of the ghost submonoid of \mathcal{M}/\mathfrak{A} under the canonical surjection $\pi_{\mathfrak{A}} : \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathfrak{A}$, cf. (2.4). Namely, it is the g -kernel of $\pi_{\mathfrak{A}}$ (Definition 3.6).

Remark 4.4. For any congruence \mathfrak{A} on a ν -monoid \mathcal{M} we have the following properties.

(i) If $a^{\nu} \equiv b^{\nu}$, then

$$[a + b] = [a] + [b] = [a]^{\nu} \Rightarrow a + b \equiv a^{\nu}.$$

This equivalence is compatible with the canonical surjection $\pi_{\mathfrak{A}} : \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathfrak{A}$, cf. Remark 2.3.(i).

(ii) Suppose $a + b = c$. If $a \equiv b$, then $a^{\nu} = a + a \equiv a + b$, and thus

$$a^{\nu} \equiv a + b = c, \quad a^{\nu} = a^{\nu} + a^{\nu} \equiv (a + b) + (a + b) = c + c = c^{\nu},$$

implying that $c \equiv c^{\nu}$. In the case that $a^{\nu} + b \in \mathcal{G}$, the same holds under the equivalence $a \equiv a^{\nu}$, since $a^{\nu} + b \equiv a + b = c$.

(iii) If $a + b = a$ and $a \equiv b$, then $a \equiv a^{\nu}$ by (i), and hence $b \equiv b^{\nu}$, since

$$b \equiv a \equiv a^{\nu} = a + a \equiv b + b = b^{\nu}.$$

In particular, this implies that if $a \equiv b$ where $b \leq_{\nu} a$, then $a \equiv a^{\nu}$ and $b \equiv b^{\nu}$.

(iv) If $a + b = a$ and $a \equiv b$, then $a \equiv c$ for every c such that $a + c = a$ and $c + b = c$ (especially when $a >_{\nu} c >_{\nu} b$). Indeed

$$a = a + c \equiv b + c = c.$$

Furthermore, $c \equiv c^{\nu}$ for each such c , since $a \equiv a^{\nu}$ by (iii).

Clearly, an inclusion of congruences implies the inclusion of their ghost clusters:

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \Rightarrow \mathbf{G}_{\text{cls}}(\mathfrak{A}_1) \subset \mathbf{G}_{\text{cls}}(\mathfrak{A}_2). \quad (4.6)$$

Also, by transitivity of congruences, the intersection $\mathfrak{A}_1 \cap \mathfrak{A}_2$ respects intersection of ghost clusters: $\mathbf{G}_{\text{cls}}(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \mathbf{G}_{\text{cls}}(\mathfrak{A}_1) \cap \mathbf{G}_{\text{cls}}(\mathfrak{A}_2)$.

Having the above insights, we next observe congruences that arise from suitable ‘‘ghost relations’’, taking place on subsets of ν -monoids. For a nonempty subset E of \mathcal{M} we define the set of congruences

$$\mathbf{G}(E) := \{\mathfrak{A} \in \text{Cong}(\mathcal{M}) \mid E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})\} \subseteq \text{Cong}(\mathcal{M}), \quad (4.7)$$

and consider the congruence determined as the intersection of all its members:

$$\mathfrak{G}_E := \bigcap_{\mathfrak{A} \in \mathbf{G}(E)} \mathfrak{A}. \quad (4.8)$$

This construction provides \mathfrak{G}_E as the minimal congruence in which the entire subset E is declared as ghost, that is $a \equiv a'$ for every $a \in E$, cf. Remark 2.1 and Lemma 4.1.

The congruence \mathfrak{G}_E respects the monoid operation of \mathcal{M} , as it is the intersection of congruences, and hence it is transitively closed. We call \mathfrak{G}_E the **ghostifying congruence** of E , while $\mathcal{M}/\mathfrak{G}_E$ is said to be the **ghostification** of E , for short. We also say that K is **ghostified** by \mathfrak{G}_E , when $K \subseteq E$.

Lemma 4.5. *The underlying equivalence \equiv of \mathfrak{G}_E can be formulated as*

$$a + b \equiv a + b' \quad \text{for all } b \in E. \quad (4.9)$$

In particular, $b \equiv b'$ for all $b \in E$.

Proof. By Lemma 4.1, the inclusion $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ is equivalent to having the relation $b \equiv b'$ satisfied for all $b \in E$. Every $\mathfrak{A} \in \mathbf{G}(E)$ satisfies the equivalence (4.9), since \mathfrak{A} respects addition, and therefore their intersection \mathfrak{G}_E also admits condition (4.9). Then, the minimality of \mathfrak{G}_E completes the proof. \square

Accordingly, the ghostification of a subset $E \subseteq \mathcal{M}$ is provided by the minimal congruence whose underlying equivalence \equiv admits the relation

$$a \equiv a' \quad \text{for all } a \in E \cup \mathcal{G}.$$

In the degenerated case that $E \subseteq \mathcal{G}$, we simply have $\mathfrak{G}_E = \Delta(\mathcal{M})$. On the other hand, if $\mathbb{1} \in E$, then \mathfrak{G}_E is a ghost congruence.

Remark 4.6. *Let $[a], [b]$ be classes of R/\mathfrak{G}_E . From (4.9) it follows that $[a] = [b]$ only if $a \cong_\nu b$, cf. (3.1). (The converse does not hold in general, take for example $a, b \notin \mathcal{G}$ such that $a \neq b$ and $a \cong_\nu b$.)*

We say that the congruence \mathfrak{G}_E is **determined** by $E \subseteq \mathcal{M}$, and define the set

$$\text{Cong}_{\mathfrak{g}}(\mathcal{M}) := \{\mathfrak{G}_E \mid E \subseteq \mathcal{M}\} \quad (4.10)$$

of all ghostifying congruences on \mathcal{M} . We call these congruences **g-congruences**, for short. $\text{Cong}_{\mathfrak{g}}(\mathcal{M})$ is a nonempty set as it contains $\Delta(\mathcal{M})$.

Remark 4.7. *For any subsets $E, E' \subseteq \mathcal{M}$ we have the following properties:*

- (i) $\mathfrak{G}_{E \cap E'} = \mathfrak{G}_E \cap \mathfrak{G}_{E'}$;
- (ii) $\mathfrak{G}_{E \cup E'} = \overline{\mathfrak{G}_E \cup \mathfrak{G}_{E'}}$ (cf. (2.8));
- (iii) $E \subset E' \Rightarrow \mathfrak{G}_E \subset \mathfrak{G}_{E'}$.

Namely, $\text{Cong}_{\mathfrak{g}}(\mathcal{M})$ is closed for intersection (and respects inclusion), in which $\mathfrak{G}_{\mathcal{M}}$ is maximal congruence. Therefore, $\text{Cong}_{\mathfrak{g}}(\mathcal{M})$ has the structure of a semilattice.

(By Remark 3.2, $a \equiv_{\mathfrak{g}} \mathbb{0}$ in \mathfrak{G}_E implies that $a = \mathbb{0}$, and thus formally the zero congruence \mathfrak{D} does not belong to $\text{Cong}_{\mathfrak{g}}(\mathcal{M})$.)

Lemma 4.8. *Let $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ be a ν -monoid homomorphism (Definition 3.4), then $\text{gker}(\varphi) = \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_{\text{gker}(\varphi)})$.*

Proof. Let $E = \text{gker}(\varphi)$. The inclusion $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$ is obvious. Conversely, suppose $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E) \setminus E$ is nonempty and take $a \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E) \setminus E$. Thus, a is not a ghost, since $E = \text{gker}(\varphi)$, and a can be written as

$$a = b + d + \sum e_i \quad \text{for some } b \in \mathcal{G}, d \in \mathcal{M} \setminus \mathcal{G}, e_i \in E \setminus \mathcal{G},$$

where b and e_i are possibly all $\mathbb{0}$. (In fact $b + \sum e_i \in E$ as E is a monoid.) By Lemma 4.5 we obtain

$$a' = d' + b + \sum e_i' = d + b + \sum e_i',$$

since $a \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$, which implies by Axiom NM3 in Definition 3.1 that $a = d' + b + \sum e_i$. Thus

$$a = b' + \sum e_i \quad \text{for some } e_i \in E, b' \in \mathcal{G}.$$

But $\mathcal{G} \subset E$, and hence $a \in E$. \square

Using ghostifying congruences, one can define quotients of ν -monoids.

Definition 4.9. *The **quotient ν -monoid** of a ν -monoid \mathcal{M} by a subset $E \subseteq \mathcal{M}$ is defined to be the ν -monoid $\mathcal{M}/\mathfrak{G}_E$. We write $\mathcal{M} // E$ for $\mathcal{M}/\mathfrak{G}_E$.*

In other words, all the elements of $E \subseteq \mathcal{M}$ are identified as ghosts in \mathcal{M}/E . In particular, for $E \subseteq \mathcal{G}$, we have $\mathfrak{G}_E = \Delta(\mathcal{M})$, and therefore, in this case, $\mathcal{M}/E = \mathcal{M}$. On the other hand, $\mathcal{M}/E \cong \mathcal{G}$ when $E = \mathcal{M}$. The process of quotienting by the means of ghsotification is canonically defined and allows a factoring oyt by substructures, as well as taking closures.

Definition 4.10. The *g -cokernel* of a homomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ of ν -monoids is defined as

$$\text{gcoker}(\varphi) := \mathcal{M}' // \text{im}(\varphi). \quad (4.11)$$

Let E be a subset of \mathcal{M} , and consider the surjective homomorphism

$$\varphi : \mathcal{M} \longrightarrow \mathcal{M}/E.$$

By Lemma, 4.8 $\text{gker}(\varphi) = \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_{\text{gker}(\varphi)})$ is a ν -submonoid of \mathcal{M} containing $E \cup \mathcal{G}$.

Lemma 4.11. Let $\mathcal{N} \subset \mathcal{M}$ be a ν -submonoid containing \mathcal{G} , and let $\phi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{N}$ be the canonical homomorphism. Then $\text{gker}(\phi) = \mathcal{N}$.

Proof. Follows from Lemma 4.8. □

Theorem 4.12. Let $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ be a ν -monoid homomorphism. There exists a unique ν -homomorphism

$$\bar{\varphi} : \mathcal{M} // \text{gker}(\varphi) \longrightarrow \mathcal{M}', \quad \bar{\varphi} : [a] \mapsto \varphi(a),$$

such that $\varphi = \bar{\varphi} \circ \pi$, where $\pi : \mathcal{M} \rightarrow \mathcal{M} // \text{gker}(\varphi)$ is the canonical homomorphism. Namely, φ factors uniquely through π and the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\pi} & \mathcal{M} // \text{gker}(\varphi) \\ & \searrow_{\varphi = \bar{\varphi} \circ \pi} & \downarrow \bar{\varphi} \\ & & \mathcal{M}' \end{array}$$

commutes.

Proof. Clearly, $\bar{\varphi}$ is a homomorphism: $\bar{\varphi}([a + b]) = \varphi(a + b) = \varphi(a) + \varphi(b) = \bar{\varphi}([a]) + \bar{\varphi}([b])$. Let \mathfrak{A}_φ be the c-kernel of φ (Remark 2.3.(ii)) and write $E = \text{gker}(\varphi)$, which is a monoid. Then, $E = \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$ by Lemma 4.8, and $\mathfrak{G}_E \subset \mathfrak{A}_\varphi$ since \mathfrak{A}_φ includes the relations $a \equiv a'$ for all $a \in E$, and their transitive closure, implying that $\bar{\varphi}$ is well defined by Remark 2.7. Also, $(\bar{\varphi} \circ \pi)(a) = \bar{\varphi}(\pi(a)) = \bar{\varphi}([a]) = \varphi(a)$, proving that $\bar{\varphi} \circ \pi = \varphi$. Suppose there exists another ψ such that $\psi : \mathcal{M} // \text{gker}(\varphi) \rightarrow \mathcal{M}'$ with $\psi \circ \pi = \varphi$, then $\psi([a]) = \psi(\pi(a)) = (\psi \circ \pi)(a) = \varphi(a)$ so that $\bar{\varphi} = \psi$, and $\bar{\varphi}$ is unique. □

4.2. Congruences on ν -semirings.

Henceforth, our ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ is always assumed to be commutative, cf. §3.3. Recall that \equiv denotes the underlying equivalence of a congruence \mathfrak{A} . We begin with the set $\text{Cong}(R)$ of all congruences on R , and later restrict to better behaved congruences which include families of \mathfrak{g} -congruences.

Remark 4.13. Any congruence $\mathfrak{A} \in \text{Cong}(R)$ satisfies the condition that if $a \equiv b$ then

$$a' = ea \equiv eb = b'.$$

Conversely, when \mathfrak{A} is cancellative (Definition 2.2), if $a' \equiv b'$ then $a \equiv b$, and moreover $a \equiv a'$ for every $a \in R$. (Indeed, $ea \equiv ea'$, and $a \equiv a'$ by cancellativity.) Therefore, one sees that cancellativity is much too restrictive for congruences on ν -semirings.

We specialize congruences on ν -monoids to ν -semirings, involving their multiplicative structure. While the additive structure induced by a congruence \mathfrak{A} on R has been addressed through ghost elements, characterized by the ghost cluster $\mathbf{G}_{\text{cls}}(\mathfrak{A})$, the multiplicative structure induced by \mathfrak{A} is approached via tangible elements.

The **tangible cluster** $\mathbf{T}_{\text{cls}}(\mathfrak{A})$ of a congruence $\mathfrak{A} \in \text{Cong}(R)$ is defined as

$$\mathbf{T}_{\text{cls}}(\mathfrak{A}) := \{(a, b) \in \mathfrak{A} \mid a \in \mathcal{T}, (a, t) \in \mathfrak{A} \text{ only for } t \in \mathcal{T}\} \subseteq \mathcal{T} \times \mathcal{T}, \quad (4.12)$$

By transitively, b must also be congruent only to tangibles. The set-theoretic complement of $\mathbf{T}_{\text{cls}}(\mathfrak{A})$ in \mathfrak{A} is denoted by

$$\mathbf{T}_{\text{cls}}^c(\mathfrak{A}) := \mathfrak{A} \setminus \mathbf{T}_{\text{cls}}(\mathfrak{A}).$$

In analogy to $G_{\text{cls}}^{-1}(\mathfrak{A})$ in (4.13), the **tangible projection** of \mathfrak{A} is the projection of the diagonal of $T_{\text{cls}}(\mathfrak{A})$ on R , defined as

$$T_{\text{cls}}^{-1}(\mathfrak{A}) := \{a \in \mathcal{T} \mid a \equiv t \text{ only for } t \in \mathcal{T}\} \subseteq \mathcal{T}, \quad (4.13)$$

that is $T_{\text{cls}}^{-1}(\mathfrak{A}) = \Delta^{-1}(T_{\text{cls}}(\mathfrak{A}))$, cf. (2.2). Accordingly, the tangible projection $T_{\text{cls}}^{-1}(\mathfrak{A})$ is the preimage of the tangible subset of R/\mathfrak{A} under the canonical surjection $\pi_{\mathfrak{A}} : R \twoheadrightarrow R/\mathfrak{A}$. By definition, we immediately see that

$$a \in T_{\text{cls}}^{-1}(\mathfrak{A}) \implies a \in \mathcal{T}. \quad (4.14)$$

For short, we write

$$T_{\text{cls}}^c(\mathfrak{A}) := (T_{\text{cls}}^{-1}(\mathfrak{A}))^c \quad (4.15)$$

for the set-theoretic complement of $T_{\text{cls}}^{-1}(\mathfrak{A})$ in R .

Remark 4.14. *Let \mathfrak{A} be a congruence on R , let $a, b \in \mathcal{T} \setminus R^\times$ be tangibles, and let $u, v \in R^\times$ be units.*

- (i) *If $u \equiv v$ then $u^{-1} \equiv v^{-1}$, since $v^{-1} = uu^{-1}v^{-1} \equiv vu^{-1}v^{-1} = u^{-1}vv^{-1} = u^{-1}$.*
- (ii) *The equivalence $u \equiv a$ implies that $[u^{-1}] = [a]^{-1}$ in R/\mathfrak{A} , since $[u^{-1}][a] = [u^{-1}][u] = [u^{-1}u] = [\mathbb{1}]$. Thus, $[a]$ is invertible in R/\mathfrak{A} .*
- (iii) *If $u \in G_{\text{cls}}^{-1}(\mathfrak{A})$, then $T_{\text{cls}}^{-1}(\mathfrak{A}) = \emptyset$. Indeed, u is then congruent to a ghost element in R , and since \mathfrak{A} respects multiplication, also $\mathbb{1} = uu^{-1} \in G_{\text{cls}}^{-1}(\mathfrak{A})$ and $\mathbb{1} \equiv e$. Consequently, every $a \in R$ is congruent to some ghost element, and thus $T_{\text{cls}}^{-1}(\mathfrak{A})$ is empty.*
- (iv) *By Remark 4.4.(iv), if $a \equiv b$ where $a + u = a$ and $b + u = u$ (in particular when $a >_\nu u >_\nu b$), then $u \equiv u^\nu$ and $T_{\text{cls}}^{-1}(\mathfrak{A}) = \emptyset$.*
- (v) *If $a + b = u$, where $a \equiv a'$ and $b \equiv b'$ such that $a' + b'$ is ghost, then u is congruent to a ghost and again $T_{\text{cls}}^{-1}(\mathfrak{A}) = \emptyset$.*

These properties will be of much use in the analysis of maximal congruences.

The ghost cluster $G_{\text{cls}}(\mathfrak{A})$ in (4.1) and the tangible cluster $T_{\text{cls}}(\mathfrak{A})$ in (4.12) of a congruence \mathfrak{A} are set theoretically disjoint. Together they induce a classification of equivalent classes as tangible, ghost, or neither tangible nor ghost. Accordingly, we sometimes refer to a congruence \mathfrak{A} as the triplet

$$\mathfrak{A} := (\mathfrak{A}, T_{\text{cls}}(\mathfrak{A}), G_{\text{cls}}(\mathfrak{A})).$$

By definition, $(a, b) \in T_{\text{cls}}(\mathfrak{A})$ implies $(a, a) \in T_{\text{cls}}(\mathfrak{A})$ and $(b, b) \in T_{\text{cls}}(\mathfrak{A})$. The same holds for $G_{\text{cls}}(\mathfrak{A})$, where it may contain a pair of tangibles $(a, b) \in \mathcal{T} \times \mathcal{T}$, if $a \equiv a^\nu$.

Set-theoretically, $T_{\text{cls}}(\mathfrak{A})$ is not necessarily the complement of $G_{\text{cls}}(\mathfrak{A})$ in \mathfrak{A} , since \mathfrak{A} may elements which are neither in $T_{\text{cls}}(\mathfrak{A})$ nor in $G_{\text{cls}}(\mathfrak{A})$. This means that $T_{\text{cls}}^c(\mathfrak{A})$ and $G_{\text{cls}}^c(\mathfrak{A})$ need not be disjoint; this happens only in certain cases. For example, when R a supertropical domain (Definition 3.24) we have $\mathfrak{A} = G_{\text{cls}}(\mathfrak{A}) \cup T_{\text{cls}}(\mathfrak{A})$.

Remark 4.15. *As \mathfrak{A} respects the multiplication of the carrier ν -semiring R , $a \in G_{\text{cls}}^{-1}(\mathfrak{A})$ implies that $ab \in G_{\text{cls}}^{-1}(\mathfrak{A})$ for any $b \in R$, since the product of any element in R with a ghost element is ghost. Hence, the ghost projection $G_{\text{cls}}^{-1}(\mathfrak{A})$ of \mathfrak{A} is a semiring ideal of R (Definition 2.14) – a ghost absorbing subset containing \mathcal{G} .*

If $G_{\text{cls}}^{-1}(\mathfrak{A})$ contains a unit $u \in R^\times$, then \mathfrak{A} is a ghost congruence. Indeed, $\mathbb{1} = u^{-1}u \in G_{\text{cls}}^{-1}(\mathfrak{A})$, and thus $b = b\mathbb{1} \in G_{\text{cls}}^{-1}(\mathfrak{A})$ for every $b \in R$. In other words, $G_{\text{cls}}^{-1}(\mathfrak{A}) = R$, which is the case of Remark 4.14.(iii).

Categorically, the projections $T_{\text{cls}}^{-1}(_)$ and $G_{\text{cls}}^{-1}(_)$ are viewed as maps.

Remark 4.16. *The subsets $T_{\text{cls}}^{-1}(\mathfrak{A})$ and $G_{\text{cls}}^{-1}(\mathfrak{A})$ of R define class forgetful maps $T_{\text{cls}}^{-1}(_): \text{Cong}(R) \twoheadrightarrow \mathcal{T}$ and $G_{\text{cls}}^{-1}(_): \text{Cong}(R) \twoheadrightarrow R$ that preserve only clusters' decomposition. That is, the property of being tangible or ghost under the canonical surjection $\pi_{\mathfrak{A}} : R \twoheadrightarrow R/\mathfrak{A}$. This data is fully recorded by restricting classes to subsets of the diagonal of \mathfrak{A} , where the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\Delta} & \mathfrak{A} \\ & \searrow \pi_{\mathfrak{A}} & \downarrow \psi \\ & & R/\mathfrak{A} \end{array}$$

respects this clustering.

In oppose to (4.6), an inclusion of congruences implies an inverse inclusion of tangible clusters:

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \quad \Rightarrow \quad \mathsf{T}_{\text{cls}}(\mathfrak{A}_1) \supset \mathsf{T}_{\text{cls}}(\mathfrak{A}_2). \quad (4.16)$$

Remark 4.17. *The clusters $\mathsf{T}_{\text{cls}}(\mathfrak{A}_i)$ and $\mathsf{G}_{\text{cls}}(\mathfrak{A}_j)$ admit the following relations for intersection:*

$$\mathsf{T}_{\text{cls}}(\mathfrak{A}_1 \cap \mathfrak{A}_2) \supseteq \mathsf{T}_{\text{cls}}(\mathfrak{A}_1) \cap \mathsf{T}_{\text{cls}}(\mathfrak{A}_2), \quad \mathsf{G}_{\text{cls}}(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \mathsf{G}_{\text{cls}}(\mathfrak{A}_1) \cap \mathsf{G}_{\text{cls}}(\mathfrak{A}_2).$$

Intersections of clusters of different congruences need not be empty, and include the following cases:

- (i) $\mathsf{T}_{\text{cls}}(\mathfrak{A}_1) \cap \mathsf{G}_{\text{cls}}(\mathfrak{A}_2) \subseteq \mathsf{T}_{\text{cls}}(\mathfrak{A}_1)$;
- (ii) $\mathsf{T}_{\text{cls}}(\mathfrak{A}_1) \cap \mathsf{T}_{\text{cls}}^c(\mathfrak{A}_2) \subset \mathsf{T}_{\text{cls}}(\mathfrak{A}_1)$;
- (iii) $\mathsf{T}_{\text{cls}}(\mathfrak{A}_1) \cap \mathsf{G}_{\text{cls}}(\mathfrak{A}_2) \not\subseteq \mathsf{G}_{\text{cls}}(\mathfrak{A}_2)$, unless it is empty;
- (iv) $\mathsf{T}_{\text{cls}}^c(\mathfrak{A}_1) \cap \mathsf{T}_{\text{cls}}^c(\mathfrak{A}_2)$ can be in $\mathsf{T}_{\text{cls}}(\mathfrak{A}_1)$, $\mathsf{G}_{\text{cls}}(\mathfrak{A}_1)$, or in the complement of their union.

The intersection of the projections $\mathsf{T}_{\text{cls}}^{-1}(_)$ and $\mathsf{G}_{\text{cls}}^{-1}(_)$ are induced by these cases.

The ghostification of subsets of ν -monoids, i.e., identifying subsets as ghosts, extends naturally to ν -semirings.

Remark 4.18. *The underlying equivalence \equiv of the ghostifying congruence \mathfrak{G}_E of a subset $E \subseteq R$ is formulated as*

$$a + b \equiv a + b^\nu, \quad ab \equiv (ab)^\nu, \quad \text{for all } b \in E, \quad (4.17)$$

cf. (4.8) and (4.9). Indeed, Lemma 4.5 gives the additive relation; the multiplicative relation is obtained from the ν -semiring multiplication and the role of the ghost ideal, since $a^\nu = ea$ for every $a \in R$.

From relations (4.17) it follows that, for any $a \in R$,

$$a \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E) \quad \Rightarrow \quad a = q + \sum c_i e_i \quad \text{for some } e_i \in E, \quad c_i \in R, \quad q \in \mathcal{G}, \quad (4.18)$$

since a is ghostified as a consequence of ghostifying E . Indeed, assume that $a = d + q + \sum c_i e_i$ for some essential term $d \notin \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$, then

$$a^\nu = d^\nu + q + \sum c_i e_i^\nu = d + q + \sum c_i e_i^\nu,$$

implying by Axiom NM3 in Definition 3.1 that $a = d^\nu + q + \sum c_i e_i$. Thus, a admits (4.18), since $d^\nu + q \in \mathcal{G}$. (See also Lemma 4.11.)

In this view, since \mathfrak{G}_E is a congruence, the passage to quotient structures by subsets $E \subset R$ is natural. A ghostifying congruence \mathfrak{G}_E with $E = \{a\}$, $a \notin \mathcal{G}$, is called a **principal congruence**.

One sees that ghostifying a subset E of a ν -semiring makes E a ‘‘ghost absorbing’’ subset, in analogy to ideals generated by subsets in ring theory. Note that the ghsotification of a reduced sum x (Definition 2.13) does not necessarily ghsotifies sums $y < x$, cf. (3.5). Using (4.18), the ghostification of E by (4.17) determines a ‘‘ghost dependence’’, in the sense that any combination of elements in E becomes ghost. Although this dependence is weaker than spanning, it often suffices to simulate the role of the latter.

Lemma 4.19. *Given $a, b \in R$, for $\mathsf{G}(a)$ as defined in (4.7) we have:*

- (i) $\mathsf{G}(a) \cap \mathsf{G}(b) \subseteq \mathsf{G}(a + b)$;
- (ii) $\mathsf{G}(a) \cap \mathsf{G}(b) \subseteq \mathsf{G}(ab)$;
- (iii) $\mathsf{G}(a) \cup \mathsf{G}(b) \subseteq \mathsf{G}(ab)$.

Proof. (i)-(ii): $\mathfrak{A} \in \mathsf{G}(a) \cap \mathsf{G}(b)$ means that both $a \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ and $b \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$, thus $a + b \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ and $ab \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$, since $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ is an ideal of R by Remark 4.15.

(iii): $\mathfrak{A} \in \mathsf{G}(a)$ means that $a \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ and thus also $ab \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$, since $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ is an ideal of R . By symmetry, the same holds for $\mathfrak{A} \in \mathsf{G}(b)$. \square

Corollary 4.20. *Let $\mathcal{A} = (a_i)_{i \in I}$ and $\mathcal{B} = (b_j)_{j \in J}$ be families of elements $a_i, b_j \in R$. Suppose $d = \sum a_i b_j$ is a finite sum, then $\mathfrak{G}_{\{d\}} \subseteq \mathfrak{G}_{\mathcal{A}} \cap \mathfrak{G}_{\mathcal{B}}$, cf. (4.8).*

Proof. Lemma 4.19 gives $\mathsf{G}(\mathcal{A}) \cap \mathsf{G}(\mathcal{B}) \subseteq \mathsf{G}(\{d\})$, implying that $\mathfrak{G}_{\{d\}} \subseteq \mathfrak{G}_{\mathcal{A}} \cap \mathfrak{G}_{\mathcal{B}}$. \square

Lemma 4.21. *Let $\mathfrak{a} \triangleleft R$ be a ν -semiring ideal (Definition 2.14) containing \mathcal{G} , and let $\varphi : \mathcal{M} \rightarrow R/\mathfrak{a}$ be the canonical homomorphism. Then $\text{gker}(\varphi) = \mathfrak{a}$.*

Proof. In ν -monoid view, $\text{gker}(\varphi) = \mathfrak{a}$, by Lemma 4.8, while $\text{gker}(\varphi)$ is a ν -semiring ideal by §3.7. \square

4.3. \mathfrak{q} -congruences and ℓ -congruences.

Not all congruences \mathfrak{A} on a ν -semiring R possess \mathfrak{q} -homomorphisms (Definition 3.11); furthermore, a quotient R/\mathfrak{A} does not necessarily preserve a tangible component of R . To retain the category $\nu\mathbf{Smr}$ of ν -semirings, we restrict to congruences which endow R/\mathfrak{A} with ν -semiring structure and are also applicable for localization. Moreover, to obtain the desired supertropical analogy of ideals in classical commutative algebra, together with their correspondences to varieties, such congruences should coincide with the notion of ghostification. Let us first address some guiding pathological cases.

Example 4.22. *Let \mathfrak{A} be a congruence on a ν -semiring R .*

- (i) *If $a + b = u$ is a unit and $a \equiv b$ in \mathfrak{A} , then $u \equiv u^\nu$ by Remark 4.4.(iv), implying by Remark 4.15 that \mathfrak{A} is a ghost congruence. Hence, R/\mathfrak{A} has no tangible component.*
- (ii) *If $a \equiv b$ for any pair of tangible elements a, b in R , and \mathcal{G} is ordered, then for $a >_\nu b$ we have*

$$a = a + b \equiv a + a = a^\nu,$$

implying that $\mathsf{T}_{\text{cls}}(\mathfrak{A})$ is empty, cf. Remark 4.4.(iii).

- (iii) *Similarly, if $a \equiv u$ for some unit $u \in R^\times$, where a is non-tangible, then R/\mathfrak{A} does not necessarily have tangible elements and $\mathsf{T}_{\text{cls}}(\mathfrak{A})$ could be empty. It might also not be a ν -semiring, since $[a]$ must be unit, but is non-tangible.*

Consequently, in all these cases R/\mathfrak{A} need not be a ν -semiring and the canonical surjection $\pi_{\mathfrak{A}} : R \longrightarrow R/\mathfrak{A}$ is not necessarily a \mathfrak{q} -homomorphism (Definition 3.31), whereas $\text{tcor}(\pi_{\mathfrak{A}}) = \emptyset$.

To avoid the above drawbacks, we first distinguish those elements in R which must be preserved as tangible in the passage to a quotient R/\mathfrak{A} .

Definition 4.23. *An element $a \in \mathcal{T}$ is called **tangible unalterable**, written **t-unalterable**, if*

$$a \equiv b \text{ for some } b \notin \mathcal{T} \implies R^\times \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}), \quad (4.19)$$

in all $\mathfrak{A} \in \text{Cong}(R)$. We denote the set of all t-unalterable elements by \mathcal{S} .

Clearly, $\mathbb{1} \in \mathcal{S}$ and thus \mathcal{S} is nonempty. Also, $\mathcal{S} \subset \mathcal{T}^\circ$, since congruences respect the ν -semiring multiplication. However, t-unalterable elements need not be units, while $R^\times \subseteq \mathcal{S}$.

For instance, every tangible element in a definite ν -semifield F is t-unalterable, as well as in the polynomial ν -semiring $F[\Lambda]$. For example, consider the polynomial ν -semiring $\mathbb{T}[\lambda_1, \lambda_2]$ from Example 3.55; its t-unalterable elements are the tangible elements of \mathbb{T} . In this case, each t-unalterable element is a unit. On the other hand, $(0, -1) \in \mathbb{T}^{(2)}$ is t-unalterable, since $(0, -1) = (-1, 0) = (1, 1) = \mathbb{1}_{\mathbb{T}^{(2)}}$, but is not a unit.

To ensure that a quotient semiring R/\mathfrak{A} is a proper ν -semiring (Definition 3.11) and that the canonical surjection $\pi_{\mathfrak{A}} : R \longrightarrow R/\mathfrak{A}$ is a (unital) \mathfrak{q} -homomorphism of ν -semirings with a nonempty tangible core, we exclude all the ghost congruences and restrict our intention to the following types of congruences. As will be seen later, the characteristic of these particular congruences, concerning localization as well, is curial for our forthcoming results.

Definition 4.24. *A congruence \mathfrak{A} on a ν -semiring R is a **\mathfrak{q} -congruence** (abbreviation for **quotienting congruence**), if $R^\times \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$. Hence, its tangible projection $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ contains a nonempty tangible submonoid $\mathsf{P}_\bullet^{-1}(\mathfrak{A}) \supseteq R^\times$. The set of all \mathfrak{q} -congruences on R is denoted by $\text{Cong}_{\mathfrak{q}}(R)$.*

*A \mathfrak{q} -congruence \mathfrak{A} is an **ℓ -congruence** (abbreviation for **localizing congruence**), if $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ by itself is a multiplicative tangible submonoid of \mathcal{T} , written $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) = \mathsf{P}_\bullet^{-1}(\mathfrak{A})$, and hence*

$$ab \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) \implies a \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) \text{ or } b \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}). \quad (4.20)$$

The set of all ℓ -congruences is denoted by $\text{Cong}_\ell(R)$.

By this definition, $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) \subseteq \mathcal{T}^\circ$ for every ℓ -congruence \mathfrak{A} , while $\mathcal{S} \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ for any \mathfrak{q} -congruence \mathfrak{A} , since otherwise $R^\times \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$.

\mathfrak{q} -congruences and ℓ -congruences are defined solely by the structure of their tangible projections $\mathsf{T}_{\text{cls}}^{-1}(_)$, cf. (4.13). They are proper congruences (Definition 2.2) whose tangible clusters and ghost clusters are disjoint and nonempty. For example, the trivial congruence $\Delta(R)$ on a ν -semiring R is a \mathfrak{q} -congruence. $\Delta(R)$ is an ℓ -congruence, if R is tangibly closed, since then $\mathsf{T}_{\text{cls}}^{-1}(\Delta) = \mathcal{T}$ is a multiplicative monoid.

Lemma 4.25. *Let \mathfrak{A} be a \mathfrak{q} -congruence on a ν -semiring R , then R/\mathfrak{A} is a ν -semiring (Definition 3.11).*

Proof. The quotient R/\mathfrak{A} is a ν -monoid by §3.1. $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{T}_{\text{cls}}(\mathfrak{A})$ is the tangible set of R/\mathfrak{A} , containing $[u]$, since $u \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, for every $u \in R^\times$, and thus each $[u]$ is \mathfrak{t} -persistent in R/\mathfrak{A} . The \mathfrak{t} -persistent set of R/\mathfrak{A} is formally defined to be all $[a] \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{T}_{\text{cls}}(\mathfrak{A})$ such that $[a]^n \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{T}_{\text{cls}}(\mathfrak{A})$ for every n , so that Axiom NS1 holds.

Suppose $[a] + [b]$ is tangible, where $[b]$ is ghost in R/\mathfrak{A} , that is $b \equiv b^\nu$ by Lemma 4.1. Then $a^\nu + b \equiv a^\nu + b^\nu$ is ghost, and thus $[a + b]$ is not tangible – a contradiction. Hence Axiom NS2 holds. \square

Corollary 4.26. *The quotient R/\mathfrak{A} of a ν -semiring R by an ℓ -congruence \mathfrak{A} is a tangibly closed ν -semiring.*

Proof. Indeed, R/\mathfrak{A} is a ν -semiring by Lemma 4.25, where $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{T}_{\text{cls}}(\mathfrak{A})$ is its \mathfrak{t} -persistent monoid. \square

It follows from Definition 4.24 that in a \mathfrak{q} -congruence \mathfrak{A} none of the units of R is congruent to a non-tangible, especially not to a ghost or 0 . Also, we have the following properties.

Remark 4.27. *For a \mathfrak{q} -congruence \mathfrak{A} , Remarks 4.14 and 4.15 can be strengthened.*

- (i) *If u is a unit, then $[u]$ is a unit of R/\mathfrak{A} , by Remark 4.14.(ii), and thus $\pi_{\mathfrak{A}}(R^\times) = (R/\mathfrak{A})^\times$, implying that $\pi_{\mathfrak{A}} : R \longrightarrow R/\mathfrak{A}$ is a local homomorphism (Definition 2.10).*
- (ii) *Any \mathfrak{q} -congruence is a proper congruence having at least three equivalence classes (Definition 2.2): a tangible class, a ghost class, and the zero class.*
- (iii) *If $u = a + b$, then a and b cannot be ghostified simultaneously by a \mathfrak{q} -congruence \mathfrak{A} , since otherwise u would be congruent to a ghost, implying that $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) = \emptyset$, cf. Remark 4.14.(v).*
- (iv) *From Remark 4.14.(iii) we learn that if a subset $E \subset R$ contains a unit, then the ghostifying congruence \mathfrak{G}_E , cf. (4.8), is not a \mathfrak{q} -congruence.*
- (v) *\mathfrak{A} is not a \mathfrak{q} -congruence whenever $\mathsf{T}_{\text{cls}}^c(\mathfrak{A})$ contains a \mathfrak{t} -unalterable element.*

We observe that for \mathfrak{q} -congruences the pathologies in Example 4.22 are dismissed, where the compatibility of \mathfrak{q} -congruences with \mathfrak{q} -homomorphisms follows obviously. The quotient R/\mathfrak{A} of a ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ by a \mathfrak{q} -congruence \mathfrak{A} is again a ν -semiring (Lemma 4.25), given as

$$R/\mathfrak{A} = (R/\mathfrak{A}, \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{T}_{\text{cls}}(\mathfrak{A}), \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A})/\mathsf{G}_{\text{cls}}(\mathfrak{A}), [\nu]), \quad (4.21)$$

where the ghost map $[\nu]$ of R/\mathfrak{A} is induced from the ghost map $\nu : R \longrightarrow \mathcal{G}$ of R via $[a]^{[\nu]} = [a^\nu]$. Moreover, the canonical surjection (Remark 2.3)

$$\pi_{\mathfrak{A}} : R \longrightarrow R/\mathfrak{A}, \quad a \longmapsto [a],$$

is a \mathfrak{q} -homomorphism (Definition 3.31), in particular $\pi_{\mathfrak{A}}(\mathbb{1}_R) = \mathbb{1}_{R/\mathfrak{A}}$, with

$$\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) = \text{tcor}(\pi_{\mathfrak{A}}) \quad \text{and} \quad \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A}) = \text{gker}(\pi_{\mathfrak{A}}). \quad (4.22)$$

(See respectively Proposition 3.33 and Lemma 3.5.)

Remark 4.28. *The intersection of two ℓ -congruences \mathfrak{A}_1 and \mathfrak{A}_2 need not be an ℓ -congruence, since $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_1 \cap \mathfrak{A}_2)$ is not necessarily closed for multiplication of tangibles. For example, take ℓ -congruences \mathfrak{A}_1 and \mathfrak{A}_2 such that $a_i \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_i)$ and $a_i \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_j)$ for $i \neq j$, where $i, j = 1, 2$. Then, both $a_1, a_2 \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_1 \cap \mathfrak{A}_2)$, but $a_1 a_2 \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_1 \cap \mathfrak{A}_2)$. However, $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}_1 \cap \mathfrak{A}_2)$ is nonempty, it contains the group R^\times , and thus $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is a \mathfrak{q} -congruence. On the other hand, the intersection of \mathfrak{q} -congruences is a \mathfrak{q} -congruence, since R^\times is contained the intersection of their tangible clusters. Therefore, $\text{Cong}_{\mathfrak{q}}(R)$ is closed for intersection.*

Suppose \mathfrak{A}' is a congruence contained in a \mathfrak{q} -congruence \mathfrak{A} , then \mathfrak{A}' is a \mathfrak{q} -congruence. Indeed, $\mathfrak{A}' \subset \mathfrak{A}$ implies by (4.16) that $\mathsf{T}_{\text{cls}}(\mathfrak{A}') \supseteq \mathsf{T}_{\text{cls}}(\mathfrak{A})$, and hence $R^\times \subseteq \mathsf{T}_{\text{cls}}(\mathfrak{A}')$.

We specialize the congruence closures from (2.8) to \mathfrak{q} -congruences $\mathfrak{A}_1, \mathfrak{A}_2$ by setting

$$\overline{\mathfrak{A}_1 \cup \mathfrak{A}_2} := \bigcap_{\substack{\mathfrak{A} \in \text{Cong}_{\mathfrak{q}}(R) \\ \mathfrak{A}_1 \cup \mathfrak{A}_2 \subseteq \mathfrak{A}}} \mathfrak{A}, \quad \overline{\mathfrak{A}_1 + \mathfrak{A}_2} := \bigcap_{\substack{\mathfrak{A} \in \text{Cong}_{\mathfrak{q}}(R) \\ \mathfrak{A}_1 + \mathfrak{A}_2 \subseteq \mathfrak{A}}} \mathfrak{A}, \quad (4.23)$$

to obtain these closures in $\text{Cong}_{\mathfrak{q}}(R)$.

Remark 4.29. *Let R and R' be ν -semirings.*

- (i) For any \mathfrak{q} -congruence \mathfrak{A} on R , the canonical \mathfrak{q} -homomorphism $\pi_{\mathfrak{A}} : R \rightarrow R/\mathfrak{A}$ induces a one-to-one order preserving correspondence between the \mathfrak{q} -congruences (resp. ℓ -congruences) on R which contain \mathfrak{A} and the \mathfrak{q} -congruences (resp. ℓ -congruences) on R/\mathfrak{A} .
- (ii) Given a \mathfrak{q} -homomorphism $\varphi : R \rightarrow R'$, the pullback $\varphi^\vee(\mathfrak{A}')$ of a \mathfrak{q} -congruence \mathfrak{A}' on R' (Remark 2.3) is a \mathfrak{q} -congruence on R . Indeed, the congruence $\varphi^\vee(\mathfrak{A}')$ on R is defined via $\varphi(a) \equiv \varphi(b)$, where $\varphi(a) \in \Gamma_{\text{cls}}^{-1}(\mathfrak{A}')$ implies $a \in \mathcal{T}$ and $a \in \Gamma_{\text{cls}}^{-1}(\varphi^\vee(\mathfrak{A}'))$, since φ is a \mathfrak{q} -homomorphism, in particular $R^\times \subseteq \Gamma_{\text{cls}}^{-1}(\varphi^\vee(\mathfrak{A}'))$. By the same argument, if \mathfrak{A}' is an ℓ -congruence, then $\varphi^\vee(\mathfrak{A}')$ is an ℓ -congruence.

In the sequel of this paper we extensively rely on \mathfrak{q} -congruences, especially to define radical, prime, and maximal congruences. To preserve our objects in the category $\nu\mathbf{Smr}$ of ν -semirings, we comply the principles:

- ✂ quotienting is done only by \mathfrak{q} -congruences,
- ✂ localization is performed only by ℓ -congruences.

As seen later, \mathfrak{q} -congruences and ℓ -congruences appear naturally in various ways. Note that a \mathfrak{g} -congruence \mathfrak{G}_E need not be a \mathfrak{q} -congruence, e.g., see Remark 4.27.(iv).

Lemma 4.30. *Given $a \in R$ where $a^k \notin \mathcal{G}$ for every k , there exists a \mathfrak{q} -congruence \mathfrak{A} such that $\mathbf{G}_{\text{cls}}^{\text{c}}(\mathfrak{A})$ is a multiplicative monoid which contains a , i.e., $a \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$.*

Proof. Note that a by itself is not ghost. Let \mathcal{J} be the set of all \mathfrak{q} -congruences \mathfrak{A} on R such that no power of a is in $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$. First, \mathcal{J} is not empty since it contains the trivial congruence $\Delta(R)$, as $a^k \notin \mathcal{G}$ for every k . Second, as ghost projections are semiring ideals (Remark 4.15), from Zorn's lemma it follows that \mathcal{J} has at least one \mathfrak{q} -congruence $\overline{\mathfrak{A}}$ with maximal ghost projection $E = \mathbf{G}_{\text{cls}}^{-1}(\overline{\mathfrak{A}})$. Moreover, $E \neq R$, since $\overline{\mathfrak{A}}$ is a \mathfrak{q} -congruence.

Suppose $b, c \notin E$, and let $E' := E \cup \{b\}$, $E'' := E \cup \{c\}$. Then, both \mathfrak{g} -congruences $\mathfrak{G}_{E'}$ and $\mathfrak{G}_{E''}$ do not belong to \mathcal{J} , and thus there exist powers m, n such that $a^m \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_{E'})$ and $a^n \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_{E''})$. Hence, by (4.18), we have $a^m = g' + \sum e'_i s'_i + bt'$ and $a^n = g'' + \sum e''_j s''_j + ct''$ for some $e'_i, e''_j \in E$, $s'_i, s''_j, t', t'' \in R$, $g', g'' \in \mathcal{G}$. Computing the product $a^m a^n$, we get

$$\begin{aligned} a^{m+n} &= (g' + \sum e'_i s'_i + bt')(g'' + \sum e''_j s''_j + ct'') \\ &= (g'g'' + g' \sum e''_j s''_j + g'ct'' + g'' \sum e'_i s'_i + g''bt') + (\sum e'_i s'_i e''_j s''_j + ct'' \sum e'_i s'_i + bt' \sum e''_j s''_j) + bct't'' \\ &= g + (\sum e'_i s'_i e''_j s''_j + ct'' \sum e'_i s'_i + bt' \sum e''_j s''_j) + bct't'', \end{aligned}$$

which belongs to $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{G}_K)$, where $K = E \cup \{bc\}$. Consequently, $\mathfrak{G}_K \notin \mathcal{J}$ and $bc \notin E$, since $\mathfrak{G}_E \subseteq \overline{\mathfrak{A}}$. Therefore, $\mathbf{G}_{\text{cls}}^{\text{c}}(\overline{\mathfrak{A}})$ is a multiplicative monoid, where $a \notin \mathbf{G}_{\text{cls}}^{-1}(\overline{\mathfrak{A}})$. \square

4.4. Interweaving congruences.

Remark 4.4 deals with equivalence relations which are derived from ghostifying elements through the underlying ν -monoid addition (Lemma 4.5). However, ghostification does not capture “direct” relations on non-ghost elements. For example, it does not encompass cases where $a' \equiv a''$ for ν -equivalent elements $a', a'' \notin \mathcal{G}$, i.e., $a' \cong_\nu a''$, namely when a' and a'' are contained in a ν -fiber $\text{fib}_\nu(a)$, cf. (3.2).

Definition 4.31. *A (tangible) interweaving congruence \mathfrak{I}_a , written i-congruence, of an element $a \in \mathcal{T}$ is a congruence whose underlying equivalence \equiv_i is determined by*

$$a \equiv_i a' \quad \text{for all } a' \in \mathcal{T} \text{ such that } a' \cong_\nu a, \quad (4.24)$$

i.e., $a + b \equiv_i a' + b$ and $ab \equiv_i a'b$ for every $b \in R$. The interweaving congruence \mathfrak{I}_E of a subset $E \subseteq \mathcal{T}$ is the congruence with the equivalence \equiv_i applies for all $a \in E$. The set of all i-congruences on R is denoted by $\text{Cong}_i(R)$.

The interweaving congruence \mathfrak{I}_a of an element $a \in \mathcal{T}$ effects directly the tangible members of the ν -fiber $\text{fib}_\nu(a)$ and unite all of them to a single tangible element.

Lemma 4.32. *\mathfrak{I}_E is a \mathfrak{q} -congruence.*

Proof. To see that \mathfrak{I}_E is a congruence, suppose that $a \cong_\nu a'$ and both a and a' are tangible belonging to E , then $[a + a'] = [a^\nu] = [a']^\nu$. On the other hand $[a] + [a'] = [a]^\nu$, since $[a] = [a']$, and hence \mathfrak{I}_E

respects the ghost map ν . The verification that \mathfrak{J}_E preserves addition and multiplication is routine. Since \mathfrak{J}_E unites only tangibles with tangibles from a same ν -fiber, then $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{J}_E)$ is nonempty, containing R^\times , and thus \mathfrak{J}_E is a \mathfrak{q} -congruence. \square

The inclusions $\mathfrak{J}_a \subset \mathfrak{J}_E \subset \mathfrak{J}_\mathcal{T}$ hold for any $a \subset E \subset \mathcal{T}$.

Lemma 4.33. *The quotient ν -semiring $R/\mathfrak{J}_\mathcal{T}$ of a ν -semiring R by the \mathfrak{i} -congruence $\mathfrak{J}_\mathcal{T}$ is a faithful ν -semiring (Definition 3.11).*

Proof. Immediate, since $\text{fib}_\nu([a])$ of any $[a] \in R/\mathfrak{J}_\mathcal{T}$ contains at most one tangible element. \square

Example 4.34. *All \mathfrak{q} -congruences on a supertropical semifield F (Definition 3.24) are interweaving congruences. Indeed, the ghost ideal of F is totally ordered, while its tangible set is an abelian group; namely, each tangible element in F is a unit. Therefore, by Remark 4.14.(iv), a \mathfrak{q} -congruence can only unite elements $a \cong_\nu b$ which are ν -equivalent.*

If, furthermore, F is faithful (Definition 3.11), then the trivial congruence $\Delta(F)$ (Definition 2.2) is the unique \mathfrak{q} -congruence on F . In this case, the only possible congruences on F are the trivial congruence, the zero congruence, and the congruence defined by $a \equiv b$ for all nonzero $a, b \in F$. This case generalizes [44, Lemma 11].

One benefit of \mathfrak{i} -congruences is that $\text{Cong}_\mathfrak{i}(R)$ contains the unique maximal \mathfrak{i} -congruence $\mathfrak{J}_\mathcal{T}$, which is determined solely by the ν -equivalence \cong_ν , applied only to tangibles. In general two non-tangible a, a' in $\text{fib}_\nu(a)$ cannot always be unified, as the congruence properties may be violated. However, one can identify all the non-tangible elements in $\text{fib}_\nu(a)$ with a' to get a congruence, but, this congruence is not necessarily a \mathfrak{q} -congruence.

Definition 4.35. *The congruence \mathfrak{C}_R is defined for each $b \in \mathcal{G}$ by the equivalence*

$$\begin{aligned} a' \equiv_{\mathfrak{c}} a & \quad \text{for all tangibles } a, a' \in \text{fib}_\nu(b), \\ b' \equiv_{\mathfrak{c}} b & \quad \text{for all non-tangibles } b' \in \text{fib}_\nu(b). \end{aligned} \tag{4.25}$$

We call \mathfrak{C}_R the **structure congruence** of R .

When \mathfrak{C}_R is a \mathfrak{q} -congruence, the quotient R/\mathfrak{C}_R is a faithful definite ν -semiring (Definition 3.11).

4.5. Congruences vs. ν -semiring localizations.

Recall that \equiv denotes the underlying equivalence of a congruence \mathfrak{A} on a ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$. Let $C \subseteq \mathcal{T}$ be a tangible multiplicative submonoid of R with $1 \in C$, and let R_C be the tangible localization of R by C as described in §3.8. As C is a tangible submonoid, all its elements are \mathfrak{t} -persistent (cf. §3.3). Using the canonical injection $\tau_C : R \rightarrow R_C$ in (3.13), a congruence \mathfrak{A}' on R_C **restricts** naturally to the congruence $\mathfrak{A}'|_R := \mathfrak{A}' \cap (R \times R)$ on R via

$$\frac{a}{1} \equiv' \frac{b}{1} \Rightarrow a \equiv'_{|_R} b,$$

for all $a, b \in R$.

Remark 4.36. *When \mathfrak{A}' is a \mathfrak{q} -congruence on R_C , so does its restriction $\mathfrak{A}'|_R$ to R . Indeed, the ghost projection of $\mathfrak{A}'|_R$ is obtained as $\{b \mid \frac{b}{c} \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{A}')\}$, since every $c \in C$ is tangible. By the same reason, $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}'|_R) = \{a \mid \frac{a}{c} \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}')\}$ is a tangible subset of R , containing R^\times . Similarly, the restriction $\mathfrak{A}'|_R$ of an ℓ -congruence \mathfrak{A}' on R_C is an ℓ -congruence on R .*

Conversely, a congruence \mathfrak{A} with equivalence \equiv on R **extends** to the congruence $C^{-1}\mathfrak{A}$ on R_C , whose underlying equivalence \equiv_C is given by

$$\frac{a}{c} \equiv_C \frac{b}{c'} \quad \text{iff} \quad ac'c'' \equiv bcc'' \quad \text{for some } c'' \in C. \tag{4.26}$$

By this definition, we see that the equivalence \sim_C in (3.12) implies the equivalence \equiv_C . We set

$$C^{-1}\mathfrak{A} := \left\{ \left(\frac{a}{c}, \frac{b}{c'} \right) \mid \frac{a}{c} \equiv_C \frac{b}{c'} \right\} \subset R_C \times R_C,$$

which is a congruence on R_C . The following diagram summarizes the structures we have so far

$$\begin{array}{ccc} R & \xrightarrow{\pi_{\mathfrak{A}}} & R/\mathfrak{A} \\ \downarrow \tau_C & & \downarrow \phi \\ R_C & \xrightarrow{\pi_{C^{-1}\mathfrak{A}}} & R_C/(C^{-1}\mathfrak{A}) \end{array}$$

We aim for additional comprehensive relations on these ν -semirings, for the case that \mathfrak{A} is a \mathfrak{q} -congruence.

Lemma 4.37. *Let R_C be the localization of R by a tangible multiplicative submonoid $C \subseteq \mathcal{T}$, and let \mathfrak{A} be a \mathfrak{q} -congruence on R .*

- (i) $\frac{a}{c} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$ iff $ac'' \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ for any $c'' \in C$.
- (ii) If $\frac{a}{c} \in \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$, then $a \in \mathsf{P}_{\bullet}^{-1}(\mathfrak{A})$.

Proof. (i): (\Rightarrow): Suppose $ac'' \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ for some $c'' \in C$, that is $a \equiv b$ for some non-tangible b in \mathfrak{A} . Then, $acc'' \equiv bcc''$ for all $c \in C$, implying that $\frac{a}{c} \equiv_C \frac{b}{c}$, which is not tangible, and hence $\frac{a}{c} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$ – a contradiction.

(\Leftarrow): Suppose $ac'' \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, then $\frac{ac''}{c''} = \frac{a}{\mathbb{1}} \in \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, and hence $a \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, implying that $\frac{a}{c} \in \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$ – a contradiction.

(ii): $\mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$ is closed for multiplication and thus $\frac{a}{\mathbb{1}}, \frac{b}{\mathbb{1}} \in \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A}) \Rightarrow \frac{ab}{\mathbb{1}} \in \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$, implying that $ab \in \mathsf{P}_{\bullet}^{-1}(\mathfrak{A})$. On the other hand, if $a \notin \mathsf{P}_{\bullet}^{-1}(\mathfrak{A})$ is \mathfrak{t} -persistent, then there exists $b \in \mathsf{P}_{\bullet}^{-1}(\mathfrak{A})$ such that $ab \notin \mathsf{P}_{\bullet}^{-1}(\mathfrak{A})$. Hence $\frac{ab}{c} \notin \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$ for every $c \in C$, and thus $\frac{a}{c} \notin \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$, since $\frac{b}{\mathbb{1}} \in \mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$. \square

Part (ii) of the lemma shows that the restriction of a tangible submonoid $\mathsf{P}_{\bullet}^{-1}(C^{-1}\mathfrak{A})$ of R_C to R is a tangible submonoid of R .

Recall that for an ℓ -congruence \mathfrak{A} the tangible projection $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ is a multiplicative monoid. The next proposition provides the converse of Remark 4.36 for ℓ -congruences and plays a central role in our forthcoming study.

Proposition 4.38. *Let R_C be the tangible localization of R by a tangible multiplicative submonoid $C \subseteq \mathcal{T}$.*

- (i) An ℓ -congruence \mathfrak{A} on R extends to an ℓ -congruence $C^{-1}\mathfrak{A}$ on R_C if and only if $C \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$.
- (ii) The restriction $\mathfrak{A}'|_R = \mathfrak{A} \cap (R \times R)$ of an ℓ -congruence \mathfrak{A}' on R_C to R satisfies $\mathfrak{A}' = C^{-1}(\mathfrak{A}'|_R)$.

Proof. Recall that $\mathbb{1} := \mathbb{1}_R \in C$, since $C \subseteq \mathcal{T}$ is a tangible multiplicative submonoid.

(i): Let \mathfrak{A} be an ℓ -congruence on R and assume that $C \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$. Namely, there exists a tangible $c \in C \setminus \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ such that $c \equiv b$ for some $b \notin \mathcal{T}$. By extending \mathfrak{A} to $C^{-1}\mathfrak{A}$ we have $\frac{c}{\mathbb{1}} \equiv_C \frac{b}{\mathbb{1}}$, since $\mathbb{1} \in C$, which implies $\frac{\mathbb{1}c}{c\mathbb{1}} \equiv_C \frac{\mathbb{1}b}{c\mathbb{1}} = \frac{b}{c}$ – a non-tangible element in R_C . On the other hand, $\frac{\mathbb{1}c}{c\mathbb{1}} = \frac{\mathbb{1}}{\mathbb{1}} = \mathbb{1}_{R_C}$ is the identity of R_C , and thus $\mathbb{1}_{R_C}$ is non-tangible. Hence $(R_C)^\times \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, and therefore $C^{-1}\mathfrak{A}$ is not an ℓ -congruence.

Conversely, suppose that $C^{-1}\mathfrak{A}$ is a congruence on R_C which is not an ℓ -congruence. We have the following two cases:

(a) $\mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A}) = \emptyset$, i.e., $C^{-1}\mathfrak{A}$ is not a \mathfrak{q} -congruence, and in particular $\mathbb{1}_{R_C} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$. As \mathfrak{A} is an ℓ -congruence, we have $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A}) \neq \emptyset$ with $\mathbb{1} \in R^\times \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$. Thus, there are $a \in R \setminus C$ and $c \in C$, where a is non-tangible, such that $\frac{a}{c} \equiv_C \frac{\mathbb{1}}{\mathbb{1}} = \mathbb{1}_{R_C}$ in $R_C/(C^{-1}\mathfrak{A})$. This means that $ac'' \equiv cc''$ for some $c'' \in C$, where ac'' is non-tangible (otherwise $\frac{ac''}{c''} = \frac{a}{\mathbb{1}}$ would be tangible, cf. Lemma 4.37.(i)). Therefore, since $ac'' \equiv cc''$, we obtain that $cc'' \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$. But, $cc'' \in C$, since C is a monoid, and hence $C \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$.

(b) $\mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A}) \neq \emptyset$. If $(R_C)^\times \not\subseteq \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, then $\mathbb{1}_{R_C} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, and we are done by part (a). Otherwise, $\mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$ is not closed for multiplication, i.e., $C^{-1}\mathfrak{A}$ is a \mathfrak{q} -congruence but not an ℓ -congruence. Let $\frac{a_1}{c_1}, \frac{a_2}{c_2} \in \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, which implies $a_1, a_2 \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ by Lemma 4.37.(i), and hence $a_1a_2 \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, since \mathfrak{A} is an ℓ -congruence.

Assume that $\frac{a_1a_2}{c_1c_2} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, which gives $\frac{c_1}{\mathbb{1}} \frac{a_1}{c_1} \frac{a_2}{c_2} \frac{c_2}{\mathbb{1}} = \frac{a_1a_2}{\mathbb{1}} \notin \mathsf{T}_{\text{cls}}^{-1}(C^{-1}\mathfrak{A})$, since $c_1c_2 \in C$. Namely, $\frac{a_1a_2}{\mathbb{1}} \equiv_C \frac{b}{c'}$ for some non-tangible $\frac{b}{c'} \in R_C$, where $b \notin \mathcal{T}$ and $c' \in C$ is tangible. This implies that $a_1a_2c'c'' \equiv bc''$ over R , for some $c'' \in C$, where bc'' is non-tangible in R (cf. Lemma 4.37.(ii)). Hence

$a_1 a_2 c' c'' \notin T_{\text{cls}}^{-1}(\mathfrak{A})$, while $a_1 a_2 \in T_{\text{cls}}^{-1}(\mathfrak{A})$, implying that $c' c'' \notin T_{\text{cls}}^{-1}(\mathfrak{A})$, since $T_{\text{cls}}^{-1}(\mathfrak{A})$ is a submonoid. Thus, $C \not\subseteq T_{\text{cls}}^{-1}(\mathfrak{A})$, since $cc'' \in C$.

(ii): To show that $\mathfrak{A}' \subseteq C^{-1}(\mathfrak{A}'|_R)$, let $\mathfrak{A}' \subset R_C \times R_C$, and take $(\frac{a}{c}, \frac{b}{c'}) \in \mathfrak{A}'$. Now assume that $(\frac{a}{c}, \frac{b}{c'}) \notin C^{-1}(\mathfrak{A}'|_R)$, which means that $ac'c'' \not\equiv_{|R} bcc''$ for all $c'' \in C$, and in particular for $c'' = \mathbb{1}$. But then $ac' \not\equiv_{|R} bc$, implying that $\frac{ac'}{\mathbb{1}} \not\equiv_{|R} \frac{bc}{\mathbb{1}}$, and thus $\frac{a}{c} \not\equiv_{|R} \frac{b}{c'}$ – a contradiction.

The opposite inclusion $\mathfrak{A}' \supseteq C^{-1}(\mathfrak{A}'|_R)$ is trivial. \square

Let \mathfrak{A} be an ℓ -congruence, and let $C_{\mathfrak{A}} := \text{tcor}(\pi_{\mathfrak{A}})$ be the tangible core (Definition 3.31) of the canonical \mathfrak{q} -homomorphism $\pi_{\mathfrak{A}} : R \rightarrow R/\mathfrak{A}$. Then $C_{\mathfrak{A}} \subseteq \mathcal{T}$ is a tangible multiplicative submonoid, by Corollaries 3.34 and 4.26, equals to $T_{\text{cls}}^{-1}(\mathfrak{A}) \subseteq \mathcal{T}$ by (4.22), which localizes R as in (3.14).

Definition 4.39. We define the *localization of a ν -semiring R by an ℓ -congruence \mathfrak{A}* to be the tangible localization (Definition 3.39)

$$R_{\mathfrak{A}} := C_{\mathfrak{A}}^{-1}R = (\text{tcor}(\pi_{\mathfrak{A}}))^{-1}R, \quad \text{where } C_{\mathfrak{A}} = T_{\text{cls}}^{-1}(\mathfrak{A}),$$

whose elements are fractions $\frac{a}{c}$ with tangible $c \in C_{\mathfrak{A}}$.

Given an ℓ -congruence \mathfrak{A} on R , we write

$$\mathfrak{A}_R := (T_{\text{cls}}^{-1}(\mathfrak{A}))^{-1}\mathfrak{A} \tag{4.27}$$

for the extension of \mathfrak{A} to the localization $R_{\mathfrak{A}}$ of R by \mathfrak{A} , cf. (4.26). Then, by Corollary 4.26, $R_{\mathfrak{A}}/\mathfrak{A}_R$ is a tangibly closed ν -semiring (but not necessarily a ν -domain), in which every tangible element is a unit. For the localized ν -semiring $R_{\mathfrak{A}} := (R_{\mathfrak{A}}, \mathcal{T}_{\mathfrak{A}}, \mathcal{G}_{\mathfrak{A}}, \nu_{\mathfrak{A}})$, this gives the diagram

$$\begin{array}{ccc} R & \xrightarrow{\tau_{C_{\mathfrak{A}}}} & R_{\mathfrak{A}} \\ & \searrow \phi & \downarrow \pi_{\mathfrak{A}_R} \\ & & R_{\mathfrak{A}}/\mathfrak{A}_R, \end{array} \tag{4.28}$$

which later helps us to construct local ν -semirings.

4.6. Prime congruences and ν -semiring localization.

Let \mathfrak{A} be a \mathfrak{q} -congruence with underlying equivalence \equiv on a ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$. Recall that we write

$$a \equiv \text{ghost} \quad \text{if} \quad a \equiv b \text{ for some } b \in \mathcal{G},$$

and that by Lemma 4.1 this condition is equivalent to $a \equiv a^\nu$. To ease the exposition, we use both notations. We write $(R/\mathfrak{A})|_{\text{tng}}^\circ$ for the \mathfrak{t} -persistent set of the quotient ν -semiring R/\mathfrak{A} , $(R/\mathfrak{A})|_{\text{tng}}$ for its tangible set, and $(R/\mathfrak{A})|_{\text{gh}}$ for its ghost ideal.

The following definition is a key definition in this paper.

Definition 4.40. A **\mathfrak{g} -prime congruence \mathfrak{P}** (alluded for **ghost prime congruence**) on a ν -semiring R is an ℓ -congruence whose underlying equivalence $\equiv_{\mathfrak{P}}$ satisfies for any $a, b \in R$ the condition

$$ab \equiv_{\mathfrak{P}} \text{ghost} \quad \Rightarrow \quad a \equiv_{\mathfrak{P}} \text{ghost} \text{ or } b \equiv_{\mathfrak{P}} \text{ghost}. \tag{4.29}$$

That is, $ab \in G_{\text{cls}}^{-1}(\mathfrak{P})$ implies $a \in G_{\text{cls}}^{-1}(\mathfrak{P})$ or $b \in G_{\text{cls}}^{-1}(\mathfrak{P})$, or equivalently $a, b \notin G_{\text{cls}}^{-1}(\mathfrak{P})$ implies $ab \notin G_{\text{cls}}^{-1}(\mathfrak{P})$. The **\mathfrak{g} -prime (congruence) spectrum** of R is defined as

$$\text{Spec}(R) := \{\mathfrak{P} \mid \mathfrak{P} \text{ is a } \mathfrak{g}\text{-prime congruence on } R\}. \tag{4.30}$$

In supertropical context, being “prime” for a congruence \mathfrak{P} essentially means that a *product of two non-ghost elements can never be equivalent to a ghost*, while its tangible projection $T_{\text{cls}}^{-1}(\mathfrak{P})$ is a (tangible) monoid. By Lemma 4.1, Condition (4.29) is stated equivalently as

$$ab \equiv_{\mathfrak{P}} (ab)^\nu \quad \Rightarrow \quad a \equiv_{\mathfrak{P}} a^\nu \text{ or } b \equiv_{\mathfrak{P}} b^\nu, \tag{4.31}$$

which means that the ghsotification of a product ab by \mathfrak{P} is obtained by ghostifying at least one of its terms. Also, from (4.20) we obtain

$$ab \notin T_{\text{cls}}^{-1}(\mathfrak{P}) \quad \Rightarrow \quad a \notin T_{\text{cls}}^{-1}(\mathfrak{P}) \text{ or } b \notin T_{\text{cls}}^{-1}(\mathfrak{P}),$$

since a \mathfrak{g} -prime congruence is an ℓ -congruence, in which $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \mathsf{P}_{\bullet}^{-1}(\mathfrak{P})$. (Thereby $a \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$ for each $a \notin \mathcal{T}^\circ$.) Accordingly, localizing by a \mathfrak{g} -prime congruence \mathfrak{P} is the same as localizing by an ℓ -congruence, written specifically $R_{\mathfrak{P}}$.

Remark 4.41. *The condition that if $ab \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ then a or b belongs to $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ shows that the ghost projection $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ of a \mathfrak{g} -prime congruence \mathfrak{P} is a \mathfrak{g} -prime ideal of R containing its ghost ideal \mathcal{G} (Definition 3.18). Furthermore, it implies that the complement $\mathsf{G}_{\text{cls}}^{-\text{c}}(\mathfrak{P}) = R \setminus \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ of $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ is a multiplicative submonoid of R , consisting of non-ghost elements. In particular, it contains the \mathfrak{t} -persistent monoid $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \mathsf{P}_{\bullet}^{-1}(\mathfrak{P})$ which is employed to execute localization.*

We derive the following observation, which is of much help later.

Remark 4.42. *The canonical surjection $\pi_{\mathfrak{A}} : R \twoheadrightarrow R/\mathfrak{A}$ and its inverse $[a] \mapsto \pi_{\mathfrak{A}}^{-1}([a])$ define a one-to-one correspondence between all \mathfrak{g} -prime congruences on R/\mathfrak{A} and the \mathfrak{g} -prime congruences on R that contain \mathfrak{A} , cf. Remark 4.29.(i).*

If $\varphi : R \rightarrow R'$ is a \mathfrak{q} -homomorphism and \mathfrak{P}' is a \mathfrak{g} -prime congruence on R' , then $\varphi^\vee(\mathfrak{P}')$ is a \mathfrak{g} -prime congruence on R . Indeed, first, $\varphi^\vee(\mathfrak{P}')$ is an ℓ -congruence on R by Remark 4.29.(ii). Second, for the ghost component, recall that $\varphi^\vee(\mathfrak{P}')$ is determined by the equivalence \equiv given by $a \equiv b$ iff $\varphi(a) \equiv_{\mathfrak{p}} \varphi(b)$. Then, $ab \equiv \text{ghost}$ iff $\varphi(ab) = \varphi(a)\varphi(b) \equiv_{\mathfrak{p}} \text{ghost}$ iff $\varphi(a) \equiv_{\mathfrak{p}} \text{ghost}$ or $\varphi(b) \equiv_{\mathfrak{p}} \text{ghost}$ iff $a \equiv \text{ghost}$ or $b \equiv \text{ghost}$. Therefore, $\varphi^\vee(\mathfrak{P}')$ is a \mathfrak{g} -prime congruence.

We start with an easy characterization of the simplest type of \mathfrak{g} -prime congruences, arising when the clusters of an ℓ -congruence are the complement of each other.

Lemma 4.43. *An ℓ -congruence \mathfrak{A} in which $\mathfrak{A} = \mathsf{T}_{\text{cls}}(\mathfrak{A}) \cup \mathsf{G}_{\text{cls}}(\mathfrak{A})$ is a \mathfrak{g} -prime congruence.*

Proof. Assume that $ab \equiv \text{ghost}$ where both $a \not\equiv \text{ghost}$ and $b \not\equiv \text{ghost}$, hence $a, b \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, as $\mathfrak{A} = \mathsf{T}_{\text{cls}}(\mathfrak{A}) \cup \mathsf{G}_{\text{cls}}(\mathfrak{A})$ is a disjoint union. But $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ is a multiplicative monoid of R , since \mathfrak{A} is an ℓ -congruence, and thus $ab \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$, implying that $ab \not\equiv \text{ghost}$ – a contradiction. \square

Example 4.44.

- (i) *The trivial congruence $\Delta(R)$ on a ν -domain R is a \mathfrak{g} -prime congruence.*
- (ii) *Any \mathfrak{g} -prime congruence on a supertropical semifield (Definition 3.24) is an interweaving congruence, cf. Example 4.34.*
- (iii) *All \mathfrak{g} -prime congruences \mathfrak{P} on a definite ν -semifield F are determined by equivalences on the tangible monoid \mathcal{T} of F , which is an abelian group (Definition 3.19). Therefore, $\mathsf{P}_{\bullet}^{-1}(\mathfrak{P}) = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \mathcal{T}$ for all \mathfrak{P} on F , where their equivalence classes are varied.*
- (iv) *Let $R := \tilde{F}[\lambda]$ be the (tangibly closed) ν -semiring of polynomial functions over a supertropical semifield F . For any $E = \{\lambda + a\}$ the \mathfrak{g} -congruence \mathfrak{G}_E (cf. (4.17)) is a \mathfrak{g} -prime congruence on R . The same holds for any factor as in Theorem 3.48, with the respective conditions.*
- (v) *The polynomial function $f = g_1g_2 = (\lambda_1 + \lambda_2 + \mathbb{1})(\lambda_1 + \lambda_2 + \lambda_1\lambda_2) = (\lambda_1 + \mathbb{1})(\lambda_2 + \mathbb{1})(\lambda_1 + \lambda_2) = h_1h_2h_3$ in Example 3.49 demonstrates a pathological behavior where unique factorization fails. Although the factor g_1 is irreducible, its ghostification congruence \mathfrak{G}_E , $E = \{g_1\}$, is not a \mathfrak{g} -prime congruence, as $f \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$, but the h_i 's are not necessarily contained in $\mathsf{G}_{\text{cls}}^{-1}(\mathfrak{G}_E)$.*

Definition 4.40 lays the first supertropical analogy to the familiar connection between prime ideals and integral domains in ring theory.

Proposition 4.45. *Let \mathfrak{A} be a congruence on a ν -semiring R . The quotient R/\mathfrak{A} is a ν -domain (Definition 3.19) if and only if \mathfrak{A} is a \mathfrak{g} -prime congruence.*

Proof. (\Rightarrow) : Let R/\mathfrak{A} be a ν -domain. As R/\mathfrak{A} is a tangibly closed ν -semiring, $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ must be a monoid, so \mathfrak{A} is an ℓ -congruence. For $ab \equiv \text{ghost}$ in R , we have $[ab] = [a][b] \in (R/\mathfrak{A})|_{\text{gh}}$, implying that $[a] \in (R/\mathfrak{A})|_{\text{gh}}$ or $[b] \in (R/\mathfrak{A})|_{\text{gh}}$, since R/\mathfrak{A} has no ghost divisors. This is equivalent to $a \equiv \text{ghost}$ or $b \equiv \text{ghost}$, and thus \mathfrak{A} is \mathfrak{g} -prime.

(\Leftarrow) : Let $\mathfrak{A} = \mathfrak{P}$ be a \mathfrak{g} -prime congruence. R/\mathfrak{P} is tangibly closed ν -semiring, by Corollary 4.26, since \mathfrak{P} is an ℓ -congruence. Suppose that $[a][b] \in (R/\mathfrak{P})|_{\text{gh}}$, then $[ab] \in (R/\mathfrak{P})|_{\text{gh}}$. Namely $ab \equiv_{\mathfrak{p}} \text{ghost}$, implying that $a \equiv_{\mathfrak{p}} \text{ghost}$ or $b \equiv_{\mathfrak{p}} \text{ghost}$, since \mathfrak{P} is \mathfrak{g} -prime. Hence, $[a] \in (R/\mathfrak{P})|_{\text{gh}}$ or $[b] \in (R/\mathfrak{P})|_{\text{gh}}$, and thus R/\mathfrak{P} has no ghost-divisors and is a ν -domain. \square

From Proposition 4.45 we deduce that a ν -semiring R is a ν -domain if and only if the trivial congruence $\Delta(R)$ on R is a \mathfrak{g} -prime congruence. When \mathfrak{P} is \mathfrak{g} -prime congruence as in Lemma 4.43, R/\mathfrak{P} is a definite ν -domain, cf. Example 3.21 and Corollary 4.26.

Example 4.46. Let $\text{STR}(\mathbb{N}_0)$ be the supertropical domain (Definition 3.24) over $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as constructed in Example 3.25, and let $\mathfrak{P} \neq \Delta(\mathbb{N}_0)$ be a nontrivial \mathfrak{g} -prime congruence on $\text{STR}(\mathbb{N}_0)$. Suppose $n \equiv_{\mathfrak{p}} n^\nu$ for some $n \in \mathbb{N}_0$, then (4.31) inductively implies that $1 \equiv_{\mathfrak{p}} 1^\nu$ or $0 \equiv_{\mathfrak{p}} 0^\nu$. But $\mathbb{1} = 0$ is the identity of $\text{STR}(\mathbb{N}_0)$, and thus in the latter case \mathfrak{P} is not a \mathfrak{q} -congruence. Hence $\text{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \{0\}$, which shows that all nontrivial \mathfrak{g} -prime congruences on $\text{STR}(\mathbb{N}_0)$ have the same tangible cluster.

The ghost classes of \mathfrak{P} can be varied, induced by the equivalences $n \equiv_{\mathfrak{p}} n^\nu$ of $n \in \mathbb{N}$. For a fixed $p \in \mathbb{N}$, the ghost classes are as follows:

$$[n^\nu] = \{n^\nu\} \text{ for each } 0 \leq n < p, \quad [p^\nu] = \{n^\nu \mid n \geq p\}.$$

Accordingly, we see that there exists an injection of \mathbb{N}_0 in the spectrum $\text{Spec}(\text{STR}(\mathbb{N}_0))$.

By similar considerations, as every tangible in $\text{STR}(\mathbb{Z})$ is invertible, the trivial congruence $\Delta(\mathbb{Z})$ is the only \mathfrak{g} -prime congruence on the supertropical semifield $\text{STR}(\mathbb{Z})$. Indeed, if $n \equiv n^\nu$ then $0 = (-n)n \equiv (-n)n^\nu = e$, which gives a ghost congruence. Furthermore, by Remark 4.4.(iii), if $n \equiv m$ for $n > m$, then $n = n + m \equiv n^\nu$, which implies again $0 \equiv e$. The same holds for the supertropical semifields $\text{STR}(\mathbb{Q})$ and $\text{STR}(\mathbb{R})$.

Recalling that $\text{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \text{P}_{\bullet}^{-1}(\mathfrak{P})$ is a tangible monoid for every \mathfrak{g} -prime congruence \mathfrak{P} , we specialize Proposition 4.38 to \mathfrak{g} -prime congruences.

Proposition 4.47. Let R be a ν -semiring, and let R_C be its localization by a tangible multiplicative submonoid $C \subset R$.

- (i) A \mathfrak{g} -prime congruence \mathfrak{P} on R such that $C \subseteq \text{T}_{\text{cls}}^{-1}(\mathfrak{P})$ extends to a \mathfrak{g} -prime congruence $C^{-1}\mathfrak{P}$ on R_C and satisfies $(C^{-1}\mathfrak{P})|_R = \mathfrak{P}$.
- (ii) The restriction $\mathfrak{P}'|_R := \mathfrak{P}' \cap (R \times R)$ of a \mathfrak{g} -prime congruence \mathfrak{P}' on R_C to R is a \mathfrak{g} -prime congruence satisfying $\mathfrak{P}' = C^{-1}(\mathfrak{P}'|_R)$. In particular $C \subseteq \text{T}_{\text{cls}}^{-1}(\mathfrak{P}'|_R)$.

Proof. (i): First, $C^{-1}\mathfrak{P}$ is an ℓ -congruence on R_C , by Proposition 4.38.(i). Assume that \mathfrak{P} is a \mathfrak{g} -prime congruence on R . Take $\frac{a}{c}, \frac{b}{c'} \in R_C$ such that $\frac{a}{c} \frac{b}{c'} \equiv_C \frac{g}{h}$ in $C^{-1}\mathfrak{P}$, where $\frac{g}{h}$ is a ghost of R_C , which implies that $g \in \mathcal{G}$, since $h \in C$ is tangible. Then,

$$ab(hc'') = abhc'' \equiv_{\mathfrak{p}} gcc'c'' = g(cc'c'')$$

for some $c'' \in C$. Since \mathfrak{P} is \mathfrak{g} -prime and $g(cc'c'') \equiv_{\mathfrak{p}} \text{ghost}$, then $ab \equiv_{\mathfrak{p}} \text{ghost}$ or $hc'' \equiv_{\mathfrak{p}} \text{ghost}$. But $hc'' \in C \subseteq \mathcal{T}$, and thus $ab \equiv_{\mathfrak{p}} \text{ghost}$. By the same argument, as \mathfrak{P} is \mathfrak{g} -prime, this implies $a \equiv_{\mathfrak{p}} \text{ghost}$ or $b \equiv_{\mathfrak{p}} \text{ghost}$, and therefore $\frac{a}{c}$ or $\frac{b}{c'}$ is ghost in R_C . Hence, $C^{-1}\mathfrak{P}$ is a \mathfrak{g} -prime congruence on R_C .

Clearly $\mathfrak{P} \subseteq (C^{-1}\mathfrak{P})|_R$. To verify the opposite inclusion, by the use of Proposition 4.38.(i), we only need to show that this restriction preserves the property of being \mathfrak{g} -prime. Suppose $\frac{a}{\mathbb{1}} \in \text{G}_{\text{cls}}^{-1}((C^{-1}\mathfrak{P})|_R)$, that is $\frac{a}{\mathbb{1}} \equiv_C \text{ghost}$ in $C^{-1}\mathfrak{P}$. As $\frac{a}{\mathbb{1}} \in R_C$, there are $a' \in R$, $c, c' \in C$, and $g \in \mathcal{G}$ such that

$$\frac{a}{\mathbb{1}} = \frac{a'}{c} \equiv_C \frac{g}{c'} . \quad (*)$$

From the right hand side, we have $a'c'c'' \equiv_{\mathfrak{p}} gcc''$ for some $c'' \in C$, where $gcc'' \in \mathcal{G}$. This implies $a' \equiv_{\mathfrak{p}} \text{ghost}$, since the product $c'c''$ is tangible and \mathfrak{P} is \mathfrak{g} -prime. From the left hand side of (*), we have $ac = a' \equiv_{\mathfrak{p}} \text{ghost}$, and thus $a \equiv_{\mathfrak{p}} \text{ghost}$, as $c \in C$ is tangible. Therefore $a \in \text{G}_{\text{cls}}^{-1}(\mathfrak{P})$, and we conclude that $a \in \text{G}_{\text{cls}}^{-1}(\mathfrak{P})$ iff $\frac{a}{\mathbb{1}} \in \text{G}_{\text{cls}}^{-1}((C^{-1}\mathfrak{P})|_R)$.

(ii): Suppose \mathfrak{P}' is a \mathfrak{g} -prime congruence on R_C , then an equivalence $\frac{a}{c} \frac{b}{c'} \equiv_{\mathfrak{p}} \frac{g}{h}$ to some ghost $\frac{g}{h}$ implies that $\frac{a}{c}$ or $\frac{b}{c'}$ is ghost. In particular, this holds when $c = c' = h = \mathbb{1}$, and thus also for the restriction $\mathfrak{P}'|_R$. Hence, $\mathfrak{P}'|_R$ is a \mathfrak{g} -prime congruence. The same argument as in the proof of Proposition 4.38 shows that $\mathfrak{P}' = C^{-1}(\mathfrak{P}'|_R)$. Finally, $C \subseteq \text{T}_{\text{cls}}^{-1}(\mathfrak{P}'|_R)$ by Proposition 4.38.(i). \square

Having the correspondences between \mathfrak{g} -prime congruences on R and \mathfrak{g} -prime congruences on R_C settled, we have the following desired result.

Theorem 4.48. *The canonical injection $\tau_C : R \hookrightarrow R_C$ of a ν -semiring R into its localization R_C by a tangible multiplicative submonoid $C \subset R$ induces a bijection*

$$\mathrm{Spec}(R_C) \xrightarrow{\sim} \{\mathfrak{P} \in \mathrm{Spec}(R) \mid C \subseteq \mathrm{T}_{\mathrm{cls}}^{-1}(\mathfrak{P})\}, \quad \mathfrak{P}_C \mapsto \mathfrak{P}_C|_R, \quad (4.32)$$

that, together with its inverse, respects inclusions between \mathfrak{g} -prime congruences on R and \mathfrak{g} -prime congruences on R_C .

Proof. Proposition 4.38 determines the map of ℓ -congruences, while Proposition 4.47 restricts to the case of \mathfrak{g} -prime congruences. \square

4.7. Deterministic \mathfrak{q} -congruences.

In classical ring theory, prime and maximal ideals are main structural ideals, where there is no significant intermediate ideal structure between them. In supertropical theory, with our analogues congruences, we do have the following special ℓ -congruences.

Definition 4.49. *A \mathfrak{q} -congruence \mathfrak{D} is called **deterministic**, if $\mathfrak{D} = \mathrm{T}_{\mathrm{cls}}(\mathfrak{D}) \cup \mathrm{G}_{\mathrm{cls}}(\mathfrak{D})$, i.e., it has only tangible and ghost classes. The set of all deterministic ℓ -congruences on R is denoted by*

$$\mathrm{Spd}(R) := \{\mathfrak{D} \mid \mathfrak{D} \text{ is a deterministic } \ell\text{-congruence}\},$$

called the **deterministic spectrum** of R .

For example, every ℓ -congruence on a supertropical domain R (Definition 3.24) is deterministic, as each element in R is either tangible or ghost. Over an arbitrary ν -semiring R deterministic \mathfrak{q} -congruences are heavily dependent on the properties of R . For example, if R has two non-tangibles a, b whose sum $a+b$ is a unit, then R cannot carry a deterministic \mathfrak{q} -congruence which unites these elements, cf. Example 4.22.(i).

Lemma 4.50. *A deterministic ℓ -congruence \mathfrak{D} is \mathfrak{g} -prime.*

Proof. Follows immediately from Lemma 4.43. \square

Accordingly, we have the inclusion $\mathrm{Spd}(R) \subseteq \mathrm{Spec}(R)$ of spectra (Definition 4.49).

Proposition 4.51. *Let R be a ν -semiring, and let \mathfrak{A} be a \mathfrak{q} -congruence on R .*

- (i) *R/\mathfrak{A} is a definite ν -semiring (Definition 3.11) iff \mathfrak{A} is a deterministic \mathfrak{q} -congruence.*
- (ii) *R/\mathfrak{A} is a definite ν -domain (Definition 3.19) iff \mathfrak{A} is a deterministic ℓ -congruence.*

Proof. (i): (\Rightarrow): As R/\mathfrak{A} is a definite ν -domain, i.e., $R/\mathfrak{A} = (R/\mathfrak{A})|_{\mathrm{tng}} \dot{\cup} (R/\mathfrak{A})|_{\mathrm{gh}}$, and \mathfrak{A} is a \mathfrak{q} -congruence, $[ab] = [a][b] \in (R/\mathfrak{A})|_{\mathrm{tng}}$ implies $[a], [b] \in (R/\mathfrak{A})|_{\mathrm{tng}}$, and thus $a, b \in \mathcal{T}$. Otherwise, $[a][b]$ belongs to the complement $(R/\mathfrak{A})|_{\mathrm{gh}}$. Hence, \mathfrak{A} is a deterministic \mathfrak{q} -congruence.

(\Leftarrow): Assume that $\mathfrak{A} = \mathfrak{D}$ is a deterministic \mathfrak{q} -congruence, i.e., $\mathfrak{D} = \mathrm{T}_{\mathrm{cls}}(\mathfrak{D}) \cup \mathrm{G}_{\mathrm{cls}}(\mathfrak{D})$. Then R/\mathfrak{D} has only tangible and ghost classes, and thus R/\mathfrak{D} is definite.

(ii) \mathfrak{A} is \mathfrak{g} -prime by Lemma 4.50, where R/\mathfrak{A} is a ν -domain iff \mathfrak{A} is a \mathfrak{g} -prime congruence by Proposition 4.45. \square

Example 4.52. *Let $\mathfrak{D}_{\mathcal{T}}$ be the congruence whose underlying equivalence $\equiv_{\mathfrak{D}}$ is determined by the relation*

$$a \equiv_{\mathfrak{D}} a^{\nu} \quad \text{for all } a \notin \mathcal{T}.$$

*In other words, $\mathfrak{D}_{\mathcal{T}} = \mathfrak{G}_{R \setminus \mathcal{T}}$ is the \mathfrak{g} -congruence that ghostifies all non-tangible elements of R . We call $\mathfrak{D}_{\mathcal{T}}$ the **ghost determination** of R . (Note that $\mathfrak{D}_{\mathcal{T}}$ is not necessarily an ℓ -congruence.)*

4.8. Maximal ℓ -congruences.

In ring theory, a maximal ideal is defined set-theoretically by inclusions, and induces equivalences on its complement. This characterization is sometimes too restrictive for ν -semirings, especially regarding tangible clusters whose equivalences are not directly induced by relations on their complements. One reason for this is that the tangible cluster and the ghost cluster in general are not complements of each other. Furthermore, maximality of a congruence does not implies maximality of its tangible projection, cf. (4.16). Therefore, we need a coarse setup that concerns tangible projections directly. We begin with the naive definition.

Definition 4.53. An ℓ -congruence (resp. \mathfrak{q} -congruence) on R is **maximal**, if it is a proper congruence and is maximal with respect to inclusion in $\text{Cong}_\ell(R)$ (resp. in $\text{Cong}_\mathfrak{q}(R)$). The **maximal spectrum** of a ν -semiring R is defined to be

$$\text{Spm}_\ell(R) := \{\mathfrak{M} \mid \mathfrak{M} \text{ is a maximal } \ell\text{-congruence}\}.$$

The classification of maximal ℓ -congruences on ν -semirings is rather complicated, involving a careful analysis, as seen from the following constraints. Remark 4.27 shows that in ℓ -congruences, or in any \mathfrak{q} -congruence on a ν -semiring, units enforce major constraints. For example, if $b + a = u$ for $u \in R^\times$, we cannot have $a \equiv b$, but it may happen that a or b is congruent to a non-tangible. Moreover, from Example 4.22 we learn that two ordered units cannot be congruent to each other, and that a unit might only be congruent to other units. (In addition, tangible projections must contain \mathfrak{t} -unalterable elements.) The maximal ℓ -congruences consisting $\text{Spm}_\ell(R)$ must obey these constraints.

Remark 4.54. From Remark 4.29.(i) it infers that R/\mathfrak{M} carries no nontrivial ℓ -congruences.

To link maximality to ν -primeness we need an extra coarse resolution, concerned with projections of ℓ -congruences, more precisely with their tangible projection.

Definition 4.55. A **\mathfrak{t} -minimal ℓ -congruence**, alluded for **tangibly minimal ℓ -congruence**, is an ℓ -congruence \mathfrak{A} whose projection $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{A})$ is minimal with respect to inclusion of tangible projections in R . The **\mathfrak{t} -minimal spectrum** of a ν -semiring R is defined to be

$$\text{Spn}_\ell(R) := \{\mathfrak{N} \mid \mathfrak{N} \text{ is a } \mathfrak{t}\text{-minimal } \ell\text{-congruence}\}.$$

Namely, $\text{Spn}_\ell(R)$ contains all ℓ -congruences $\mathfrak{A} \in \text{Cong}_\ell(R)$ for which the complement $\mathbb{T}_{\text{cls}}^c(\mathfrak{A})$ of the tangible projection $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{A})$ is maximal in R .

In other words, \mathfrak{t} -minimality of \mathfrak{A} is equivalent to maximality of non-tangible elements in R/\mathfrak{A} – the analogy of maximal ideals in ring theory. As only ℓ -congruences are considered, for which $R^\times \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{A})$, the trivial minimality of $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{A}) = \emptyset$ is excluded, and our setup is properly defined.

A \mathfrak{t} -minimal ℓ -congruence \mathfrak{N} need not be maximal in the sense of Definition 4.53, since it might be contained in some $\mathfrak{M} \in \text{Spm}_\ell(R)$ with $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{N}) = \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{M})$, but $\mathbb{T}_{\text{cls}}(\mathfrak{N}) \subset \mathbb{T}_{\text{cls}}(\mathfrak{M})$. On the other hand, recalling from (4.16) that $\mathfrak{A}_1 \subset \mathfrak{A}_2$ implies $\mathbb{T}_{\text{cls}}(\mathfrak{A}_1) \supset \mathbb{T}_{\text{cls}}(\mathfrak{A}_2)$, a maximal ℓ -congruence is \mathfrak{t} -minimal and $\text{Spm}_\ell(R) \subseteq \text{Spn}_\ell(R)$.

Example 4.56. All maximal ℓ -congruences on a supertropical semifield $F := (F, \mathcal{T}, \mathcal{G}, \nu)$ share the same tangible projection \mathcal{T} , while their equivalence classes are varied over $\mathcal{T} \times \mathcal{T}$ and $R \setminus \mathcal{T} \times R \setminus \mathcal{T}$. The same holds for every \mathfrak{t} -minimal ℓ -congruence.

The ν -semiring $\tilde{F}[\lambda]$ of polynomial functions (e.g., $\mathbb{T}[\lambda]$ in Example 3.26) carries many maximal ℓ -congruences and \mathfrak{t} -minimal ℓ -congruences. On the other hand, it has the unique maximal ideal $\tilde{F}[\lambda] \setminus \mathcal{T}$. A maximal ℓ -congruence \mathfrak{M} on $\tilde{F}[\lambda]$ has tangible equivalence classes of the form $[a]_{\mathfrak{m}}$ or $[a\lambda + b]_{\mathfrak{m}}$, with $a, b \in \mathcal{T}$, as follows from Theorem 3.48.

A deterministic ℓ -congruence (Definition 4.49) need not be maximal nor \mathfrak{t} -minimal, and vice versa, despite its tangible cluster is the complement of its ghost cluster. To deal with ν -primeness, we mostly employ \mathfrak{t} -minimal ℓ -congruences.

Proposition 4.57. Every ℓ -congruence on a ν -semiring is contained in a maximal ℓ -congruence. Hence, any ν -semiring R with $\text{Cong}_\ell(R) \neq \emptyset$ carries at least one maximal ℓ -congruence.

Proof. Let \mathfrak{A}' be an ℓ -congruence, and let $\mathcal{A} := \{\mathfrak{A} \in \text{Cong}_\ell(R) \mid \mathfrak{A}' \subset \mathfrak{A}\}$, which is nonempty since $\mathfrak{A}' \in \mathcal{A}$. Let \mathcal{C} be an inclusion chain in \mathcal{A} , and set $\mathfrak{K} = \bigcup_{\mathfrak{A} \in \mathcal{C}} \mathfrak{A}$. Suppose $(a_1, b_1), (a_2, b_2) \in \mathfrak{K}$, then there are $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$ such that $(a_i, b_i) \in \mathfrak{A}_i$, where either $\mathfrak{A}_1 \subset \mathfrak{A}_2$ or $\mathfrak{A}_2 \subset \mathfrak{A}_1$; say the former. Thus $(a_1, b_1) \in \mathfrak{A}_2$, so $(a_1 + a_2, b_1 + b_2) \in \mathfrak{A}_2 \subset \mathfrak{K}$ and $x(a_i, b_i) \in \mathfrak{A}_2$ for each i and every $x \in R$. So \mathfrak{K} is a congruence. Clearly, $R^\times \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{C}$; thus $R^\times \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{K})$ and \mathfrak{K} is a \mathfrak{q} -congruence. Furthermore, if $ab \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{K})$ for $a, b \in \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{K})$, then the same holds for some $\mathfrak{A} \in \mathcal{C}$ – contradicting the fact that \mathfrak{A} is an ℓ -congruence. Hence, \mathfrak{K} is an ℓ -congruence.

Therefore, \mathcal{A} is an inclusion poset of ℓ -congruences containing \mathfrak{A}' , where every chain has as upper bound the ℓ -congruence that is the union of all the ℓ -congruences in the chain. By Zorn's Lemma, we deduce that \mathcal{A} has a maximal element, say \mathfrak{M} , which is an ℓ -congruence containing \mathfrak{A}' , and there exists no ℓ -congruence on R that strictly contains \mathfrak{M} . \square

In particular, by the proposition, any \mathfrak{t} -minimal ℓ -congruence is contained in a maximal ℓ -congruence.

Corollary 4.58. *Any ν -semiring R with $\text{Spec}(R) \neq \emptyset$ carries at least one (nontrivial) \mathfrak{t} -minimal ℓ -congruence.*

Proof. $\text{Cong}_\ell(R) \supset \text{Spec}(R) \neq \emptyset$ and thus $\text{Spm}_\ell(R) \neq \emptyset$, by Proposition 4.57, implying that $\text{Spn}_\ell(R) \neq \emptyset$, since $\text{Spn}_\ell(R) \supseteq \text{Spm}_\ell(R)$. \square

Maximality and \mathfrak{t} -minimality of congruence can be considered equivalently for \mathfrak{q} -congruences. However, we are mostly interested in elements of $\text{Spec}(R)$ and restrict to ℓ -congruences.

Proposition 4.59. *If R/\mathfrak{A} is a ν -semifield, then \mathfrak{A} is a \mathfrak{t} -minimal \mathfrak{g} -prime congruence.*

Proof. Assume R/\mathfrak{A} is a ν -semifield, namely $(R/\mathfrak{A})|_{\text{tng}} = (R/\mathfrak{A})^\times$, where every tangible $u \in (R/\mathfrak{A})|_{\text{tng}}$ is a unit. This means that $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{A})$ cannot be reduced further by additional equivalences of type $u \equiv b$ with a non-tangible b , since otherwise we would get an improper ℓ -congruence by Remark 4.14.(v). Hence, \mathfrak{A} is \mathfrak{t} -minimal. As any ν -semifield is a ν -domain, Proposition 4.45 implies that \mathfrak{A} is a \mathfrak{g} -prime congruence. \square

To overcome the discrepancy of maximality in the sense of congruences on ν -semirings, we drive the next definition which generalizes the classical notion of a “local ring”.

Definition 4.60. *A ν -semiring R is called **local**, if $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}) = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}')$ for all $\mathfrak{N}, \mathfrak{N}' \in \text{Spn}_\ell(R)$. A quotient R/\mathfrak{N} is called **residue ν -semiring** of the local ν -semiring R .*

When R is local, despite $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}) = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}')$ for all $\mathfrak{N}, \mathfrak{N}' \in \text{Spn}_\ell(R)$, we may have $(R/\mathfrak{N})|_{\text{tng}} \neq (R/\mathfrak{N}')|_{\text{tng}}$, since \mathfrak{N} and \mathfrak{N}' can have different tangible classes. The same holds for their ghost clusters. By the above discussion we see that any ν -semiring R with $\mathcal{T} \subseteq \mathcal{S}$ is local; for example, when R is a supertropical semifield.

Remark 4.61. *A \mathfrak{q} -homomorphism $\varphi : R/\mathfrak{N} \rightarrow R'/\mathfrak{N}'$ of residue ν -semirings, where R and R' are local ν -semirings, is a local homomorphism, i.e., $\varphi^{-1}((R'/\mathfrak{N}')^\times) = (R/\mathfrak{N})^\times$ (Definition 2.10).*

Similarly to (4.27), we write \mathfrak{P}_R for $C^{-1}\mathfrak{P}$, where $C = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$.

Corollary 4.62. *The localization $R_{\mathfrak{P}}$ of a ν -semiring R by a \mathfrak{g} -prime congruence \mathfrak{P} (Definition 4.39) is a local ν -semiring with \mathfrak{t} -minimal ℓ -congruence $\mathfrak{N} = \mathfrak{P}_R$, for which the residue ν -semiring $R_{\mathfrak{P}}/\mathfrak{N}$ is a ν -semifield.*

Proof. Observe that $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_R) = (R_{\mathfrak{P}})^\times$, that is the tangible projection of \mathfrak{P}_R consists of the units in $R_{\mathfrak{P}}$; thus $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_R)$ cannot be reduced further and is \mathfrak{t} -minimal. Hence, $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}) = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_R)$ for every $\mathfrak{N} \in \text{Spn}_\ell(R)$, implying that $R_{\mathfrak{P}}$ is local.

The ℓ -congruence \mathfrak{P}_R on $R_{\mathfrak{P}}$ is \mathfrak{g} -prime by Proposition 4.47, and thus $R_{\mathfrak{P}}/\mathfrak{P}_R$ is a ν -domain by Proposition 4.45. Furthermore, since \mathfrak{P}_R unites each $a \notin \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_R)$ with a non-tangible element, we obtain that every tangible in $R_{\mathfrak{P}}/\mathfrak{P}_R$ is a unit. Therefore $R_{\mathfrak{P}}/\mathfrak{P}_R$ is a ν -semifield. \square

To indicate that \mathfrak{P}_R is the \mathfrak{t} -minimal ℓ -congruence on $R_{\mathfrak{P}}$ determined by \mathfrak{P} , we write

$$\mathfrak{N}_{\mathfrak{P}} := \mathfrak{P}_R, \tag{4.33}$$

and call it the **central \mathfrak{t} -minimal ℓ -congruence** of $R_{\mathfrak{P}}$. By the proof of Corollary 4.62, every element in $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}_{\mathfrak{P}})$ is a unit, and hence $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}_{\mathfrak{P}}) = \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{N}_{\mathfrak{P}})$.

Notation 4.63. *Given $f \in R$, we write $\bar{f}(\mathfrak{P})$ for the equivalence class $[\frac{f}{1}]$ in $\mathfrak{N}_{\mathfrak{P}}$ of the localization of f by \mathfrak{P} . Namely, $\bar{f}(\mathfrak{P})$ is an element of the residue ν -semiring $R_{\mathfrak{P}}/\mathfrak{N}_{\mathfrak{P}}$ – the image of f under the map composition $R \xrightarrow{\tau} R_{\mathfrak{P}} \xrightarrow{\pi} R_{\mathfrak{P}}/\mathfrak{N}_{\mathfrak{P}}$, cf. (3.13) and (2.4), respectively.*

For a subset $E \subseteq \text{Spec}(R)$, we write $\bar{f}|_E = \text{ghost}$, if $\bar{f}(\mathfrak{P})$ is ghost in $R_{\mathfrak{P}}/\mathfrak{N}_{\mathfrak{P}}$, i.e., $\bar{f}(\mathfrak{P}) \in \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{N}_{\mathfrak{P}})$, for all $\mathfrak{P} \in E$.

4.9. Radical congruences.

To approach the interplay between ghostified subsets and congruences on ν -semirings we employ several types of radicals, which later are shown to coincide. (Radical congruences initially need not be ℓ -congruences, but they are \mathfrak{q} -congruences.)

4.9.1. Congruence radicals.

Definition 4.64. A \mathfrak{q} -congruence \mathfrak{R} on R is **\mathfrak{g} -radical**, alluded for **ghost radical**, if its underlying equivalence \equiv_{τ} satisfies for any $a \in R$ the condition

$$a^k \equiv_{\tau} \text{ghost for some } k \in \mathbb{N} \Rightarrow a \equiv_{\tau} \text{ghost.} \quad (4.34)$$

We define the **\mathfrak{g} -radical spectrum** of R to be

$$\text{Spr}(R) := \{ \mathfrak{R} \mid \mathfrak{R} \text{ is a } \mathfrak{g}\text{-radical congruence on } R \}.$$

The **congruence radical**, written **\mathfrak{c} -radical**, of a congruence $\mathfrak{A} \in \text{Cong}(R)$ is defined as

$$\text{rad}_{\mathfrak{c}}(\mathfrak{A}) := \bigcap_{\substack{\mathfrak{R} \in \text{Spr}(R) \\ \mathfrak{R} \supseteq \mathfrak{A}}} \mathfrak{R}. \quad (4.35)$$

When $\text{rad}_{\mathfrak{c}}(\mathfrak{A}) = \mathfrak{A}$, we say that \mathfrak{A} is **\mathfrak{g} -radically closed**.

Law (4.34) can be equivalently stated by Lemma 4.1 as:

$$a^k \equiv_{\tau} (a^k)^{\nu} \Rightarrow a \equiv_{\tau} a^{\nu}. \quad (4.36)$$

The \mathfrak{c} -radical is defined for any congruence \mathfrak{A} , not necessarily a \mathfrak{q} -congruence, and \mathfrak{A} may not be contained in any \mathfrak{q} -congruence. In this case we formally set $\text{rad}_{\mathfrak{c}}(\mathfrak{A})$ to be the empty set.

Lemma 4.65. A (nonempty) \mathfrak{c} -radical $\text{rad}_{\mathfrak{c}}(\mathfrak{A})$ is a \mathfrak{q} -congruence satisfying property (4.36).

Accordingly, a \mathfrak{c} -radical cannot be a ghost congruence, and a \mathfrak{g} -radically closed congruence must be a \mathfrak{q} -congruence.

Proof. $\text{rad}_{\mathfrak{c}}(\mathfrak{A}) \neq \emptyset$ is a nonempty intersection of \mathfrak{q} -congruences; hence it is a \mathfrak{q} -congruence by Remark 4.28. $\mathbf{G}_{\text{cls}}(\mathfrak{A}) \subseteq \mathbf{G}_{\text{cls}}(\mathfrak{R})$ for every \mathfrak{g} -radical congruence \mathfrak{R} containing \mathfrak{A} , thus (4.36) holds for each $a^k \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$, with $k \in \mathbb{N}$. \square

Clearly, any \mathfrak{g} -prime congruence (Definition 4.40) is \mathfrak{g} -radical, and therefore there are the inclusions

$$\text{Spd}(R) \subseteq \text{Spec}(R) \subseteq \text{Spr}(R). \quad (4.37)$$

By definition, $\text{Spr}(R)$ does not contain congruences that are not \mathfrak{q} -congruences, e.g., ghost congruences. Yet, \mathfrak{g} -congruences, which could be ghost congruences, are involved in our framework and should be considered.

Remark 4.66. In fact, formula (4.35) can be applied to an arbitrary subset of $R \times R$, not necessarily to congruences, and in particular to sums $\mathfrak{A}_1 + \cdots + \mathfrak{A}_k$ of congruences. In this way we directly obtain a congruence which is the \mathfrak{c} -radical of the closure of a sum of congruences (4.23).

Lemma 4.67. Let \mathfrak{R} be a \mathfrak{g} -radical congruence on R , then

$$\mathfrak{R} = \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(R) \\ \mathfrak{P} \supseteq \mathfrak{R}}} \mathfrak{P}.$$

Proof. (\subseteq): Immediate by the inclusion (4.37).

(\supseteq): Each $(a, b) \in \mathfrak{R}$ is contained in every $\mathfrak{P} \supseteq \mathfrak{R}$, and therefore it belongs to their intersection. \square

From Lemma 4.67 we deduce the following.

Corollary 4.68. Let \mathfrak{A} be a \mathfrak{q} -congruence on R , then

$$\text{rad}_{\mathfrak{c}}(\mathfrak{A}) := \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(R) \\ \mathfrak{P} \supseteq \mathfrak{A}}} \mathfrak{P}. \quad (4.38)$$

This setup leads to an abstract form of a Nullstellensatz, analogous to the Hilberts Nullstellensatz, now taking place over ν -semirings.

Theorem 4.69 (Abstract Nullstellensatz). *Let \mathfrak{A} be a \mathfrak{q} -congruence on a ν -semiring R , and define $\mathcal{V}(\mathfrak{A}) := \{\mathfrak{P} \in \text{Spec}(R) \mid \mathfrak{P} \supseteq \mathfrak{A}\}$.¹³ For any $f \in R$ we have*

$$\bar{f}|_{\mathcal{V}(\mathfrak{A})} = \text{ghost} \iff f \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_c(\mathfrak{A})), \quad (4.39)$$

cf. Notation 4.63.

Proof. Recall that $\mathfrak{N}_{\mathfrak{P}} = \mathfrak{P}_R$ is the central \mathfrak{t} -minimal ℓ -congruence of $R_{\mathfrak{P}}$, cf. (4.33). The following hold

- $\bar{f}|_{\mathcal{V}(\mathfrak{A})} = \text{ghost}$ iff (by notation)
- $\bar{f}(\mathfrak{P}) = \text{ghost}$ for all $\mathfrak{P} \in \mathcal{V}(\mathfrak{A})$ iff (by definition)
- $[f/\mathbb{1}] \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{N}_{\mathfrak{P}}) \subset R_{\mathfrak{P}}$ for all \mathfrak{g} -prime congruences $\mathfrak{P} \supseteq \mathfrak{A}$ iff (by Proposition 4.47)
- $f \in \mathbf{G}_{\text{cls}}^{-1}(\bigcap_{\mathfrak{P} \in \mathcal{V}(\mathfrak{A})} \mathfrak{P}) \subset R$ iff (by Corollary 4.68)
- $f \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_c(\mathfrak{A}))$.

□

4.9.2. Set radicals.

We turn to our second type of radicals, applied to subsets $E \subseteq R$, possibly empty, via their ghostifying congruences \mathfrak{G}_E , or equivalently throughout ghost clusters.

Definition 4.70. *The **set-radical congruence**, written **s-radical**, of a subset $E \subseteq R$ is defined as*

$$\text{rad}_s(E) := \text{rad}_c(\mathfrak{G}_E) = \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(R) \\ \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P}) \supseteq E}} \mathfrak{P}. \quad (4.40)$$

When $E = \{a\}$, we write $\text{rad}_s(a)$ for $\text{rad}_s(\{a\})$ and say that $\text{rad}_s(a)$ is a **principal s-radical**.

The **set-radical closure** $\text{rcl}(E)$ of $E \subseteq R$ is the subset

$$\text{rcl}(E) := \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E)) \subseteq R.$$

A subset E is called **radically g-closed** if $\text{rcl}(E) = E \cup \mathcal{G}$.

It may happen that E is not contained in any ghost projection $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, e.g. when $E \cap R^\times \neq \emptyset$. In this case, we formally set $\text{rad}_s(E)$ and $\text{rcl}(E)$ to be the empty set, for example $\text{rad}_s(R) = \emptyset$ and $\text{rcl}(R^\times) = \emptyset$. Otherwise, an **s-radical congruence** $\text{rad}_s(E)$ is identified with the **c-radical** of the **g-congruence** \mathfrak{G}_E of E , cf. (4.8). By Lemma 4.65, this implies that $\text{rad}_s(E)$ is a **q-congruence** (and thus is not a ghost congruence) which obeys condition (4.36). Therefore, the correspondence between **c-radicals** and **s-radicals** is established.

We also see that by definition

$$\text{rad}_s(E) = \text{rad}_s(E \cup \mathcal{G}), \quad (4.41)$$

for every $E \subseteq R$, in particular $\text{rad}_s(\emptyset) = \text{rad}_s(E) = \text{rad}_s(\mathcal{G})$ for all $E \subseteq \mathcal{G}$. When $\text{rad}_s(E)$ is not empty, $\text{rad}_s(E)$ decomposes E to ghost classes, while its ghost projection $\text{rcl}(E) \subset R$ dismisses this decomposition and only care of being ghost or not (Remark 4.16). Moreover, we always have $\text{rcl}(E) \subseteq R \setminus \mathcal{T}^\times$, and $\text{rcl}(E) \neq R$ for any E . Also $E \subseteq \text{rcl}(E)$, where $\text{rcl}(E)$ records the membership in the ghost projection of \mathfrak{G}_E , linking it to a ν -semiring ideal.

Lemma 4.71. *A nonempty subset $\text{rcl}(E)$ is a **g-radical ideal** of R (Definition 3.18).*

Proof. The subset $\text{rcl}(E)$ is the ghost kernel of the surjection $\pi : R \twoheadrightarrow R/\text{rad}_s(E)$, which is an ideal by Remark 4.15. If $a^k \equiv \text{ghost}$, then $a^k \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E))$, implying that $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E))$, since a^k lies in the intersection of **g-prime congruences**, cf. (4.40). □

Remark 4.72. *For any congruence \mathfrak{A} we have*

$$\text{rad}_s(\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})) \subseteq \text{rad}_c(\mathfrak{A}),$$

since $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ dismisses the decomposition into equivalence classes. On the other hand, viewing a subset $E \subseteq R$ as a partial congruence $\Delta(E)$ on R (Definition 2.4), we have

$$\text{rad}_s(E) = \text{rad}_c(\overline{\Delta(E)}) = \text{rad}_c(\mathfrak{G}_E).$$

We use both forms to distinguish between the different types of radicals.

¹³This set of congruences will be studied in much details later in §6.

Considering radically \mathfrak{g} -closed subsets which contain no ghosts, we define

$$\mathbf{RSet}(R) := \{E \subseteq R \setminus \mathcal{G} \mid E \text{ is radically } \mathfrak{g}\text{-closed}\},$$

for which the map

$$\vartheta : \mathbf{RSet}(R) \xrightarrow{\sim} \mathbf{Spr}(R), \quad E \longmapsto \text{rad}_s(E), \quad (4.42)$$

is bijective. Indeed, each radically \mathfrak{g} -closed subset $E \subseteq R \setminus \mathcal{G}$ is uniquely mapped to $\text{rad}_s(E)$, since $E \cup \mathcal{G} = \text{rcl}(E) = \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E))$. Conversely, a \mathfrak{g} -radical congruence $\mathfrak{R} \in \mathbf{Spr}(R)$ is mapped to its ghost projection $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{R})$.

For a \mathfrak{g} -prime congruence \mathfrak{P} and a subset $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, we have $\text{rad}_s(E) \subseteq \mathfrak{P}$ and

$$E \subseteq E \cup \mathcal{G} \subseteq \text{rcl}(E) \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P}). \quad (4.43)$$

This leads to a notion of primeness for subsets of a ν -semiring.

Definition 4.73. *A radically \mathfrak{g} -closed subset $E \in \mathbf{RSet}(R)$ is called **c-prime**, if $\text{rcl}(E) = \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for some \mathfrak{g} -prime congruence \mathfrak{P} .*

The study of \mathfrak{c} -prime subsets and their role in arithmetic geometry is left for future work.

4.9.3. Properties of set radicals.

Properties of \mathfrak{s} -radical congruences (Definition 4.70) may classify the generators of their ghost clusters, or at least determine dependence relations on these generators. These relations are useful for the passage from subsets to congruences and vice versa. We first specialize (4.18) in Remark 4.18 to tangibles.

Lemma 4.74. *Suppose $a \in R$ is \mathfrak{t} -persistent, i.e., $a \in \mathcal{T}^\circ$, then $\text{rad}_s(a) \subseteq \text{rad}_s(b)$ if and only if $a^n = bc$ for some $c \in R$ and $n \in \mathbb{N}$.*

Proof. (\Rightarrow): Assume that $\text{rad}_s(a) \subseteq \text{rad}_s(b)$, hence $a^n \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(b))$ for some $n \in \mathbb{N}$, and thus $a^n = bc + g$, where $g \in \mathcal{G}$, by Remark 4.18. But, a^n is tangible, whereas a is \mathfrak{t} -persistent, where g must be inessential by (3.8), as follows from Axiom NS2 in Definition 3.11. Thus $a^n = bc$.

(\Leftarrow): The inclusion $b \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ gives $b \equiv_{\mathfrak{p}} b^\nu$, by Lemma 4.1, and thus $a^n \equiv_{\mathfrak{p}} (a^n)^\nu = (bc)^\nu$. Since \mathfrak{P} is \mathfrak{g} -prime, we deduce that $a \equiv_{\mathfrak{p}} a^\nu$. Taking the intersection of all such \mathfrak{g} -prime congruences, we obtain $\text{rad}_s(a) \subseteq \text{rad}_s(b)$. \square

Corollary 4.75. *Let $b \in \mathcal{T}^\circ \setminus \mathfrak{g}\text{div}(R)$ be a \mathfrak{t} -persistent element in a tame ν -semiring R .*

- (i) *If $\text{rad}_s(a) \subseteq \text{rad}_s(b)$, then a is \mathfrak{t} -persistent.*
- (ii) *If $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(b))$, then $a^n = bc$, for some $c \in R$ and $n \in \mathbb{N}$, and thus $a \in \mathcal{T}$.*
- (iii) *If $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E))$, then there exists a finite subset $E' \subset E$ such that $a^n = \sum_j e'_j c_j$ for some $e'_j \in E'$, $c_j \in R$, and $n \in \mathbb{N}$.*

Proof. (i): As a is \mathfrak{t} -persistent, also a^n is \mathfrak{t} -persistent. Since $\text{rad}_s(a) \subseteq \text{rad}_s(b)$, from Lemma 4.74 we obtain that $a^n = bc$ for some $c \in R$. This forces that $b, c \in \mathcal{T}$, as R is tame (Definition 3.11), and moreover that b and c are \mathfrak{t} -persistents, by Lemma 3.23.(iii).

(ii): Follows from (i), since $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(b))$ implies that $\text{rad}_s(a) \subseteq \text{rad}_s(b)$.

(iii): By (ii), $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_s(E))$ means that $a^n = bc$ for some b that is generated by a subset E' of E and possibly an extra ghost term, cf. Remark 4.18. But a^n is tangible, so, by (3.8), no extra ghost element can be included. \square

4.9.4. Ghostpotent radicals.

We turn to our third type of radical, emerging in a global sense.

Definition 4.76. *An element $a \in R$ is called **ghsotpotent**, if $a^k \in \mathcal{G}$ for some $k \in \mathbb{N}$. The set*

$$\mathcal{N}(R) := \{a \in R \mid a \text{ is ghsotpotent}\}$$

*is called the **ghsotpotent ideal of R** (see Remark 4.77 below). A ν -semiring R is said to be **ghost reduced**, if $\mathcal{N}(R) = \mathcal{G}$.*

*The **ghsotpotent radical congruence of R** , written **gp-radical**, is defined to be the \mathfrak{q} -congruence*

$$\text{rad}_{\mathfrak{g}}(R) := \text{rad}_s(\mathcal{N}(R)),$$

determined by (4.40).

As every ghost element $a \in \mathcal{G}$ is ghostpotent, we see that $\mathcal{G} \subseteq \mathcal{N}(R)$. On the other hand, a \mathfrak{t} -persistent element cannot be ghostpotent, i.e., $\mathcal{T}^\circ \cap \mathcal{N}(R) = \emptyset$, but we may have tangibles which are ghostpotents. Therefore, we immediately deduce that $\text{rad}_{\mathfrak{g}}(R)$ is indeed a \mathfrak{q} -congruence on R . (Alternatively, it follows from (4.40) and Lemma 4.65.)

Remark 4.77. $\mathcal{N}(R)$ is an ideal of the ν -semiring R (Definition 2.14). Indeed, if $a, b \in \mathcal{N}(R)$, i.e., $a^{k_a} \in \mathcal{G}$ and $b^{k_b} \in \mathcal{G}$ for some $k_a, k_b \in \mathbb{N}$, then

$$(a + b)^{k_a + k_b} = \sum_{i=0}^{k_a + k_b} \binom{k_a + k_b}{i} a^i b^{k_a + k_b - i},$$

where either $i \geq k_a$ or $k_a + k_b - i \geq k_b$. So, each term in the sum is a ghost, and $\mathcal{N}(R)$ is closed for addition. For $c \in R$, we have $(ac)^{k_a} = a^{k_a} c^{k_a} \in \mathcal{G}$, since \mathcal{G} is an ideal, showing that $ac \in \mathcal{N}(R)$.

Clearly, when R is reduced $\text{rad}_{\mathfrak{g}}(R) = \Delta(R)$ is the trivial congruence.

Lemma 4.78. A congruence \mathfrak{A} is the \mathfrak{gp} -radical of R if and only if R/\mathfrak{A} is ghost reduced.

Proof. \mathfrak{A} is the \mathfrak{gp} -radical of R iff for every $a \in R$ such that $a^k \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ also $a \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$. Passing to the quotient ν -semiring $R' := R/\mathfrak{A}$, this is obviously equivalent to saying that $a^k \in \mathcal{G}'$ implies $a \in \mathcal{G}'$, i.e., that R/\mathfrak{A} has no ghostpotents except pure ghosts. \square

Remark 4.79. An element $b \in R$ is a ghostpotent if and only if $b \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})$ for every \mathfrak{g} -radical congruence $\mathfrak{A} \in \text{Spr}(R)$ (Definition 4.64). In particular $b \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for every \mathfrak{g} -prime congruence \mathfrak{P} on R . Therefore, when b is ghostpotent, we always have $\text{rad}_{\mathfrak{s}}(b) \subseteq \text{rad}_{\mathfrak{s}}(a)$ for any $a \in R$.

Clearly, any ghostpotent which is not ghost by itself is a ghost divisor (Definition 3.19), implying that $\mathcal{N}(R) \setminus \mathcal{G} \subset \text{gdiv}(R)$.

Lemma 4.80. $\text{rad}_{\mathfrak{g}}(R) = \text{rad}_{\mathfrak{s}}(\mathcal{G}) = \text{rad}_{\mathfrak{s}}(\emptyset)$.

Proof. (\subseteq): Suppose that $a \in \mathcal{N}(R)$ is non-ghost, then $a^k = \text{ghost}$ for some $k \in \mathbb{N}$. In any \mathfrak{g} -prime congruence, $a^k \equiv_{\mathfrak{p}} \text{ghost}$ implies $a \equiv_{\mathfrak{p}} \text{ghost}$, and thus $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_{\mathfrak{s}}(\mathcal{G}))$ by (4.40).

(\supseteq): Immediate by (4.41), since $\mathcal{G} \subseteq \mathcal{N}(R)$.

The equality $\text{rad}_{\mathfrak{s}}(\mathcal{G}) = \text{rad}_{\mathfrak{s}}(\emptyset)$ is given by (4.41). \square

From the above exposition we derive the following theorem.

Theorem 4.81 (Krull). For any ν -semiring R we have

$$\text{rad}_{\mathfrak{g}}(R) = \bigcap_{\mathfrak{P} \in \text{Spec}(R)} \mathfrak{P}. \quad (4.44)$$

Proof. A direct consequence of Lemma 4.80, in the view of Definition 4.70. \square

Corollary 4.82. Let $a \in R$, then $a \in \mathbf{G}_{\text{cls}}^{-1}(\text{rad}_{\mathfrak{g}}(R))$ if and only if a is a ghostpotent, i.e., $a \in \mathcal{N}(R)$.

Proof. (\Rightarrow): By Theorem 4.81, a belongs to the ghost projection $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ of every \mathfrak{g} -prime congruence \mathfrak{P} and thus is a ghostpotent.

(\Leftarrow): Clear by definition, since $a \in \mathcal{N}(R)$. \square

Corollary 4.83. For any $a \notin \mathcal{N}(R)$ there exists a \mathfrak{g} -prime congruence \mathfrak{P} such that $a \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$

The corollary is a strengthening of Lemma 4.30, applied there to \mathfrak{q} -congruences, and it can be enhanced further for \mathfrak{t} -persistent elements.

Lemma 4.84. For each $a \in \mathcal{T}^\circ$ there exists a \mathfrak{g} -prime congruence \mathfrak{P} such that $a \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$.

Proof. The multiplicative monoid $C = \langle a \rangle$ is tangible, since a is \mathfrak{t} -persistent. Take the tangible localization R_C of R by C , in which $\frac{a}{\mathbb{1}}$ is a tangible unit. By Corollary 4.83 there exists a \mathfrak{g} -prime congruence \mathfrak{P}' on R_C for which $\frac{a}{\mathbb{1}} \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P}')$, and furthermore $\frac{a}{\mathbb{1}} \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P}')$, since $\frac{a}{\mathbb{1}}$ is a unit and \mathfrak{P}' is \mathfrak{g} -prime. Then, by Proposition 4.47.(ii), the restriction of \mathfrak{P}' to R gives a \mathfrak{g} -prime congruence \mathfrak{P} with $C \subseteq \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$, and thus $a \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$. \square

For the \mathfrak{g} -radical, Remark 4.18 can be strengthened as follows.

Remark 4.85.

- (i) A \mathfrak{t} -persistent element $a \in \mathcal{T}^\circ$ cannot be written as a sum which involves a ghostpotent term b . Indeed, otherwise, if $a = b + c$ such that $b^n \in \mathcal{G}$, then $a^n = (b + c)^n = b^n + \sum_{i=1}^n \binom{n}{i} b^{n-i} c^i$, which contradicts Axiom NS2 in Definition 3.11, since $a^n \in \mathcal{T}$, cf. (3.8).
- (ii) If R is a persistent full (resp. tangibly closed) ν -semiring, then $R/\text{rad}_{\mathfrak{g}}(R)$ is also persistent full (resp. tangibly closed). Indeed, each member of $\mathbb{T}_{\text{cls}}^{-1}(\text{rad}_{\mathfrak{g}}(R))$ is \mathfrak{t} -persistent, since $\mathcal{T} = \mathcal{T}^\circ$ (resp. $\mathcal{T} = \mathcal{T}^\bullet$), where a \mathfrak{t} -persistent element cannot be ghostpotent, implying by (i) that $\mathbb{T}_{\text{cls}}^{-1}(\text{rad}_{\mathfrak{g}}(R)) = \mathcal{T}$ is a monoid. Hence, $R/\text{rad}_{\mathfrak{g}}(R)$ is persistent full (resp. tangibly closed).
- (iii) If R is a tame ν -semiring, then $R/\text{rad}_{\mathfrak{g}}(R)$ is also tame, since $\text{rad}_{\mathfrak{g}}(R)$ is a \mathfrak{q} -congruence which respects the ν -semiring operations.

4.9.5. *Jacobson radical.*

Finally, we reach the last type of radical, defined in terms of maximal ℓ -congruences (Definition 4.53).

Definition 4.86. The **Jacobson radical** of a congruence \mathfrak{A} on a ν -semiring R is defined as

$$\text{jac}(\mathfrak{A}) := \bigcap_{\substack{\mathfrak{M} \in \text{Spm}_{\ell}(R) \\ \mathfrak{M} \supseteq \mathfrak{A}}} \mathfrak{M}.$$

Lemma 4.87. Let $\pi = \pi_{\mathfrak{A}} : R \rightarrow R/\mathfrak{A}$ be the canonical surjective homomorphism. Then,

- (i) $\text{jac}(\mathfrak{A}) = \pi^{-1}(\text{jac}(R/\mathfrak{A}))$;
- (ii) $\text{rad}_{\mathfrak{c}}(\mathfrak{A}) = \pi^{-1}(\text{rad}_{\mathfrak{g}}(R/\mathfrak{A}))$;

Proof. (i): Observe that the map $\mathfrak{M} \mapsto \pi^{-1}(\mathfrak{M})$ defines a bijection between all maximal ℓ -congruences of R/\mathfrak{A} and the maximal ℓ -congruences on R that contain \mathfrak{A} , cf. Remark 4.29.(i). As inverse images with respect to π commute with intersections, (i) follows from Definition 4.86 and Theorem 4.81.

(ii): Follows from the fact that a power a^n of an element $a \in R$ belongs to $\mathbb{G}_{\text{cls}}^{-1}(\mathfrak{A})$ if and only if $\pi(a)^n \in (R/\mathfrak{A})|_{\text{gh}}$. \square

4.10. **Krull dimension.**

The development of dimension theory is left for future work. To give the flavor of its basics, we bring some basic definitions.

Definition 4.88. A ν -semiring R is called **noetherian** (resp. **\mathfrak{q} -noetherian**), if any ascending chain

$$\mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \cdots$$

of congruences (resp. \mathfrak{q} -congruences) on R stabilizes after finitely many steps, i.e., $\mathfrak{A}_n = \mathfrak{A}_{n+1} = \cdots$ for some n .

R is called **artinian** (resp. **\mathfrak{q} -artinian**), if any descending chain

$$\mathfrak{A}_0 \supsetneq \mathfrak{A}_1 \supsetneq \mathfrak{A}_2 \supsetneq \cdots$$

of congruences (resp. \mathfrak{q} -congruences) stabilizes after finitely many steps.

The use of \mathfrak{g} -prime congruences, supported by the results of this section, allows for a natural definition of dimension of ν -semirings.

Definition 4.89. The **Krull dimension** of a ν -semiring R , denoted $\dim(R)$, is defined to be the supremum of lengths of chains

$$\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_0) \supsetneq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_1) \supsetneq \cdots \supsetneq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_n),$$

such that $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_n$ are \mathfrak{g} -prime congruences on R .

For example, any supertropical semifield (and more generally any ν -semifield) F has dimension 0, while the ν -semiring $\hat{F}[\lambda]$ of polynomial functions over F has dimension 1, cf. Example 4.44. The case of general ν -semirings is more subtle.

Definition 4.90. The *height* $\text{ht}(\mathfrak{P})$ (or *codimension*) of a \mathfrak{g} -prime congruence \mathfrak{P} , is the supremum of the lengths of all chains of \mathfrak{g} -prime congruences contained in \mathfrak{P} , meaning that $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_n = \mathfrak{P}$ such that $\Gamma_{\text{cls}}^{-1}(\mathfrak{P}_0) \supsetneq \Gamma_{\text{cls}}^{-1}(\mathfrak{P}_1) \supsetneq \cdots \supsetneq \Gamma_{\text{cls}}^{-1}(\mathfrak{P}_n)$.

By Theorem 4.48, we see that the height of \mathfrak{P} is the Krull dimension of the localization $R_{\mathfrak{P}}$ of R by \mathfrak{P} . A \mathfrak{g} -prime congruence has height zero if and only if it is a minimal \mathfrak{g} -prime congruence. The Krull dimension of a ν -semiring is the supremum of the heights of all \mathfrak{g} -prime congruences that it carries.

The study of \mathfrak{q} -noetherian ν -semirings, \mathfrak{q} -artinian ν -semirings, and Krull dimension is left for future work.

5. ν -MODULES

Modules over ν -semirings are a specialization of modules over semirings (Definition 2.17), playing the similar role to that of modules over rings. As before, ν -semirings in this section are all assumed to be commutative.

Definition 5.1. A *left R - ν -module* $M := (M, \mathcal{H}, \mu)$ over a (commutative) ν -semiring $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ is a *smeimodule* (Definition 2.17) having the structure of an additive ν -monoid (Definition 3.1) whose ghost map $\mu : M \rightarrow \mathcal{H}$ satisfies for all $a \in R$, $u, v \in M$ the additional axioms:

$$\begin{aligned} \text{MD1: } & \mu(au) = a\mu(u); \\ \text{MD2: } & \mu(u + v) = \mu(u) + \mu(v). \end{aligned} \quad {}^{14}$$

\mathcal{H} is called the *ghost submodule* of M .

An *R - ν -module congruence* is a congruence on a ν -monoid which also respects multiplication by elements of R , i.e., if $u \equiv v$, then $au \equiv av$ for all $a \in R$, $u, v \in M$.

A *homomorphism* of R - ν -modules is a ν -monoid homomorphism (Definition 3.4)

$$\varphi : (M, \mathcal{H}, \mu) \longrightarrow (M', \mathcal{H}', \mu'),$$

in which $\varphi(au) = a\varphi(u)$ for any $a \in R$, $u \in M$. In particular, $\varphi(0_M) = 0_{M'}$. The *\mathfrak{g} -kernel* of φ is defined as

$$\text{gker}(\varphi) := \{u \in M \mid \varphi(u) \in \mathcal{H}'\} \subset M.$$

When R is clear from the context, to simplify notations, we write ν -module for R - ν -module. As in the case of ν -semirings, we write u^μ for the ghost image $\mu(u)$ of an element $u \in M$ in the submodule $\mathcal{H} \subseteq M$. Since M has the structure of a ν -monoid, the ghost submodule \mathcal{H} is partially ordered and it induces a (partial) μ -ordering on the whole M , i.e., $u >_\mu v$ iff $u^\mu > v^\mu$. We say that M is a **ghost ν -module**, if $\mathcal{H} = M$.

By definition, for every $u \in M$,

$$u^\mu = u + u = (\mathbf{1} + \mathbf{1})u = eu \quad (5.1)$$

i.e., $\mu(u) = eu$. Axiom MD1 implies that $\mu(eu) = e\mu(u)$, and thus $\mu(au) = a^\nu \mu(u)$ for every $a \in R$. One observes that the action of R on M respects the μ -ordering (3.4) of M , i.e.,

$$u >_\mu v \implies au >_\mu av \quad \text{for all } u, v \in M, a \neq 0 \text{ in } R. \quad (5.2)$$

Indeed, for $u >_\mu v$ in M , by the ν -monoid properties, $a(u + v) = au = (au + av)$.

For a homomorphism $\varphi : M \rightarrow M'$ of ν -modules, we have $\varphi(u^\mu) = \varphi(u)^{\mu'}$ by Lemma 3.5, providing the ghost inclusion $\varphi(\mathcal{H}) \subseteq \mathcal{H}'$.

Definition 5.2. Let M be an R - ν -module, and let $S \subseteq M$ be a subset. The *ghost annihilator* $\text{Ann}_R(S)$ of S is the set of all $a \in R$ such that au is a ghost for every $u \in S$, i.e.,

$$\text{Ann}_R(S) = \{a \in R \mid au \in \mathcal{H} \text{ for all } u \in S\}.$$

The *Krull dimension* of M is defined as

$$\dim_R(M) := \dim(R/\text{Ann}_R(M)),$$

where the quotient $R/\text{Ann}_R(M)$ is as given in Definition 4.9.

¹⁴This axiom is part of the ν -monoid structure that makes the ghost map a monoid homomorphism.

Clearly, $\mathcal{G} \subseteq \text{Ann}_R(S)$ for any subset $S \subseteq M$, while $\text{Ann}_R(\mathcal{H}) = R$. When $M = R$ is a ν -semiring, $\text{Ann}_R(S)$ is a ν -semiring ideal (Definition 2.14).

The usual verification shows that the direct sum of ν -modules is a ν -module. Thus, one can construct the free ν -module as a direct sum of copies of R .

Definition 5.3. An R - ν -module M is **free**, if it is isomorphic to $R^{(I)}$ for some index set I . M is **projective**, if there is a split epic from $R^{(I)}$ to M for some I (which can be taken to have order m , if M is finitely generated by m elements).

Ghostifying congruences \mathfrak{G}_N of ν -submodules N are naturally defined in terms of ν -monoids, cf. (4.8).

Definition 5.4. The **quotient** of a ν -module M by a ν -submodule $N \subseteq M$ is defined as $M//N := M/\mathfrak{G}_N$, where \mathfrak{G}_N is the ghostifying congruence of N .

In fact, this process of quotienting is applicable for a general subset $S \subseteq M$, and not only for ν -submodules.

Given two R - ν -modules M and N , we denote by $\text{Hom}_R(M, N)$ the set of all ν -module homomorphisms $\phi : M \rightarrow N$. A routine check shows that $\text{Hom}_R(M, N)$, with R a commutative ν -semiring, is an R - ν -module where $a\phi$ is defined via $(a\phi)(u) := a\phi(u)$, cf. Proposition 3.7.

The category of ν -module over a ν -semiring R is denoted by $\nu\mathbf{Mod}_R$, its morphisms are ν -module homomorphisms. As usual, we have the following functors.

Definition 5.5. The **covariant functor**

$$\text{Hom}_R(M, _) : \nu\mathbf{Mod}_R \longrightarrow \nu\mathbf{Mod}_R$$

is given by sending N to $\text{Hom}_R(M, N)$ and sending $\varphi : N_1 \rightarrow N_2$ to $\widehat{\varphi} : \text{Hom}_R(M, N_1) \rightarrow \text{Hom}_R(M, N_2)$ by $\widehat{\varphi}(\phi) = \phi\varphi$ for $\phi : M \rightarrow N_1$.

The **contravariant functor**

$$\text{Hom}(_ , N) : \nu\mathbf{Mod}_R \longrightarrow \nu\mathbf{Mod}_R$$

is given by sending M to $\text{Hom}_R(M, N)$ and sending the ν -module homomorphism $\varphi : M_1 \rightarrow M_2$ to $\widehat{\varphi} : \text{Hom}_R(M_2, N) \rightarrow \text{Hom}_R(M_1, N)$ by $\widehat{\varphi}(\phi) = \phi\varphi$ for $\phi : M_2 \rightarrow N$.

5.1. Exact sequences.

A sequence of R - ν -modules is a chain of morphisms of R - ν -modules

$$\cdots \xrightarrow{\phi_{n-2}} M_{n-1} \xrightarrow{\phi_{n-1}} M_n \xrightarrow{\phi_n} M_{n+1} \xrightarrow{\phi_{n+1}} \cdots$$

with indices varying over a finite or an infinite part of \mathbb{Z} . The sequence satisfies the **complex ghost property** at M_n if $\phi_n \circ \phi_{n-1} = \mathcal{H}_{n+1}$ or, in equivalent terms, $\text{im}(\phi_{n-1}) \subseteq \text{gker}(\phi_n)$, cf. Definition 3.6. The sequence is said to be **ghost exact** (written **g-exact**) at M_n , if $\text{im}(\phi_{n-1}) = \text{gker}(\phi_n)$. When the sequence satisfies the complex ghost property at every M_n , it is called **g-complex**. Likewise, the sequence is called **g-exact**, if it is exact at all places. **Short g-exact sequences** are sequences of type

$$\mathcal{H}' \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow \mathcal{H}'' ,$$

such that

- (a) $\text{im}(\phi) = \text{gker}(\psi)$;
- (b) ψ is surjective.

Often we also require that ϕ is injective. Then, for short g-exact sequences, M' can be viewed as a ν -submodule of M via ϕ , where ψ induces a homomorphism $M//M' \rightarrow M''$. Conversely, for any ν -submodule $N \subseteq M$ the sequence

$$\mathcal{H}_N \longrightarrow N \hookrightarrow M \twoheadrightarrow M//N \longrightarrow \mathcal{H}_{M/N} (= \mathcal{H}_M)$$

is a short g-exact sequence. A further study of g-exact sequences is left for future work.

5.2. Tensor products.

The tensor product of modules over semirings has appeared in the literature; for example in [61]. Since the theory closely parallels tensor products over algebras, we briefly review it for the reader's convenience. Patchkoria has built an extensive theory of derived functors in [55], but assumes that the modules are (additively) cancellative, in order to be able to use factor modules. Here, since factorization by submodule is performed by means of ghostification, as indicated in Definition 5.4, we can avoid cancellativity conditions on ν -modules.

Suppose M and N are R - ν -modules, and $\mathcal{M} := (\mathcal{M}, \mathcal{G}, +)$ is an additive ν -monoid (Definition 3.1). A map

$$\psi : M \times N \longrightarrow \mathcal{M}$$

is **bilinear** (also called **balanced**), if

$$\begin{aligned} \psi(u + u', v) &= \psi(u, v) + \psi(u', v), \\ \psi(u, v + v') &= \psi(u, v) + \psi(u, v'), \\ \psi(ua, v) &= \psi(u, av), \end{aligned} \tag{5.3}$$

for all $a \in R$, $u, u' \in M$, $v, v' \in N$.

Definition 5.6. A *tensor product* of R - ν -modules M and N (over R) is an R - ν -module T , together with an bilinear map $\psi : M \times N \longrightarrow T$, such that the universal property holds: For each bilinear map $\phi : M \times N \longrightarrow L$ to some ν -module L , there is a unique linear map $\varphi : T \longrightarrow L$ such that $\phi = \varphi \circ \psi$, i.e., that renders the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\psi} & T \\ \phi \downarrow & \swarrow \varphi & \\ L & & \end{array}$$

commutative.

Given R - ν -modules $M := (M, \mathcal{H}_M, \mu_M)$ and $N := (N, \mathcal{H}_N, \mu_N)$, a routine verification shows that $R(M \times N)$ is again an R - ν -module with ghost submodule $\mathcal{H}_{M \times N} := R(\mathcal{H}_M \times \mathcal{H}_N) \cong \mathcal{G}(\mathcal{H}_M \times \mathcal{H}_N)$. Its μ -ordering is induced jointly from the μ -orderings of M and N by coordinate-wise addition. By the ν -module properties one can identify $R(M \times N)$ with $M \times N$. But, to stress the fact that $\mathcal{H}_{M \times N}$ is determined as $e \cdot (M \times N)$, we retain the notation $R(M \times N)$.

Let \mathfrak{T} be the congruence on $R(M \times N)$ whose underlying equivalence $\sim_{\mathfrak{T}}$ is given by the generating relations

$$\begin{aligned} (i) \quad & (u + u', v) \sim_{\mathfrak{T}} (u, v) + (u', v), \\ (ii) \quad & (u, v + v') \sim_{\mathfrak{T}} (u, v) + (u, v'), \\ (iii) \quad & a \cdot (u, v) \sim_{\mathfrak{T}} (au, v) \sim_{\mathfrak{T}} (u, av), \end{aligned} \tag{5.4}$$

for $a \in R$, $u, u' \in M$, $v, v' \in N$. We define the **tensor product** of M and N to be

$$M \otimes_R N := R(M \times N) / \mathfrak{T},$$

and set $\mathbb{0}_{\otimes} := \mathbb{0}_M \otimes \mathbb{0}_N$, which is the class $[\mathbb{0}_{R(M \times N)}]$. The equivalence classes $[(u, v)]$ of \mathfrak{T} are denoted as customary by $u \otimes v$.

Observe that $f \sim_{\mathfrak{T}} f'$ and $g \sim_{\mathfrak{T}} g'$ in \mathfrak{T} implies $f + g \sim_{\mathfrak{T}} f' + g'$; therefore \mathfrak{T} induces a binary operation $[f] + [g] := [f + g]$ on $M \otimes_R N$. So, it suffices to check that $f + g \sim_{\mathfrak{T}} f' + g$, which is built from the three relations (5.4). The operation $+$ of $M \otimes_R N$ is commutative, associative, and unital with respect to $\mathbb{0}_{\otimes}$, as this is the case in $R(M \times N)$ and \mathfrak{T} is a congruence.

The ghost submodule $\mathcal{H}_{M \otimes_R N}$ of $M \otimes_R N$ is defined as

$$\mathcal{H}_{M \otimes_R N} := \{u \otimes v \mid u \in \mathcal{H}_M \text{ or } v \in \mathcal{H}_N\}.$$

When no confusion arises, we write u^{μ} and v^{μ} for $\mu_M(u)$ and $\mu_N(v)$, respectively. Recalling from (5.1) that $u^{\mu} = e \cdot u$, by the third generating relation in (5.4) we have

$$(u^{\mu}, v) = (eu, v) \sim_{\mathfrak{T}} e \cdot (u, v) \sim_{\mathfrak{T}} (u, ev) = (u, v^{\mu}),$$

moreover

$$(u^{\mu}, v) = (eu, v) = (e^2u, v) \sim_{\mathfrak{T}} e \cdot (eu, v) \sim_{\mathfrak{T}} (eu, ev) = (u^{\mu}, v^{\mu}),$$

providing the equivalence

$$(u^\mu, v^\mu) \sim_t e \cdot (u, v). \quad (5.5)$$

Therefore

$$\mathcal{H}_{M \otimes_R N} = \{e \cdot (u, v) \mid u \in M, v \in N\} = \mathcal{H}_M \otimes_R \mathcal{H}_N.$$

To define the ghost map μ_\otimes of $M \otimes_R N$, first observe by (5.5) that the ghost map of $R(M \times N)$ is given by $\mu_{R(M \times N)} = (\mu_M, \mu_N)$, and it respects the defining relations (5.4) of \mathfrak{T} . Indeed, writing $\tilde{\mu}$ for $\mu_{R(M \times N)}$, for the first generating relation we have:

$$\begin{aligned} \tilde{\mu}(u + u', v) &\sim_t e \cdot (u + u', v) \sim_t (u + u', ev) \\ &\sim_t (u, ev) + (u', ev) \sim_t e \cdot (u, v) + e \cdot (u', v) \\ &\sim_t \tilde{\mu}(u, v) + \tilde{\mu}(u', v). \end{aligned}$$

The second relation is checked similarly, where for the third relation we have

$$\tilde{\mu}(a \cdot (u, v)) \sim_t ea \cdot (u, v) \sim_t (eau, v) \sim_t e \cdot (au, v) \sim_t \tilde{\mu}(au, v).$$

We assign $M \otimes_R N$ with the ghost map

$$\mu_\otimes : M \otimes_R N \longrightarrow \mathcal{H}_M \otimes_R \mathcal{H}_N, \quad \text{defined by } u \otimes v \longmapsto u^\mu \otimes v^\mu,$$

for which

$$(u \otimes v)^{\mu_\otimes} = [(u, v)^{\tilde{\mu}}] = [(u^\mu, v^\mu)] = [e \cdot (u, v)] = u^\mu \otimes v^\mu \quad (5.6)$$

for all $u, u' \in \mathcal{H}_M, v, v' \in \mathcal{H}_N$.

The partial ordering $<_\otimes$ of $\mathcal{H}_{M \otimes_R N}$ is induced from the partial ordering of $\mathcal{H}_{R(M \times N)}$, determined as

$$u \otimes v <_\otimes u' \otimes v' \iff u < u' \text{ and } v < v', \quad (5.7)$$

for $u, u' \in \mathcal{H}_M, v, v' \in \mathcal{H}_N$. Equivalently, the partial ordering $<_\otimes$ can be extracted directly from the additive structure of ghost submonoid $\mathcal{H}_{M \otimes_R N}$, that is

$$u \otimes v <_\otimes u' \otimes v' \iff u \otimes v + u' \otimes v' = u' \otimes v'.$$

This formulation shows that the ordering $<_\otimes$ does not depend on the representatives of classes $u \otimes v$ in $M \otimes N$, so we only need to verify that $<_\otimes$ respects the third generating relation in (5.4). But this is clear by choosing $a \cdot (u, v)$ as representatives.

We define a map

$$(\cdot) : R \times (M \otimes_R N) \xrightarrow{(a, [f]) \longmapsto [a \cdot f]} M \otimes_R N,$$

which by routine check seen to be well defined, namely $f \sim_t g \Rightarrow a \cdot f \sim_t a \cdot g$, for any $a \in R$.

Proposition 5.7. *($M \otimes_R N, \mathcal{H}_M \otimes_R \mathcal{H}_N, \mu_\otimes$) is an R - ν -module.*

Proof. First, observe that $M \otimes_R N$ is a ν -monoid (Definition 3.1) and its ghost map μ_\otimes coincides with the ν -monoid operation. Indeed, let $L := R(M \times N)$, write $\tilde{\mu}$ for μ_L , and for elements $f, g \in L$ use (5.6) to obtain

$$[f + g]^{\mu_\otimes} = [(f + g)^{\tilde{\mu}}] = [f^{\tilde{\mu}} + g^{\tilde{\mu}}] = [f^{\tilde{\mu}}] + [g^{\tilde{\mu}}] = [f]^{\mu_\otimes} + [g]^{\mu_\otimes}$$

and $(0_\otimes)^{\mu_\otimes} = [0_L^{\tilde{\mu}}] = [0_L] = 0_\otimes$. Since $[f]^{\mu_\otimes} = [f^{\tilde{\mu}}]$ and $\tilde{\mu}$ is an idempotent map on L , then μ_\otimes is idempotent as well.

Having the partial ordering $>_\otimes$ on $\mathcal{H}_M \otimes \mathcal{H}_N$ defined in (5.7), we verify the axioms of a ν -monoid (Definition 3.1).

NM1: Assume $\mu_\otimes([f]) >_\otimes \mu_\otimes([g])$, that is $[f^{\tilde{\mu}}] > [g^{\tilde{\mu}}]$, implying that $f^{\tilde{\mu}} > g^{\tilde{\mu}}$. Thus $[f] + [g] = [f + g] = [f]$.

NM2: If $\mu_\otimes(f) = \mu_\otimes(g)$, that is $[f^{\tilde{\mu}}] = [g^{\tilde{\mu}}]$, then $f^{\tilde{\mu}} = g^{\tilde{\mu}}$ and $[f] + [g] = [f + g] = [f^{\tilde{\mu}}] = [f]^{\mu_\otimes}$.

NM3: The condition $[f] + [g] \notin \mathcal{H}_M \otimes_R \mathcal{H}_N$ and $[f] + [g]^{\mu_\otimes} \in \mathcal{H}_M \otimes_R \mathcal{H}_N$ are equivalent to $f + g \notin \mathcal{H}_L$ and $f + g^{\tilde{\mu}} \in \mathcal{H}_L$, which implies $f + g = f^{\tilde{\mu}} + g$. But then $[f] + [g] = [f + g] = f^{\tilde{\mu}} + g = [f^{\tilde{\mu}}] + [g] = [f]^{\mu_\otimes} + [g]$, since $[f]^{\mu_\otimes} = [f^{\tilde{\mu}}]$.

Finally, a routine check shows that $M \otimes_R N$ is an R - ν -module via (\cdot) , since L by itself is an R - ν -module. \square

Remark 5.8. Since $R(M \times N)$ is generated by $\{(u, v) \mid u \in M, v \in N\}$, we see that $M \otimes_R N$ is generated as an ν -module by $\{u \otimes v \mid u \in M, v \in N\}$. Therefore, the ghost submodule $\mathcal{H}_{M \otimes_R N}$ of $M \otimes_R N$ is generated by $\{u^\mu \otimes v^\mu \mid u \in M, v \in N\}$, since $u^\mu \otimes v^\mu = [e \cdot (u, v)] = e[(u, v)]$ by (5.6). Thus, if $[f] = \sum_i a_i \cdot (u_i \otimes v_i)$, then

$$[f]^{\mu \otimes} = e \cdot [f] = \sum_i a_i \cdot e \cdot e \cdot [(u_i, v_i)] = \sum_i a_i \cdot [(eu_i, ev_i)] = \sum_i a_i \cdot [(u_i^\mu, v_i^\mu)].$$

Hence, the ghost map μ_\otimes is induced by the ghost maps μ_M and μ_N of M and N , respectively.

Corollary 5.9. The tensor product $T = M \otimes_R N$ exists for any R - ν -modules M and N .

Proof. Follows from Proposition 5.7. □

For two ν -semirings R and R' , an (R, R') - ν -**bimodule** is a ν -module M such that:

- (a) M is a left R - ν -module and also a right R' - ν -module;
- (b) $(au)b' = a(ub')$ for all $a \in R, b' \in R'$ and $u \in M$.

An (R, R) - ν -bimodule is shortly termed R - ν -bimodule.

Theorem 5.10. There is a left adjoint functor $\text{Ten}_N(_)$ to the contravariant functor $\text{Hom}(_, N)$ in Definition 5.5.

The theorem is a restatement of the **adjoint isomorphism**, writing $\text{Ten}_N(M)$ as $M \otimes_R N$, that is

$$\text{Hom}(W, \text{Hom}(M, N)) = \text{Hom}(W \otimes M, N).$$

$\text{Ten}_N(_)$ is called the **tensor product functor** $_ \otimes N$, where all constructions of $\text{Ten}_N(_)$ are naturally isomorphic, by the uniqueness of the left adjoint functor.

Proof of Theorem 5.10. The proof follows from the above construction of the tensor product $M \otimes_R N$, based on congruences instead of ν -submodules, and a verification of the adjoint isomorphism.

Given an abelian semigroup $\mathcal{S} := (\mathcal{S}, +)$, any bilinear map $\psi : M \times N \rightarrow \mathcal{S}$ gives rise to a semigroup homomorphism determined by $u \otimes v \mapsto \psi(u, v)$. Having gone this far, one follows the program spelled out in [40, §3.7] to show that if M is an (R', R) - ν -bimodule and N is an (R, R'') - ν -bimodule, then $M \otimes_R N$ is an (R', R'') - ν -bimodule under the natural operations

$$a'(u \otimes v) = a'u \otimes v, \quad (u \otimes v)a'' = u \otimes va''.$$

The verification of the adjoint isomorphism now is exactly as in the proof of [40, Proposition 3.2]. □

Remark 5.11. Let R, M , and N be ν -semirings, with \mathfrak{q} -homomorphisms $\phi : R \rightarrow M$ and $\psi : R \rightarrow N$, that make M and N into R - ν -modules. For every ν -semiring A and \mathfrak{q} -homomorphisms $\alpha : M \rightarrow A$, $\beta : N \rightarrow A$, rendering a commutative diagram with ϕ and ψ , there is a unique \mathfrak{q} -homomorphism $\xi : M \otimes_R N \rightarrow A$ such that the whole diagram commutes:

$$\begin{array}{ccccc}
 & & A & & \\
 & & \swarrow \beta & & \\
 & & & & N \\
 & & \searrow \xi & \longleftarrow & \\
 & & & & M \otimes_R N \\
 & & \swarrow \alpha & & \uparrow \psi \\
 & & & & R \\
 & & & \longleftarrow \phi & \\
 & & & & M
 \end{array}$$

(the maps $M \rightarrow M \otimes_R N$ and $N \rightarrow M \otimes_R N$ are the obvious maps $u \mapsto u \otimes 1$ and $v \mapsto 1 \otimes v$). For \mathfrak{q} -homomorphisms $\alpha : M \rightarrow A$ and $\beta : N \rightarrow A$, the map $\xi : M \otimes_R N \rightarrow A$ is determined by $u \otimes v \mapsto \alpha(u) \cdot \beta(v)$.

Once we have the notion of a tensor product at our disposal, we can recover classical structures, especially localization of ν -modules.

Definition 5.12. The **tangible localization** of an R - ν -module M by a tangible submonoid $C \subseteq R$ is defined as

$$M_C := R_C \otimes M,$$

where R_C is the tangible localization of R by C (Definition 3.39).

When C is generated by a \mathfrak{t} -persistent element $f \in \mathcal{T}^\circ$, we write M_f for M_C .

5.3. F - ν -algebra.

Let F be a ν -semifield, and let A be an F - ν -module equipped with an additional binary operation $(\cdot) : A \times A \rightarrow A$. A is an algebra over F , if the following properties hold for every elements $x, y, z \in A$, $a, b \in F$:

- (a) Right distributivity: $(x + y) \cdot z = x \cdot z + y \cdot z$,
- (b) Left distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$,
- (c) Multiplications by scalars: $(ax) \cdot (by) = (ab)(x \cdot y)$.

In other words, the binary operation (\cdot) is bilinear. An algebra over F is called F - ν -**algebra**, while F is said to be the base ν -semifield of A .

A **homomorphism** of F - ν -algebras A, B is a homomorphism $\phi : A \rightarrow B$ of ν -modules (i.e., an F -linear map) such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. The set of all F - ν -algebra homomorphisms from A to B is denoted by $\text{Hom}_F(A, B)$.

Definition 5.13. An F - ν -algebra is said to be **finitely generated**, if there exist $f_1, \dots, f_n \in A$ such that any $f \in A$ can be presented in the form

$$f = \sum_{j=1}^n a_j f_j, \quad \text{with } a_1, \dots, a_n \in F.$$

The elements f_1, \dots, f_n are called **generators** of A .

Tensor products and ν -algebras lay a foundation for developing new types of algebra, parallel to known algebras, e.g., exterior ν -algebra, Lie ν -algebra, or Clifford ν -algebra.

Part II: Supertropical Algebraic Geometry

6. VARIETIES

Recall that all our underlying ν -semirings (Definition 3.11) are assumed to be commutative. Henceforth, since ν -semirings are referred to also as ν -algebras whose elements are functions, unless otherwise is specified, $A := (A, \mathcal{T}, \mathcal{G}, \nu)$ denotes a commutative ν -semiring. For a clearer exposition, given a \mathfrak{q} -congruence \mathfrak{A} on A , we explicitly write $(A/\mathfrak{A})|_{\text{tng}}^\circ$ (resp. $(A/\mathfrak{A})|_{\text{tng}}^\bullet$) for the \mathfrak{t} -persistent set (resp. a \mathfrak{t} -persistent monoid) of the quotient ν -semiring A/\mathfrak{A} , and $(A/\mathfrak{A})|_{\text{ual}}$ for the set of its \mathfrak{t} -unalterable elements. Similarly, we write $(A/\mathfrak{A})|_{\text{tng}}$ for the tangible set of A/\mathfrak{A} , and $(A/\mathfrak{A})|_{\text{gh}}$ for its ghost ideal. To unify notations, we write $A|_{\text{ual}}$, $A|_{\text{tng}}^\bullet$, $A|_{\text{tng}}^\circ$, $A|_{\text{tng}}$, and $A|_{\text{gh}}$ for \mathcal{S} , \mathcal{T}^\bullet , \mathcal{T}° , \mathcal{T} , and \mathcal{G} , respectively.

Notation 6.1. A \mathfrak{g} -prime congruence is denoted by \mathfrak{P} , where its underlying equivalence is denoted by $\equiv_{\mathfrak{p}}$. We let $X = \text{Spec}(A)$ be the \mathfrak{g} -prime spectrum of A whose formal elements are \mathfrak{g} -prime congruences (Definition 4.40). Later, X is realized as a topological space. To designate this view, we denote a point of X by x , where \mathfrak{P}_x stands for \mathfrak{g} -prime congruence assigned with x . We write A_x for the localization $A_{\mathfrak{P}_x}$ of A by \mathfrak{P}_x (Definition 4.39).

Elements of A are denoted by the letters f, g, h , as from now on they are realized also as functions (as explained below). We write $\langle f_1, \dots, f_\ell \rangle$ for the subset generated by elements $f_1, \dots, f_\ell \in A$, i.e., all finite sums of the form $\sum_i g_i f_i$ with $g_i \in A$.

Recall from Remark 4.16 that $\mathbb{G}_{\text{cls}}^{-1}(_)$ and $\mathbb{T}_{\text{cls}}^{-1}(_)$ provide respectively the class-forgetful maps $\mathbb{G}_{\text{cls}}(_) : \text{Cong}_{\mathfrak{q}}(A) \rightarrow A$ and $\mathbb{T}_{\text{cls}}(_) : \text{Cong}_{\mathfrak{q}}(A) \rightarrow A$ that encode clusters' decomposition. Recall also that a \mathfrak{g} -prime congruence \mathfrak{P} is an ℓ -congruence, and thus its tangible projection $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$ is a multiplicative monoid, written $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}) = \mathbb{P}^{-1}(\mathfrak{P})$.

Comment 6.2. A subset $E \subseteq A$ can be realized as the trivial partial congruence $\Delta(E)$, which set theoretically is contained in $A \times A$ (Definition 2.4). We are mainly concerned with the case that $\Delta(E) \subseteq \mathbb{G}_{\text{cls}}(\mathfrak{P})$, written equivalently as $E \subseteq \mathbb{G}_{\text{cls}}^{-1}(\mathfrak{P})$, where \mathfrak{P} is a \mathfrak{g} -prime congruence. When possible, to simplify notations, we use the latter form. Equivalently, this setup is formulated in terms of \mathfrak{g} -congruences as $\mathfrak{E}_E \subseteq \mathfrak{P}$, cf. (4.8), which reads as $f \equiv_{\mathfrak{p}} f^\nu$ in \mathfrak{P} for all $f \in E$. We alternate between these equivalent descriptions, for a clearer exposition.

Our forthcoming exposition includes some equivalent definitions, relying on different structural views, that later help for a better understanding of the interplay among the involved objects.

6.1. Varieties over ν -semirings.

Let $A = (A, \mathcal{T}, \mathcal{G}, \nu)$ be a ν -semiring, and let $\text{Spec}(A)$ be its \mathfrak{g} -prime spectrum (Definition 4.40). The elements of A can be interpreted as functions on $\text{Spec}(A)$ by defining $f(\mathfrak{P})$ to be the residue class $[f]$ of f in A/\mathfrak{P} , that is

$$f : \mathfrak{P} \longmapsto [f] \in A/\mathfrak{P}. \quad (6.1)$$

Thereby, every element $f \in A$ determines a map

$$\text{Spec}(A) \longrightarrow \coprod_{\mathfrak{P} \in \text{Spec}(A)} A/\mathfrak{P} \subseteq \coprod_{\mathfrak{P} \in \text{Spec}(A)} Q(A/\mathfrak{P}),$$

where $f(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{gh}}$ if $f \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, and likewise $f(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{tng}}$ when $f \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$. Namely, on a \mathfrak{g} -prime congruence \mathfrak{P} the function f possesses ghost or tangible, respectively; but it could also result in a class which is neither ghost nor tangible.

Remark 6.3. *A function $f \notin A|_{\text{tng}}$ cannot not be congruent to any $h \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$, in each $\mathfrak{P} \in \text{Spec}(A)$, but it is not necessarily evaluated as ghost. When $f \notin A|_{\text{gh}}$, it may still be considered as tangibly evaluated on some \mathfrak{P} .*

Since $\mathfrak{P} \in \text{Spec}(A)$ is a \mathfrak{g} -prime congruence, a ghost product $(fg)(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{gh}}$ of functions $f, g \in A$ implies that $f(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{gh}}$ or $g(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{gh}}$. Equivalently, this reads as $fg \equiv_{\mathfrak{p}}$ ghost implies $f \equiv_{\mathfrak{p}}$ ghost or $g \equiv_{\mathfrak{p}}$ ghost, where $\equiv_{\mathfrak{p}}$ is the underlying equivalence of \mathfrak{P} . Hence, $f \equiv_{\mathfrak{p}} f^\nu$ or $g \equiv_{\mathfrak{p}} f^\nu$ by Lemma 4.1. We identify $f \in A$ with the pair $(f, f) \in A \times A$.

We define the **ghost locus** of a nonempty subset $E \subseteq A$ to be

$$\mathcal{V}(E) := \{\mathfrak{P} \in \text{Spec}(A) \mid f(\mathfrak{P}) \in (A/\mathfrak{P})|_{\text{gh}} \text{ for all } f \in E\}.$$

For $f \in A$ we therefore equivalently define (cf. Comment 6.2):

$$\begin{aligned} \mathcal{V}(f) &:= \{\mathfrak{P} \in \text{Spec}(A) \mid f \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})\} \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid (f, f) \in \mathbf{G}_{\text{cls}}(\mathfrak{P})\}, \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid \text{s.t. } f \equiv_{\mathfrak{p}} f^\nu \text{ in } \mathfrak{P}\}, \\ \mathcal{D}(f) &:= \{\mathfrak{P} \in \text{Spec}(A) \mid f \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})\} \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid (f, f) \notin \mathbf{G}_{\text{cls}}(\mathfrak{P})\} = \text{Spec}(A) \setminus \mathcal{V}(f) \end{aligned} \quad (6.2)$$

A set of the form $\mathcal{D}(f)$ is called a **principal subset**. Using the bijection $\iota : E \xrightarrow{\sim} \Delta(E)$, a subset $\mathcal{V}(E)$ can be written as

$$\begin{aligned} \mathcal{V}(E) &:= \{\mathfrak{P} \in \text{Spec}(A) \mid E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})\} \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid \Delta(E) \subseteq \mathbf{G}_{\text{cls}}(\mathfrak{P})\}. \end{aligned} \quad (6.3)$$

We use these equivalent forms of $\mathcal{V}(E)$ and $\mathcal{D}(f)$ to ease the exposition.

Comment 6.4. *Note that $\mathfrak{P} \in \mathcal{D}(f)$ does not imply that $f \in \mathbf{T}_{\text{cls}}^{-1}(\mathfrak{P})$, which means that as a function f need not take tangible values on \mathfrak{P} . However, it takes non-ghost values on the entire $\mathcal{D}(f)$.*

A ν -**variety** in $\text{Spec}(A)$ is a subset of type $\mathcal{V}(E)$ for some $E \subseteq A$; it is called ν -**hypersurface** when $E = \{f\}$ for some $f \in A$, in other words \mathfrak{G}_E is a principal \mathfrak{q} -congruence. We write $\mathcal{V}_X(f)$ and $\mathcal{D}_X(f)$ when we want to keep track of the ambient space $X = \text{Spec}(A)$.

Recall that an element $f \in A$ is ghostpotent, belonging to $\mathcal{N}(A)$, if $f^n \in A|_{\text{gh}}$ for some $n \in \mathbb{N}$ (Definition 4.76), where f by itself could be ghost.

Remark 6.5. *Since $A|_{\text{gh}} := \mathcal{G} \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for any \mathfrak{g} -prime congruence \mathfrak{P} , from (6.3) we obtain that $\mathcal{V}(E) = \mathcal{V}(E \cup \mathcal{G})$ for any $E \subseteq A$. Therefore, $\mathcal{V}(E) = \mathcal{V}(\mathcal{G})$ when $E \subseteq \mathcal{G}$, and $\mathcal{V}(g) = \mathcal{V}(\mathcal{G})$ for any $g \in A|_{\text{gh}}$. Clearly, $\mathcal{V}(A) = \emptyset$, since $A^\times \not\subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for every $\mathfrak{P} \in \text{Spec}(A)$, by Remark 4.15. On the other hand, for principal subsets, we immediately see that $\mathcal{D}(f) = \emptyset$ for every $f \in A|_{\text{gh}}$, and more generally for all $f \in \mathcal{N}(A)$, while $\mathcal{D}(f) = \text{Spec}(A)$ for any unit $f \in A^\times$ and any $f \in A|_{\text{ual}}$, in particular $\mathcal{D}(1) = \text{Spec}(A)$.*

Recall that $f \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ means that $f \equiv g$ for some $g \in A|_{\text{gh}}$, yet f need not be ghost in A .

Proposition 6.6. *Let E, E' be subsets of A , and let $(E_i)_{i \in I}$ be a family of subsets of A . Then,*

- (i) $\mathcal{V}(h) = \text{Spec}(A)$ for any $h \in A|_{\text{gh}}$, and for any ghostpotent $h \in \mathcal{N}(A)$;
- (ii) $\mathcal{V}(f) = \emptyset$ for every unit $f \in A^\times$;
- (iii) $E \subset E' \Rightarrow \mathcal{V}(E) \supset \mathcal{V}(E')$;
- (iv) $\mathcal{V}(\bigcup_{i \in I} E_i) = \bigcap_{i \in I} \mathcal{V}(E_i) = \mathcal{V}(\sum_{i \in I} E_i)$;
- (v) $\mathcal{V}(EE') = \mathcal{V}(E) \cup \mathcal{V}(E')$, where $EE' = \{ff' \mid f \in E, f' \in E'\}$.

Proof. (i)–(iii) are obvious, since each $\mathfrak{P} \in \text{Spec}(A)$ is an ℓ -congruence, where $h \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for every $h \in \mathcal{N}(A)$, cf. Remarks 4.79 and 6.5. In addition, $A^\times \subseteq \mathbf{G}_{\text{cls}}^{-c}(\mathfrak{P})$ for each \mathfrak{P} , since otherwise \mathfrak{P} would be a ghost congruence, and thus $\mathcal{V}(f) = \emptyset$.

(iv): The set $\mathcal{V}(\bigcup_{i \in I} E_i)$ consists of all $\mathfrak{P} \in \text{Spec}(A)$ with $\bigcup_{i \in I} E_i \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, hence $E_i \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for every $i \in I$, and therefore coincides with $\bigcap_{i \in I} \mathcal{V}(E_i)$. Given $\mathfrak{P} \in \text{Spec}(A)$, $\bigcup_{i \in I} E_i$ is contained in $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ iff the set $\sum_{i \in I} E_i$ generated by $\bigcup_{i \in I} E_i$ (which is a semiring ideal) is contained in $\mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$. Therefore, we see that, in addition, $\mathcal{V}(\bigcup_{i \in I} E_i)$ coincides with $\mathcal{V}(\sum_{i \in I} E_i)$.

(v): Take $\mathfrak{P} \in \text{Spec}(A)$ such that $\mathfrak{P} \notin \mathcal{V}(E)$ and $\mathfrak{P} \notin \mathcal{V}(E')$. Then $E \not\subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ and $E' \not\subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, and there are elements $f \in E$ and $f' \in E'$ such that $f, f' \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$. But then, $ff' \notin \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ since \mathfrak{P} is a \mathfrak{g} -prime congruence, and thus $\mathfrak{P} \notin \mathcal{V}(EE')$ so that $\mathcal{V}(EE') \subseteq \mathcal{V}(E) \cup \mathcal{V}(E')$. Conversely, $\mathfrak{P} \in \mathcal{V}(E)$ implies $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ and hence $EE' \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ by Remark 4.41, thus $\mathfrak{P} \in \mathcal{V}(EE')$. This shows that $\mathcal{V}(E) \subseteq \mathcal{V}(EE')$ and, likewise, $\mathcal{V}(E') \subseteq \mathcal{V}(EE')$. \square

For the set-radical closure $\text{rcl}(E)$ of a subset $E \subseteq A$, determined by \mathfrak{s} -radical (Definition 4.70), we receive a one-to-one correspondence.

Proposition 6.7. $\mathcal{V}(E) = \mathcal{V}(\text{rcl}(E))$ for any $E \subseteq A$.

Proof. Clearly, $E \subseteq \text{rcl}(E)$, and thus $\mathcal{V}(E) \supseteq \mathcal{V}(\text{rcl}(E))$ by Proposition 6.6.(iii). Conversely, by (4.43), $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ is equivalent to $\text{rcl}(E) \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$, since \mathfrak{P} is \mathfrak{g} -prime. This holds for any $\mathfrak{P} \in \mathcal{V}(E)$, and thus $\mathcal{V}(E) \subseteq \mathcal{V}(\text{rcl}(E))$. \square

With the above setting, the \mathfrak{g} -prime spectrum $\text{Spec}(A)$ is endowed with a Zariski type topology, defined in the obvious way.

Corollary 6.8. *Let A be a ν -semiring, and let $X = \text{Spec}(A)$ be its spectrum. There exists a Zariski topology on X whose closed sets are subsets of type $\mathcal{V}(E) \subseteq X$, where $E \subseteq A$. Moreover:*

- (i) *The sets of type $\mathcal{D}(f)$ with $f \in A$ are open and satisfy $\mathcal{D}(f) \cap \mathcal{D}(g) = \mathcal{D}(fg)$ for $f, g \in A$.*
- (ii) *Every open subset of X is a union of sets of type $\mathcal{D}(f)$, which form a basis of the topology on X .*

Proof. The characteristic properties of closed sets of a topology are given by parts (i), (iii), and (iv) of Proposition 6.6, where (iv) is generalized by induction to finite unions of sets of type $\mathcal{V}(E)$. The sets of type $\mathcal{D}(f)$ are open, as they are complements of sets of type $\mathcal{V}(f)$, and thus satisfy the intersection property by Proposition 6.6.(iv). Finally, an arbitrary open subset $U \subseteq X = \text{Spec}(A)$ is the complement of a closed set of type $\mathcal{V}(E)$ for some subset $E \subseteq A$. Therefore, $\mathcal{V}(E) = \bigcap_{f \in E} \mathcal{V}(f)$ and hence $U = \bigcup_{f \in E} \mathcal{D}(f)$, which says that every open subset in X is a union of sets of type $\mathcal{D}(f)$. \square

Note that, in contrast to the familiar spectra of rings, arbitrary open sets in the Zariski topology on $\text{Spec}(A)$ need not be dense.

Proposition 6.9. $f \in A \setminus (A|_{\text{gh}} \cup \text{gdiv}(A))$ iff $\mathcal{D}(f)$ is dense in X (Definition 3.19).

Proof. $\mathcal{D}(f) \neq \emptyset$ is dense iff $\mathcal{V}(g) = X$ for every closed set $\mathcal{V}(g)$ that contains $\mathcal{D}(f)$. The inclusion $\mathcal{D}(f) \subset \mathcal{V}(g)$ means that $g(x) \in (A/\mathfrak{P}_x)|_{\text{gh}}$ for every $x \in X$ for which $f(x) \notin (A/\mathfrak{P}_x)|_{\text{gh}}$. This holds iff $g(x)f(x) \in (A/\mathfrak{P}_x)|_{\text{gh}}$ for every $x \in X$ iff gf is a ghostpotent (Corollary 4.83) iff g is ghostpotent and $f \notin \text{gdiv}(A)$ is not ghost. The former condition holds, since $\mathcal{V}(g) = X$ for every g such that $\mathcal{V}(g)$ contains $\mathcal{D}(f)$. \square

Remark 6.10. *If A is a tame ν -semiring, then $\mathcal{D}(f)$ is dense iff $f \in A|_{\text{tng}} \setminus \text{gdiv}(A)$, since otherwise, either, $f \in A|_{\text{gh}}$ and thus $\mathcal{D}(f) = \emptyset$, or f is a ghost divisor by Lemma 3.23.(i) and thus $\mathcal{D}(f)$ is not dense by Proposition 6.9.*

We strengthen Remark 6.5 and Proposition 6.6.

Lemma 6.11. *Let A be a ν -semiring.*

- (i) $\mathcal{V}(f) = \text{Spec}(A)$ iff f is a ghostpotent.
- (ii) If $\mathcal{V}(f) \subseteq \mathcal{V}(g)$ where f is a ghostpotent, then g is a ghostpotent.
- (iii) $\mathcal{D}(f) = \emptyset$ iff f is a ghostpotent.
- (iv) $\mathcal{D}(f) = \mathcal{D}(g)$ iff $\text{rad}_s(f) = \text{rad}_s(g)$, cf. Definition 4.70.
- (v) If f is a unit (or \mathfrak{t} -unalterable) in A , then $\mathcal{D}(f) = \text{Spec}(A)$.
- (vi) $\mathcal{D}(f^n) = \mathcal{D}(f)$ for any $f \in A$.
- (vii) If $\mathcal{D}(f) \subseteq \mathcal{D}(g)$ where $f \in A|_{\text{tnng}}^\circ$ and A is a tame ν -semiring, then $g \in A|_{\text{tnng}}^\circ$.
- (viii) $E' \subseteq E$ iff $\mathcal{D}(E') \subseteq \mathcal{D}(E)$ iff $\mathcal{V}(E') \supseteq \mathcal{V}(E)$ iff $\text{rad}_s(E') \subseteq \text{rad}_s(E)$.

Proof. (i): $\mathcal{V}(f) = \text{Spec}(A) \Leftrightarrow f \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for every \mathfrak{g} -prime congruence \mathfrak{P} on $A \Leftrightarrow f$ is ghostpotent, cf. Remark 4.79.

(ii): Follows immediately from part (i).

(iii): $\mathcal{D}(f) = \emptyset \Leftrightarrow \mathcal{V}(f) = \text{Spec}(A)$, and apply part (i).

(iv): Using set theoretic considerations we see that

$$\mathcal{D}(f) \subseteq \mathcal{D}(g) \Leftrightarrow \mathcal{V}(f) \supseteq \mathcal{V}(g) \Leftrightarrow \text{rad}_s(f) \subseteq \text{rad}_s(g). \quad (*)$$

By symmetry we conclude that $\text{rad}_s(f) = \text{rad}_s(g)$.

(v): (\Rightarrow): $\mathcal{D}(f) = \text{Spec}(A) = \mathcal{D}(g)$ for some unit $g \in A^\times \Rightarrow \text{rad}_s(g) \subseteq \text{rad}_s(f)$ by (*) $\Rightarrow g^n = fh$ by Lemma 4.74. But g^n is a unit, and thus fh is also a unit, implying that f is a unit by Remark 3.14.

(\Leftarrow): $f \in A^\times$ is a unit of $A \Rightarrow \mathcal{V}(f) = \emptyset \Rightarrow \mathcal{D}(f) = \text{Spec}(A)$, by Lemma 6.6.(ii).

(vi): $\mathcal{V}(f^n) = \mathcal{V}(f) \cap \dots \cap \mathcal{V}(f) = \mathcal{V}(f)$ by Lemma 6.6, taking complements we get $\mathcal{D}(f^n) = \mathcal{D}(f)$.

(vii): Follows from Corollary 4.75, applied to (*), as f is \mathfrak{t} -persistent and A is tame.

(viii): It is an obvious generalization of (iv), combined with Proposition 6.6.(iii). \square

By Comment 6.2, a subset E of a ν -semiring A is identified with $\Delta(E)$, realized as a partial ghost concurrence. In addition, E is canonically associated to the congruence \mathfrak{G}_E by the means of ghostification (4.7), which gives the inclusion $\Delta(E) \subseteq \mathbf{G}_{\text{cls}}(\mathfrak{G}_E)$. In this view, $\mathcal{V}(_)$ defines a map $A \rightarrow \text{Spec}(A)$ via (6.3), which extends naturally to the map

$$\mathcal{V}(_): \text{Cong}(A) \longrightarrow \text{Spec}(A), \quad \mathfrak{A} \longmapsto \mathcal{V}(\mathfrak{A}),$$

where $\mathcal{V}(\mathfrak{A})$ is defined as

$$\mathcal{V}(\mathfrak{A}) := \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \mathfrak{A}\}. \quad (6.4)$$

Using the same notation, we apply $\mathcal{V}(_)$ to both subsets E of A and arbitrary congruences on A , no confusion arises. More generally, $\mathcal{V}(_)$ can be applied to subsets $S \subset A \times A$, e.g., to $\mathfrak{A}_1 \cap \mathfrak{A}_2$ and $\mathfrak{A}_1 + \mathfrak{A}_2$. We write \mathcal{V}_A , when want to stress the fact that \mathcal{V} is taken over A .

Remark 6.12. *With this notation, we have $\mathcal{V}(\mathfrak{G}_E) = \mathcal{V}(E)$ for any $E \subseteq A$. Indeed,*

$$\begin{aligned} \mathcal{V}(\mathfrak{G}_E) &= \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \mathfrak{G}_E\} \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \bigcap \mathfrak{A} \text{ such that } E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{A})\} \\ &= \{\mathfrak{P} \in \text{Spec}(A) \mid E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})\} = \mathcal{V}(E). \end{aligned}$$

Note that \mathfrak{G}_E need not be a \mathfrak{q} -congruence, e.g., if $E \cap A^\times \neq \emptyset$; in such case $\mathcal{V}(\mathfrak{G}_E) = \emptyset$ by definition.

We define the converse map of (6.1) to be the map

$$\mathcal{K}(_): \text{Spec}(A) \longrightarrow \text{Cong}_{\mathfrak{q}}(A) \quad [\subset \text{Cong}(A)]$$

that sends a subset $Y \subseteq \text{Spec}(A)$ to the \mathfrak{q} -congruence

$$\mathcal{K}(Y) := \bigcap_{\mathfrak{P} \in Y} \mathfrak{P}. \quad (6.5)$$

$\mathcal{K}(Y)$ is indeed a \mathfrak{q} -congruence, since it is an intersection of ℓ -congruences, cf. Remark 4.28. In particular, $\mathcal{K}(\{\mathfrak{P}\}) = \mathfrak{P}$ for any $\mathfrak{P} \in \text{Spec}(A)$, while $\mathcal{K}(\text{Spec}(A)) = \text{rad}_{\mathfrak{g}}(A)$ by Theorem 4.81.

The map $\mathcal{K}(_)$ determines the restricted map from $\text{Spec}(A)$ to A , given in terms of projections:

$$\mathcal{K}_{\text{gh}}(_) : \text{Spec}(A) \longrightarrow A, \quad Y \longmapsto \bigcap_{\mathfrak{P} \in Y} \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P}) = \mathbf{G}_{\text{cls}}^{-1}(\mathcal{K}(Y)).$$

For $Y = \mathcal{V}(\mathfrak{A})$, the inclusion $(f, f) \in \mathcal{K}(Y)$ implies $f \in \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P})$ for each $\mathfrak{P} \in Y$. Therefore, $\mathcal{K}_{\text{gh}}(Y)$ is the subset of all functions in A that take ghost values over the entire Y .

Remark 6.13. *From (6.5) it follows immediately that*

- (i) *If $Y \subset Y' \subset \text{Spec}(A)$, then $\mathcal{K}(Y) \supset \mathcal{K}(Y')$,*
- (ii) *$\mathcal{K}(\bigcup_{j \in J} Y_j) = \bigcap_{j \in J} \mathcal{K}(Y_j)$ for any family of subsets $(Y_j)_{j \in J}$ of $\text{Spec}(A)$.*

Applying $\mathcal{V}(_)$ to congruences, we get the following.

Proposition 6.14. *Let A be a ν -semiring, and let $X = \text{Spec}(A)$.*

- (i) *$\mathcal{K}(\mathcal{V}(E)) = \text{rad}_{\mathfrak{s}}(E)$ for any subset $E \subseteq A$, cf. Definition 4.70. In particular, $\mathcal{K}(\mathcal{V}(E)) = \mathfrak{G}_E$ when $\mathfrak{G}_E = \text{rad}_{\mathfrak{c}}(\mathfrak{G}_E)$, cf. Definition 4.64.*
- (ii) *$\mathcal{V}(\mathcal{K}(Y))$ coincides with the closure $\overline{Y} \subset X$ with respect to the Zariski topology on X , and $Y = \mathcal{V}(\mathcal{K}(Y))$ for closed subsets $Y \subset X$.*

Proof. (i): Using Proposition 6.7 and Lemma 4.67, write

$$\mathcal{K}(\mathcal{V}(E)) = \bigcap_{\mathfrak{P} \in \mathcal{V}(E)} \mathfrak{P} = \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(A) \\ \mathbf{G}_{\text{cls}}(\mathfrak{P}) \supseteq \Delta(E)}} \mathfrak{P} = \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(A) \\ \mathbf{G}_{\text{cls}}^{-1}(\mathfrak{P}) \supseteq E}} \mathfrak{P} = \text{rad}_{\mathfrak{s}}(E).$$

(This assertion is also obtained from Lemma 6.15 by considering $\Delta(E)$ as a partial ghost congruence.)

(ii): First $Y \subseteq \mathcal{V}(\mathcal{K}(Y))$, since

$$\begin{aligned} \mathcal{V}(\mathcal{K}(Y)) &= \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \mathcal{K}(Y)\} \\ &= \left\{ \mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \bigcap_{\mathfrak{P} \in Y} \mathfrak{P} \right\}, \end{aligned}$$

and clearly $\mathfrak{P} \in \mathcal{V}(\mathcal{K}(Y))$ for every $\mathfrak{P} \in Y$. Moreover, $\overline{Y} \subseteq \mathcal{V}(\mathcal{K}(Y))$, since the closure \overline{Y} of Y in X is the smallest closed subset in X that contains Y , i.e., the intersection of all closed subsets $\mathcal{V}(E) \subseteq \text{Spec}(A)$ such that $Y \subseteq \mathcal{V}(E)$. To see that $\overline{Y} = \mathcal{V}(\mathcal{K}(Y))$, it remains to check that $Y \subseteq \mathcal{V}(E)$ implies $\mathcal{V}(\mathcal{K}(Y)) \subseteq \mathcal{V}(E)$. Indeed, from the inclusion $Y \subseteq \mathcal{V}(E)$ we conclude that $\Delta(E) \subseteq \mathbf{G}_{\text{cls}}(\mathfrak{P})$ for all $\mathfrak{P} \in Y$, hence $\Delta(E) \subseteq \mathbf{G}_{\text{cls}}(\mathcal{K}(Y))$, which is equivalent to $E \subseteq \mathbf{G}_{\text{cls}}^{-1}(\mathcal{K}(Y))$. Then, $\mathcal{V}(\mathcal{K}(Y)) \subseteq \mathcal{V}(E)$ by Proposition 6.6.(iii). \square

Proposition 6.14, together with Lemma 6.11, shows that $\mathcal{V}(f) = \mathcal{V}(g)$ is equivalent to $\text{rad}_{\mathfrak{s}}(f) = \text{rad}_{\mathfrak{s}}(g)$, for $f, g \in A$, and therefore also to $\mathcal{D}(f) = \mathcal{D}(g)$.

Lemma 6.15. *Let $\mathfrak{A} \in \text{Cong}(A)$ be an arbitrary congruence on A , then:*

- (i) *$\mathcal{V}(\text{rad}_{\mathfrak{c}}(\mathfrak{A})) = \mathcal{V}(\mathfrak{A})$,*
- (ii) *$\mathcal{K}(\mathcal{V}(\mathfrak{A})) = \text{rad}_{\mathfrak{c}}(\mathfrak{A})$.*

Proof. Write explicitly to obtain the following.

$$\begin{aligned} (i) : \quad \mathcal{V}(\text{rad}_{\mathfrak{c}}(\mathfrak{A})) &= \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \text{rad}_{\mathfrak{c}}(\mathfrak{A})\} \\ &= \left\{ \mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(A) \\ \mathfrak{P} \supseteq \mathfrak{A}}} \mathfrak{P} \right\} = \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq \mathfrak{A}\} = \mathcal{V}(\mathfrak{A}), \\ (ii) : \quad \mathcal{K}(\mathcal{V}(\mathfrak{A})) &= \bigcap_{\mathfrak{P} \in \mathcal{V}(\mathfrak{A})} \mathfrak{P} = \bigcap_{\substack{\mathfrak{P} \in \text{Spec}(A) \\ \mathfrak{P} \supseteq \mathfrak{A}}} \mathfrak{P} = \text{rad}_{\mathfrak{c}}(\mathfrak{A}). \end{aligned} \quad \square$$

Accordingly, we conclude that, if \mathfrak{A} is a \mathfrak{q} -congruence, then also $\mathcal{K}(\mathcal{V}(\mathfrak{A}))$ is a \mathfrak{q} -congruence, unless $\mathcal{V}(\mathfrak{A})$ is empty.

Remark 6.16. *The usual straightforward inverse Zariski correspondence holds for $\mathfrak{A}_1, \mathfrak{A}_2 \in \text{Cong}(A)$:*

- (i) $\mathcal{V}(\overline{\mathfrak{A}_1 \cup \mathfrak{A}_2}) = \mathcal{V}(\mathfrak{A}_1) \cap \mathcal{V}(\mathfrak{A}_2)$, cf. (4.23);
- (ii) If $\mathfrak{A}_1 \supseteq \mathfrak{A}_2$, then $\mathcal{V}(\mathfrak{A}_1) \subseteq \mathcal{V}(\mathfrak{A}_2)$;
- (iii) $\mathcal{V}(\mathfrak{A}_1 \cap \mathfrak{A}_2) \supseteq \mathcal{V}(\mathfrak{A}_1) \cup \mathcal{V}(\mathfrak{A}_2)$.

By this remark and Lemma 6.15, using Notation 6.1, we conclude the following.

Corollary 6.17. *Let A be a ν -semiring, and let $X = \text{Spec}(A)$ be its spectrum.*

- (i) For $x \in X$ we have $\overline{\{x\}} = \mathcal{V}(\mathfrak{P}_x)$ and the closure of x consists of all $\mathfrak{P} \supseteq \mathfrak{P}_x$.
- (ii) A point $x \in X$ is closed if and only if \mathfrak{P}_x is a maximal ℓ -congruence in $\text{Spec}(A)$.

(Note that part (ii) does not hold for a \mathfrak{t} -minimal ℓ -congruence (Definition 4.55).)

Corollary 6.18. *The mappings between $X = \text{Spec}(A)$ and $\text{Cong}_{\mathfrak{q}}(A)$, given by*

$$\left\{ \begin{array}{l} \text{closed} \\ \text{subsets in } X \end{array} \right\} \begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow{\mathcal{V}} \end{array} \left\{ \begin{array}{l} \mathfrak{c}\text{-radical congruences} \\ \mathfrak{R} = \text{rad}_{\mathfrak{c}}(\mathfrak{R}) \text{ on } A \end{array} \right\}, \quad \begin{array}{c} Y \\ \mathcal{V}(\mathfrak{R}) \end{array} \begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow{\mathcal{V}} \end{array} \begin{array}{c} \mathcal{K}(Y) \\ \mathfrak{R} \end{array},$$

are inclusion-reversing, bijective, and inverse to each other. Furthermore,

$$\mathcal{K}\left(\bigcup_{j \in J} Y_j\right) = \bigcap_{j \in J} \mathcal{K}(Y_j), \quad \mathcal{K}\left(\bigcap_{j \in J} Y_j\right) = \text{rad}_{\mathfrak{c}}\left(\sum_{j \in J} \mathcal{K}(Y_j)\right),$$

for a family $(Y_j)_{j \in J}$ of subsets $Y_i \subset X$, where in the latter equation the Y_i 's are closed in X .

Proof. The assertions infer from Proposition 6.14, except the latter equation. Since $Y_j = \mathcal{V}(\mathcal{K}(Y_j))$ by Proposition 6.14.(ii), from Proposition 6.6.(iv) we obtain

$$\bigcap_{j \in J} Y_j = \mathcal{V}\left(\sum_{j \in J} \mathcal{K}(Y_j)\right), \quad \text{and thus} \quad \mathcal{K}\left(\bigcap_{j \in J} Y_j\right) = \text{rad}_{\mathfrak{c}}\left(\sum_{j \in J} \mathcal{K}(Y_j)\right)$$

by Remark 4.66 and Proposition 6.14.(i). □

Quasi compactness occurs for open sets in the Zariski topology of $\text{Spec}(A)$.

Proposition 6.19. *Let A be a ν -semiring, and let $X = \text{Spec}(A)$ be its spectrum. Every set $\mathcal{D}(g) \subset X$, with $g \in A$, is quasi-compact. In particular, $X = \mathcal{D}(\mathfrak{1})$ is quasi-compact.*

Proof. Since the sets $\mathcal{D}(f)$ form a basis of the Zariski topology on X (Corollary 6.8), it is enough to show that every covering of $\mathcal{D}(g)$ admits a finite subcover. We may assume that $g \notin A|_{\text{gh}}$ is not a ghospotent, since otherwise $\mathcal{D}(g) = \emptyset$ by Lemma 6.11. Let $\mathcal{F} := (f_i)_{i \in I}$ be a family of elements in A such that $\mathcal{D}(g) \subseteq \bigcup_{i \in I} \mathcal{D}(f_i)$. Taking complements, this is equivalent to

$$\mathcal{V}(g) \supseteq \bigcap_{i \in I} \mathcal{V}(f_i) = \mathcal{V}(\mathfrak{G}_{\mathcal{F}}),$$

where $\mathfrak{G}_{\mathcal{F}}$ is the ghostifying congruence of \mathcal{F} . Then, $\text{rad}_{\mathfrak{s}}(g) \subset \text{rad}_{\mathfrak{s}}(\mathcal{F}) = \text{rad}_{\mathfrak{c}}(\mathfrak{G}_{\mathcal{F}})$ by Proposition 6.14, and there exists $n \in \mathbb{N}$ such that $g^n \in \mathfrak{G}_{\text{cls}}^{-1}(\mathfrak{G}_{\mathcal{F}})$.

$\mathfrak{G}_{\mathcal{F}}$ is the minimal congruence determined by the ghost relations on \mathcal{F} . Its ghost projection $\mathfrak{G}_{\text{cls}}^{-1}(\mathfrak{G}_{\mathcal{F}})$ is an ideal of A (Remark 4.15), generated by elements from \mathcal{F} and elements of $A|_{\text{gh}}$ (Remark 4.18). Thus, there exists a finite set of indices $i_1, \dots, i_m \in I$ such that $g^n = \sum_{j=1}^m a_{i_j} g_{i_j}$, with $g_{i_j} \in \mathfrak{G}_{\text{cls}}^{-1}(\mathfrak{G}_{\mathcal{F}})$. In terms of ideals, this implies $\langle g^n \rangle \subseteq \langle g_{i_1}, \dots, g_{i_m} \rangle$.

Since g^n is not a ghost, the set $\{g_{i_1}, \dots, g_{i_m}\}$ contains a subset K of non-ghost elements $\{f_{k_1}, \dots, f_{k_\ell}\}$ from \mathcal{F} . Hence $g^n \in \mathfrak{G}_{\text{cls}}^{-1}(\mathfrak{G}_K)$, and $\mathcal{V}(g) = \mathcal{V}(g^n) \supseteq \mathcal{V}(\{f_{k_1}, \dots, f_{k_\ell}\})$. Therefore $\mathcal{D}(g) \subseteq \bigcup_{j=1}^{\ell} \mathcal{D}(f_{k_j})$, and $\mathcal{D}(g)$ admits a finite subcover. □

Recall from Remark 2.3.(iii) that a \mathfrak{q} -homomorphism $\varphi : A \longrightarrow B$ of ν -semirings induces the congruence pull-back map

$${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A), \quad \mathfrak{P}' \longmapsto \varphi^\vee(\mathfrak{P}'), \quad (6.6)$$

given by $(a, b) \in \mathfrak{P}$ if $(\varphi(a), \varphi(b)) \in \mathfrak{P}'$. The map ${}^a\varphi$ is well defined for \mathfrak{g} -prime congruences, preserving \mathfrak{g} -primes as well, by Remark 4.42.

Proposition 6.20. *Let \mathfrak{A} be a \mathfrak{q} -congruence on A , and let $\pi : A \rightarrow A/\mathfrak{A}$ be the canonical surjection. The map*

$${}^a\pi : \text{Spec}(A/\mathfrak{A}) \longrightarrow \text{Spec}(A), \quad \mathfrak{P}' \longmapsto \pi^\vee(\mathfrak{P}'),$$

induces a homeomorphism of topological spaces $\text{Spec}(A/\mathfrak{A}) \xrightarrow{\sim} \mathcal{V}(\mathfrak{A})$, where $\mathcal{V}(\mathfrak{A})$ is equipped with the subspace topology obtained from the Zariski topology of $\text{Spec}(A)$.

Proof. The map ${}^a\pi$ defines a bijection between $\text{Spec}(A/\mathfrak{A})$ and the subset of $\text{Spec}(A)$ consisting of all \mathfrak{g} -prime congruences which contain \mathfrak{A} , cf. Remark 4.42. Hence, it provides a bijection $\text{Spec}(A/\mathfrak{A}) \xrightarrow{\sim} \mathcal{V}(\mathfrak{A})$. Considering this map as an identification, for elements $f \in A$, we obtain

$$\text{Spec}(A) \supseteq \mathcal{D}(f) \cap \mathcal{V}(\mathfrak{A}) = \mathcal{D}(\pi(f)) \subseteq \text{Spec}(A/\mathfrak{A}).$$

Since π is surjective, for $\bar{f} \in A/\mathfrak{A}$, the sets $\mathcal{D}(\bar{f}) \subseteq \text{Spec}(A/\mathfrak{A})$ correspond bijectively to the restrictions $\mathcal{V}(\mathfrak{A}) \cap \mathcal{D}(f) \subseteq \text{Spec}(A)$ with $f \in A$, which proves the assertion. \square

Corollary 6.21. *The spectrum of a ν -semiring A and the spectrum of its reduction $A/\text{rad}_{\mathfrak{g}}(A)$ are canonically homeomorphic.*

Proof. Recall that the $\mathfrak{g}\mathfrak{p}$ -radical $\text{rad}_{\mathfrak{g}}(A)$ of A is defined as the \mathfrak{s} -radical of the ghostpotent ideal $\mathcal{N}(R) \supseteq A|_{\text{gh}}$ (Definition 4.76). Therefore, $\mathcal{V}(\text{rad}_{\mathfrak{g}}(A)) = \mathcal{V}(A|_{\text{gh}}) = \text{Spec}(A)$ by Lemma 4.80, and Proposition 6.20 applies. \square

6.2. Irreducible varieties.

To deal directly with irreducibility of ν -varieties, analogous to irreducibility over rings, in this subsection we assume that our underlining ν -semiring A is tame.

We recall the following standard definitions.

Definition 6.22. *Let $X \neq \emptyset$ be an arbitrary topological space.*

- (i) X is called **irreducible**, if any decomposition $X = X_1 \cup X_2$ into closed subsets X_1, X_2 implies $X_1 = X$ or $X_2 = X$. A subset $Y \subset X$ is irreducible, if it is irreducible under the topology induced from X on Y .
- (ii) A point $x \in X$ is **closed**, if the set $\{x\}$ is closed; x is **generic**, if $\overline{\{x\}} = X$; x is a **generalization** of a point $y \in X$, if $y \in \overline{\{x\}}$.
- (iii) A point $x \in X$ is called a **maximal point**, if its closure $\overline{\{x\}}$ is an irreducible component of X .

Thus, a point $x \in X$ is generic if and only if it is a generalization of every point of X . Since the closure of an irreducible set is again irreducible, the existence of a generic point implies that X is irreducible.

Proposition 6.23. *For the spectrum $X = \text{Spec}(A)$ of a tame ν -semiring A the following conditions are equivalent:*

- (i) X is irreducible as a topological space under the Zariski topology,
- (ii) $A/\text{rad}_{\mathfrak{g}}(A)$ is a ν -domain,
- (iii) $\text{rad}_{\mathfrak{g}}(A)$ is a \mathfrak{g} -prime congruence.

Proof. (i) \Rightarrow (ii): The ν -semiring $A/\text{rad}_{\mathfrak{g}}(A)$ is tame by Remark 4.85.(iii), so we may replace A by its reduction $A/\text{rad}_{\mathfrak{g}}(A)$, cf. Corollary 6.21, to get a reduced tame ν -semiring with $\mathcal{G} := \text{rad}_{\mathfrak{g}}(A) = A|_{\text{gh}}$. Let X be irreducible. Suppose there exist non-ghost elements $f, g \in A \setminus \mathcal{G}$ such that $fg \equiv \text{ghost}$, then, by Remark 6.5 and Proposition 6.6.(v),

$$X = \mathcal{V}(\mathcal{G}) = \mathcal{V}(fg) = \mathcal{V}(f) \cup \mathcal{V}(g).$$

Namely, X decomposes into the closed subsets $\mathcal{V}(f), \mathcal{V}(g) \subseteq X$, implying $X = \mathcal{V}(f)$ or $X = \mathcal{V}(g)$, since X is irreducible. If $\mathcal{V}(f) = X$ [= $\mathcal{V}(\mathcal{G})$], then we conclude by Proposition 6.14 that $\text{rad}_{\mathfrak{s}}(f)$ coincides with $\text{rad}_{\mathfrak{s}}(\mathcal{G}) = \text{rad}_{\mathfrak{g}}(A)$, cf. Lemma 4.80, and, hence, that $f \equiv \text{ghost}$. Similarly, $\mathcal{V}(g) = X$ implies that $g \equiv \text{ghost}$. This shows that A has no ghost divisors, i.e. $\text{gdiv}(A) = \emptyset$. Moreover, since A is tame, $A|_{\text{tng}}^{\circ} \setminus \text{gdiv}(A)$ is a tangible monoid by Lemma 3.23.(ii), and therefore A is a ν -domain.

(ii) \Leftrightarrow (iii): Immediate by Proposition 4.45.

(ii) \Rightarrow (i): Assume that A is a ν -domain and $X = X_1 \cup X_2$ is a proper decomposition of X into closed subsets X_1, X_2 . Then, for $\mathcal{G} := A|_{\text{gh}}$, Remark 6.13 and Proposition 6.14 give

$$\text{rad}_s(\mathcal{G}) = \mathcal{K}(X) = \mathcal{K}(X_1) \cap \mathcal{K}(X_2), \quad \mathcal{K}(X_1) \neq \text{rad}_s(\mathcal{G}) \neq \mathcal{K}(X_2).$$

Now, for $f_i \in \mathbb{G}_{\text{cls}}^{-1}(\mathcal{K}(X_i)) \setminus \mathcal{G}$, $i = 1, 2$, we get $f_1 f_2 \equiv \text{ghost}$ – a contradiction as A was assumed to be a ν -domain. Therefore X must be irreducible. \square

Theorem 6.24. *Let $Y \subset X = \text{Spec}(A)$ be a closed subset, where A a tame ν -semiring. Then, Y is irreducible if and only if $\mathcal{K}(Y)$ is a \mathfrak{g} -prime congruence.*

Proof. Write $\mathfrak{A} = \mathcal{K}(Y)$, then $Y = \mathcal{V}(\mathfrak{A})$ and $\mathfrak{A} = \text{rad}_c(\mathfrak{A})$, cf. (6.5) and Lemma 6.15. Thus, the homeomorphism $\text{Spec}(A/\mathfrak{A}) \xrightarrow{\sim} Y$ of Proposition 6.20 is applicable. Accordingly, Y is irreducible iff $\text{Spec}(A/\mathfrak{A})$ is irreducible iff A/\mathfrak{A} is a ν -domain (by Proposition 6.23) iff \mathfrak{A} is a \mathfrak{g} -prime congruence (by Proposition 4.45). \square

Corollary 6.25. *The mappings \mathcal{K} and \mathcal{V} between $X = \text{Spec}(A)$ and $\text{Cong}_{\mathfrak{g}}(A)$, given in Corollary 6.18, with A a tame ν -semiring, yield mutually inverse and inclusion-reversing bijections*

$$\left\{ \begin{array}{l} \text{irreducible closed} \\ \text{subsets of } X \end{array} \right\} \begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow{\mathcal{V}} \end{array} \left\{ \begin{array}{l} \mathfrak{g}\text{-prime congruences} \\ \text{on } A \end{array} \right\}, \quad \begin{array}{c} Y \\ \overline{\{y\}} \end{array} \begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow{\mathcal{V}} \end{array} \begin{array}{c} \mathcal{K}(Y) \\ \mathfrak{P}_y \end{array}.$$

Proof. This is an immediate consequence of Corollary 6.18 and Theorem 6.24, since $\mathcal{V}(\mathfrak{P}_y) = \overline{\{y\}}$ by Corollary 6.17. \square

Let $x \in X$ be a point of $X = \text{Spec}(A)$ corresponding to the \mathfrak{g} -prime congruence \mathfrak{P}_x on A . The topological notions of Definition 6.22 have the following algebraic meanings.

- (a) x is closed iff \mathfrak{P}_x is a maximal ℓ -congruence.
- (b) x is a generic point of X iff \mathfrak{P}_x is the unique minimal \mathfrak{g} -prime congruence, which exists iff the \mathfrak{gp} -radical $\text{rad}_{\mathfrak{g}}(A)$ of (a tangibly closed ν -semiring) A is a \mathfrak{g} -prime congruence. Thus, Proposition 6.23 shows that X is irreducible iff $\text{rad}_{\mathfrak{g}}(A)$ is a \mathfrak{g} -prime congruence.
- (c) x is a generization of a point $y \in X$ iff $\mathfrak{P}_x \subset \mathfrak{P}_y$.
- (d) x is a maximal point iff \mathfrak{P}_x is a minimal \mathfrak{g} -prime congruence.

The next example recovers the traditional correspondence between irreducible polynomials and irreducible hypersurfaces. This correspondence is not so evident in standard tropical geometry, whose objects are polyhedral complexes satisfying certain constraints, cf. §3.11.

Example 6.26 (Tangible hypersurfaces). *Let f be a tangible polynomial function in the tame ν -semiring $\tilde{F}[\Lambda]$, with F a ν -semifield, cf. §3.9. Write $Y := \mathcal{V}(f) = \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{G}_f \subseteq \mathfrak{P}\}$, and let $\mathcal{K}(Y) = \bigcap_{\mathfrak{P} \in Y} \mathfrak{P}$, which is the \mathfrak{s} -radical $\mathfrak{R} := \text{rad}_{\mathfrak{s}}(f)$ of f (Definition 4.70 and Proposition 6.14).*

Recall that an element f is irreducible, if it cannot be factored as a nontrivial product gh with $g, h \in \tilde{F}[\Lambda]$. If f factorises into irreducible polynomial functions $f = g_1^{m_1} \cdots g_s^{m_s}$, then $\text{rad}_{\mathfrak{s}}(f) = \text{rad}_{\mathfrak{s}}(g_1^{m_1} \cdots g_s^{m_s})$, cf. Lemma 4.74, which is \mathfrak{g} -prime iff $s = 1$, i.e., $f = g_1^{m_1}$, where f is not irreducible if $m_1 > 1$. Therefore, when f is irreducible, $\text{rad}_{\mathfrak{s}}(f)$ is a \mathfrak{g} -prime congruence, and by Proposition 6.24 we conclude that Y is an irreducible ν -variety.

The example assumes that the polynomial function f is tangible, which thereby captures all standard tropical hypersurfaces via Example 3.26, see also parts (i) and (ii) of Example 3.55. However, in general, irreducibility of a function $f \in \tilde{F}[\Lambda]$ does not imply irreducibility of its ν -variety, for example take f as in Example 3.49.(ii).

Example 6.27. *The spectrum $\mathbb{A}_R = \text{Spec}(A)$ of the ν -semiring $A = \tilde{R}[\lambda]$ of polynomial functions in variable λ over a ν -semiring R is referred to as the **affine line** over R , viewed as an object over $\text{Spec}(R)$ via the projection $\mathbb{A}_R \rightarrow \text{Spec}(R)$ induced by the canonical injection $R \hookrightarrow \tilde{R}[\lambda]$.*

6.3. Homomorphisms and functorial properties.

Let $\varphi : A \longrightarrow B$ be a \mathfrak{q} -homomorphism of ν -semirings. By Remark 2.3.(iii), φ determines for any \mathfrak{q} -congruence \mathfrak{A}_b on B a \mathfrak{q} -homomorphism $A/\varphi^\vee(\mathfrak{A}_b) \hookrightarrow B/\mathfrak{A}_b$. In the case of a \mathfrak{g} -prime congruence $\mathfrak{P}_y \in \text{Spec}(B)$ on a ν -semiring B , the factor ν -semiring B/\mathfrak{P}_y is a ν -domain (Proposition 4.45). Moreover, since φ is a \mathfrak{q} -homomorphism, φ induces the map φ^\vee from \mathfrak{q} -congruences on B to \mathfrak{q} -congruences on A which preserves intersection, i.e.,

$$\varphi^\vee(\mathfrak{A}'_b \cap \mathfrak{A}''_b) = \varphi^\vee(\mathfrak{A}'_b) \cap \varphi^\vee(\mathfrak{A}''_b). \quad (6.7)$$

Recall that φ^\vee defines the map (6.6) of spectra, written shortly as

$${}^a\varphi : Y \longrightarrow X, \quad \mathfrak{P}_y \longmapsto \varphi^\vee(\mathfrak{P}_y), \quad (6.8)$$

where $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$.

Lemma 6.28. *The following diagram commutes for any $y \in \text{Spec}(B)$*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow \pi & & \downarrow \pi_{\mathfrak{P}_y} \\ A/\varphi^\vee(\mathfrak{P}_y) & \xrightarrow{\varphi_x} & B/\mathfrak{P}_y \\ \downarrow & & \downarrow \\ Q(A/\varphi^\vee(\mathfrak{P}_y)) & \xrightarrow{\varphi_x} & Q(B/\mathfrak{P}_y) \end{array}$$

In particular, for $f \in A$, we have

$$\varphi_x(f({}^a\varphi(y))) = \varphi(f)(y),$$

i.e., $f \circ {}^a\varphi = \varphi(f)$.

Recall that $f \in A$ is viewed as a map $f : \mathfrak{P} \longmapsto [f]$ on $\text{Spec}(A)$, cf. (6.1).

Proof. The map ${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ in (6.8) gives the upper part of the diagram, whereas the lower part is the canonical extension to residue ν -semiring, cf. the beginning of §6.1. \square

Proposition 6.29. *Let $\varphi : A \longrightarrow B$ be a \mathfrak{q} -homomorphism, and let ${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ be the associated map of spectra. Then,*

- (i) $({}^a\varphi)^{-1}(\mathcal{V}(E)) = \mathcal{V}(\varphi(E))$ for any subset $E \subset A$,
- (ii) $\overline{{}^a\varphi(\mathcal{V}(\mathfrak{A}_b))} = \mathcal{V}(\varphi^\vee(\mathfrak{A}_b))$ for any congruence \mathfrak{A}_b on B .

Proof. (i): The relation $y \in ({}^a\varphi)^{-1}(\mathcal{V}(E))$ for $y \in \text{Spec}(B)$ is equivalent to ${}^a\varphi(y) \in \mathcal{V}(E)$, and hence to $E \subseteq \mathbf{G}_{\text{cls}}^{-1}({}^a\varphi(y))$. Thus $f \in \mathbf{G}_{\text{cls}}^{-1}({}^a\varphi(y))$ for all $f \in E$. In functional interpretation, cf. the beginning of §6.1, this means that $f({}^a\varphi(y)) \in (A/{}^a\varphi(y))|_{\text{gh}}$ for all $f \in E$, which by Lemma 6.28 is equivalent to $\varphi(f)(y) \in (A/{}^a\varphi(y))|_{\text{gh}}$. Thus, $y \in ({}^a\varphi)^{-1}(\mathcal{V}(E))$ is equivalent to $y \in \mathcal{V}(\varphi(E))$.

(ii): By Proposition 6.14.(ii), $\overline{{}^a\varphi(\mathcal{V}(\mathfrak{A}_b))}$ can be written as $\mathcal{V}(\mathcal{K}({}^a\varphi(\mathcal{V}(\mathfrak{A}_b))))$. Using (6.7), we have

$$\mathcal{K}({}^a\varphi(\mathcal{V}(\mathfrak{A}_b))) = \bigcap_{\mathfrak{P}_y \in \mathcal{V}(\mathfrak{A}_b)} \varphi^\vee(\mathfrak{P}_y) = \varphi^\vee \left(\bigcap_{\mathfrak{P}_y \in \mathcal{V}(\mathfrak{A}_b)} \mathfrak{P}_y \right) = \varphi^\vee(\text{rad}_c(\mathfrak{A}_b)) = \text{rad}_c(\varphi^\vee(\mathfrak{A}_b)),$$

and, in view of Lemma 6.15, the claim follows by applying $\mathcal{V}(_)$. \square

Corollary 6.30. *The map ${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ associated to a \mathfrak{q} -homomorphism $\varphi : A \longrightarrow B$ is continuous with respect to Zariski topologies on $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, i.e., the preimage of any open (resp. closed) subset in X is open (resp. closed) in Y with*

$$({}^a\varphi)^{-1}(\mathcal{D}(f)) = \mathcal{D}(\varphi(f)), \quad ({}^a\varphi)^{-1}(\mathcal{V}(f)) = \mathcal{V}(\varphi(f)),$$

for every $f \in A$.

Proof. The former holds as the preimages of ${}^a\varphi$ is compatible with passing to complements. The latter follows from Proposition 6.29. \square

Next, we focus on a \mathfrak{q} -homomorphism $\varphi : A \rightarrow B$ whose associated map ${}^a\varphi$ is injective, and therefore defines an isomorphism between $\text{Spec}(B)$ and $\text{im}({}^a\varphi)$.

Proposition 6.31. *Let $\varphi : A \rightarrow B$ be a \mathfrak{q} -homomorphism such that every $f' \in B$ can be written as $f' = \varphi(f)h'$ for some $f \in A$ and a unit $h' \in B^\times$. Then, ${}^a\varphi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is injective and defines a homomorphism $\text{Spec}(B) \xrightarrow{\sim} \text{im}({}^a\varphi) \subset \text{Spec}(A)$, where $\text{im}({}^a\varphi)$ is equipped with the subspace topology induced from the Zariski topology on $\text{Spec}(A)$.*

Proof. Write $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Assume that $y', y'' \in Y$ satisfy ${}^a\varphi(y') = {}^a\varphi(y'')$, namely $\varphi^\vee(\mathfrak{P}_{y'}) = \varphi^\vee(\mathfrak{P}_{y''})$. We claim that $\mathfrak{P}_{y'} = \mathfrak{P}_{y''}$, that is $y' = y''$. Indeed, given $(f'_1, f'_2) \in \mathfrak{P}_{y'}$, there exist $f_1, f_2 \in A$ and $h'_1, h'_2 \in B^\times$ such that $(f'_1, f'_2) = (\varphi(f_1)h'_1, \varphi(f_2)h'_2)$. Then, $(\varphi(f_1), \varphi(f_2)) = (f'_1 h'^{-1}_1, f'_2 h'^{-1}_2) \in \mathfrak{P}_{y'}$, and hence $(f_1, f_2) \in \varphi^\vee(\mathfrak{P}_{y'}) = \varphi^\vee(\mathfrak{P}_{y''})$. This implies $(\varphi(f_1), \varphi(f_2)) \in \mathfrak{P}_{y''}$ and therefore $(f'_1, f'_2) = (\varphi(f_1)h'_1, \varphi(f_2)h'_2) \in \mathfrak{P}_{y''}$, so that $\mathfrak{P}_{y'} \subseteq \mathfrak{P}_{y''}$. By symmetry $\mathfrak{P}_{y''} \subseteq \mathfrak{P}_{y'}$, implying that $\mathfrak{P}_{y'} = \mathfrak{P}_{y''}$. Hence $y' = y''$, and thus ${}^a\varphi$ is injective.

Recall that a subset $U \subseteq \text{im}({}^a\varphi)$ is closed (resp. open) with respect to the subspace topology of $\text{im}({}^a\varphi)$ iff there is a closed (resp. open) set $\hat{U} \subseteq X$ such that $U = \hat{U} \cap \text{im}({}^a\varphi)$. Since ${}^a\varphi : Y \rightarrow X$ is continuous by Corollary 6.30, clearly also the induced bijection ${}^a\tilde{\varphi} : Y \rightarrow \text{im}({}^a\varphi)$ is continuous. We next verify that ${}^a\tilde{\varphi}$ is a homeomorphism, i.e., that for any closed subset $V \subset Y$ there is a closed subset $U \subset X$ such that $V = ({}^a\varphi)^{-1}(U)$. If $V = \mathcal{V}(E')$ for some subset $E' \subseteq B$, adjusting the elements of E' by suitable units in B^\times , we may assume that $E' \subseteq \varphi(A)$. Then, taking $E' = \varphi(E)$ for some $E \subseteq A$, Proposition 6.29 gives

$$V = \mathcal{V}(E') = \mathcal{V}(\varphi(E)) = ({}^a\varphi)^{-1}(\mathcal{V}(E)) = ({}^a\varphi)^{-1}(U),$$

with $U = \mathcal{V}(E)$, as required. \square

Corollary 6.32. *Let \mathfrak{A} be a \mathfrak{q} -congruence on A , and let $\pi : A \rightarrow A/\mathfrak{A}$ be the canonical projection. The map ${}^a\pi : \text{Spec}(A/\mathfrak{A}) \rightarrow \text{Spec}(A)$ defines a **closed immersion of spectra**, i.e., a homeomorphism*

$$\text{Spec}(A/\mathfrak{A}) \xrightarrow{\sim} \mathcal{V}(\mathfrak{A}) \subset \text{Spec}(A).$$

Proof. Proved in Proposition 6.20. Alternatively, $\text{im}({}^a\pi) = \mathcal{V}(\mathfrak{A})$ as follows from Proposition 6.31. \square

Recall that the tangible cluster and the ghost cluster of a congruence on A are disjoint, but need not be the complement of each other, and so does their projections. Recall also that an open set $\mathcal{D}(f)$ in $\text{Spec}(A)$ may contain points over which f is not tangibly evaluated. To cope with this discrepancy, we designate several special subsets within open sets of $\text{Spec}(A)$, as described below.

The **tangible support** of an element $f \in A$ is defined as

$$\mathcal{E}(f) := \{\mathfrak{P} \in \text{Spec}(A) \mid f \in \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})\} \subseteq \mathcal{D}(f). \quad (6.9)$$

That is, $\mathcal{E}(f)$ is the subset of $\mathcal{D}(f)$ on which, as a function (6.1), f takes tangible values, cf. §6.1. $\mathcal{E}(f)$ is nonempty for every \mathfrak{t} -persistent $f \in A|_{\text{tng}}^\circ$, by Lemma 4.84, where $\mathcal{E}(f) = \text{Spec}(A)$ for every $f \in A^\times$. The latter also holds for all \mathfrak{t} -unalterable elements $f \in A|_{\text{ual}}$ (Definition 4.23), for which $\mathcal{D}(f) = \mathcal{E}(f)$. On the other hand, by definition, $\mathcal{E}(f) = \emptyset$ for any $f \notin A|_{\text{tng}}^\circ$, since every \mathfrak{P} is an ℓ -congruence. Furthermore, $\mathcal{D}(f) \cap \mathcal{D}(g) = \mathcal{D}(fg)$ by Corollary 6.8, and hence

$$\mathcal{E}(f) \cap \mathcal{E}(g) = \mathcal{E}(fg) \quad (6.10)$$

for any $f, g \in A$. Tangible supports are generalized to (nonempty) subsets $E \subseteq A$ by defining

$$\mathcal{E}(E) := \{\mathfrak{P} \in \text{Spec}(A) \mid E \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})\} = \bigcap_{h \in E} \mathcal{E}(h). \quad (6.11)$$

For these tangible supports, Corollary 6.30 specializes further.

Proposition 6.33. *Let $\varphi : A \rightarrow B$ be a \mathfrak{q} -homomorphism, and let ${}^a\varphi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the associated map of spectra. Then, $({}^a\varphi)^{-1}(\mathcal{E}(E)) = \mathcal{E}(\varphi(E))$, for any subset $E \subseteq A$,*

Proof. The relation $y \in ({}^a\varphi)^{-1}(\mathcal{E}(E))$ for $y \in \text{Spec}(B)$ is equivalent to ${}^a\varphi(y) \in \mathcal{E}(E)$, and hence to $E \subseteq \mathbb{T}_{\text{cls}}^{-1}({}^a\varphi(y))$. Therefore, $f \in \mathbb{T}_{\text{cls}}^{-1}({}^a\varphi(y))$ for all $f \in E$. This means that $f({}^a\varphi(y)) \in (A/{}^a\varphi(y))|_{\text{tng}}$ for all $f \in E$, which by Lemma 6.28 is equivalent to $\varphi(f)(y) \in (A/{}^a\varphi(y))|_{\text{tng}}$. Thus, $y \in ({}^a\varphi)^{-1}(\mathcal{E}(E))$ is equivalent to $y \in \mathcal{E}(\varphi(E))$, since φ is a \mathfrak{q} -homomorphism. \square

Given an open set $\mathcal{D}(f)$ and a tangible multiplicative monoid $C \subseteq A|_{\text{tng}}$ of A (accordingly C must consist of \mathfrak{t} -persistent elements and thus $C \subseteq A|_{\text{tng}}^\circ$), we define the **restricted subset**

$$\mathcal{D}(C, f) := \{\mathfrak{P} \in \mathcal{D}(f) \mid C \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})\} \subseteq \mathcal{D}(f). \quad (6.12)$$

In other words, $\mathcal{D}(C, f)$ can be written in terms of (6.11) as

$$\mathcal{D}(C, f) = \mathcal{D}(f) \cap \mathcal{E}(C) = \mathcal{D}(f) \cap \bigcap_{h \in C} \mathcal{E}(h), \quad (6.13)$$

i.e., the restriction of $\mathcal{D}(f)$ to the tangible support of C , which could be empty. In general, f need not be contained in C nor in the tangible projections $\mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$ of $\mathfrak{P} \in \mathcal{D}(C, f)$.

Properties 6.34. *For a restricted subset $\mathcal{D}(C, f)$ we have the following properties.*

- (a) $\mathcal{D}(C, f) = \mathcal{D}(f)$ for $C \subset A|_{\text{ual}}$.
- (b) $\mathcal{D}(C, f) = \mathcal{E}(C)$ for any C and $f \in A|_{\text{ual}}$.
- (c) $\mathcal{D}(C, f) = \mathcal{E}(C) = \mathcal{D}(f) = \text{Spec}(A)$ for $f \in A|_{\text{ual}}$ and $C \subset A|_{\text{ual}}$.
- (d) If $f \in C$, then $\mathcal{D}(C, f) = \mathcal{E}(C)$.
- (e) $\mathcal{D}(C, f) = \mathcal{D}(C', f)$ if the generating sets of C and C' are different by units.

Although, in general, f is independent on C , we still have the following observation.

Lemma 6.35. *$\mathcal{D}(C, f)$ is nonempty for any non-ghostpotent $f \notin \mathcal{N}(A)$ and $C \subseteq A|_{\text{tng}}^\circ$.*

Proof. Let A_C be the tangible localization of A by C , in which $\frac{f}{1}$ is not a ghost. By Corollary 4.83 there exists a \mathfrak{g} -prime congruence \mathfrak{P}' on A_C for $\frac{f}{1} \notin \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P}')$. Then, by Proposition 4.47.(ii), the restriction of \mathfrak{P}' to A gives a \mathfrak{g} -prime congruence \mathfrak{P} with $C \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$ and $f \notin \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P})$. Hence, \mathfrak{P} belongs to $\mathcal{D}(C, f)$ by (6.12), and $\mathcal{D}(C, f)$ is nonempty. \square

Using the restricted subsets (6.12), we have the next analogy to Corollary 6.32, now referring to localization.

Corollary 6.36. *Let $C \subseteq A|_{\text{tng}}^\circ$ be a tangible multiplicative submonoid of a ν -semiring A . The canonical \mathfrak{q} -homomorphism $\tau : A \rightarrow A_C$ induces a homomorphism*

$${}^a\tau : \text{Spec}(A_C) \xrightarrow{\sim} \bigcap_{h \in C} \mathcal{D}(C, h) = \bigcap_{h \in C} \mathcal{E}(h) = \mathcal{E}(C). \quad (6.14)$$

Thus, ${}^a\tau : \text{Spec}(A_C) \hookrightarrow \bigcap_{h \in C} \mathcal{D}(h)$ is an injective homeomorphism.¹⁵

Proof. Using Proposition 6.31, we only need to determine the image $\text{im}({}^a\tau)$ of ${}^a\tau$. Recall from Propositions 4.38 and 4.47 that taking inverse images with respect to $\tau : A \rightarrow A_C$ yields a bijective correspondence between all \mathfrak{g} -prime congruences on A_C and the \mathfrak{g} -prime congruences \mathfrak{P} on A satisfying $C \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$. But, for a point $x \in \text{Spec}(A)$, we have $C \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ iff $x \in \mathcal{D}(C, h)$ for all $h \in C$, so that $\text{im}({}^a\tau) = \bigcap_{h \in C} \mathcal{D}(C, h)$. The latter equalities follow from (6.11) and (6.13). \square

If $\text{im}({}^a\tau)$ is open in $\text{Spec}(A)$, i.e., when $\text{im}({}^a\tau) = \bigcap_{h \in C} \mathcal{D}(h)$, then ${}^a\tau$ is called an **open immersion** of spectra. For example, ${}^a\tau$ is an open immersion for a finitely generated tangible monoid $C \subseteq A|_{\text{ual}}$ of \mathfrak{t} -unalterable elements, say by $h_1, \dots, h_\ell \in A^\times$, since $\bigcap_{h \in C} \mathcal{D}(C, h) = \bigcap_{h \in C} \mathcal{D}(h) = \mathcal{D}(h_1 \cdots h_\ell)$.

Remark 6.37. *The occurrence of open immersions of \mathfrak{g} -prime spectra might be rare, as it appears when $\mathcal{D}(C, h) = \mathcal{D}(h)$ for all $h \in C$, e.g., for units of A , cf. Properties 6.34.(b). An open subset $\mathcal{D}(h)$ may contain \mathfrak{g} -primes over which h does not possess tangibles, even if h is \mathfrak{t} -persistent. In contrary, localization is performed only upon \mathfrak{t} -persistents. As a consequence of that a \mathfrak{g} -prime congruence in $\bigcap_{h \in C} \mathcal{D}(h)$ which includes equivalences of $h \in C$ to non-tangibles (or non- \mathfrak{t} -persistents) does not necessarily have a pre-image in $\text{Spec}(A_C)$ under the map ${}^a\tau$ in (6.14).*

From Remark 4.27 it follows that (6.14) is an open immersion for any subgroup $C \subseteq A^\times \subseteq A|_{\text{ual}}$ of units, since any \mathfrak{g} -prime congruence \mathfrak{P} is an ℓ -congruence, in which \mathfrak{t} -unalterables are congruent only to \mathfrak{t} -persistent elements; in particular $A|_{\text{ual}} \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P})$.

¹⁵This is the place where our theory slightly deviates from classical theory in which this homeomorphism is bijective. However, later we are mainly interested in the converse homomorphism, and show that it is a surjective homeomorphism.

In the extreme case that $C = \mathcal{T}$, where $\mathcal{T} := A|_{\text{tng}}$ is a monoid, then the ν -semiring A is tangibly closed, $A_{\mathcal{T}} = Q(A)$, and

$${}^a\tau : \text{Spec}(Q(A)) \xrightarrow{\sim} \bigcap_{h \in \mathcal{T}} \mathcal{D}(\mathcal{T}, h) = \{\mathfrak{P} \in \text{Spec}(A) \mid \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}) = A|_{\text{tng}}\}.$$

We denote the subset $\{\mathfrak{P} \in \text{Spec}(A) \mid \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}) = A|_{\text{tng}}\}$ by $\text{Spt}(A)$.

7. SHEAVES

We recall the classical setup of sheaves over a topological space X , applied here to ν -semirings. Later, some of the below objects are slightly generalized, making them applicable for our framework. In this framework standard objects may have different interpretations. Yet, our forthcoming abstraction well-captures the familiar notions, as described next.

Definition 7.1. A **presheaf** \mathcal{F} of ν -semirings on a topological space X consists of the data $(U, V, W \subseteq X$ are open sets):

- (a) A ν -semiring $\mathcal{F}(U)$ for every U ;
- (b) A restriction map $\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ for each pair $U \subset V$, such that $\rho_U^U = \text{id}_{\mathcal{F}(U)}$ for any U , and $\rho_U^W = \rho_U^V \circ \rho_V^W$ for any $U \subseteq V \subseteq W$.

The elements σ of $\mathcal{F}(U)$ are called the **sections** of \mathcal{F} over U .

A presheaf \mathcal{F} is a **sheaf**, if for any coverings $U = \bigcup_i U_i$ by open sets $U_i \subset X$ the following hold:

- (c) If $\sigma, \sigma' \in \mathcal{F}(U)$ with $\sigma|_{U_i} = \sigma'|_{U_i}$ for all i , then $\sigma = \sigma'$.
- (d) If $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for any $\sigma_i \in \mathcal{F}(U_i)$, $\sigma_j \in \mathcal{F}(U_j)$, then there exists $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ (which is unique by (c)).

A **morphism** of presheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a family of maps $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, for all $U \subseteq X$ open, such that $\rho_U^V \circ \phi_V = \phi_U \circ \rho_U^V$ for all pairs $U \subset V$ from X .

The **stalk** of a sheaf \mathcal{F} at a point $x \in X$ is the inductive limit (i.e., colimit)

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U). \quad (7.1)$$

That is, \mathcal{F}_x is the set of equivalence classes of pairs (U, σ) , where U is an open neighborhood of x and $\sigma \in \mathcal{F}(U)$, such that (U_1, σ_1) and (U_2, σ_2) are equivalent if $\sigma_1|_V = \sigma_2|_V$ for some open neighborhood $V \subseteq U_1 \cap U_2$ of x . For each open neighborhood U of x there is the canonical map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_x, \quad \sigma \longmapsto \sigma_x,$$

sending $\sigma \in \mathcal{F}(U)$ to the class σ_x of (U, σ) in \mathcal{F}_x , called the **germ** of σ at x . A standard proof shows that the map

$$\mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x, \quad f \longmapsto (f_x)_{x \in U},$$

is injective for any open set $U \subset X$.

A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on X induces a map of the stalks at x

$$\varphi_x := \varinjlim_{U \ni x} \varphi_U : \mathcal{F}_x \longrightarrow \mathcal{G}_x,$$

providing a functor $\mathcal{F} \rightarrow \mathcal{F}_x$ from the category of sheaves on X to the category of sets.

Given a continuous map $\phi : X \rightarrow Y$ of topological spaces and a sheaf \mathcal{F} on X , the **direct image** of \mathcal{F} is the sheaf $\phi_*\mathcal{F}$ on Y , defined for open subsets $V \subset Y$ by

$$(\phi_*\mathcal{F})(V) = \mathcal{F}(\phi^{-1}(V)),$$

whose restriction maps are inherited from \mathcal{F} . For a morphism $\psi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves, the family of maps $\phi_*(\psi)_V := \psi_{\phi^{-1}(V)}$, with $V \subset Y$ open, is a morphism $\phi_*(\psi) : \phi_*\mathcal{F} \rightarrow \phi_*\mathcal{G}$. Therefore, ϕ_* is a functor from the category of sheaves on X to the category of sheaves on Y that admits composition $\psi_*(\phi_*\mathcal{F}) = (\psi \circ \phi)_*\mathcal{F}$ for a continuous map $\psi : Y \rightarrow Z$.

Henceforth, unless otherwise is indicated, **we assume that A is a tame ν -semiring**, i.e., every $f \in A \setminus (A|_{\text{tng}} \cup A|_{\text{gh}})$ can be written as $f = p + eq$ for some $p, q \in A|_{\text{tng}}$ (Definition 3.11). This is a very mild assumption, holds in all of our examples, especially in Examples 3.25 and 3.26. Recall from

Lemma 3.23 that, since A is tame, $A|_{\text{tng}}^{\circ} \setminus \text{gdiv}(A)$ is a monoid of \mathfrak{t} -persistent elements. We denote this monoid by $A|_{\text{tng}}^*$, for short. Recall from §3.4 that $A^{\times} \subseteq A|_{\text{tng}}^*$.

7.1. Functoriality towards sheaves.

Sheaves have characteristic functorial properties which are shown later to fit well our constructions. We begin with a \mathfrak{t} -persistent element $f \in A|_{\text{tng}}^{\circ}$, and write

$$C(f) := \{\mathbb{1}, f, f^2, \dots\} \quad (7.2)$$

for the tangible monoid generated by f . For an element $f \notin A|_{\text{tng}}^{\circ}$, i.e., is not \mathfrak{t} -persistent, we formally set $C(f) = \{\mathbb{1}\}$. For this monoid we have the inclusion $C(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$ in every $\mathfrak{P} \in \mathcal{E}(f)$ where $\mathcal{E}(f) = \mathcal{D}(C(f), f)$ if $f \in C(f)$, cf. (6.9) and (6.12). We write A_f for the tangible localization $A_{C(f)}$ of A by $C(f)$, cf. §3.8.

Remark 7.2. *Given $f \in A|_{\text{tng}}^{\circ}$, we assign the set $\mathcal{D}(f) \subset \text{Spec}(A)$ with the localization A_f of A by $C(f)$. Then, for $h \in A|_{\text{tng}}^{\circ}$, the inclusion $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ gives rise to a ν -semiring \mathfrak{q} -homomorphism $\tau_f^h : A_h \rightarrow A_f$. Indeed, $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ is equivalent to $\mathcal{V}(f) \supseteq \mathcal{V}(h)$ and to $\text{rad}_s(f) \subseteq \text{rad}_s(h)$ by Lemma 6.11.(iv). Furthermore, $h \in A|_{\text{tng}}^{\circ}$, and hence, by Lemma 4.74, $f^n = hg$ for some $n \in \mathbb{N}$ and $g \in A$.*

The canonical map $\tau_f : A \rightarrow A_f$ shows that $\tau_f(f)^n = \tau_f(h)\tau_f(g)$, and hence $\tau_f(h)$ is a unit in A_f . Thus, by the universal property of localization (Proposition 3.41), the \mathfrak{q} -homomorphism $\tau_f : A \rightarrow A_f$ factorizes uniquely as $\tau_f = \tau_h \circ \tau_f^h$ through the \mathfrak{q} -homomorphisms $\tau_h : A \rightarrow A_h$ and $\tau_f^h : A_h \rightarrow A_f$, i.e., the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tau_f} & A_f \\ \tau_h \downarrow & \dashrightarrow & \tau_f^h \\ A_h & & \end{array}$$

commutes. Then, in functorial sense, $\tau_f^h : A_h \rightarrow A_f$ is assigned to the inclusion $\mathcal{D}(f) \subseteq \mathcal{D}(h)$. Also, for every $f \in A|_{\text{tng}}^{\circ}$ the map $\tau_f^f : A_f \rightarrow A_f$ is the identity map, and for any inclusions $\mathcal{D}(f) \subset \mathcal{D}(g) \subset \mathcal{D}(h)$, the map composition $A_h \xrightarrow{\tau_g^h} A_g \xrightarrow{\tau_f^g} A_f$ coincides with $\tau_f^h : A_h \rightarrow A_f$.

Besides basic open sets $\mathcal{D}(f)$ with f \mathfrak{t} -persistent, our topological space also includes sets $\mathcal{D}(f)$ which are determined by elements $f \notin A|_{\text{tng}}^{\circ}$. Each such $\mathcal{D}(f)$ should be allocated with a ν -semiring. To cope with this type of open sets, in a way that is compatible with the case of $f \in A|_{\text{tng}}^{\circ}$, for every $f \in A$ we define the subset

$$S^*(f) := \{h \in A|_{\text{tng}}^* \mid \mathcal{D}(f) \subseteq \mathcal{D}(h)\} \subseteq A|_{\text{tng}}^*, \quad (7.3)$$

which consists of \mathfrak{t} -persistent elements which are not ghost divisors. We then let

$$S(f) := \langle C(f), S^*(f) \rangle \subseteq A|_{\text{tng}}^{\circ}, \quad (7.4)$$

be the set multiplicatively generated by $C(f)$ and $S^*(f)$. Accordingly, $S^*(f) = S(f)$ when $f \notin A|_{\text{tng}}^{\circ}$, or when $f \in A|_{\text{tng}}^*$.

Lemma 7.3. *For any $f \in A$, the subset $S(f)$ is a tangible multiplicative submonoid of A*

Proof. $S(f)$ is nonempty, as $\mathbb{1} \in A|_{\text{tng}}^*$ and $\mathcal{D}(f) \subseteq \mathcal{D}(\mathbb{1})$ for every $f \in A$. If $\mathcal{D}(f) \subseteq \mathcal{D}(h')$ and $\mathcal{D}(f) \subseteq \mathcal{D}(h'')$ for $h', h'' \in S(f)$, then $\mathcal{D}(f) \subseteq \mathcal{D}(h') \cap \mathcal{D}(h'') = \mathcal{D}(h'h'')$ by Corollary 6.8.(i).

If $f \notin A|_{\text{tng}}^{\circ}$, then both $h', h'' \in S^*(f)$. But $A|_{\text{tng}}^*$ is a tangible monoid by Lemma 3.23.(ii), since A is a tame ν -semiring, and thus $h'h'' \in S^*(f) \subseteq A|_{\text{tng}}^*$, implying that $S^*(f) = S(f)$ is a multiplicative submonoid of $A|_{\text{tng}}^* \subseteq A|_{\text{tng}}^{\circ}$. (The same holds for $f \in A|_{\text{tng}}^{\circ}$.)

If $f \in A|_{\text{tng}}^{\circ}$, i.e., f is \mathfrak{t} -persistent, then $h'h'' \in A|_{\text{tng}}^{\circ}$ by Lemma 6.11.(vii), since $\mathcal{D}(f) \subseteq \mathcal{D}(h'h'')$. Thus, $S(f)$ is a multiplicative submonoid of $A|_{\text{tng}}^{\circ}$. \square

Properties 7.4. *Let f be an element of A .*

- (a) *All units $h \in A^{\times}$ are contained in $S(f)$, in particular $\mathbb{1} \in S(f)$.*
- (b) *$S(f) = A|_{\text{tng}}^*$ for any ghostpotent $f \in \mathcal{N}(A)$, since $\mathcal{D}(f) = \emptyset$ (see Lemma 3.23.(ii)).*

- (c) $S(f) = A^\times$ for every unit $f \in A^\times \cap A|_{\text{tng}}^\circ$, since $\mathcal{D}(f) = \text{Spec}(A)$ (cf. Lemma 6.11.(v)).
- (d) When $f \in A|_{\text{tng}}^\circ$ is \mathfrak{t} -persistent, $C(f) \subseteq S(f)$, and $S(f)$ contains all powers of f .
- (e) If $f \notin A|_{\text{tng}}^\circ$, then $S(f)$ does not contain f , even if $f \in A|_{\text{tng}}$. But, it contains every $h \in A|_{\text{tng}}^*$ such that $h \cong_\nu f$, if exist.
- (f) $S(h) \subseteq S(f)$ for every $h \in S(f)$.

Note that $\mathcal{D}(f) \subseteq \mathcal{D}(g)$ is equivalent to $S(f) \supseteq S(g)$, and hence to

$$g \equiv_{\mathfrak{p}} \text{ghost} \quad \Rightarrow \quad f \equiv_{\mathfrak{p}} \text{ghost} , \quad (7.5)$$

in every $\mathfrak{P} \in \text{Spec}(A)$.

Remark 7.5. $S(f) \cap S(g) \subseteq S(fg)$ for any $f, g \in A$. Indeed, $h \in S(f) \cap S(g)$ means that $\mathcal{D}(h) \supseteq \mathcal{D}(f)$ and $\mathcal{D}(h) \supseteq \mathcal{D}(g)$, and thus $\mathcal{D}(h) \supseteq \mathcal{D}(f) \cup \mathcal{D}(g) \supseteq \mathcal{D}(f) \cap \mathcal{D}(g) = \mathcal{D}(fg)$, by Corollary 6.8.(i). Moreover, $S(fg) \supseteq S(f)$, since $\mathcal{D}(fg) \subseteq \mathcal{D}(f)$, and, by symmetry, also $S(fg) \supseteq S(g)$, hence $S(fg) \supseteq S(f) \cup S(g)$.

The inclusion $C(f) \subseteq S(f)$ for \mathfrak{t} -persistent elements from Properties 7.4.(d) gives rise to an isomorphism of localized ν -semirings.

Lemma 7.6. For $f \in A|_{\text{tng}}^\circ$, the map $\iota_f : A_f \longrightarrow A_{S(f)}$ is an isomorphism.

Proof. Let $\tau_f : A \longrightarrow A_f$ be the canonical injection, and let $h \in S(f)$. Then, $\tau_f(h) \in A_f^\times$ is a unit, since $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ by Remark 7.2. Thus, τ_f maps $S(f)$ into the group of units in A_f , and hence $\iota_f : A_f \longrightarrow A_{S(f)}$ is an isomorphism by the universal property of localization (Proposition 3.41). \square

Remark 7.7. Given $f \in \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, and hence $f \in A|_{\text{tng}}^\circ$, we have the injection $A_f \hookrightarrow A_{\mathfrak{P}}$. Then Lemma 7.6 gives the injection $A_{S(f)} \hookrightarrow A_{\mathfrak{P}}$, which implies that $S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$.

We can now extend Remark 7.2 to all elements in A , to obtain functoriality of arbitrary open sets $\mathcal{D}(f)$.

Lemma 7.8. The inclusion $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ gives rise to a \mathfrak{q} -homomorphism

$$\tilde{\tau}_f^h : A_{S(h)} \longrightarrow A_{S(f)}.$$

Furthermore, if $f, h \in A|_{\text{tng}}^\circ$, then $\tilde{\tau}_f^h = \iota_f \circ \tau_f^h$.

Proof. $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ implies $S(h) \subseteq S(f)$, whose elements are all \mathfrak{t} -persistents. Thereby, the canonical \mathfrak{q} -homomorphism $\tilde{\tau}_f : A \longrightarrow A_{S(f)}$ maps $S(h)$ to units of $A_{S(f)}$, i.e., $\tilde{\tau}_f(S(h)) \subseteq (A_{S(f)})^\times$. Then, by the universal property of localization (Proposition 3.41), $\tilde{\tau}_f$ factorizes uniquely as $\tilde{\tau}_f = \tilde{\tau}_h \circ \tilde{\tau}_f^h$ through the \mathfrak{q} -homomorphisms $\tilde{\tau}_h : A \longrightarrow A_{S(h)}$ and $\tilde{\tau}_f^h : A_{S(h)} \longrightarrow A_{S(f)}$, rendering the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\tau}_f} & A_{S(f)} \\ \tilde{\tau}_h \downarrow & \nearrow \tilde{\tau}_f^h & \\ A_{S(h)} & & \end{array}$$

commutative. The relation $\tilde{\tau}_f^h = \iota_f \circ \tau_f^h$ is obtained from Remark 7.2 and Lemma 7.6. \square

We see that for every $f \in A$ the map $\tilde{\tau}_f^f : A_{S(f)} \longrightarrow A_{S(f)}$ is the identity map, and for any inclusions $\mathcal{D}(f) \subset \mathcal{D}(g) \subset \mathcal{D}(h)$, the map compositions

$$A_{S(h)} \xrightarrow{\tilde{\tau}_g^h} A_{S(g)} \xrightarrow{\tilde{\tau}_f^g} A_{S(f)}$$

coincides with $\tilde{\tau}_f^h : A_{S(h)} \longrightarrow A_{S(f)}$.

For a ghostpotent $f \in \mathcal{N}(A)$ we have $\mathcal{D}(f) = \emptyset$ and thus $S(f) = A|_{\text{tng}}^*$, cf. Properties 7.4.(b). Thereby, the localized ν -semiring $A_{S(f)}$ is the same for all $f \in \mathcal{N}(A)$, and for any $h \in A$ the \mathfrak{q} -homomorphism $\tilde{\tau}_f^h : A_{S(h)} \longrightarrow A_{S(f)}$ can be taken to be the identity map. However, we are mostly interested in elements $f \notin \mathcal{N}(A)$, for which $\mathcal{D}(f) \neq \emptyset$.

Let A be a tame ν -semiring, and let $X = \text{Spec}(A)$ be its \mathfrak{g} -prime spectrum (Definition 4.40). Denote by $\mathcal{D}(X)$ the category whose objects are open subsets $\mathcal{D}(f) \subseteq X$, with $f \in A$, and its morphisms are inclusions $\mathcal{D}(f) \subseteq \mathcal{D}(h)$. By the above construction (Lemma 7.8) we obtain a well defined functor

$$\mathcal{O}_X : \mathcal{D}(X) \longrightarrow \nu\mathbf{Smr}$$

from $\mathcal{D}(X)$ to the category $\nu\mathbf{Smr}$ of ν -semirings (Definition 3.37), given by

$$\begin{aligned} \mathcal{D}(f) &\longmapsto A_{S(f)}, \\ \mathcal{D}(f) \subseteq \mathcal{D}(h) &\longmapsto A_{S(h)} \longrightarrow A_{S(f)}. \end{aligned} \quad (7.6)$$

The image of \mathcal{O}_X restricts to the subcategory of tame ν -semirings in $\nu\mathbf{Smr}$. From our preceding discussion we conclude the following.

Corollary 7.9. \mathcal{O}_X is a presheaf of (tame) ν -semirings on $\mathcal{D}(X)$.

Given an element $f \in A$, assumed not to be ghostpotent, with $S(f)$ as defined in (7.4), we write

$$X_f := \text{Spec}(A_{S(f)}).$$

We define the **(tangible) cover set** of $\mathcal{D}(f)$ to be

$$\mathcal{C}(f) := \bigcap_{h \in S(f)} \mathcal{D}(h), \quad (7.7)$$

for which we have $\mathcal{D}(f) \subseteq \mathcal{C}(f)$. When $f \in A|_{\text{tng}}^\circ$ is \mathfrak{t} -persistent, $f \in S(f)$ and thus $\mathcal{C}(f) = \mathcal{D}(f)$, where the map $\iota_f : A_f \longrightarrow A_{S(f)}$ is isomorphism by Lemma 7.6. If $f \in \mathcal{N}(A)$, then $\mathcal{C}(f) = \bigcap_{h \in A|_{\text{tng}}^*} \mathcal{D}(h)$, cf. Properties 7.4.(b).

Recall from (6.12) that $\mathcal{D}(C, f)$ denotes the restriction of $\mathcal{D}(f)$ to those \mathfrak{g} -primes $\mathfrak{P} \in \text{Spec}(A)$ satisfying $C \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, where C is a multiplicative tangible submonoid of A . Obviously, $\mathcal{D}(f) \subseteq \mathcal{D}(h)$ implies $\mathcal{D}(C, f) \subseteq \mathcal{D}(C, h)$. Furthermore, by Lemma 6.35, the restricted set $\mathcal{D}(C, f)$ is nonempty for $C = S(f)$ when $f \notin \mathcal{N}(A)$, cf. Corollary 4.83.

Remark 7.10. Letting $S := S(f)$ with $f \in A$, Corollary 6.36 asserts that the canonical \mathfrak{q} -homomorphism $\tau : A \longrightarrow A_{S(f)}$ induces an isomorphism

$${}^a\tau : X_f \xrightarrow{\sim} \mathcal{C}(S, f) := \bigcap_{h \in S} \mathcal{D}(S, h) = \mathcal{E}(S), \quad \mathfrak{P}'_x \longmapsto \mathfrak{P}'_x|_A,$$

by restricting \mathfrak{g} -prime congruences on $A_{S(f)}$ to \mathfrak{g} -prime congruences on A , cf. Propositions 4.38 and 4.47. (This correspondence relies on the bijection between \mathfrak{g} -prime congruences on $A_{S(f)}$ and those \mathfrak{g} -prime congruences on A with $S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$.) The latter equality to $\mathcal{E}(S)$ –the tangible support of $S(f)$ – is by definition, see (6.11).

Note that when $f \notin A|_{\text{tng}}^\circ$, $\mathcal{D}(f)$ does not necessarily contain the entire restriction $\mathcal{C}(S, f)$ of the cover set $\mathcal{C}(f)$ to $S := S(f)$, but their intersection $\mathcal{C}(S, f) \cap \mathcal{D}(f)$ is nonempty, again by Lemma 6.35. Also, we always have $\mathcal{E}(f) \subseteq \mathcal{C}(S, f)$, cf. (6.9). Otherwise, if $f \in A|_{\text{tng}}^\circ$, then $\mathcal{C}(S, f) \subseteq \mathcal{D}(S, f) \subseteq \mathcal{D}(f)$, since $f \in S := S(f)$.

We denote by $\mathcal{D}_f(h)$ the open set $\mathcal{D}(h) \subseteq \text{Spec}(A_{S(f)})$ associated to $h \in A_{S(f)}$. To summarize, in this setting, for the case of \mathfrak{t} -persistent elements $f \in A|_{\text{tng}}^\circ$ we have the following.

Lemma 7.11. Let A be a (tame) ν -semiring and let $f \in A|_{\text{tng}}^\circ$ be a \mathfrak{t} -persistent element.

- (i) The canonical \mathfrak{q} -homomorphism $\tau : A \longrightarrow A_{S(f)}$ induces an injective homeomorphism

$${}^a\tau : X_f \hookrightarrow \mathcal{D}(f) \subset X$$

and, conversely, a surjective homeomorphism

$${}^a\tau : \mathcal{D}(f) \twoheadrightarrow X_f.$$

- (ii) $({}^a\tau)^{-1}(\mathcal{D}(g)) = \mathcal{D}_f(\tau(g))$ for any $g \in A|_{\text{tng}}^\circ$ with $\mathcal{D}(g) \subseteq \mathcal{D}(f)$. Furthermore, $({}^a\tau)^{-1}$ induces a bijective correspondence between sets $\mathcal{D}(g) \subseteq \mathcal{D}(f)$ and the open sets $\mathcal{D}_f(h)$, where $h \in A_{S(f)}$.
- (iii) The restriction of the functor $\mathcal{O}_X : \mathcal{D}(X) \longrightarrow \nu\mathbf{Smr}$ to the subcategory induced on $\mathcal{D}(f)$ is equivalent to the functor $\mathcal{O}_{X_f} : \mathcal{D}(X_f) \longrightarrow \nu\mathbf{Smr}$.

Proof. (i): Corollary 6.36 gives ${}^a\tau : X_f \xrightarrow{\sim} \mathcal{C}(S, f) = \mathcal{D}(S, f) \subseteq \mathcal{D}(f)$, for $S = S(f)$, which proves the first part and also induces the converse map ${}_{a\tau}$, showing that ${}_{a\tau}$ is onto.

(ii): Corollary 6.30 gives the first part. The map $\iota_f : A_f \longrightarrow A_{S(f)}$ is isomorphism by Lemma 7.6, so that $A_{S(f)}$ can be replaced by A_f . Then, the second part is clear, as $h = \frac{fg}{f^k} \in A_f$ yields

$$\mathcal{D}_f(h) = \mathcal{D}_f\left(\frac{fg}{f^k}\right) = \mathcal{D}_f(\tau(fg)) = ({}^a\tau)^{-1}(\mathcal{D}(fg)),$$

where $\mathcal{D}(fg) \subseteq \mathcal{D}(f)$.

(iii): Obtained by the canonical isomorphism $A_{S(fg)} \xrightarrow{\sim} (A_{S(f)})_{\tau(S(g))}$. \square

Lemma 7.11 applies only to \mathfrak{t} -persistent elements $f \in A|_{\text{tng}}^\circ$, while Remark 7.10 refers to the tangible support $\mathcal{E}(S(f)) = \mathcal{C}(S(f), f)$ of the tangible cover $\mathcal{C}(f)$ of $\mathcal{D}(f)$, rather than to their (nonempty) intersection. This type of intersection is dealt next.

Definition 7.12. Let $f \in A$, and take $S(f)$ from (7.4) to be the multiplicative monoid C in (6.12). The subset

$$\check{\mathcal{D}}(f) := \mathcal{D}(S(f), f) = \{\mathfrak{P} \in \mathcal{D}(f) \mid S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})\} \subseteq \mathcal{D}(f) \quad (7.8)$$

is called the **focal zone** of $\mathcal{D}(f)$. An element $f \in A$ is said to be **strict**, if $\check{\mathcal{D}}(f) = \mathcal{D}(f)$.

Clearly, every ghostpotent $f \in \mathcal{N}(A)$ is strict, whereas $\check{\mathcal{D}}(f) = \mathcal{D}(f) = \emptyset$. On the other edge, every \mathfrak{t} -unalterable element $f \in A|_{\text{ual}}$, and in particular every unit, is strict with $\check{\mathcal{D}}(f) = \mathcal{D}(f) = \text{Spec}(A)$.

Example 7.13. Monomials having \mathfrak{t} -persistent coefficients in the polynomial ν -semiring $A = F[\Lambda]$ over a ν -semifield F are strict, cf. Example 3.17.(i) and §3.9.

As $S(f) \subseteq A|_{\text{tng}}^\circ$ is a tangible monoid for any $f \in A$ (Lemma 7.3), the focal zone $\check{\mathcal{D}}(f) \subseteq \mathcal{D}(f)$ is canonically defined for every $f \in A$, and by Lemma 6.35 is nonempty whenever $f \notin \mathcal{N}(A)$. Moreover, $\mathcal{E}(f) \subseteq \check{\mathcal{D}}(f)$, since $C(f) \subseteq S(f)$ by Properties 7.4.(d), where $\mathcal{E}(f)$ could be empty, e.g., when $f \notin A|_{\text{tng}}^\circ$.

The focal zone $\check{\mathcal{D}}(f)$ can be written in terms of tangible support (6.11) as

$$\check{\mathcal{D}}(f) := \mathcal{D}(S(f), f) = \mathcal{D}(f) \cap \bigcap_{h \in S(f)} \mathcal{E}(h) = \mathcal{D}(f) \cap \mathcal{E}(S(f)) \subseteq \mathcal{D}(f), \quad (7.9)$$

providing a useful form.

Remark 7.14. Focal zones respect intersections of open sets in the sense that

$$\check{\mathcal{D}}(fg) \subseteq \check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g) \subseteq \mathcal{D}(f) \cap \mathcal{D}(g) = \mathcal{D}(fg)$$

for any $f, g \in A$, since $S(f) \cup S(g) \subseteq S(fg)$ by Remark 7.5. This also shows that if $\mathcal{D}(f) \cap \mathcal{D}(g) \neq \emptyset$, then $\check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g) \neq \emptyset$, whereas $\check{\mathcal{D}}(fg) \neq \emptyset$ by Lemma 6.35, since $S(fg)$ is a tangible monoid.

Also, the inclusion $\mathcal{D}(f) \subset \mathcal{D}(h)$ implies $\check{\mathcal{D}}(f) \subset \check{\mathcal{D}}(h)$, since $S(h) \subset S(f)$ for every $h \in S(f)$, cf. Properties 7.4.(f).

This remark can be strengthened further for \mathfrak{t} -persistent elements, specializing Corollary 6.8 to focal zones.

Lemma 7.15. $\check{\mathcal{D}}(fg) = \check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g)$, when $f, g \in A|_{\text{tng}}^\circ$.

Proof. $\check{\mathcal{D}}(fg) \subseteq \check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g)$ by Remark 7.14. Suppose $\check{\mathcal{D}}(fg) \subsetneq \check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g)$, and let $\mathfrak{P} \in (\check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g)) \setminus \check{\mathcal{D}}(fg)$, i.e., $S(fg) \not\subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, which means that there exists $h \in S(fg)$ with $h \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$. On the other hand, $\mathcal{D}(fg) \subseteq \mathcal{D}(h)$ iff $\mathcal{V}(fg) \supseteq \mathcal{V}(h)$ iff $\text{rad}_s(fg) \subseteq \text{rad}_s(h)$ by Lemma 6.11.(iv), and thus $(fg)^n = hb + c$ with $b \in A$, $c \in \mathcal{G}$, by Remark 4.18. Then, $(fg)^n \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, implying that $(fg) \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, and furthermore that $f \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$ or $g \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$, since $\mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$ is a monoid. Say $f \notin \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$. But, $f \in S(f)$, since $f \in A|_{\text{tng}}^\circ$, and thus $S(f) \not\subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$. Hence, $\mathfrak{P} \notin \check{\mathcal{D}}(f)$, so $\mathfrak{P} \notin \check{\mathcal{D}}(f) \cap \check{\mathcal{D}}(g)$. \square

Note that by definition $\check{\mathcal{D}}(f) \subseteq \mathcal{C}(S(f), f)$ for any $f \in A$, cf. (7.7), while $\check{\mathcal{D}}(f) = \mathcal{C}(S(f), f)$ when $f \in A|_{\text{tng}}^\circ$. In this view Corollary 6.30 specializes to focal zones.

Lemma 7.16. Let $\varphi : A \longrightarrow B$ a be \mathfrak{q} -homomorphism, and let ${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ be the induced map (6.8). Then, $\check{V} = ({}^a\varphi)^{-1}(\check{U})$, where $V := ({}^a\varphi)^{-1}(U)$, for any open set $U \subset \text{Spec}(A)$.

Proof. Let $U = \mathcal{D}(f)$. Then, $U \subset \mathcal{D}(h)$ for each $h \in S(f)$, and by Corollary 6.30 we get

$$V = \mathcal{D}(\varphi(f)) = ({}^a\varphi)^{-1}(\mathcal{D}(f)) \subset ({}^a\varphi)^{-1}(\mathcal{D}(h)) = \mathcal{D}(\varphi(h)).$$

Hence $\varphi(h) \in S(\varphi(f))$, which gives $\varphi(S(f)) \subseteq S(\varphi(f))$. By (7.9) and the specialization in Proposition 6.33, i.e., $({}^a\varphi)^{-1}(\mathcal{E}(S(f))) = \mathcal{E}(\varphi(S(f)))$, we obtain the inclusion $\check{V} \subseteq ({}^a\varphi)^{-1}(\check{U})$.

On the other hand, if $\check{V} \subsetneq ({}^a\varphi)^{-1}(\check{U})$, then there exists $h' \in S(\varphi(f)) \subset B$ such that $({}^a\varphi)^{-1}(\check{U}) \not\subseteq \mathcal{E}(h') \cap \mathcal{E}(\varphi(h))$ for some $h \in S(f) \subset A$. Then, $W := \mathcal{D}(h') \cap \mathcal{D}(\varphi(h))$ is an open subset of $\text{Spec}(B)$, which is contained in $\mathcal{D}(\varphi(h))$ and contains $\mathcal{D}(\varphi(f))$ as well, by the first part. Therefore, ${}^a\varphi(W)$ is an open set of $\text{Spec}(A)$ which is contained in $\mathcal{D}(h)$, so there exists $g \in A|_{\text{tng}}^\circ$ such that ${}^a\varphi(W) = \mathcal{D}(g)$. As $\mathcal{D}(\varphi(f)) \subseteq W$, we get that $\mathcal{D}(f) \subseteq \mathcal{D}(g)$, and hence $g \in S(f)$ with $\mathcal{D}(\varphi(g)) = W$. But, this shows that $({}^a\varphi)^{-1}(\check{U}) \subseteq \mathcal{E}(h') \cap \mathcal{E}(\varphi(h))$, and hence $\check{V} = ({}^a\varphi)^{-1}(\check{U})$. \square

The setup (7.9) allocates each open set $\mathcal{D}(f)$ in the Zariski topology on $X := \text{Spec}(A)$ with a distinguished (nonempty) subset – its focal zone $\check{\mathcal{D}}(f)$. By the lemma, focal zones respect contiguity of maps between spectra, induced by \mathfrak{q} -homomorphisms of ν -semirings.

Having gone this far with functorial properties, one can prove that, restricting to basic open coverings $\mathcal{D}(f) = \bigcup_{i \in I} \mathcal{D}(f_i)$, subject to certain constrainers concerning their focal zones, the functor $\mathcal{O}_X : \mathcal{D}(X) \rightarrow \nu\mathbf{Smr}$ as defined in (7.6) is a sheaf of (tame) ν -semirings. It extends to the entire spectrum $X = \text{Spec}(A)$ of A by letting

$$\overline{\mathcal{O}}_X(U) = \varprojlim_{\mathcal{D}(f) \subset U} \mathcal{O}_X(\mathcal{D}(f)) = \varprojlim_{\substack{f \in A, \\ \mathcal{D}(f) \subset U}} A_{S(f)}$$

for open subsets $U \subset X$, where the first projective limit runs over all open subsets $\mathcal{D}(f) \subset X$ that are contained in U , and the second over all $f \in A$ such that $\mathcal{D}(f) \subset U$. By the universal property of projective limits $\overline{\mathcal{O}}_X$ is a functor on the category of open subsets in X . Moreover, if $U = \mathcal{D}(g)$ is an open set, for some $g \in A$, then $\overline{\mathcal{O}}_X(U)$ is canonically isomorphic to $\mathcal{O}_X(\mathcal{D}(g)) = A_{S(g)}$ and $\overline{\mathcal{O}}_X$ restricts to a functor on $\mathcal{D}(X)$, isomorphic to \mathcal{O}_X , yielding $\overline{\mathcal{O}}_X$ as a sheaf of ν -semirings on X .

Nevertheless, the above level of generality is not required for the purpose of the present paper, and below we provide an explicit construction of a structure sheaf, obtained directly in terms of sections which are coincident with stalk structures, subject to focal zones.

Let M be an A - ν -module (Definition 5.1), and let $X = \text{Spec}(A)$. Using \mathcal{O}_X one can define the map

$$\mathcal{F} : \mathcal{D}(f) \longrightarrow M_{S(f)},$$

where $M_{S(f)} = M \otimes_A A_{S(f)}$ is the localization of M by the multiplicative system $S(f)$ determined by f in A (Definition 5.12). Then, \mathcal{F} is a functor from the category $\mathcal{D}(X)$ of all open subsets $\mathcal{D}(f) \subset X$ to the category of A - ν -modules, and it extends to a sheaf of A - ν -modules on all open subsets of X . Since $M_{S(f)}$ is an $A_{S(f)}$ - ν -module for every $f \in A$, the ν -module structure on \mathcal{F} extends canonically to a \mathcal{O}_X - ν -module structure, called the \mathcal{O}_X - ν -module associated to M .

7.2. Structure sheaves.

We begin with an explicit construction of local sections which are patched together to a sheaf \mathcal{O}_X of (tame) ν -semirings on $X := \text{Spec}(A)$, and are compatible with functoriality as described previously in §7.1. Recall that A_x denotes the localization of A by the \mathfrak{g} -prime congruence \mathfrak{P}_x , corresponding to a point x in X (Notation 6.1). By Definition 4.39, such localization is defined for \mathfrak{P}_x by letting $A_{\mathfrak{P}_x} := A_C$, where $C = \text{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$. Recall also from Definition 7.12 that $\check{\mathcal{D}}(f)$ denotes the focal zone of $\mathcal{D}(f) \subseteq X$. With these notations, by (7.8) we can write

$$S(f) \subseteq \text{T}_{\text{cls}}^{-1}(\mathfrak{P}_x) \iff x \in \check{\mathcal{D}}(f), \tag{7.10}$$

which easily restates our correspondence.

In classical algebraic geometry over rings, which employs ideals, the inclusion of a prime ideal \mathfrak{p}_x in a basic open set $\mathcal{D}(f)$ automatically implies that $f \in A \setminus \mathfrak{p}_x$, where $A \setminus \mathfrak{p}_x$ is a multiplicative system. The same holds for a collection of elements $f_1, \dots, f_n \in A \setminus \mathfrak{p}_x$, where now \mathfrak{p}_x belongs to the intersection of the corresponding $\mathcal{D}(f_i)$. These inclusions in a multiplicative system are curtail for localization, as these

elements become units in A_x . Furthermore, they establish the correspondence $A_f \longrightarrow A_x$ having the required universal property that coincides with inductive limits.

In the supertropical setting, since the complement of a ghost projection $\mathbb{G}_{\text{cls}}^{-1}(\mathfrak{P})$ in A may contain elements which are not \mathfrak{t} -persistent and thus cannot determine units, maps of type $A_{S(f)} \longrightarrow A_x$ are not always properly accessible for every $x \in \mathcal{D}(f)$. Therefore, a direct supertropical analogy to classical approach does not suit for this setting and our construction of sections basically relies on those points of $\mathcal{D}(f)$ that admit the right behaviour, i.e. $S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P})$. These points are the points of the focal zone $\check{\mathcal{D}}(f)$ of $\mathcal{D}(f)$, defined for every $f \in A$ which is not ghostpotnet. As we shall see, this specialization answers all our needs, especially concerning computability with inductive limits.

Given an open set $U = \mathcal{D}(f)$, where $f \in A$ is assumed not to be ghostpotnet, we write $\check{U} = \check{\mathcal{D}}(f)$ for its (nonempty) focal zone (7.8).

Definition 7.17. *Let $U \subseteq X$ be an open set. We say that a map*

$$\sigma : U \longrightarrow \prod_{x \in \check{U}} A_x, \quad \sigma = (\sigma_x)_{x \in \check{U}}, \quad \sigma_x \in A_x, \quad (7.11)$$

is **locally quotient of A on U** , if for every $x \in \check{U}$ there is a neighborhood V in U and elements $f, g \in A$ with $g \in \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_y)$ such that $\sigma_y = \frac{f}{g}$ in A_y for all $y \in \check{V}$. Such maps are the **(local) sections of the structure ν -sheaf \mathcal{O}_X of X** , defined via

$$\mathcal{O}_X(U) := \{ \sigma \mid \sigma \text{ is locally quotient of } A \text{ on } U \}$$

for all open sets U in X .

The conditions in this definition imply that the elements of $\mathcal{O}_X(U)$ are local, and that these sections indeed form a sheaf \mathcal{O}_X on X (Definition 7.1) due to the following structure over all open sets U, V of X . The restriction maps $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ are induced from the inclusions $\iota : U \hookrightarrow V$ by sending $\sigma \longrightarrow \sigma \circ \iota$. Given sections $\sigma_1, \sigma_2 \in \mathcal{O}_X(U)$, their addition $\sigma_1 + \sigma_2$ is the section that sends x to $\sigma_1(x) + \sigma_2(x)$ in A_x for every $x \in \check{U}$. Similarly, their multiplication $\sigma_1 \cdot \sigma_2$ sends x to the product $\sigma_1(x)\sigma_2(x)$ in A_x for every $x \in \check{U}$. Associativity and distributivity of these operations follow from the point-wise operations of A_x . The \mathfrak{t} -persistent (resp. tangible, ghost) elements of $\mathcal{O}_X(U)$ are the sections with $\sigma(x) \in A_x|_{\text{tng}}$ (resp. $\sigma(x) \in A_x|_{\text{tng}}$, $\sigma(x) \in A_x|_{\text{gh}}$) for all $x \in \check{U}$. Accordingly, $\mathcal{O}_X(U)$ is endowed with a structure of a (tame) ν -semiring. Note that for any $f \notin \mathcal{N}(A)$ the local sections (7.11) on $U = \mathcal{D}(f)$ are subject to elements within its (nonempty) focal zone $\check{U} = \check{\mathcal{D}}(f)$. Therefore, having Lemmas 7.15 and 7.16, these sections coincide with the functor (7.6) given by $\mathcal{O}_X : U \longmapsto A_{S(f)}$, since $S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ for each $x \in \check{U}$.

To customize the stalk \mathcal{F}_x at a point $x \in X$ (cf. (7.1)) to our setup, we introduce the family

$$\begin{aligned} \mathfrak{D}_x &:= \{ \mathcal{D}(h) \mid h \in A|_{\text{tng}}^\circ, \check{\mathcal{D}}(h) \ni x \} \\ &= \{ \mathcal{D}(h) \mid h \in A|_{\text{tng}}^\circ, \mathcal{D}(h) \ni x \text{ with } S(h) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_x) \}. \end{aligned} \quad (7.12)$$

of open sets of X , determining by \mathfrak{t} -persistent elements $h \in A|_{\text{tng}}^\circ$, that contain x . In particular, $X = \mathcal{D}(1) \in \mathfrak{D}_x$ for every $x \in X$, and thus \mathfrak{D}_x is nonempty. An open neighborhood U of x which belongs to \mathfrak{D}_x is denoted by U_x .

Remark 7.18. \mathfrak{D}_x is closed for intersection, as follows from Lemma 7.15. Namely,

$$\mathcal{D}(f), \mathcal{D}(g) \in \mathfrak{D}_x \implies \mathcal{D}(f) \cap \mathcal{D}(g) \in \mathfrak{D}_x,$$

where $\mathcal{D}(f) \cap \mathcal{D}(g) = \mathcal{D}(fg)$ by Corollary 6.8.(i).

For \mathfrak{t} -persistent elements $f \in A|_{\text{tng}}^\circ$ we have the correspondence

$$x \in \check{\mathcal{D}}(f) \iff S(f) \subseteq \mathbb{T}_{\text{cls}}^{-1}(\mathfrak{P}_x) \iff \mathcal{D}(f) \in \mathfrak{D}_x, \quad (7.13)$$

cf. (7.8) and (7.12), respectively.

Definition 7.19. *The ν -stalk $\mathcal{O}_{X,x}$ of \mathcal{O}_X at a point $x \in X$ is defined to be the inductive limit*

$$\mathcal{O}_{X,x} := \varinjlim_{U_x \in \mathfrak{D}_x} \mathcal{O}_X(U_x) \quad (7.14)$$

of sections $\sigma \in \mathcal{O}_X(U_x)$ over \mathfrak{D}_x .

For each open neighborhood $U_x \in \mathfrak{D}_x$ of x we have the well-defined canonical map

$$\mathcal{O}_X(U_x) \longrightarrow \mathcal{O}_{X,x}, \quad \sigma \longmapsto \sigma_x,$$

sending $\sigma \in \mathcal{O}_X(U_x)$ to the germ σ_x in the ν -stalk $\mathcal{O}_{X,x}$.

Theorem 7.20. *Let A be a (tame) ν -semiring and let $X = \text{Spec}(A)$ be its spectrum.*

- (i) *For any $x \in X$ the ν -stalk $\mathcal{O}_{X,x}$ of the ν -sheaf \mathcal{O}_X is isomorphic to the local ν -semiring A_x .*
- (ii) *The ν -semiring $\mathcal{O}_X(\mathcal{D}(f))$, with strict $f \in A|_{\text{tnng}}^\circ$, is isomorphic to the localized ν -semiring $A_{S(f)}$.*
- (iii) *In particular, $\mathcal{O}_X(X) \cong A$.*

Proof. (i): The map sending a local section σ in a neighborhood $U_x \in \mathfrak{D}_x$ of x to $\sigma_x \in A_x$ provides a well-defined \mathfrak{q} -homomorphism of ν -semirings

$$\varphi : \mathcal{O}_{X,x} \longrightarrow A_x, \quad (U_x, \sigma) \longmapsto \sigma_x (\in A_x),$$

which we claim is a bijection.

Surjectivity of φ : Each element of A_x has the form $\frac{f}{g}$ with $f, g \in A$ where $g \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ is \mathfrak{t} -persistent, thus $S(g) \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ (cf. Remark 7.7) and $\mathcal{D}(g) \in \mathfrak{D}_x$. Hence, the fraction $\frac{f}{g}$ is well-defined on the focal zone $\check{\mathcal{D}}(g) \subseteq \mathcal{D}(g)$, and $(\mathcal{D}(g), \frac{f}{g})$ defines an element in $\mathcal{O}_{X,x}$ (cf. Lemma 7.15) that is mapped by φ to the given element.

Injectivity of φ : Let $\sigma_1, \sigma_2 \in \mathcal{O}_X(U_x)$ for some neighborhood $U_x \in \mathfrak{D}_x$ of x , and assume that σ_1 and σ_2 have the same value at x , namely $(\sigma_1)_x = (\sigma_2)_x$. We will show that σ_1 and σ_2 coincide over \check{V}_x in a neighborhood $V_x \in \mathfrak{D}_x$ of x , so that they define the same element in $\mathcal{O}_{X,x}$. Shrinking U_x if necessary, we may assume that $\sigma_i = \frac{f_i}{g_i}$ on \check{U}_x , for $i = 1, 2$, where $f_i, g_i \in A$ with $g_i \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$. As σ_1 and σ_2 have the same image in A_x , it follows that $hf_1g_2 = hf_2g_1$ in A for some $h \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$. Therefore, we also have $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ in every local ν -semiring A_y such that $g_1, g_2, h \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_y)$. But, the set of such y 's (cf. (7.10)) is the set $\check{\mathcal{D}}(g_1) \cap \check{\mathcal{D}}(g_2) \cap \check{\mathcal{D}}(h)$, laying in the open set $\mathcal{D}(g_1) \cap \mathcal{D}(g_2) \cap \mathcal{D}(h)$, and belongs to \mathfrak{D}_x , since g_1, g_2, h are \mathfrak{t} -persistents (Remark 7.18). Hence, $\sigma_1 = \sigma_2$ on \check{V}_x for some neighborhood $V_x \in \mathfrak{D}_x$ of x , and thus on the entire set V_x , as required. Therefore, φ is injective.

(ii): Recall from Lemma 7.6 that, for a \mathfrak{t} -persistent element $f \in A|_{\text{tnng}}^\circ$, the ν -semiring $A_{S(f)}$ is isomorphic to A_f , where $f \in S(f)$. We define the \mathfrak{q} -homomorphism of ν -semirings

$$\psi : A_{S(f)} \longrightarrow \mathcal{O}_X(\mathcal{D}(f)), \quad \frac{g}{h^\ell} \longmapsto \frac{g}{h^\ell}, \quad h \in S(f),$$

given by sending $\frac{g}{h^\ell}$ to the section of $\mathcal{O}_X(\mathcal{D}(f))$ that assigns to any $x \in \check{\mathcal{D}}(f)$ the image of $\frac{g}{h^\ell}$ in A_x . (The elements $\frac{g}{h^\ell}$ are well defined in A_x , since $S(f) \subseteq \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ for $x \in \check{\mathcal{D}}(f)$.)

Injectivity of ψ : Assume that $\psi\left(\frac{g_1}{h_1^{\ell_1}}\right) = \psi\left(\frac{g_2}{h_2^{\ell_2}}\right)$, with $h_1, h_2 \in S(f)$. Then, for every $x \in \check{\mathcal{D}}(f)$ there is a \mathfrak{t} -persistent element $h_x \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ such that $h_x g_1 h_2^{\ell_2} = h_x g_2 h_1^{\ell_1}$. Namely h_x is contained in the equaliser $E := \text{Eq}(g_1 h_2^{\ell_2}, g_2 h_1^{\ell_1})$ of $g_1 h_2^{\ell_2}$ and $g_2 h_1^{\ell_1}$ (Definition 2.15) – a ν -semiring ideal. Let \mathfrak{G}_E be its ghostifying congruence (4.8). Then, for any $x \in \check{\mathcal{D}}(f)$, we have $E \not\subseteq \mathsf{G}_{\text{cls}}^{-1}(\mathfrak{P}_x)$, whereas $h_x \in E$ with $h_x \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$. As this holds for any $x \in \mathcal{D}(f)$, where $\mathcal{D}(f) = \check{\mathcal{D}}(f)$ since f is strict, we have $\mathcal{V}(\mathfrak{G}_E) \cap \mathcal{D}(f) = \emptyset$, or in other words $\mathcal{V}(E) = \mathcal{V}(\mathfrak{G}_E) \subseteq \mathcal{V}(f)$. Then, $\text{rad}_s(f) \subseteq \text{rad}_s(E)$ by Lemma 6.11.(viii), implying that $f^m \in E$ for some m (Corollary 4.75.(iii)). Therefore, $f^m g_1 h_2^{\ell_2} = f^m g_2 h_1^{\ell_1}$ in A , and hence $\frac{g_1}{h_1^{\ell_1}} = \frac{g_2}{h_2^{\ell_2}}$ in $A_{S(f)}$.

Surjectivity of ψ : Let $\sigma \in \mathcal{O}_X(U)$, where $U = \mathcal{D}(f)$. By definition, U can be covered by open sets U_i on which σ is represented as a quotient $\frac{q_i}{f_i}$, with $f_i \in \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)$ for all $x \in \check{U}_i$, i.e., $U_i \subseteq \mathcal{D}(f_i)$. To emphasize, $f_i \in A|_{\text{tnng}}^\circ$, namely it is \mathfrak{t} -persistent. As the open sets of type $\mathcal{D}(h_i)$ form a base for the topology of X , we may assume that $U_i = \mathcal{D}(h_i)$ for some h_i .

We may also assume that $f_i = h_i$, and therefore $h_i \in A|_{\text{tnng}}^\circ$ is \mathfrak{t} -persistent as well. Indeed, since $\mathcal{D}(h_i) \subseteq \mathcal{D}(f_i)$, by taking complements we obtain $\mathcal{V}(f_i) \subseteq \mathcal{V}(h_i)$, and thus $\text{rad}_s(h_i) \subseteq \text{rad}_s(f_i)$ by Lemma 6.11.(viii). Therefore $h_i \in \mathsf{G}_{\text{cls}}^{-1}(\text{rad}_s(f_i))$ and, since A is tame, $h_i^\ell = c f_i$ for some $c \in A$ by Corollary 4.75.(ii). Thus $\frac{q_i}{f_i} = \frac{c q_i}{h_i^\ell}$. Replacing h_i by h_i^ℓ (since $\mathcal{D}(h_i) = \mathcal{D}(h_i^\ell)$ by Lemma 6.11.(vi)) and g_i

by cg_i , we can assume that $\mathcal{D}(f)$ is covered by open sets of type $\mathcal{D}(h_i)$, and that σ is represented by $\frac{g_i}{h_i}$ on $\check{\mathcal{D}}(h_i) \subset \mathcal{D}(h_i)$.

More precisely, $\mathcal{D}(f)$ can be covered by finitely many such sets $\mathcal{D}(h_i)$. Indeed, $\mathcal{D}(f) \subseteq \bigcap_i \mathcal{D}(h_i)$ iff $\mathcal{V}(f) \supseteq \bigcap_i \mathcal{V}(h_i) = \mathcal{V}(\sum_i \langle h_i \rangle)$, where $\langle h_i \rangle$ is the ideal generated by h_i , cf. Proposition 6.6.(iv). By Lemma 6.11.(viii) and Corollary 4.75.(iii) this is equivalent to $f^n \in \sum_i \langle h_i \rangle$ for some n , which means that f can be written as a finite sum $f^n = \sum_i b_i h_i$ with $b_i \in A$. Hence, we may assume that only finitely many h_i are involved.

Over the intersection $\mathcal{D}(h_i) \cap \mathcal{D}(h_j) = \mathcal{D}(h_i h_j)$ we have two elements $\frac{g_i}{h_i}$ and $\frac{g_j}{h_j}$ representing σ , subject to $\check{\mathcal{D}}(h_i) \cap \check{\mathcal{D}}(h_j) = \check{\mathcal{D}}(h_i h_j)$ (Lemma 7.15), which contains the focal zone $\check{\mathcal{D}}(h_i h_j)$ of $\mathcal{D}(h_i h_j)$. Then, by the injectivity proven above, it follows that $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ in $A_{S(h_i h_j)}$. But the product $h_i h_j$ is \mathfrak{t} -persistent, i.e., $h_1 h_2 A|_{\text{tng}}^\circ$, implying that $A_{S(h_i h_j)}$ is isomorphic to $A_{h_i h_j}$ by Lemma 7.6, and hence $(h_i h_j)^m g_i h_j = (h_i h_j)^m g_j h_i$ for some m . As we have only finitely many h_i , we may choose one m which holds for all i, j . Replacing g_i by $g_i h_i^m$ and h_i by h_i^{m+1} for all i , we remain with σ represented by $\frac{g_i}{h_i}$ on $\mathcal{D}(h_i)$, subject to $\check{\mathcal{D}}(h_i)$, and furthermore $g_i h_j = g_j h_i$ for any i, j .

Write $f^n = \sum_i b_i h_i$ as above, which is possible since the $\mathcal{D}(h_i)$'s cover $\mathcal{D}(f)$, and let $g = \sum_i b_i g_i$, with $b_i \in A$. Then, for every h_j we have

$$gh_j = \sum_i b_i g_i h_j = \sum_i b_i h_i g_j = f^n g_j,$$

thus $\frac{f}{g} = \frac{h_j}{g_j}$ on $\check{\mathcal{D}}(h_j) \subseteq \mathcal{D}(h_j)$. Hence, σ is represented on $\mathcal{D}(f)$, subject to $\check{\mathcal{D}}(f)$, by $\frac{g}{f^n} \in A_{S(f)}$, and therefore $\psi : A_{S(f)} \rightarrow \mathcal{O}_X(\mathcal{D}(f))$ is surjective.

(iii): Every unit $f \in A^\times$, in particular $\mathbb{1}$, is strict, and thus $A = A_f \cong \mathcal{O}_X(X) = \mathcal{O}_X(\mathcal{D}(f))$ by part (ii). \square

7.3. Locally ν -semiringed spaces.

Let A and B be ν -semirings, having the spectra $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, respectively. Recall that a \mathfrak{q} -homomorphism $\varphi : A \rightarrow B$ induces the pull-back map ${}^a\varphi : Y \rightarrow X$ of congruences, given by sending $\mathfrak{P}_b \in \text{Spec}(B)$ to $\varphi^\vee(\mathfrak{P}_b)$ in $\text{Spec}(A)$, cf. (6.6) and Remark 2.3.(iii).

Definition 7.21. A ν -semiringed space (X, \mathcal{O}_X) is a topological space X together with a **structure ν -sheaf** \mathcal{O}_X of (commutative) ν -semirings. A **morphism** of ν -semiringed spaces is a pair of maps

$$(\phi, \phi^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

where $\phi : X \rightarrow Y$ is a continuous map of topological spaces and $\phi^\# : \mathcal{O}_Y \rightarrow \phi_* (\mathcal{O}_X)$ is a morphism of ν -sheaves on Y (Definition 7.1).

The ν -sheaf $\phi_* (\mathcal{O}_X)$ on Y is defined by $V \mapsto \mathcal{O}_X(\phi^{-1}(V))$ for open subsets $V \subset Y$, and $\phi^\#$ is a system of ν -semiring \mathfrak{q} -homomorphisms

$$\phi^\#(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\phi^{-1}(V)), \quad V \subset Y \text{ open}, \quad (7.15)$$

which agree with restriction morphisms. In addition, by Lemma 7.16, $\phi^{-1}(V)$ respects focal zones, that is $\check{U} = \phi^{-1}(\check{V})$ for $U = \phi^{-1}(V)$.

The map $\phi : X \rightarrow Y$ induces for any $x \in X$ a map

$$\check{\phi} : \mathfrak{D}_x \longrightarrow \mathfrak{D}_{\phi(x)}, \quad U_x \longmapsto V_{\phi(x)}, \quad (7.16)$$

of sets of the form (7.12). This is seen by (7.13), as a membership of $\mathcal{D}(f)$ in \mathfrak{D}_x corresponds to containment of x in focal zones $\check{\mathcal{D}}(f)$, while ϕ respects focal zones by Lemma 7.16.

Compositions of morphisms of ν -semiringed spaces are defined in the natural way. A ν -semiringed space (X, \mathcal{O}_X) restricts to an open subset $U \subset X$, yielding the ν -semiringed space $(U, \mathcal{O}_X|_U)$. The injection $(U, \mathcal{O}_X|_U) \hookrightarrow (X, \mathcal{O}_X)$ is then a morphism of ν -semiringed spaces, called **open immersion** of ν -semiringed spaces.

Remark 7.22. A morphism $(\phi, \phi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of ν -semiringed spaces canonically induces for each $x \in X$ a ν -semiring \mathfrak{q} -homomorphism $\phi_x^\# : \mathcal{O}_{Y, \phi(x)} \rightarrow \mathcal{O}_{X, x}$. Indeed, for open subsets $V_y \subset Y$

with $V_y \in \mathfrak{D}_{\phi(x)}$, i.e., $y = \phi(x)$, the compositions

$$\mathcal{O}_Y(V_y) \xrightarrow{\phi^\#(V_y)} \mathcal{O}_X(\phi^{-1}(V_y)) \longrightarrow \mathcal{O}_{X,x}$$

agree with the restriction morphisms of \mathcal{O}_Y , respecting focal zones as well, and therefore induce a \mathfrak{q} -homomorphism $\phi_x^\# : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$.

Recall that $\text{Spn}_\ell(A)$ denotes the set of all \mathfrak{t} -minimal ℓ -congruences (Definition 4.55) on a ν -semiring A , and that A is local, if all \mathfrak{t} -minimal ℓ -congruences $\mathfrak{N} \in \text{Spn}_\ell(A)$ have the same tangible projection $\Gamma_{\text{cls}}^{-1}(\mathfrak{N})$ (Definition 4.60). (For example, any ν -semifield is local.) In particular, by Corollary 4.62, the localization $A_{\mathfrak{P}}$ of A by a \mathfrak{g} -prime congruence \mathfrak{P} is a local ν -semiring with central \mathfrak{t} -minimal ℓ -congruence $\mathfrak{N}_{\mathfrak{P}}$, cf. (4.33).

As $\mathcal{O}_{X,x}$ is isomorphic to the local ν -semiring $A_x := A_{\mathfrak{P}_x}$ (Theorem 7.20), to allocate the central \mathfrak{t} -minimal ℓ -congruence of $\mathcal{O}_{X,x}$ we identify $\mathcal{O}_{X,x}$ with its isomorphic image, and take the ℓ -congruence $\mathfrak{N}_{\mathfrak{P}_x}$ on A_x , which we denote by \mathfrak{N}_x . A \mathfrak{q} -homomorphism $\varphi : A \longrightarrow B$ of local ν -semirings is called **local \mathfrak{q} -homomorphism**, if for any $\mathfrak{N}_B \in \text{Spn}_\ell(B)$ there exists $\mathfrak{N}_A \in \text{Spn}_\ell(A)$ such that $\varphi^\vee(\mathfrak{N}_B) = \mathfrak{N}_A$. Namely, ${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ maps \mathfrak{t} -minimal ℓ -congruences to \mathfrak{t} -minimal ℓ -congruences. For example, given a \mathfrak{q} -homomorphism $\varphi : A \longrightarrow B$ and a \mathfrak{g} -prime congruence \mathfrak{P}_b on B , the congruence $\mathfrak{P}_a = \varphi^\vee(\mathfrak{P}_b)$ is a \mathfrak{g} -prime congruence on A (Remark 4.42), then it follows that the induced \mathfrak{q} -homomorphism $A_{\mathfrak{P}_a} \longrightarrow B_{\mathfrak{P}_b}$ is local.

Definition 7.23. A ν -semiringed space (X, \mathcal{O}_X) is a **locally ν -semiringed space**, if all its ν -stalks $\mathcal{O}_{X,x}$ are local ν -semirings. A **morphism** $(\phi, \phi^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of locally ν -semiringed spaces is a **morphism of ν -semiringed spaces** such that for all $x \in X$ the morphisms

$$\phi_x^\# : (\phi^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$$

of ν -stalks are local \mathfrak{q} -homomorphisms.

We write $\text{Hom}(B, A)$ for the set of all ν -semiring \mathfrak{q} -homomorphisms $B \longrightarrow A$, and $\text{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ for the set of all morphisms of locally ν -semiringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$.

Theorem 7.24. Let A and B be ν -semirings, with spectra $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and structure ν -sheaves \mathcal{O}_X , \mathcal{O}_Y , respectively.

- (i) (X, \mathcal{O}_X) is a locally ν -semiringed space with ν -stalk $\mathcal{O}_{X,x} \cong A_x$ at every $x \in X$.
- (ii) The canonical map

$$\Upsilon : \text{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \longrightarrow \text{Hom}(B, A), \quad (\phi, \phi^\#) \longmapsto \phi^\#(Y),$$

is a bijection.

- (iii) For any morphism $(\phi, \phi^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ as in (i), and a point $x \in X$, the associated map of ν -stalks

$$\phi_x^\# : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$$

coincides canonically with the map $B_{\phi(x)} \longrightarrow A_x$ obtained from $\phi^\#(Y) : B \longrightarrow A$ by localization.

Proof. (i): Follows from Theorem 7.20.(i).

(ii): Let $\varphi : B \longrightarrow A$ be a \mathfrak{q} -homomorphism of ν -semirings, and let

$$\phi = {}^a\varphi : X \longrightarrow Y, \quad \mathfrak{P} \longmapsto \varphi^\vee(\mathfrak{P}),$$

be the induced map of spectra, given by $\phi(x) = \varphi^\vee(x)$. First, $\phi : X \longrightarrow Y$ is continuous by Corollary 6.30. Given $x \in X$, we can localize φ to obtain the local \mathfrak{q} -homomorphism $\varphi_x : B_{\varphi^\vee(x)} \longrightarrow A_x$ of local ν -semirings. Then, for any open set $V \subset Y$ we obtain the \mathfrak{q} -homomorphism of ν -semirings

$$\phi^\#(V) : \mathcal{O}_Y(V) = \coprod_{\phi^{-1}(x) \in \check{V}} B_{\phi^{-1}(x)} \longrightarrow \coprod_{x \in \check{U}} A_x = \mathcal{O}_X(\phi^{-1}(V)), \quad U = \phi^{-1}(V),$$

which gives a morphism of ν -sheaves $\phi^\# : \mathcal{O}_Y \longrightarrow \phi_*\mathcal{O}_X$. The restriction of $\phi^\#$ to focal zones \check{U} and \check{V} coincides with ϕ , since $\check{U} = \phi^{-1}(\check{V})$ by Lemma 7.16. By compatibility of sections with the continuity of ϕ , sending $x \in X$ to $\phi(x) \in Y$, Theorem 7.20 shows that the morphism of ν -stalks $\phi_x^\# : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$ coincides with the canonical map $B_{\phi(x)} = B_{\varphi^\vee(x)} \longrightarrow A_x$, which is local. Hence, $(\phi, \phi^\#)$ is a morphism of locally ν -semiringed spaces.

Conversely, let $(\phi, \phi^\#) : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ be a morphism of locally ν -semiringed spaces. On global sections, by Theorem 7.20.(iii), the map $\phi^\#$ induces a \mathfrak{q} -homomorphism of ν -semirings

$$\varphi : B \cong \mathcal{O}_Y(Y) \longrightarrow \phi_* \mathcal{O}_X(X) = \mathcal{O}_X(X) \cong A.$$

Furthermore, for any $x \in X$ there is the induced map of ν -stalks $\phi_x^\# : \mathcal{O}_{Y, \phi(x)} \longrightarrow \mathcal{O}_{X, x}$, which must coincide with φ on global sections, and thus rendering the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi = \Upsilon((\phi, \phi^\#)) = \phi^\#(Y)} & A \\ \downarrow & & \downarrow \\ B_{\phi(x)} \cong \mathcal{O}_{Y, \phi(x)} & \xrightarrow{\phi_x^\#} & A_x \cong \mathcal{O}_{X, x} \end{array}$$

commutative. (The homomorphism $\phi_x^\#$ is local, by assumption.) Since $\phi(x) = (\phi_x^\#)^{-1}(x)$, it follows that $\phi_x^\#$ is the localization of φ , which shows that ϕ coincides with ${}^a\varphi$. Hence, $\phi^\#$ is induced from φ , and the morphism $(\phi, \phi^\#)$ of locally ν -semiringed spaces derives from φ .

(iii): As $(\phi, \phi^\#)$ is a morphism of locally ν -semiringed spaces, by part (i) it coincides with the local map $B_{\phi(x)} \longrightarrow A_x$. \square

7.4. Local ν -semirings.

The ν -stalk $\mathcal{O}_{X, x}$ (Definition 7.19) of a locally ν -semiringed space (X, \mathcal{O}_X) is called the **local ν -semiring** of (X, \mathcal{O}_X) at x , cf. Definition 4.60. Its central \mathfrak{t} -minimal ℓ -congruence (4.33) is obtained from A_x as

$$\mathfrak{N}_x := \mathsf{T}_{\text{cls}}^{-1}(\mathfrak{P}_x)\mathfrak{P}_x,$$

and $K(x) := \mathcal{O}_{X, x}/\mathfrak{N}_x$ is the corresponding residue ν -semifield, cf. Corollary 4.62. The tangible (resp. ghost) cluster of the central \mathfrak{t} -minimal ℓ -congruence \mathfrak{N}_x on the ν -semiring $\mathcal{O}_{X, x}$ is the set of all sections that possess tangible (resp. ghost) values at the point $x \in X$.

If $U_x \in \mathfrak{D}_x$ is an open neighborhood of $x \in X$, cf. (7.12), and $f \in \mathcal{O}_X(U_x)$, we write $f(x) \in K(x)$ for the image of f under the composition of the canonical \mathfrak{q} -homomorphisms $\mathcal{O}_X(U_x) \longrightarrow \mathcal{O}_{X, x} \longrightarrow K(x)$.

Writing X for a locally ν -semiringed space (X, \mathcal{O}_X) , the following definitions of geometric notions are now directly accessible in terms of local ν -semirings.

- (a) The **local dimension** $\dim_x(X)$ of X at a point $x \in X$ is defined to be the Krull dimension (Definition 4.89) of the local ν -semiring $\mathcal{O}_{X, x}$. The **dimension** $\dim(X)$ of the whole X is the supremum of local dimensions $\dim_x(X)$ over all $x \in X$.
- (b) The **Zariski cotangent space** to X at a point x is defined as $\mathfrak{N}_x/\mathfrak{N}_x^2$, realized as a ν -module over the residue ν -semifield $K(x) = \mathcal{O}_{X, x}/\mathfrak{N}_x$ (cf. Definition 2.6 and (2.7)), whose dual is called the **Zariski tangent space** to X at x .
- (c) X is called **nonsingular** at a point $x \in X$, if the Zariski tangent space to X at x has dimension equal to $\dim_x(X)$; otherwise, X is said to be **singular** at x (i.e., the dimension of the Zariski tangent space is larger).

The study of these notions requires a further development of dimension theory, this is left for future work.

8. ν -SCHEMES

8.1. Affine ν -schemes.

Having the structure of locally ν -semiringed spaces settled, we employ these objects as prototypes for the so-called schemes, introduced by A. Grothendieck.

Definition 8.1. An **affine ν -scheme** is a locally ν -semiringed space (X, \mathcal{O}_X) which is isomorphic to a locally ν -semiringed space over a ν -semiring, i.e., $(X, \mathcal{O}_X) \cong (\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ for some ν -semiring A .

A **ν -scheme** is a locally ν -semiringed space (X, \mathcal{O}_X) that has an open covering by affine ν -schemes $(U_i, \mathcal{O}_X|_{U_i})_{i \in I}$. A **morphism of ν -schemes** is a (local) morphism of locally ν -semiringed spaces.

Often, for short, we write X for the ν -scheme (X, \mathcal{O}_X) , and $\phi : X \longrightarrow Y$ for a morphism $(\phi, \phi^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of ν -schemes. The sheaf \mathcal{O}_X is called the **structure ν -sheaf** of X . $\mathcal{O}_X(U)$ is called

the ν -**semiring of sections** of \mathcal{O}_X over U , with $U \subseteq X$ open, and is sometimes denoted $\Gamma(U, \mathcal{O}_X)$. The notation $\Gamma(X)$ stands for $\Gamma(X, \mathcal{O}_X)$, where $\Gamma(A)$ denotes $\Gamma(X)$ with $X = \text{Spec}(A)$.

Example 8.2 (Affine spaces). *Let $A = \tilde{R}[\lambda_1, \dots, \lambda_n]$ be the ν -semiring of polynomial functions over a (tame) ν -semiring R , cf. §3.9. Then, $\mathbb{A}_R^n := \text{Spec}(A)$ is the affine space of relative dimension n over R .*

The restriction of a morphism of ν -schemes $\phi : X \rightarrow Y$ to a subset $U \subset X$ gives the morphism $\phi|_U : U \rightarrow Y$ of ν -schemes, obtained by composing the open immersion $(U, \mathcal{O}_X|_U) \hookrightarrow (X, \mathcal{O}_X)$ with the morphism $\phi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$. Then, an open subset $V \subset Y$ determines the open subset $\phi^{-1}(V) \subset X$, together with a unique morphism of ν -schemes $\phi' : \phi^{-1}(V) \rightarrow V$ that renders the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \uparrow & & \uparrow \\ \phi^{-1}(V) & \xrightarrow{\phi'} & V \end{array}$$

commutative. All together, ν -schemes and their morphisms (Definition 8.1) establish the category $\nu\mathbf{Sch}$, containing the full subcategory $\nu\mathbf{ASch}$ of affine ν -schemes (the morphisms between affine ν -schemes are the same in $\nu\mathbf{ASch}$ and in $\nu\mathbf{Sch}$).

A morphism $\phi : X \rightarrow Y$ of ν -schemes is injective (resp. surjective, open, closed, homeomorphism), if the continuous map $X \rightarrow Y$ of the underlying topological spaces has this property.

Recall from (6.8) that for a \mathfrak{q} -homomorphism $\varphi : A \rightarrow B$ we have the induced map

$${}^a\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(A), \quad \mathfrak{P}' \longmapsto \varphi^\vee(\mathfrak{P}'). \quad (8.1)$$

For a second \mathfrak{q} -homomorphism $\psi : B \rightarrow C$, we then have ${}^a(\psi \circ \varphi) = {}^a\psi \circ {}^a\varphi$. With this view, Spec can be viewed as a contravariant functor

$$\text{Spec} : \nu\mathbf{Smr} \longrightarrow \nu\mathbf{ASch}$$

from the category of ν -semirings to the category of affine ν -schemes, assigning to a ν -semiring A the corresponding affine ν -scheme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ and to a \mathfrak{q} -homomorphism $B \rightarrow A$ the corresponding morphism $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)}) \rightarrow (\text{Spec}(B), \mathcal{O}_{\text{Spec}(B)})$ as characterized in Theorem 7.24.(i).

On the other hand, if $\phi : X \rightarrow Y$ is a morphism of ν -semiringed spaces, using the notation $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$, we obtain a \mathfrak{q} -homomorphism of ν -semirings

$$\Gamma(\phi) := \phi^\# : \Gamma(Y, \mathcal{O}_Y) = \mathcal{O}_Y(Y) \longrightarrow \Gamma(X, \mathcal{O}_X) = (\phi_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X). \quad (8.2)$$

In this way, Γ sets up a contravariant functor

$$\begin{array}{ccc} X & \longmapsto & \mathcal{O}_X(X), \\ X \longrightarrow Y & \longmapsto & \phi^\#(Y) : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X), \end{array}$$

from the category of ν -semiringed spaces to the category $\nu\mathbf{Smr}$ of ν -semirings, which restricts to a contravariant functor

$$\Gamma : \nu\mathbf{ASch} \longrightarrow \nu\mathbf{Smr}.$$

on the category of affine ν -schemes.

Recall that by Theorem 7.24 morphisms of affine ν -schemes correspond bijectively to \mathfrak{q} -homomorphisms of ν -semirings, and thus we can state the following.

Proposition 8.3. *The category of affine ν -schemes $\nu\mathbf{ASch}$ is equivalent to the opposite of the category $\nu\mathbf{Smr}$ of ν -semirings.*

Proof. The functor $\text{Spec} : \nu\mathbf{Smr} \rightarrow \nu\mathbf{ASch}$ is surjective by definition, and $\Gamma \circ \text{Spec}$ is clearly isomorphic to id on ν -semirings. It suffices to show that for any two ν -semirings A and B the maps

$$\text{Hom}(A, B) \xrightleftharpoons[\Gamma]{\text{Spec}} \text{Hom}(\text{Spec}(B), \text{Spec}(A))$$

are mutually inverse bijections. But, $\Gamma \circ \text{Spec} = \text{id}$ by (8.2), while it follows from Theorem 7.24 that $\text{Spec} \circ \Gamma = \text{id}$. \square

8.2. Open ν -subschemes.

The restriction $(U, \mathcal{O}_X|_U)$ of a ν -scheme (X, \mathcal{O}_X) to an open set $U \subset X$, called **open ν -subscheme** of X , is by itself is a ν -scheme by Lemma 7.11.(iii). If U is an affine ν -scheme, then U is an **affine open ν -subscheme**.

Proposition 8.4. *Let X be a ν -scheme, and let $U \subset X$ be an open set.*

- (i) *The locally ν -semiringed space $(U, \mathcal{O}_X|_U)$ is a ν -scheme.*
- (ii) *The open subsets that give rise to affine open ν -subschemes are a basis of the topology.*

Proof. By definition, the locally ν -semiringed space X can be covered by affine ν -schemes. By Proposition 6.19 each of these affine ν -schemes has a basis of its topology which consists of affine ν -schemes. This yields both parts of the proposition. \square

Let $U \subseteq X$ be an open subset, considered as an open ν -scheme of X , with the inclusion $\iota : U \hookrightarrow X$. For $V \subseteq X$ open, the restriction map of the structure ν -sheaf \mathcal{O}_X gives a \mathfrak{q} -homomorphism of ν -semirings

$$\Gamma(V, \mathcal{O}_X) \longrightarrow \Gamma(V \cap U, \mathcal{O}_X) = \Gamma(\iota^{-1}(V), \mathcal{O}_X|_U) = \Gamma(V, \iota_* \mathcal{O}_X|_U).$$

These maps determine a morphism $\iota^\# : \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_X|_U$ of ν -sheaves of ν -semirings and, hence, by the inclusion $U \subseteq X$, a morphism $U \longrightarrow X$ of ν -schemes. An **affine open covering** of a ν -scheme X is an open covering $X = \bigcup_i U_i$ by affine open ν -subschemes U_i of X .

Lemma 8.5. *Let X be a ν -scheme, and let U, V be affine open ν -subschemes of X . For every $x \in U \cap V$ there exists an open ν -subscheme $W \subseteq U \cap V$ containing x , such that W is a principal open, cf. (6.2), both in U and in V .*

Proof. Replacing V by a principal open set of V containing x , we may assume that $V \subseteq U$. Choose \mathfrak{t} -persistent $f \in \Gamma(U, \mathcal{O}_X)$ such that $x \in \tilde{\mathcal{D}}(f)$, where $\mathcal{D}(f) \subseteq V$, i.e. an element $f \in \Gamma_{\text{cls}}^{-1}(\mathfrak{P}_x)$. Let $f|_V$ be the restriction of the image of f under the \mathfrak{q} -homomorphism $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(V, \mathcal{O}_X)$. Then, $\mathcal{D}_U(f) = \mathcal{D}_V(f|_V)$, which also implies that $\Gamma(U, \mathcal{O}_X)_f = \Gamma(V, \mathcal{O}_X)_{f|_V}$, by the sheaf axioms (Definition 7.1). \square

8.3. Gluing ν -schemes.

With the notion of morphisms of ν -schemes at our disposal (Definition 8.1), the gluing procedure of ν -schemes becomes applicable in the usual way. That is, we start with a given collection of ν -schemes $(X_i)_{i \in I}$, and an open set $U_{ij} \subseteq X_i$ for each $i \neq j$, together with a family of isomorphisms of ν -schemes

$$\psi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$$

such that for all $i, j, k \in I$

- (a) $\psi_{ji} = \psi_{ij}^{-1}$;
- (b) $\psi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$;
- (c) $(\psi_{jk} \circ \psi_{ij})|_{U_{ij} \cap U_{ik}} = \psi_{ik}|_{X_{ij} \cap U_{ik}}$ (compatibility condition).

With this data, we define the ν -scheme X by gluing the $(X_i)_{i \in I}$ along the ψ_{ij} in the obvious way, i.e., the unique ν -scheme X covered by open ν -subschemes isomorphic to the X_i whose identity maps on $X_i \cap X_j \subseteq X$ correspond to the isomorphisms ψ_{ij} . When the X_i are affine ν -schemes with structure ν -sheaves \mathcal{O}_{X_i} , $\mathcal{O}_{X_i}(X_i \cap X_j)$ is naturally identified with $\mathcal{O}_{X_j}(X_i \cap X_j)$. Accordingly, as any ν -scheme X admits a covering by affine open ν -subschemes $(X_i)_{i \in I}$, X can be viewed as a gluing of the affine ν -schemes X_i along their intersections $X_i \cap X_j$.

Example 8.6. *Let R be a ν -semiring. The **projective space** \mathbb{P}_R^n over R is obtained by gluing $n + 1$ copies $U_i = \mathbb{A}_R^n$, $i = 0, \dots, n$, of affine space \mathbb{A}_R^n (cf. Example 6.27). Each $U_i = \text{Spec}(A_i)$ is the \mathfrak{g} -prime spectrum of a ν -semiring A_i of polynomial functions in n indeterminates over R , cf. §3.9, written as*

$$A_i = \tilde{R} \left[\frac{\lambda_0}{\lambda_i}, \dots, \frac{\check{\lambda}_i}{\lambda_i}, \dots, \frac{\lambda_n}{\lambda_i} \right],$$

where $\check{\lambda}_i$ means that λ_i is to be discarded. The ν -semirings A_i are viewed as ν -subsemirings of the Laurent polynomials $\tilde{R}[\lambda_0, \dots, \lambda_n, \lambda_0^{-1}, \dots, \lambda_n^{-1}]$.

We define a gluing datum with index set $\{0, \dots, n\}$ as follows: for $0 \leq i, j \leq n$, let

$$U_{ij} = \begin{cases} \mathcal{D}_{U_i}(\frac{\lambda_j}{\lambda_i}) \subseteq U_i & \text{if } i \neq j, \\ U_i & \text{if } i = j. \end{cases}$$

Furthermore, let $\varphi_{ii} = \text{id}_{U_i}$, and for $i \neq j$ let

$$\varphi_{ji} : U_{ij} \longrightarrow U_{ji}$$

be the isomorphism defined by the equality (as ν -subsemirings of $\tilde{R}[\lambda_0, \dots, \lambda_n, \lambda_0^{-1}, \dots, \lambda_n^{-1}]$)

$$\tilde{R}\left[\frac{\lambda_0}{\lambda_i}, \dots, \frac{\check{\lambda}_i}{\lambda_i}, \dots, \frac{\lambda_n}{\lambda_i}\right]_{\frac{\lambda_j}{\lambda_i}} \longrightarrow \tilde{R}\left[\frac{\lambda_0}{\lambda_i}, \dots, \frac{\check{\lambda}_j}{\lambda_j}, \dots, \frac{\lambda_n}{\lambda_i}\right]_{\frac{\lambda_j}{\lambda_i}}$$

of the affine ν -schemes U_{ij} and U_{ji} . Since the isomorphisms φ_{ij} are defined by equalities, the cocycle condition holds trivially, and we obtain a gluing datum. This gives a ν -scheme, called the **projective space** \mathbb{P}_R^n of relative dimension n over R . The ν -schemes U_i are considered as open ν -subschemes of \mathbb{P}_R^n .

A morphism from a glued ν -scheme X to another ν -scheme Y can be determined by a given collection of morphisms $X_i \rightarrow Y$ that coincide on the overlaps in the obvious sense. In this gluing view, Theorem 7.24 generalizes as follows.

Proposition 8.7. *Let $X = \text{Spec}(A)$ be any ν -scheme and let $Y = \text{Spec}(B)$ be an affine ν -scheme. There is a one-to-one correspondence between morphisms $X \rightarrow Y$ and ν -semiring \mathfrak{q} -homomorphisms $B \cong \Gamma(B) = \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X) = \Gamma(A) \cong A$.*

Proof. Let $\{U_i\}$ be an open affine cover of X , and let $\{U_{i,j,k}\}$ be an open affine cover of $U_i \cap U_j$. Giving a morphism $\phi : X \rightarrow Y$ is the same as giving morphisms $\phi_i : U_i \rightarrow Y$ such that ϕ_i and ϕ_j agree on $U_i \cap U_j$, i.e., such that $\phi_i|_{U_{i,j,k}} = \phi_j|_{U_{i,j,k}}$ for all i, j, k . Since the U_i and $U_{i,j,k}$ are affine, by Theorem 7.24, the morphisms ϕ_i and $\phi_i|_{U_{i,j,k}}$ correspond exactly to \mathfrak{q} -homomorphisms of ν -semirings $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_{U_i}(U_i) = \mathcal{O}_X(U_i)$ and $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_{U_{i,j,k}}(U_{i,j,k}) = \mathcal{O}_X(U_{i,j,k})$, respectively. Hence, a morphism $\phi : X \rightarrow Y$ is the same as a collection of ν -semiring \mathfrak{q} -homomorphisms $\phi_i^\# : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(U_i)$ such that the compositions $\rho_{U_i, U_{i,j,k}} \circ \phi_i^\# : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(U_{i,j,k})$ and $\rho_{U_j, U_{i,j,k}} \circ \phi_j^\# : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(U_{i,j,k})$ agree for all i, j, k . By the sheaf axiom for \mathcal{O}_X , this is exactly the data of a \mathfrak{q} -homomorphism $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$ of ν -semirings. \square

8.4. Properties of ν -schemes.

In this subsection, unless otherwise is specified, we assume that all ν -schemes are built over tame ν -semirings (Definition 3.11).

8.4.1. Generic points.

Let X be a ν -scheme, and let $Z \subseteq X$ be a subset. A point $z \in Z$ is a **generic point** of Z , if the set $\{z\}$ is dense in Z (Definition 6.22). Topologically, if Z admits a generic point, then Z is irreducible. In the case of underlying topological spaces of ν -schemes, the converse relation holds:

Proposition 8.8. *The map*

$$X \longrightarrow \{Z \subseteq X \mid Z \text{ closed, irreducible}\}, \quad x \longrightarrow \overline{\{x\}},$$

is a bijection, i.e., every irreducible closed subset contains a unique generic point.

Proof. The correspondence holds for affine ν -schemes by Corollary 6.25, as $X = \text{Spec}(A)$ where A is tame. Let $Z \subseteq X$ be closed irreducible, and let $U \subseteq X$, $Z \cap U \neq \emptyset$, be an affine open subset. The closure of $Z \cap U$ in X is Z , since Z is irreducible, and $Z \cap U$ is irreducible. Hence, the generic point in $Z \cap U$ is a generic point of Z . A generic point $z \in Z$ is contained in every open subset of X that meets Z , and thus in every U . The uniqueness of generic points is then derived from the affine uniqueness. \square

Any point x in a ν -scheme X as above has a generalization by a maximal point q , that is, the generic point of an irreducible component of X such that $x \in \{q\}$. On the other hand, specializations to closed points are more subtle, as a nonempty ν -scheme X may have no closed points, even if it is irreducible. But, when X is affine, this cannot happen (as any \mathfrak{g} -prime congruence is contained in a maximal ℓ -congruence by Proposition 4.57), which implies that in a quasi-compact ν -scheme X the closure $\overline{\{x\}}$ of any $x \in X$ contains a closed point.

Proposition 8.9. *Let $\phi : X \rightarrow Y$ be an open morphism of ν -schemes, where Y is irreducible with generic point q . Then, X is irreducible if and only if the fiber $\phi^{-1}(q)$ is irreducible.*

Proof. ϕ is open, and thus $\overline{\phi^{-1}(q)} = \phi^{-1}(\overline{\{q\}}) = \phi^{-1}(Y) = X$. Then the claim follows from the topological property that a subspace is irreducible iff its closure is irreducible. \square

Recall that a ν -domain is a tangibly closed ν -semiring that has no ghost divisors (Definition 3.19).

Remark 8.10. *Let A be a ν -domain with spectrum $X = \text{Spec}(A)$, and let $Q(A)$ be its ν -semifield of fractions. The trivial congruence $\Delta(A)$ of A is a \mathfrak{g} -prime congruence, corresponding to the point $q \in X$, where X is the closure of $\{q\}$. Therefore, q is contained in every nonempty open set of X , i.e., q is a generic point of X . The local ν -semiring $\mathcal{O}_{X,q}$ is the localization of A by $\Delta(A)$, and thus, by Theorem 7.20.(i),*

$$\mathcal{O}_{X,q} \cong Q(A).$$

For all tangible multiplicative monoids $T \subseteq C$ of A the canonical \mathfrak{q} -homomorphism $T^{-1}A \rightarrow C^{-1}A$ is injective, and the localizations $T^{-1}A$ are considered as ν -subsemirings of $Q(A)$.

For every $f \notin \mathcal{N}(A)$, we have $\mathcal{O}_X(\mathcal{D}(f)) = \coprod_{x \in \check{U}} A_x$, $U = \mathcal{D}(f)$, by definition (7.11) of structure ν -sheaves. If $V \subseteq X$ is an arbitrary open subset, then $\mathcal{O}_X(V) = \bigcap_U \mathcal{O}_X(U)$ where $U = \mathcal{D}(f)$ runs through the open sets $U \subseteq V$. Equivalently, $\mathcal{O}_X(V) = \bigcap_f \mathcal{O}_X(U)$ with $U = \mathcal{D}(f) \subseteq V$. By Theorem 7.24 $A_x \cong \mathcal{O}_{X,x}$ for every $x \in X$, and thus, for any nonempty open subset $V \subseteq X$ we have

$$\mathcal{O}_X(V) \cong \bigcap_{x \in \check{V}} \mathcal{O}_{X,x},$$

where $\check{V} = \bigcap_{U \subseteq V} \check{U}$.

8.4.2. Reduced and integral schemes.

We generalize the notion of being reduced from ν -semirings to ν -schemes.

Definition 8.11. *A ν -scheme X is called **reduced**, if all its local ν -semirings $\mathcal{O}_{X,x}$ are ghost reduced ν -semirings (Definition 4.76). X is **integral**, if it is reduced and irreducible.*

Proposition 8.12. *Let X be a ν -scheme.*

- (i) *X is reduced if and only if for every open subset $U \subseteq X$ the ν -semiring $\Gamma(U, \mathcal{O}_X)$ is reduced.*
- (ii) *X is integral if and only if for every open subset $\emptyset \neq U \subseteq X$ the ν -semiring $\Gamma(U, \mathcal{O}_X)$ is a ν -domain.*
- (iii) *If X is an integral ν -scheme, then for each $x \in X$ the local ν -semiring $\mathcal{O}_{X,x}$ is a ν -domain.¹⁶*

Proof. (i): Suppose X is reduced and $U \subseteq X$ is open. Assume $f \in \Gamma(U, \mathcal{O}_X)$ such that $f^n = \text{ghost}$ for some n . If we had $f \neq \text{ghost}$, then there would exist $x \in U$ with $f_x \neq \text{ghost}$ in $\mathcal{O}_{X,x}$, but $f_x^n = \text{ghost}$. Conversely, given a ghostpotent $\bar{f} \in \mathcal{O}_{X,x}$ (Definition 4.76), there exists an open $U \subseteq X$ and a lift $f \in \Gamma(U, \mathcal{O}_X)$ of \bar{f} . Shrinking U if necessary, we may assume that f is ghostpotent, and hence is ghost.

(ii): Let X be integral. All open ν -subschemes of X are integral, so it is enough to show that $\Gamma(X, \mathcal{O}_X)$ is a ν -domain. Taking $f, g \in \Gamma(X, \mathcal{O}_X)$ such that $fg = \text{ghost}$, we have $X = \mathcal{V}(f) \cup \mathcal{V}(g)$, and by irreducibility, say, $X = \mathcal{V}(f)$. Checking this locally on X , which we may assume is affine, we claim that f must be ghost. Indeed, f lies in the intersection of the ghost projections of all \mathfrak{g} -prime congruences, i.e., in the ghostpotent ideal of the affine ν -semiring of X . Since X is reduced, by (i), the ghost projection of its ghostpotent radical congruence is the ghost ideal of $\Gamma(X, \mathcal{O}_X)$. Namely, $\Gamma(X, \mathcal{O}_X)$ is a ν -domain, cf. Lemma 4.78 and Lemma 6.23.

Conversely, if all $\Gamma(U, \mathcal{O}_X)$ are ν -domains, then X is reduced by (i). For nonempty affine open subsets $U_1, U_2 \subseteq X$ with empty intersection, if exists, the sheaf axioms imply that

$$\Gamma(U_1 \cup U_2, \mathcal{O}_X) = \Gamma(U_1, \mathcal{O}_X) \times \Gamma(U_2, \mathcal{O}_X).$$

Obviously, the product on the right contains ghost divisors.

(iii): Follows from (ii), since any tangible localization of a ν -domain is a ν -domain. \square

¹⁶The converse does not hold.

An affine ν -scheme over $X = \text{Spec}(A)$ is integral if and only if the corresponding ν -semiring A is a ν -domain. Then, the generic point q of X corresponds to the trivial congruence $\Delta(A)$ of A , and the local ν -semiring $\mathcal{O}_{X,q}$ is the localization of A by $\Delta(A)$, i.e., by $A|_{\text{tng}}^\bullet = A|_{\text{tng}}$, which is a monoid as A is a ν -domain. But, this localization is just the ν -semifield of fractions $Q(A)$ of A (Definition 3.39). This also shows that the local ν -semiring at the generic point of an arbitrary integral ν -scheme is a ν -semifield.

Definition 8.13. *Let $q \in X$ be the generic point of an integral ν -scheme X . The local ν -semiring $\mathcal{O}_{X,q}$ is a ν -semifield, denoted by $K(X)$ and called the **function ν -semifield** of X .*

For an integral ν -scheme all “ ν -semirings of functions” are contained in its function ν -semifield.

Proposition 8.14. *Let X be an integral ν -scheme with generic point q , and let $K(X)$ be its function ν -semifield.*

- (i) *If $U = \text{Spec}(A)$ is a nonempty open affine ν -subscheme of X , then $K(X) = Q(A)$. Furthermore, $Q(\mathcal{O}_{X,x}) = K(X)$ for $x \in X$.*
- (ii) *For nonempty open subsets $U \subseteq V \subseteq X$, the maps*

$$\Gamma(V, \mathcal{O}_X) \xrightarrow{\rho_V^\nu} \Gamma(U, \mathcal{O}_X) \xrightarrow{f \mapsto f_q} K(X)$$

are injective.

- (iii) *For every nonempty open subset $U \subseteq X$ and for every open covering $U = \bigcup_i U_i$ the following holds*

$$\Gamma(U, \mathcal{O}_X) = \bigcap_i \Gamma(U_i, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x},$$

where the intersection occurs in $K(X)$.

Proof. (i): For $x \in U = \text{Spec}(A) \subseteq X$ we have $q \in U$, where q corresponds to the trivial congruence on the ν -domain A . Since $\mathcal{O}_{X,x} \cong A_x$, we have $K(X) = \mathcal{O}_{U,q} = Q(A) = Q(A_x)$.

(ii): Given $\emptyset \neq U \subseteq X$, it suffices to prove that if $f \in \Gamma(U, \mathcal{O}_X)$ with $f_q = \text{ghost}$ in $K(X)$, then f is a ghost. Since $f = \text{ghost}$ is equivalent to $f|_V = \text{ghost}$ for all open nonempty affine ν -subschemes $V \subseteq U$, we may assume that $U = \text{Spec}(A)$ is affine. Then, the map $\Gamma(U, \mathcal{O}_X) \rightarrow K(X)$ is just the canonical inclusion $A \hookrightarrow Q(A) = K(X)$.

(iii): The injectivity of the restriction maps $\rho_{U_i}^U : U \rightarrow U_i$, and the fact that \mathcal{O}_X is a sheaf, implies left equality. The analogous assertion for affine integral ν -schemes in Remark 8.10 gives the right equality. \square

8.5. Fiber products.

Let Y be a ν -scheme. A **ν -scheme over Y** is a ν -scheme X with a morphism $\phi : X \rightarrow Y$. A morphism of ν -schemes X, Z over Y is a morphism of ν -schemes $X \rightarrow Z$ that renders the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Z \\ & \searrow & \swarrow \\ & Y & \end{array}$$

commutative. A ν -scheme over $Y = \text{Spec}(B)$ is termed a ν -scheme over B , for short. A ν -scheme X over a ν -semifield F is of **finite type**, if X has a finite cover by open affine subsets $U_i = \text{Spec}(A_i)$, where each A_i is a finitely generated F - ν -algebra, cf. §5.3. We then have the following observations for an affine ν -scheme $X = \text{Spec}(A)$.

- (a) X is a ν -scheme over F if and only if there exists a morphism $F \rightarrow A$, i.e., if A is an F - ν -algebra.
- (b) A morphism $X \rightarrow Y = \text{Spec}(B)$ is a morphism of ν -schemes over F if and only if the corresponding \mathfrak{q} -homomorphism $B \rightarrow A$ of ν -semirings is a morphism of F - ν -algebras.
- (c) X is of finite type over F if and only if F is a finitely generated F - ν -algebra (Definition 5.13).
- (d) X is reduced and irreducible if and only if $fg = \text{ghost}$ in A implies $f = \text{ghost}$ or $g = \text{ghost}$, i.e., if and only if A is a ν -domain. Indeed, suppose that $fg = \text{ghost}$ where $f \neq \text{ghost}$ and $g \neq \text{ghost}$. If $f = g^n$ or $g = f^m$ for some $m, n \in \mathbb{N}$, then A has a ghostpotent, namely X is not reduced. Otherwise, X decomposes into two proper closed subsets $\mathcal{V}(f)$ and $\mathcal{V}(g)$, and thus X is not irreducible.

Definition 8.15. Let $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ be morphisms of ν -schemes. Their **fiber product** $X \times_S Y$ is defined to be a ν -scheme together with “projection” morphisms $\pi_X : X \times_S Y \rightarrow X$ and $\pi_Y : X \times_S Y \rightarrow Y$ such that the square in (8.3) below commutes, and such that for any ν -scheme Z with morphisms $Z \rightarrow X$ and $Z \rightarrow Y$ rendering (8.3) commutative with ϕ and ψ there is a unique morphism $\xi : Z \rightarrow X \times_S Y$ which renders the whole diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & & \searrow & & \\
 & & \xi & & \\
 & & \searrow & & \\
 & & X \times_S Y & \xrightarrow{\pi_Y} & Y \\
 & & \downarrow \pi_X & & \downarrow \psi \\
 & & X & \xrightarrow{\phi} & S
 \end{array} \tag{8.3}$$

commutative.

A routine proof shows that the fiber product is uniquely determined by its property.

Lemma 8.16. If the fiber product $X \times_S Y$ exists, then it is unique. (Namely, if two fiber products satisfy the above property, then they are canonically isomorphic.)

Proof. Let F_1 and F_2 be two fiber products satisfying the property of Definition 8.15, i.e., each F_i is assigned with morphisms to X and Y . Since F_i is a fiber product, the morphism $\xi_{ij} : F_i \rightarrow F_j$, $i, j \in \{1, 2\}$, renders the diagram

$$\begin{array}{ccccc}
 & & F_i & & \\
 & & \searrow & & \\
 & & \xi_{ij} & & \\
 & & \searrow & & \\
 & & F_j & \xrightarrow{\pi_Y} & Y \\
 & & \downarrow \pi_X & & \downarrow \psi \\
 & & X & \xrightarrow{\phi} & S
 \end{array}$$

commutative. Composing together, by the uniqueness part of Definition 8.15, it follows that $\xi_{ij} \circ \xi_{ji} = \text{id}$, and thus F_1 and F_2 are canonically isomorphic. \square

Remark 8.17. From Definition 8.15 we obtain the following properties.

- (i) $X \times_S Y = X \times_T Y$ for any open subset $S \subset T$ (morphisms from any Z to X and to Y that commute with ϕ and ψ are the same, independently if the base ν -scheme is S or T).
- (ii) Given open subsets $U \subseteq X$ and $V \subseteq Y$, the fiber product

$$U \times_S V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \subseteq X \times_S Y$$

is an open subset of the fiber product $X \times_S Y$.

We next show that fiber products of ν -schemes always exist, where in the affine case they should correspond to tensor products in commutative ν -algebra. Using the tensor product of ν -modules, cf. §5.2, we can construct the fiber product of ν -schemes.

Proposition 8.18. Let $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ be morphisms of ν -schemes. Then, the fiber product $X \times_S Y$ exists.

Proof. Assume first that $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, and $S = \text{Spec}(R)$ are affine ν -schemes. The morphisms $X \rightarrow S$ and $Y \rightarrow S$ provide A and B as R - ν -modules by Theorem 7.24, yielding the tensor product $A \otimes_R B$. We claim that $\text{Spec}(A \otimes_R B)$ is the fiber product $X \times_S Y$. Indeed, a morphism $Z \rightarrow \text{Spec}(A \otimes_R B)$ corresponds to a \mathfrak{q} -homomorphism $A \otimes_R B \rightarrow \mathcal{O}_Z(Z)$ by Proposition 8.7, which by Remark 5.11 is the same as \mathfrak{q} -homomorphisms $A \rightarrow \mathcal{O}_Z(Z)$ and $B \rightarrow \mathcal{O}_Z(Z)$. The latter induce the same \mathfrak{q} -homomorphism on R , which again by Proposition 8.7 corresponds to morphisms $Z \rightarrow X$ and $Z \rightarrow Y$, which in turn give rise to the same morphism from $Z \rightarrow S$. Therefore, $\text{Spec}(A \otimes_R B)$ is the desired product.

Assume that X , Y and S are general ν -schemes, and take coverings by open affine ν -schemes, first of S , and then of $\phi^{-1}(S_i)$, $\psi^{-1}(S_i)$ by $X_{i,j}$, $Y_{i,k}$, respectively. The fiber products $X_{i,j} \times_{S_i} Y_{i,k}$ exist by the construction of tensor products, which are also fiber products over S by Remark 8.17.(i). If we had another such product $X_{i',j'} \times_S Y_{i',k'}$, both of them would contain the (unique) fiber product $(X_{i,j} \cap X_{i',j'}) \times_S (Y_{i,k} \cap Y_{i',k'})$ as an open subset by Remark 8.17.(ii), hence they can be glued along these isomorphic open subsets. Thus, the ν -scheme $X \times_S Y$ obtained by gluing these patches satisfies the fiber product property. \square

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