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The centre-quotient property and weak centrality for C^*-algebras

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We give a number of equivalent conditions (including weak centrality) for a general C^*-algebra to have the centre-quotient property. We show that every C^*-algebra A has a largest weakly central ideal $J_{wc}(A)$. For an ideal I of a unital C^*-algebra A, we find a necessary and sufficient condition for a central element of A/I to lift to a central element of A. This leads to a characterisation of the set V_A of elements of an arbitrary C^*-algebra A which prevent A from having the centre-quotient property. The complement $\mathrm{CQ}(A) := A \setminus V_A$ always contains $Z(A) + J_{wc}(A)$ (where $Z(A)$ is the centre of A), with equality if and only if $A/J_{wc}(A)$ is abelian. Otherwise, $\mathrm{CQ}(A)$ fails spectacularly to be a C^*-subalgebra of A.

1 Introduction

Let A be a C^*-algebra with centre $Z(A)$. If I is a closed two-sided ideal of A, it is immediate that

$$(Z(A) + I)/I = q_I(Z(A)) \subseteq Z(A/I), \tag{1.1}$$

where $q_I : A \to A/I$ is the canonical map. A C^*-algebra A is said to have the *centre-quotient property* ([44], [4, Section 2.2] and [8, p. 2671]) if for any closed two-sided ideal I of A, equality holds in (1.1). For the sake of brevity we shall usually refer to the centre-quotient property as the *CQ-property*.

In 1971, Vesterstrøm [44] proved the following theorem.

Theorem 1.1 (Vesterstrøm)**.** If A is a unital C^*-algebra, then the following conditions are equivalent:

(i) A has the CQ-property.

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(ii) A is weakly central, that is for any pair of maximal ideals M and N of A, $M \cap Z(A) = N \cap Z(A)$ implies $M = N$.

□

Weakly central C^*-algebras were introduced by Misonou and Nakamura in [36, 37] in the unital context. The most prominent examples of weakly central C^*-algebras A are those satisfying the Dixmier property, that is for each $x \in A$ the closure of the convex hull of the unitary orbit of x intersects $Z(A)$ [7, p. 275]. In particular, von Neumann algebras are weakly central (see [20, Théorème 7] and [37, Theorem 3]). It was shown by Haagerup and Zsidó in [26] that a unital simple C^*-algebra satisfies the Dixmier property if and only if it admits at most one tracial state. In particular, weak centrality does not imply the Dixmier property. However, in [35] Magajna gave a characterisation of weak centrality in terms of averaging involving unital completely positive elementary operators. Recently, Robert, Tikuisis and the first-named author found the exact gap between weak centrality and the Dixmier property for unital C^*-algebras [8, Theorem 2.6] and showed that a postliminal C^*-algebra has the (singleton) Dixmier property if and only if it has the CQ-property [8, Theorem 2.12]. Also, in a recent paper [16], Brešar and the second-named author studied an analogue of the CQ-property in a wider algebraic setting (so called 'centrally stable algebras').

In this paper we study weak centrality, the CQ-property and several equivalent conditions for general C^*-algebras that are not necessarily unital. We then investigate the failure of weak centrality in two different ways. Firstly, we show that every C^*-algebra A has a largest weakly central ideal $J_{wc}(A)$, which can be readily determined in several examples. Secondly, we study the set V_A of individual elements of A which prevent the weak centrality (or the CQ-property) of A. The set V_A is contained in the complement of $J_{wc}(A)$ and, in certain cases, is somewhat smaller than one might expect. In the course of this, we address a fundamental lifting problem that is closely linked to the CQ-property: for a fixed ideal I of a unital C^*-algebra A, we find a necessary and sufficient condition for a central element of A/I to lift to a central element of A.

The paper is organised as follows. After some preliminaries in Section 2, the main results are obtained in Section 3 and Section 4. In Section 3, we study weak centrality and the CQ-property for arbitrary C^*-algebras. In the non-unital context, the appropriate maximal ideals for the definition of weak centrality are the modular maximal ideals (Definition 3.5). In Theorem 3.16, we give a number of conditions (including the CQ-property) that are equivalent to the weak centrality of a C^*-algebra A. In Theorem 3.22, we show that every C^*-algebra A has a largest ideal $J_{wc}(A)$ that is weakly central. In doing so, we obtain a formula for $J_{wc}(A)$ in terms of the set T_A of those modular maximal ideals of A which witness the failure of the weak centrality of A. This formula leads easily to the explicit description of $J_{wc}(A)$ in a number of examples. For example, $J_{wc}(A) = \{0\}$ when either A is the rotation algebra (the C^*-algebra of the discrete three-dimensional Heisenberg group, Example 3.24) or $A = C^*(\mathbb{F}_2)$ (the full C^*-algebra of the free group on two generators, Example 3.25), and $J_{wc}(A) = \mathrm{K}(\mathcal{H})$ for Dixmier's classic example of a C^*-algebra in which the Dixmier property fails (Example 3.28). We also obtain the stability of weak centrality and the CQ-property in the context of arbitrary C^*-tensor products (Theorem 3.29).

In Section 4, we undertake the more difficult task of describing the individual elements which prevent a C^*-algebra A from having the CQ-property. We say that an element $a \in A$ is a *CQ-element* if for every closed two-sided ideal I of A, $a + I \in Z(A/I)$ implies $a \in Z(A) + I$ (Definition 4.1). We denote by $\mathrm{CQ}(A)$ the set of all CQ-elements of A.

Clearly, A has the CQ-property if and only if $\mathrm{CQ}(A) = A$ and the complement $V_A := A \setminus \mathrm{CQ}(A)$ is precisely the set of elements which prevent the CQ-property for A. For an ideal I of a unital C^*-algebra A, we use the complete regularization map, the Tietze extension theorem and the Dauns-Hofmann theorem to obtain a necessary and sufficient condition for a central element of A/I to lift to a central element of A (Theorem 4.7). This then leads to a description of V_A (and hence $\mathrm{CQ}(A)$) for an arbitrary C^*-algebra A in terms of the subset T_A (Theorem 4.8).

We show that $\mathrm{CQ}(A)$ contains $Z(A) + J_{wc}(A)$ (Corollary 4.4), all commutators $[a, b]$ $(a, b \in A)$ and all products ab, ba where $a \in A$ and b is a quasi-nilpotent element of A (Proposition 4.5). It follows from this, together with a result of Pop [40, Theorem 1], that if A is not weakly central (so that $\mathrm{CQ}(A) \neq A$), $\mathrm{CQ}(A)$ contains the norm-closure $\overline{[A, A]}$ of $[A, A]$ (the linear span of all commutators in A) if and only if all quotients A/M $(M \in T_A)$ admit tracial states (Theorem 4.10). In particular, if A is postliminal or an AF-algebra, then $\overline{[A, A]} \subseteq \mathrm{CQ}(A)$ (Corollary 4.11). On the other hand, if the tracial condition is not satisfied, then $\mathrm{CQ}(A)$ does not even contain $[A, A]$ (Theorem 4.10 (b)).

Further, we show that for any C^*-algebra A the following conditions are equivalent:

(i) $A/J_{wc}(A)$ is abelian.

(ii) $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$.

(iii) $\mathrm{CQ}(A)$ is closed under addition.

(iv) $\mathrm{CQ}(A)$ is closed under multiplication.

(v) $\mathrm{CQ}(A)$ is norm-closed.

(Theorem 4.12). If A is postliminal or an AF-algebra, then the conditions (i)-(v) are also equivalent to the condition

(vi) For any $x \in \mathrm{CQ}(A)$, $x^n \in \mathrm{CQ}(A)$ for all positive integers n.

(Corollary 4.16). We also show that (vi) does not have to imply (i)-(v) for general (separable nuclear) C^*-algebras (Example 4.21). The methods for these results involve the lifting of nilpotent elements, commutators and simple projectionless C^*-algebras.

We finish with an example of a separable continuous-trace C^*-algebra A for which $J_{wc}(A) = Z(A) = \{0\}$ but $\mathrm{CQ}(A)$ is norm-dense in A (Example 4.25). In other words, although no non-zero ideal of A has the CQ-property, the set V_A of elements which prevent the CQ-property of A has empty interior in A.

2 Preliminaries

Throughout this paper A will be a C^*-algebras with the centre $Z(A)$. By $\mathcal{S}(A)$ we denote the set of all states on A. As usual, if $x, y \in A$ then $[x, y]$ stands for the commutator $xy - yx$. If A is non-unital, we denote the (minimal) unitization of A by $A^\sharp$. If A is unital we assume that $A^\sharp = A$.

By an ideal of A we shall always mean a closed two-sided ideal. If X is a subset of A, then $\mathrm{Id}_A(X)$ denotes the ideal of A generated by X. An ideal I of A is said to be *modular* if the quotient A/I is unital. If I is an ideal of A then it is well-known that $Z(I) = I \cap Z(A)$.

The set of all primitive ideals of A is denoted by $\mathrm{Prim}(A)$. As usual, we equip $\mathrm{Prim}(A)$ with the Jacobson topology. It is well-known that any proper modular ideal of A (if such exists) is contained in some modular maximal ideal of A (see e.g. [30, Lemma 1.4.2]) and that all modular maximal ideals of A are primitive. We denote the set of all modular maximal ideals of A by $\mathrm{Max}(A)$, so that $\mathrm{Max}(A) \subseteq \mathrm{Prim}(A)$. Note that $\mathrm{Max}(A)$ can be empty (e.g. the algebra $A = \mathrm{K}(\mathcal{H})$ of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$).

Remark 2.1. Every non-modular primitive ideal of a C^*-algebra A contains $Z(A)$. Indeed, if $z \in Z(A)$ then for $P \in \mathrm{Prim}(A)$, $z + P \in Z(A/P)$ and so is either zero or a multiple of the identity in A/P if A/P has one. Therefore, if P is non-modular, we have $z \in P$ for all $z \in Z(A)$, so $Z(A) \subseteq P$. In particular, if the set of all non-modular primitive ideals of A is dense in $\mathrm{Prim}(A)$, then $Z(A) = \{0\}$. □

For any subset $S \subseteq \mathrm{Prim}(A)$ we define its kernel $\ker S$ as the intersection of all elements of S. For the case $S = \emptyset$, we define $\ker S = A$. Note that S is closed in $\mathrm{Prim}(A)$ if and only if for any $P \in \mathrm{Prim}(A)$, $\ker S \subseteq P$ implies $P \in S$.

For any ideal I of A we define the following two subsets of $\mathrm{Prim}(A)$:

$$\mathrm{Prim}_I(A) := \{P \in \mathrm{Prim}(A) : \ I \not\subseteq P\} \quad \text{and} \quad \mathrm{Prim}^I(A) := \{P \in \mathrm{Prim}(A) : \ I \subseteq P\}.$$

Then $\mathrm{Prim}_I(A)$ is an open subset of $\mathrm{Prim}(A)$ homeomorphic to $\mathrm{Prim}(I)$ via the map $P \mapsto P \cap I$, while $\mathrm{Prim}^I(A)$ is a closed subset of $\mathrm{Prim}(A)$ homeomorphic to $\mathrm{Prim}(A/I)$ via the map $P \mapsto P/I$ (see e.g. [41, Proposition A.27]). Similarly, we introduce the following subsets of $\mathrm{Max}(A)$:

$$\mathrm{Max}_I(A) := \{M \in \mathrm{Max}(A) : \ I \not\subseteq M\} \quad \text{and} \quad \mathrm{Max}^I(A) := \{M \in \mathrm{Max}(A) : \ I \subseteq M\}.$$

We shall frequently use the next simple fact which is probably well-known but as we have been unable to find a reference we include a proof for completeness.

Lemma 2.2. Let A be a C^*-algebra and let I be an arbitrary ideal of A. Then the assignment $M \mapsto M \cap I$ defines a homeomorphism from the set $\mathrm{Max}_I(A)$ onto the set $\mathrm{Max}(I)$.

□

Proof. For $M \in \mathrm{Max}_I(A)$ set $\psi(M) := M \cap I$.

Let $M \in \mathrm{Max}_I(A)$. Since $I \not\subseteq M$, by maximality and modularity of M we have $I + M = A$ and A/M is a simple unital C^*-algebra. Using the canonical isomorphism $I/(M \cap I) \cong (I+M)/M = A/M$, we conclude that $I/(M \cap I)$ is also a simple unital C^*-algebra, so $\psi(M) = M \cap I \in \mathrm{Max}(I)$.

The injectivity of the map ψ follows directly from the injectivity of the assignment $\mathrm{Prim}_I(A) \to \mathrm{Prim}(I)$, $P \mapsto P \cap I$, and the fact that all modular maximal ideals of A are primitive.

To show the surjectivity of ψ, choose an arbitrary $N \in \mathrm{Max}(I)$. Then there is $M \in \mathrm{Prim}_I(A)$ such that $N = M \cap I$. Since $I/N \cong (I+M)/M$, it follows that $(I+M)/M$ is a unital ideal of the primitive C^*-algebra A/M. This forces $(I+M)/M = A/M$, so A/M is a unital simple C^*-algebra. Thus, $M \in \mathrm{Max}_I(A)$.

Finally, since ψ is a restriction of a canonical homeomorphism $\mathrm{Prim}_I(A) \to \mathrm{Prim}(I)$ on $\mathrm{Max}_I(A)$ with the image $\mathrm{Max}(I)$, it is itself a homeomorphism. ■

Remark 2.3. It is also easy to see that for any ideal I of a C^*-algebra A, the assignment $M \mapsto M/I$ defines a homeomorphism from the set $\mathrm{Max}^I(A)$ onto the set $\mathrm{Max}(A/I)$, but we shall not use this fact in this paper. □

If A is a C^*-algebra and P a primitive ideal of A such that $Z(A) \not\subseteq P$, then $P \cap Z(A)$ is a maximal ideal of $Z(A)$. In particular, if A is unital, then the map

$$\mathrm{Prim}(A) \to \mathrm{Max}(Z(A)) \qquad \text{defined by} \qquad P \mapsto P \cap Z(A)$$

if a well-defined continuous surjection. We shall continue to assume that A is unital in this paragraph and the next. For all $P, Q \in \mathrm{Prim}(A)$ we define

$$P \approx Q \qquad \text{if} \qquad P \cap Z(A) = Q \cap Z(A).$$

By the Dauns-Hofmann theorem [41, Theorem A.34], there exists an isomorphism

$$\Psi_A : Z(A) \to C(\mathrm{Prim}(A)) \qquad \text{such that} \qquad z + P = \Psi_A(z)(P)1 + P$$

for all $z \in Z(A)$ and $P \in \mathrm{Prim}(A)$ (note that $\mathrm{Prim}(A)$ is compact, as A is unital [14, II.6.5.7]). Hence, for all $P, Q \in \mathrm{Prim}(A)$ we have

$$P \approx Q \qquad \iff \qquad f(P) = f(Q) \quad \text{for all } f \in C(\mathrm{Prim}(A)).$$

Note that $\approx$ is an equivalence relation on $\mathrm{Prim}(A)$ and the equivalence classes are closed subsets of $\mathrm{Prim}(A)$. It follows there is one-to-one correspondence between the quotient set $\mathrm{Prim}(A)/\approx$ and a set of ideals of A given by

$$[P]_{\approx} \longleftrightarrow \ker[P]_{\approx} \quad (P \in \mathrm{Prim}(A)),$$

where $[P]_{\approx}$ denotes the equivalence class of P. The set of ideals obtained in this way is denoted by Glimm(A), and its elements are called *Glimm ideals* of A. The quotient map

$$\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A), \qquad \phi_A(P) := \ker[P]_{\approx}$$

is known as the *complete regularization map*. We equip Glimm(A) with the quotient topology, which coincides with the complete regularization topology, since A is unital. In this way Glimm(A) becomes a compact Hausdorff space. In fact, Glimm(A) is homeomorphic to Max($Z(A)$) via the assignment $G \mapsto G \cap Z(A)$ (see [9] for further details).

Definition 2.4. A C^*-algebra A is said to be *quasi-central* if no primitive ideal of A contains $Z(A)$. □

Quasi-central C^*-algebras were introduced by Delaroche in [19]. We have the following useful characterisation of quasi-central C^*-algebras.

Proposition 2.5. [5, Proposition 1] Let A be a C^*-algebra. The following conditions are equivalent:

(i) A is quasi-central.

(ii) A admits a central approximate unit, i.e. there exists an approximate unit (e_α) of A such that $e_\alpha \in Z(A)$ for all α.

□

Remark 2.6. It is easily seen from Proposition 2.5 (ii) that quasi-centrality passes to quotients and tensor products. □

We have the following prominent examples of quasi-central C^*-algebras.

Example 2.7. (a) Every unital C^*-algebra is obviously quasi-central.

(b) Every abelian C^*-algebra is quasi-central. More generally, a C^*-algebra A is said to be *n-homogeneous* if all irreducible representations of A have the same finite dimension n (note that the abelian C^*-algebras are precisely 1-homogeneous C^*-algebras). Then by [32, Theorem 4.2] Prim(A) is a (locally compact) Hausdorff space and by a well-known theorem of Fell [22, Theorem 3.2] and Tomiyama-Takesaki [43, Theorem 5] there is a locally trivial bundle $\mathcal{E}$ over Prim(A) with fibre $\mathrm{M}_n(\mathbb{C})$ and structure group $\mathrm{Aut}(\mathrm{M}_n(\mathbb{C})) \cong PU(n)$ (the projective unitary group) such that A is isomorphic to the C^*-algebra $\Gamma_0(\mathcal{E})$ of continuous sections of $\mathcal{E}$ that vanish at infinity. Using the local triviality of the underlying bundle $\mathcal{E}$ one can now easily check that $A \cong \Gamma_0(\mathcal{E})$ is quasi-central (see also [32, p. 236]).

(c) For a locally compact group G, the following conditions are equivalent:

(i) The full group C^*-algebra $C^*(G)$ is quasi-central.

(ii) The reduced group C^*-algebra $C^*_r(G)$ is quasi-central.

(iii) G is an SIN-group (that is, the identity has a base of neighbourhoods that are invariant under conjugation by elements of G).

(See [34, Corollary 1.3] and the remark which follows it.)

□

Remark 2.8. Let A be an arbitrary C^*-algebra.

(a) Using the Hewitt-Cohen factorization theorem (see e.g. [15, Theorem A.6.2]) we have

$$K_A := \mathrm{Id}_A(Z(A)) = Z(A)A = \{za : z \in Z(A), a \in A\}$$

(finite sums are not needed). In particular, A is quasi-central if and only if $A = Z(A)A$ ([23, Proposition 3.2]).

(b) The ideal K_A is in fact the largest quasi-central ideal of A [19]. Indeed, K_A contains $Z(A)$, so $Z(K_A) = K_A \cap Z(A) = Z(A)$. Therefore, $Z(K_A)K_A = K_A$, so K_A is quasi-central. On the other hand, if K is an arbitrary quasi-central ideal of A, then (a) implies $K = Z(K)K$. Since $Z(K) = K \cap Z(A) \subseteq Z(A)$, $K \subseteq Z(A)A = K_A$.

(c) Since $P \in \mathrm{Prim}(A)$ contains $Z(A)$ if and only if P contains K_A, it follows

$$K_A = \ker\{P \in \mathrm{Prim}(A) : Z(A) \subseteq P\}.$$

(d) If A is quasi-central, then all primitive ideals of A are modular. This follows directly from Remark 2.1.

□

The following well-known example shows that the converse of Remark 2.8 (d) is not true in general. First recall that a C^*-algebra A is called *n-subhomogeneous* ($n \in \mathbb{N}$) if all irreducible representations of A have dimension at most n and A also admits an n-dimensional irreducible representation.

Example 2.9. Consider the C^*-algebra A that consists of all continuous functions $f : [0,1] \to \mathrm{M}_2(\mathbb{C})$ such that $f(1) = \mathrm{diag}(\lambda(f), 0)$, for some scalar $\lambda(f) \in \mathbb{C}$. Since A is 2-subhomogeneous, all primitive ideals of A are modular. On the other hand,

$$Z(A) = \{\mathrm{diag}(f, f) : f \in C([0,1]), f(1) = 0\}$$

is contained in the kernel of the one-dimensional (hence irreducible) representation $\lambda : A \to \mathbb{C}$, defined by the assignment $\lambda : f \mapsto \lambda(f)$. Hence, A is not quasi-central. In fact the largest quasi-central ideal K_A of A consists of all $f \in A$ such that $f(1) = 0$, since the primitive ideals of this ideal have the form

$$\{f \in A : f(t) = 0 \text{ and } f(1) = 0\}$$

for $t \in [0,1)$. □

3 Characterisations of C^*-algebras with the CQ-property

We begin this section with the following C^*-algebraic version of [16, Proposition 2.1] in which for $a \in A$,

$$[a, A] := \{[a, x] : x \in A\}.$$

Since the proof requires only obvious changes, we omit it.

Proposition 3.1. Let A be a C^*-algebra. The following conditions are equivalent:

(i) A has the CQ-property.

(ii) For every $*$-epimorphism $\phi : A \to B$, where B is another C^*-algebra, $\phi(Z(A)) = Z(B)$.

(iii) For every $a \in A$, $a \in Z(A) + \mathrm{Id}_A([a, A])$.

□

The next fact was obtained in [4, Lemma 2.2.3] but we include the details here for completeness.

Proposition 3.2. If a C^*-algebra A has the CQ-property, so do all ideals and quotients of A. □

Proof. Assume that A has the CQ-property and let I be an ideal of A.

If J is an ideal of I, then J is an ideal of A and I/J is an ideal of A/J. The CQ-property of A implies

$$Z(I/J) = (I/J) \cap Z(A/J) = (I/J) \cap ((Z(A) + J)/J). \tag{3.1}$$

Let $a \in I$ such that $a + J \in (Z(A) + J)/J$. Then there is $z \in Z(A)$ such that $a - z \in J \subseteq I$, so $z \in I \cap Z(A) = Z(I)$. It follows that

$$(I/J) \cap ((Z(A) + J)/J) = (Z(I) + J)/J,$$

so by (3.1),

$$Z(I/J) = (Z(I) + J)/J.$$

Therefore, I has the CQ-property.

We now show that A/I has the CQ-property. Let $q_I : A \to A/I$ be the canonical map and $\phi : A/I \to B$ any $*$-epimorphism, where B is another C^*-algebra. Then $\phi \circ q_I : A \to B$ is a $*$-epimorphism, so the CQ-property of A implies

$$Z(B) = Z((\phi \circ q_I)(A)) = (\phi \circ q_I)(Z(A)) = \phi(q_I(Z(A))) = \phi(Z(A/I)).$$

Therefore, A/I has the CQ-property. ■

Proposition 3.3. For a C^*-algebra A the following conditions are equivalent:

(i) $Z(A) = \{0\}$ and A has the CQ-property.

(ii) Every primitive ideal of A is non-modular.

□

Proof. (i) $\Longrightarrow$ (ii). Assume that $Z(A) = \{0\}$ and that A has the CQ-property. Then for any $P \in \mathrm{Prim}(A)$ we have $Z(A/P) = (Z(A) + P)/P = \{0\}$, so P is non-modular.

(ii) $\Longrightarrow$ (i). Assume that all primitive ideals of A are non-modular. By Remark 2.1 $Z(A) = \{0\}$. Also, for any ideal I of A, all primitive ideals of A/I are non-modular, so Remark 2.1 again implies $Z(A/I) = \{0\}$. Thus, A has the CQ-property. ■

The following result was obtained in [4, Proposition 2.2.4] but we give a shorter argument in one direction by using the method of [16, Proposition 2.15].

Proposition 3.4. For a non-unital C^*-algebra A the following conditions are equivalent:

(i) A has the CQ-property.

(ii) $A^\sharp$ has the CQ-property.

□

Proof. (i) $\Longrightarrow$ (ii). Suppose that A has the CQ-property and let $\lambda 1 + a \in A^\sharp$, where $a \in A$ and $\lambda \in \mathbb{C}$. Then, by Proposition 3.1, we have $a \in Z(A) + \mathrm{Id}_A([a, A])$. Since

$$\mathrm{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp]) = \mathrm{Id}_A([a, A]),$$

it follows that $a \in Z(A) + \mathrm{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp])$. Since $Z(A^\sharp) = \mathbb{C}1 + Z(A)$ we conclude that

$$\lambda 1 + a \in Z(A^\sharp) + \mathrm{Id}_{A^\sharp}([\lambda 1 + a, A^\sharp]).$$

Therefore, by Proposition 3.1, $A^\sharp$ has the CQ-property.

(ii) $\Longrightarrow$ (i). Since A is an ideal of $A^\sharp$, this follows directly from Proposition 3.2. ■

We now extend the notion of weak centrality to arbitrary C^*-algebras.

Definition 3.5. We say that a C^*-algebra A is *weakly central* if the following two conditions are satisfied:

(a) No modular maximal ideal of A contains $Z(A)$.

(b) For each pair of modular maximal ideals M_1 and M_2 of A, $M_1 \cap Z(A) = M_2 \cap Z(A)$ implies $M_1 = M_2$.

□

Note that if A is unital then the above definition agrees with the standard notion of weak centrality.

Remark 3.6. Since all modular maximal ideals of a C^*-algebra A are primitive and since each modular primitive ideal of A is contained in a modular maximal ideal of A, the condition (a) in Definition 3.5 can be restated as:

(a') No modular primitive ideal of A contains $Z(A)$.

□

The justification of Definition 3.5 will be given in the following series of results. First consider one example.

Example 3.7. Let X be a compact Hausdorff space, $\mathcal{H}$ a separable infinite-dimensional Hilbert space and $A := C(X, \mathrm{K}(\mathcal{H}))$. Then each primitive ideal of A is of the form $P_t := \{f \in A : f(t) = 0\}$ for some $t \in X$. Since $A/P_t \cong \mathrm{K}(\mathcal{H})$ for all $t \in X$, all primitive ideals of A are maximal and non-modular. It follows from Proposition 3.3 that $Z(A) = \{0\}$ and A has the CQ-property. Secondly, $\mathrm{Max}(A) = \emptyset$ so that A is trivially weakly central even though $P_t \cap Z(A) = \{0\}$ for all of the maximal ideals P_t. On the other hand, $A^\sharp$ can be identified with the C^*-subalgebra of $B := C(X, \mathrm{B}(\mathcal{H}))$ that consists of all functions $f \in B$ for which there exists a scalar λ such that $f(t) - \lambda 1 \in \mathrm{K}(\mathcal{H})$ for all $t \in X$. Then A is the unique maximal ideal of $A^\sharp$ and hence $A^\sharp$ is weakly central. □

Proposition 3.8. For a non-unital C^*-algebra A the following conditions are equivalent:

(i) A is weakly central.

(ii) $A^\sharp$ is weakly central.

□

Proof. (i) $\implies$ (ii). Assume that A is weakly central and let $M_1, M_2 \in \mathrm{Max}(A^\sharp)$ such that $M_1 \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$.

If one of M_1 or M_2 is A, so is the other. Indeed, assume for example that $M_1 = A$. If $M_2 \neq A$ then, by Lemma 2.2, $M_2 \cap A$ is a modular maximal ideal of A. We have $Z(A) = A \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$ and so $Z(A)$ is contained in $M_2 \cap A$, contradicting the weak centrality of A.

Therefore assume that both M_1 and M_2 are not A. Again, by Lemma 2.2, $N_1 := M_1 \cap A$ and $N_2 := M_2 \cap A$ are modular maximal ideals of A such that $N_1 \cap Z(A) = N_2 \cap Z(A)$. The weak centrality of A forces $N_1 = N_2$, so Lemma 2.2 implies $M_1 = M_2$. Hence, $A^\sharp$ is weakly central.

(ii) $\implies$ (i). Suppose that $A^\sharp$ is weakly central. Let $M \in \mathrm{Max}(A)$. By Lemma 2.2, there exists $N \in \mathrm{Max}(A^\sharp)$ such that $M = N \cap A$. Since $N \neq A$, it follows that

$$N \cap Z(A^\sharp) \neq A \cap Z(A^\sharp) = Z(A).$$

But $Z(A)$ is a maximal ideal in $Z(A^\sharp)$. Thus $Z(A)$ is not contained in N and consequently neither in M.

Now suppose that $M_1, M_2 \in \operatorname{Max}(A)$ and $M_1 \cap Z(A) = M_2 \cap Z(A)$. By Lemma 2.2, there exist $N_1, N_2 \in \operatorname{Max}(A^\sharp)$ such that $M_1 = N_1 \cap A$ and $M_2 = N_2 \cap A$. We have

$$(N_1 \cap Z(A^\sharp)) \cap Z(A) = M_1 \cap Z(A) = (N_2 \cap Z(A^\sharp)) \cap Z(A).$$

By the previous paragraph, M_1 and M_2 do not contain $Z(A)$. It follows that the maximal ideals $N_1 \cap Z(A^\sharp)$ and $N_2 \cap Z(A^\sharp)$ of $Z(A^\sharp)$ do not contain $Z(A)$ and hence must be equal by Lemma 2.2 (applied to the ideal $Z(A)$ of $Z(A^\sharp)$). By the weak centrality of $A^\sharp$, we have $N_1 = N_2$ and hence $M_1 = M_2$. Thus A is weakly central. ∎

As a direct consequence of Vesterstrøm's theorem (Theorem 1.1) and Propositions 3.8 and 3.4 we get the next characterisation.

Corollary 3.9. For a C^*-algebra A the following conditions are equivalent:

(i) A has the CQ-property.

(ii) A is weakly central.

□

Remark 3.10. An immediate consequence of Proposition 3.2 and Corollary 3.9 is that the class of weakly central C^*-algebras is closed under forming ideals and quotients. □

The next simple fact follows directly from Corollary 3.9, Remark 2.8 (d) and Remark 3.6.

Proposition 3.11. For a C^*-algebra A the following conditions are equivalent:

(i) A is quasi-central and weakly central.

(ii) A has the CQ-property and every primitive ideal of A is modular.

□

Part (b) of the next result overlaps with [8, Corollary 2.13], but the proof here avoids the use of a composition series.

Corollary 3.12. Let A be a postliminal C^*-algebra.

(a) A has the (singleton) Dixmier property if and only if A is weakly central.

(b) If every irreducible representation of A is infinite-dimensional, then A has the CQ-property and the (singleton) Dixmier property and is weakly central.

□

Before proving Corollary 3.12 we record the next simple fact which will be also used in Section 4.

Remark 3.13. Let A be a postliminal C^*-algebra. If $\operatorname{Max}(A) \neq \emptyset$, then for each $M \in \operatorname{Max}(A)$, A/M is a unital simple postliminal C^*-algebra and thus $A/M \cong \mathrm{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. □

Proof of Corollary 3.12. (a) By [8, Theorem 2.12] a postliminal C^*-algebra A has the (singleton) Dixmier property if and only if it has the CQ-property. It remains to apply Corollary 3.9.

(b) Assume that A contains a modular maximal ideal M. By Remark 3.13 $A/M \cong \mathrm{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Therefore, A has a finite-dimensional irreducible representation; a contradiction. Thus all primitive ideals of A are non-modular (Remark 3.6), so by Proposition 3.3 A has the CQ-property. The other properties follow from [8, Theorem 2.12] and Corollary 3.9. ■

For the main results of this section (Theorems 3.16 and 3.22), we shall need to consider the following subsets of $\mathrm{Max}(A)$ for an arbitrary C^*-algebra A:

- T_A^1 as the set of all $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$.
- T_A^2 as the set of all $M \in \mathrm{Max}(A)$ for which exists $N \in \mathrm{Max}(A)$ such that $M \neq N$, $Z(A) \nsubseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$.
- $T_A := T_A^1 \cup T_A^2$.

The set T_A^1 is obviously closed in $\mathrm{Max}(A)$. The next example shows that this is not generally true for the set T_A^2 (and consequently T_A).

Example 3.14. Let A be the C^*-algebra consisting of all functions $a \in C([0,1], \mathrm{M}_2(\mathbb{C}))$ which are diagonal at $1/n$ for all $n \in \mathbb{N}$ and scalar at zero. Then A is a unital 2-subhomogeneous C^*-algebra, so $\mathrm{Max}(A) = \mathrm{Prim}(A)$ and

$$T_A = T_A^2 = \{P \in \mathrm{Prim}(A) : \exists Q \in \mathrm{Prim}(A),\ P \neq Q,\ P \cap Z(A) = Q \cap Z(A)\}.$$

Let $\lambda_n(a)$, $\mu_n(a)$ and $\eta(a)$ be complex numbers such that

$$a(1/n) = \mathrm{diag}(\lambda_n(a), \mu_n(a)) \quad (n \in \mathbb{N}) \quad \text{and} \quad a(0) = \mathrm{diag}(\eta(a), \eta(a)).$$

If we denote by λ_n, μ_n and η the 1-dimensional (irreducible) representations of A defined respectively by the assignments $a \mapsto \lambda_n(a)$, $a \mapsto \mu_n(a)$ and $a \mapsto \eta(a)$, it is easy to verify that

$$T_A = \{\ker \lambda_n : n \in \mathbb{N}\} \cup \{\ker \mu_n : n \in \mathbb{N}\}.$$

Then $\ker T_A$ consists of all functions in A which vanish at $1/n$ ($n \in \mathbb{N}$), and hence vanish at 0 too. Therefore, $\ker \eta \in \overline{T_A} \setminus T_A$, so T_A is not closed in $\mathrm{Max}(A) = \mathrm{Prim}(A)$. □

Lemma 3.15. If A is a C^*-algebra, then $\ker T_A$ contains any weakly central ideal of A. □

Proof. Let J be a weakly central ideal of A. Suppose that $M \in T_A^1$, so that $Z(A) \subseteq M$. Then $J \subseteq M$, for otherwise $M \cap J \in \mathrm{Max}(J)$ (Lemma 2.2) and $M \cap J$ contains $Z(J)$, contradicting the weak centrality of

J. Secondly, suppose that $M \in T_A^2$. Then there exists $N \in \mathrm{Max}(A)$ such that $N \neq M$, $Z(A) \not\subseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$. We have

$$(M \cap J) \cap Z(J) = (M \cap Z(A)) \cap J = (N \cap J) \cap Z(J). \tag{3.2}$$

Suppose that M does not contain J. Then $M \cap J \in \mathrm{Max}(J)$ (Lemma 2.2) and $M \cap J$ does not contain $Z(J)$ by the weak centrality of J. By (3.2), $N \cap J$ does not contain $Z(J)$ and hence N does not contain J. Since J is weakly central, it follows from (3.2) that $M \cap J = N \cap J$ and hence (again by Lemma 2.2) $M = N$; a contradiction. Thus $J \subseteq M$ as required. ∎

Given a C^*-algebra A we also define

$$S_A := \{P \in \mathrm{Prim}(A) : \ P \text{ is non-modular}\} \qquad \text{and} \qquad J_A := \ker S_A.$$

By Remarks 2.1 and 2.8 (c), J_A contains the largest quasi-central ideal K_A of A, so in particular $Z(J_A) = Z(A)$. Example 2.9 shows that J_A can strictly contain K_A (in this case $J_A = A$).

Theorem 3.16. For a C^*-algebra A the following conditions are equivalent:

(i) A has the CQ-property.

(ii) A is weakly central.

(iii) S_A is a closed subset of $\mathrm{Prim}(A)$ and $J_A = K_A$ is a quasi-central weakly central C^*-algebra.

(iv) There is a weakly central ideal J of A such that all primitive ideals of A that contain J are non-modular.

(v) There is an ideal J of A such that both J and A/J have the CQ-property and $Z(A/J) = (Z(A) + J)/J$.

□

In the proof of implication (v) $\Longrightarrow$ (i) of Theorem 3.16 we shall use the next simple fact.

Lemma 3.17. Let A be a C^*-algebra and J an ideal of A such that A/J has the CQ-property and $Z(A/J) = (Z(A) + J)/J$. Then $Z(A/I) = (Z(A) + I)/I$ for any ideal I of A that contains J. □

Proof. Let $a \in A$ and suppose that $a + I \in Z(A/I)$. Let $\phi : A/I \to (A/J)/(I/J)$ be the canonical isomorphism. Then

$$\phi(a + I) \in Z((A/J)/(I/J)) = \frac{Z(A/J) + I/J}{I/J} = \frac{(Z(A) + J)/J + I/J}{I/J},$$

so there exists $z \in Z(A)$ such that $\phi(a + I) = \phi(z + I)$. Hence $a + I = z + I$ and so $a \in Z(A) + I$, as required. ∎

Proof of Theorem 3.16. (i) $\Longleftrightarrow$ (ii). This is Corollary 3.9.

(ii) $\Longrightarrow$ (iii). Assume that A is weakly central. Let $P \in \mathrm{Prim}(A)$ be in the closure of S_A in $\mathrm{Prim}(A)$, that is $J_A \subseteq P$. Then $Z(A) = Z(J_A) \subseteq P$. Since A is weakly central, P must be non-modular (Remark 3.6), so $P \in S_A$. Therefore S_A is closed in $\mathrm{Prim}(A)$.

Since J_A is an ideal of A and A is weakly central, so is J_A by Remark 3.10. It remains to show that $J_A = K_A$. Since $K_A \subseteq J_A$, it suffices to show that J_A is quasi-central. Assume there exists $R \in \mathrm{Prim}(J_A)$ that contains $Z(A) = Z(J_A)$ and let $P \in \mathrm{Prim}_{J_A}(A)$ such that $R = P \cap J_A$. Obviously $Z(A) \subseteq P$. Since S_A is closed in $\mathrm{Prim}(A)$, the set $\mathrm{Prim}_{J_A}(A)$ consists of all modular primitive ideals of A. In particular, P is a modular primitive ideal of A that contains $Z(A)$, which (together with Remark 3.6) contradicts the weak centrality of A.

(iii) $\Longrightarrow$ (iv). Choose $J = J_A = K_A$.

(iv) $\Longrightarrow$ (v). Let J be a weakly central ideal of A such that all primitive ideals in $\mathrm{Prim}^J(A)$ are non-modular. By Corollary 3.9 J has the CQ-property. Also, all primitive ideals of A/J are non-modular, so by Proposition 3.3 $Z(A/J) = \{0\}$ and A/J has the CQ-property. Further, since $J = \ker \mathrm{Prim}^J(A)$ and $Z(A)$ is contained in each $P \in \mathrm{Prim}^J(A)$ (Remark 2.1), $Z(A) \subseteq J$. Thus

$$(Z(A) + J)/J = \{0\} = Z(A/J).$$

(v) $\Longrightarrow$ (i). Assume that A does not have the CQ-property. By Corollary 3.9 this is equivalent to say that A is not weakly central. Since J has the CQ-property, it is weakly central (Corollary 3.9), so by Lemma 3.15 J is contained in $\ker T_A$. We have the following two possibilities.

Case 1. There is $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$. Then $M \in T_A^1$ so $J \subseteq \ker T_A \subseteq M$. Thus, by Lemma 3.17,

$$\mathbb{C} \cong Z(A/M) = (Z(A) + M)/M = \{0\};$$

a contradiction.

Case 2. There are distinct $M, N \in \mathrm{Max}(A)$ such that $Z(A) \nsubseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$. Then

$$Z\left(\frac{A}{M \cap N}\right) \cong Z(A/M) \oplus Z(A/N) \cong \mathbb{C} \oplus \mathbb{C}.$$

On the other hand, $M, N \in T_A^2$, so $J \subseteq \ker T_A \subseteq M \cap N$. Since $Z(A) \nsubseteq M$, $M \cap Z(A)$ is a maximal ideal of $Z(A)$, so using Lemma 3.17 we get

$$Z\left(\frac{A}{M \cap N}\right) = \frac{Z(A) + (M \cap N)}{M \cap N} \cong \frac{Z(A)}{(M \cap N) \cap Z(A)} = \frac{Z(A)}{M \cap Z(A)} \cong \mathbb{C};$$

a contradiction. ■

Recall that a C^*-algebra A is said to be *central* if A is quasi-central and for all $P_1, P_2 \in \mathrm{Prim}(A)$, $P_1 \cap Z(A) = P_2 \cap Z(A)$ implies $P_1 = P_2$ (see [31, Section 9]).

Remark 3.18. It is well-known that a quasi-central C^*-algebra A is central if and only if $\text{Prim}(A)$ is a Hausdorff space (see e.g. [18, Proposition 3]). In particular, by Example 2.7 (b), all homogeneous C^*-algebras are central. Further, all central C^*-algebras are obviously weakly central. □

Corollary 3.19. A liminal C^*-algebra A has the CQ-property if and only if the set of all kernels of infinite-dimensional irreducible representations of A is closed in $\text{Prim}(A)$ and the intersection of these kernels is a central C^*-algebra. □

Proof. Since A is liminal, an irreducible representation of A is infinite-dimensional if and only if its kernel is a non-modular primitive ideal of A. Thus

$$S_A = \{\ker \pi : [\pi] \in \hat{A},\ \pi \text{ infinite-dimensional}\},$$

where $\hat{A}$ denotes the spectrum of A. Hence, by Theorem 3.16, A has the CQ-property if and only if S_A is closed in $\text{Prim}(A)$ and $J_A = \ker S_A$ is a quasi-central weakly central C^*-algebra. Suppose that S_A is closed in $\text{Prim}(A)$. Then all irreducible representations of J_A are finite-dimensional. In particular, all primitive ideals of J_A are modular and maximal, so weak centrality and quasi-centrality of J_A in this case is equivalent to centrality. ■

We also record the following special case of Corollary 3.19.

Corollary 3.20. If all irreducible representations of a C^*-algebra A are finite-dimensional, then A has the CQ-property if and only if A is central. □

Remark 3.21. In contrast to Proposition 3.8 the multiplier algebras of weakly central C^*-algebras do not have to be weakly central. In fact, Somerset and the first-named author exhibited an example of a homogeneous (hence central) C^*-algebra A such that $\text{Prim}(M(A))$ is not Hausdorff [10, Theorem 1]. Specifically, the subhomogeneous C^*-algebra $M(A)$ (see e.g. [14, Proposition IV.1.4.6]) is not (weakly) central. □

It is possible to show that every C^*-algebra contains a largest ideal with the CQ-property by using Zorn's lemma and the fact that the sum of two ideals with the CQ-property has the CQ-property. However, in view of Corollary 3.9, we are able to take a different approach that has the merit of obtaining a formula for this ideal in terms of the set T_A of those modular maximal ideals of A which witness the failure of the weak centrality of A.

Theorem 3.22. Let A be a C^*-algebra. Then $\ker T_A$ is the largest weakly central ideal of A. □

Proof. Set $J := \ker T_A$. By Lemma 3.15 it suffices to prove that J is weakly central. For this, we begin by assuming that A is unital (so that $T_A = T_A^2$) and that J is not weakly central. We have two possibilities.

Case 1. There is $M_0 \in \text{Max}(J)$ such that $Z(J) \subseteq M_0$. By Lemma 2.2 there exists $N_0 \in \text{Max}_J(A)$ such that $M_0 = N_0 \cap J$. Since

$$\ker\{N \cap Z(A) : N \in T_A\} = Z(J) \subseteq N_0 \cap Z(A) \in \text{Max}(Z(A)),$$

there is a net (N_α) in T_A such that

$$\lim_{\alpha} N_\alpha \cap Z(A) = N_0 \cap Z(A). \tag{3.3}$$

in $\mathrm{Max}(Z(A))$. Since A is unital, $\mathrm{Max}(A)$ is a compact subspace of $\mathrm{Prim}(A)$, so there is a subnet $(N_{\alpha(\beta)})$ of (N_α) that converges to some $N_0' \in \mathrm{Max}(A)$. Then the continuity of the map $\mathrm{Max}(A) \to \mathrm{Max}(Z(A))$, defined by $M \mapsto M \cap Z(A)$, implies that

$$\lim_{\beta} N_{\alpha(\beta)} \cap Z(A) = N_0' \cap Z(A). \tag{3.4}$$

Since $\mathrm{Max}(Z(A))$ is Hausdorff, (3.3) and (3.4) imply

$$N_0 \cap Z(A) = N_0' \cap Z(A). \tag{3.5}$$

Obviously N_0' lies in the closure of T_A, so $J \subseteq N_0'$. Since $N_0 \in \mathrm{Max}_J(A)$, $N_0 \neq N_0'$, so (3.5) implies $N_0, N_0' \in T_A$. In particular, $J \subseteq N_0$; a contradiction.

Case 2. There are $M_1, M_2 \in \mathrm{Max}(J)$ such that $M_1 \neq M_2$, $Z(J) \not\subseteq M_1, M_2$ and $M_1 \cap Z(J) = M_2 \cap Z(J)$. By Lemma 2.2 there are $N_1, N_2 \in \mathrm{Max}_J(A)$ such that $M_1 = N_1 \cap J$ and $M_2 = N_2 \cap J$. Since $Z(J) = J \cap Z(A)$ is an ideal of $Z(A)$,

$$N_1 \cap Z(A), N_2 \cap Z(A) \in \mathrm{Max}_{Z(J)}(Z(A))$$

and

$$(N_1 \cap Z(A)) \cap Z(J) = M_1 \cap Z(J) = (N_2 \cap Z(A)) \cap Z(J),$$

Lemma 2.2 (applied to $Z(A)$ and its ideal $Z(J)$) implies that $N_1 \cap Z(A) = N_2 \cap Z(A)$. Since $N_1 \neq N_2$, we conclude that $N_1, N_2 \in T_A$, so $J \subseteq N_1 \cap N_2$; a contradiction.

We have now established that $\ker T_A$ is weakly central in the case that A is unital. We suppose next that A is non-unital. Then, by the above arguments, $\ker T_{A^\sharp}$ is a weakly central ideal of $A^\sharp$. Since $\ker T_{A^\sharp} \cap A$ is an ideal of $\ker T_{A^\sharp}$, it is weakly central by Remark 3.10. Hence, it suffices to show that

$$J := \ker T_A \subseteq \ker T_{A^\sharp} \cap A.$$

So let $M \in T_{A^\sharp}$. We only have to show that M contains J. Since $A^\sharp$ is unital, $T_{A^\sharp} = T^2_{A^\sharp}$, so there is $M' \in \mathrm{Max}(A^\sharp)$ such that $M' \neq M$ and $M \cap Z(A^\sharp) = M' \cap Z(A^\sharp)$. We distinguish three possibilities.

- $M = A$. Then clearly M contains J.
- $M' = A$. Then M does not contain A, so by Lemma 2.2 $M \cap A$ is a modular maximal ideal of A containing $Z(A)$ (since M' does). Therefore, $M \cap A$ is in T^1_A and hence contains J.
- Both M and M' are not A. Again, by Lemma 2.2, $M \cap A$ and $M' \cap A$ are distinct modular maximal ideals of A having the same intersection with $Z(A)$. So either $M \cap A$ is in T^1_A or it is in T^2_A. In either case M

contains J.

■

In the sequel, for any C^*-algebra A by $J_{wc}(A)$ we denote the largest weakly central ideal of A. By Corollary 3.9, $J_{wc}(A)$ is precisely the largest ideal of A with the CQ-property.

Corollary 3.23. Let A be a C^*-algebra.

(a) For any ideal I of A we have $J_{wc}(I) = I \cap J_{wc}(A)$.

(b) The sum of any two weakly central ideals of A is a weakly central ideal of A.

□

Proof. (a) Since $J_{wc}(I)$ is a weakly central ideal of I, it is also a weakly central ideal of A, so $J_{wc}(I) \subseteq I \cap J_{wc}(A)$.

Conversely, since $I \cap J_{wc}(A)$ is an ideal of $J_{wc}(A)$, it is weakly central by Remark 3.10. Hence, $I \cap J_{wc}(A) \subseteq J_{wc}(I)$.

(b) If I_1 and I_2 are weakly central ideals of A, then by Theorem 3.22 both I_1 and I_2 are contained in $J_{wc}(A)$, so $I_1 + I_2 \subseteq J_{wc}(A)$. Thus, $I_1 + I_2$ is weakly central by Remark 3.10. ■

The next two examples demonstrate that there are non-trivial C^*-algebras whose largest weakly central ideal is zero.

Example 3.24. Let A be the rotation algebra (the C^*-algebra of the discrete three-dimensional Heisenberg group, see [2] and the references therein). For each t in the unit circle $\mathbb{T}$, there is an ideal J_t of A such that, with $A_t := A/J_t$, A is $*$-isomorphic to a continuous field of C^*-algebras $(A_t)_{t\in\mathbb{T}}$ via the assignment $a \mapsto (a + J_t)_{t\in\mathbb{T}}$. This isomorphism maps $Z(A)$ onto $C(\mathbb{T})$. If $t \in \mathbb{T}$ is a root of unity then A_t is a non-simple homogeneous C^*-algebra. If P is a primitive ideal of A that contains J_t then $P \in \mathrm{Max}(A)$, $P \cap Z(A) = J_t \cap Z(A) \neq Z(A)$ and hence $P \in T_A^2$. It follows that $J_{wc}(A) = \ker T_A \subseteq J_t$. Since the roots of unity form a dense subset of $\mathbb{T}$ and the field $(A_t)_{t\in\mathbb{T}}$ is continuous, $J_{wc}(A) = \{0\}$. Consequently, no non-zero ideal of A has the CQ-property. □

Example 3.25. Consider the C^*-algebra $A = C^*(\mathbb{F}_2)$ (the full C^*-algebra of the free group $\mathbb{F}_2$ on two generators). Then by [17], A is a unital primitive residually finite-dimensional C^*-algebra (that is, the intersection of the kernels of the finite-dimensional irreducible representations of A is $\{0\}$). As A is unital and primitive, $Z(A) = \mathbb{C}1$, so $T_A = \mathrm{Max}(A)$. In particular, $J_{wc}(A) = \ker T_A$ is contained in the intersection of the kernels of the finite-dimensional irreducible representations of A which is zero. Therefore $J_{wc}(A) = \{0\}$, as with the rotation algebra. □

Remark 3.26. Both C^*-algebras in Examples 3.24 and 3.25 are antiliminal. In Example 4.25 we shall also give an example of a (separable) continuous-trace C^*-algebra A for which $J_{wc}(A) = \{0\}$.

On the other hand, if A is a C^*-algebra for which all irreducible representations have finite dimension, then it follows from [24, Corollary 3.8] that the ideal $J_{wc}(A)$ is essential. □

We record next a slightly surprising result which can be used for a direct argument that the sum of two ideals with the CQ-property has the CQ-property.

Proposition 3.27. Let A be a C^*-algebra and let J and K be ideals of A. If one of J or K has the CQ-property, then $Z(J+K) = Z(J) + Z(K)$. □

Proof. Assume that J has the CQ-property and let $z \in Z(J+K)$. Then $z + K \in Z((J+K)/K)$. Let $\phi : (J+K)/K \to J/(J \cap K)$ be the canonical isomorphism. Then, since J has the CQ-property and $J \cap K$ is an ideal of J,

$$\phi(z+K) \in Z\left(\frac{J}{J \cap K}\right) = \frac{Z(J) + (J \cap K)}{J \cap K}.$$

So there exists $y \in Z(J)$ such that

$$\phi(z+K) = y + (J \cap K) = \phi(y+K).$$

Hence $z + K = y + K$ and so $z - y \in K \cap Z(A) = Z(K)$. It follows that $Z(J+K) \subseteq Z(J) + Z(K)$. For the reverse inclusion, observe that

$$(J \cap Z(A)) + (K \cap Z(A)) \subseteq (J+K) \cap Z(A).$$

■

The next example shows that if both ideals J and K of a C^*-algebra A fail to satisfy the CQ-property, then $Z(J+K)$ can strictly contain $Z(J) + Z(K)$.

Example 3.28. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and let $p \in \mathrm{B}(\mathcal{H})$ be any projection with infinite-dimensional kernel and image. Set

$$A := \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p) \subset \mathrm{B}(\mathcal{H})$$

[20, NOTE 1, p.257]. Then A has precisely two maximal ideals, namely

$$J := \mathrm{K}(\mathcal{H}) + \mathbb{C}p \qquad \text{and} \qquad K := \mathrm{K}(\mathcal{H}) + \mathbb{C}(1-p).$$

Obviously $Z(J) = Z(K) = \{0\}$, but $Z(J+K) = Z(A) = \mathbb{C}1$.

For later use, we also note that

$$J_{wc}(A) = \ker T_A = J \cap K = \mathrm{K}(\mathcal{H})$$

and hence $A/J_{wc}(A)$ is abelian. □

We finish this section with a generalization of [3, Theorem 3.1] for arbitrary C^*-algebras. For C^*-algebras A_1 and A_2, we denote their algebraic tensor product by $A_1 \odot A_2$. If β is any C^*-norm on $A_1 \odot A_2$, we denote the completion of $A_1 \odot A_2$ with respect to β by $A_1 \otimes_\beta A_2$.

Theorem 3.29. Let A_1 and A_2 be C^*-algebras. The following conditions are equivalent:

(i) Both A_1 and A_2 have the CQ-property.

(ii) $A_1 \otimes_\beta A_2$ has the CQ-property for every C^*-norm β.

(iii) $A_1 \otimes_\beta A_2$ has the CQ-property for some C^*-norm β.

□

Proof. (i) $\implies$ (ii). Suppose that A_1 and A_2 have the CQ-property and that β is a C^*-norm on $A_1 \odot A_2$. Since $A_i^\sharp \subseteq A_i^{**}$ $(i = 1, 2)$, it follows from [6, Theorem 2] that there is a C^*-norm β' on $A_1^\sharp \odot A_2^\sharp$ extending β (recall that by our convention $A_i^\sharp = A_i$ if A_i is unital). Since A_i $(i = 1, 2)$ has the CQ-property if and only if $A_i^\sharp$ is weakly central, by [3, Theorem 3.1] $A_1^\sharp \otimes_{\beta'} A_2^\sharp$ is weakly central. Hence, $A_1^\sharp \otimes_{\beta'} A_2^\sharp$ has the CQ-property and so does its ideal $A_1 \otimes_\beta A_2$ (Proposition 3.2).

(ii) $\implies$ (iii). This is trivial.

(iii) $\implies$ (i). Assume that $A_1 \otimes_\beta A_2$ has the CQ-property for some C^*-norm β. Since the minimal tensor product $A_1 \otimes_{\min} A_2$ is $*$-isomorphic to a quotient of $A_1 \otimes_\beta A_2$, it follows from Proposition 3.2 that $A_1 \otimes_{\min} A_2$ has the CQ-property. We show that A_1 has the CQ-property (a similar argument applies to A_2). So let I be an ideal of A_1 and let $q_I : A_1 \to A_1/I$ be the canonical map. Using the canonical $*$-epimorphism

$$q_I \otimes \mathrm{id}_{A_2} : A_1 \otimes_{\min} A_2 \to (A/I) \otimes_{\min} A_2$$

and two applications of [27, Corollary 1], we have

$$\begin{aligned} Z(A_1/I) \otimes Z(A_2) &= Z((A_1/I) \otimes_{\min} A_2) = (q_I \otimes \mathrm{id}_{A_2})(Z(A_1) \otimes Z(A_2)) \\ &= q_I(Z(A_1)) \otimes Z(A_2). \end{aligned} \tag{3.6}$$

Assume that $q_I(Z(A_1))$ is strictly contained in $Z(A_1/I)$. Then there is a non-zero functional $\varphi \in Z(A_1/I)^*$ that annihilates $q_I(Z(A_1))$. If ψ is any non-zero functional on $Z(A_2)$, then $\varphi \otimes \psi$ is a non-zero functional on $Z(A_1/I) \otimes Z(A_2)$ that annihilates $q_I(Z(A_1)) \otimes Z(A_2)$, contradicting (3.6). Thus

$$Z(A_1/I) = q_I(Z(A_1)) = (Z(A_1) + I)/I$$

as desired.

■

4 CQ-elements in C^*-algebras

Motivated by [16], and with a view to identifying the individual elements which prevent the CQ-property, we now introduce a local version of the CQ-property.

Definition 4.1. Let A be a C^*-algebra. We say that an element $a \in A$ is a *CQ-element* of A if for every ideal I of A, $a + I \in Z(A/I)$ implies $a \in Z(A) + I$. □

By CQ(A) we denote the set of all CQ-elements of A. Obviously A has the CQ-property if and only if CQ$(A) = A$. We also define $V_A := A \setminus \mathrm{CQ}(A)$, which is the set of elements which prevent A from having the CQ-property.

We state the following C^*-algebraic version of [16, Proposition 2.2].

Proposition 4.2. Let A be a C^*-algebra and let $a \in A$. The following conditions are equivalent:

(i) $a \in \mathrm{CQ}(A)$.

(ii) For every $*$-epimorphism $\phi : A \to B$, where B is another C^*-algebra, $\phi(a) \in Z(B)$ implies $a \in Z(A) + \ker \phi$.

(iii) $a \in Z(A) + \mathrm{Id}_A([a, A])$.

□

Proposition 4.3. Let A be a C^*-algebra.

(a) CQ(A) is a self-adjoint subset of A that is closed under scalar multiplication.

(b) $Z(A) + \mathrm{CQ}(A) \subseteq \mathrm{CQ}(A)$.

(c) If I is an ideal of A then $\mathrm{CQ}(I) = I \cap \mathrm{CQ}(A)$. In particular, I has the CQ-property if and only if $I \subseteq \mathrm{CQ}(A)$.

(d) If A is unital, then for any $a \in \mathrm{CQ}(A)$ and invertible $x \in A$ we have $xax^{-1} \in \mathrm{CQ}(A)$.

□

Proof. (a) This is trivial.

(b) Let $a \in \mathrm{CQ}(A)$ and $z \in Z(A)$. By Proposition 4.2, $a \in Z(A) + \mathrm{Id}_A([a, A])$, so

$$z + a \in Z(A) + \mathrm{Id}_A([a, A]) = Z(A) + \mathrm{Id}_A([z + a, A]).$$

Using again Proposition 4.2 it follows that $z + a \in \mathrm{CQ}(A)$.

(c) Let $a \in \mathrm{CQ}(I)$. By Proposition 4.2, $a \in Z(I) + \mathrm{Id}_I([a, I])$. Since $Z(I) = I \cap Z(A) \subseteq Z(A)$ and $\mathrm{Id}_I([a, I]) \subseteq \mathrm{Id}_A([a, A])$, we get $a \in Z(A) + \mathrm{Id}_A([a, A])$. Therefore $a \in \mathrm{CQ}(A)$, so $\mathrm{CQ}(I) \subseteq I \cap \mathrm{CQ}(A)$.

Conversely, let $a \in I \cap \mathrm{CQ}(A)$ and $\varepsilon > 0$. By Proposition 4.2 there is a finite number of elements $u_i, v_i, x_i \in A$ and $z \in Z(A)$ such that

$$\left\| a - z - \sum_i u_i[a, x_i]v_i \right\| < \frac{\varepsilon}{3}. \tag{4.1}$$

In particular, $\|z + I\| < \varepsilon/3$, so using the canonical isomorphism $(Z(A) + I)/I \cong Z(A)/(I \cap Z(A))$ we can find an element $z' \in I \cap Z(A) = Z(I)$ such that

$$\|z - z'\| < \frac{\varepsilon}{3}. \tag{4.2}$$

Let (e_α) be an approximate identity for I. Then for all indices i

$$\lim_\alpha u_i e_\alpha [a, e_\alpha x_i] e_\alpha v_i = u_i[a, x_i]v_i. \tag{4.3}$$

Indeed, since the multiplication on A is continuous, it suffices to show that for any $x \in A$,

$$\lim_\alpha e_\alpha [a, e_\alpha x] e_\alpha = [a, x].$$

But this follows directly from the estimate

$$\begin{aligned} \|e_\alpha[a, e_\alpha x]e_\alpha - [a, x]\| &\leq \|e_\alpha([a, e_\alpha x] - [a, x])e_\alpha\| + \|e_\alpha([a, x]e_\alpha - [a, x])\| + \|e_\alpha[a, x] - [a, x]\| \\ &\leq \|[a, e_\alpha x] - [a, x]\| + \|[a, x]e_\alpha - [a, x]\| + \|e_\alpha[a, x] - [a, x]\|. \end{aligned}$$

Hence, by (4.3) there are $u_i', v_i', x_i' \in I$ such that

$$\left\| \sum_i u_i[a, x_i]v_i - \sum_i u_i'[a, x_i']v_i' \right\| < \frac{\varepsilon}{3}. \tag{4.4}$$

Then by (4.1), (4.2) and (4.4)

$$\left\| a - z' - \sum_i u_i'[a, x_i']v_i' \right\| < \varepsilon.$$

Invoking again Proposition 4.2, we conclude that $a \in \mathrm{CQ}(I)$, so $I \cap \mathrm{CQ}(A) \subseteq \mathrm{CQ}(I)$.

(d) Assume that A is unital, $a \in \mathrm{CQ}(A)$ and $x \in A$ invertible. If I is an arbitrary ideal of A such that $xax^{-1} + I \in Z(A/I)$, then $a + I \in Z(A/I)$. Since $a \in \mathrm{CQ}(A)$, this implies $a \in Z(A) + I$. Then also $xax^{-1} \in Z(A) + I$, so $xax^{-1} \in \mathrm{CQ}(A)$. ■

Corollary 4.4. If A is a C^*-algebra, then $Z(A) + J_{wc}(A) \subseteq \mathrm{CQ}(A)$. □

Proof. By Theorem 3.22 and Corollary 3.9 $J_{wc}(A) = \ker T_A$ has the CQ-property, so by Proposition 4.3 (c), $J_{wc}(A) \subseteq \mathrm{CQ}(A)$. It remains to apply Proposition 4.3 (b). ■

Proposition 4.5. Let A be a C^*-algebra.

(a) All commutators $[a, b]$ $(a, b \in A)$ belong to $\mathrm{CQ}(A)$. In particular, $\mathrm{CQ}(A) = Z(A)$ if and only if A is abelian.

(b) All quasi-nilpotent elements of A belong to $\mathrm{CQ}(A)$. Moreover if $a \in A$ is quasi-nilpotent, then $ab, ba \in \mathrm{CQ}(A)$ for any $b \in A$.

□

Proof. If $x \in A$ is a commutator, quasi-nilpotent, or a product by a quasi-nilpotent element, we claim that for any primitive ideal P of A, $x + P \in Z(A/P)$ implies $x \in P$. It then follows that $x \in \mathrm{CQ}(A)$. Indeed, assume that I is an ideal of A such that $x + I \in Z(A/I)$. Then $x + P \in Z(A/P)$ for any $P \in \mathrm{Prim}^I(A)$, so $x \in P$. As $\ker \mathrm{Prim}^I(A) = I$, it follows that $x \in I$ and thus $x \in \mathrm{CQ}(A)$ as claimed.

So assume that P is a primitive ideal of A such that $x + P \in Z(A/P)$. If P is non-modular, then $Z(A/P) = \{0\}$, so trivially $x \in P$. Hence, assume that P is modular, so that $Z(A/P) \cong \mathbb{C}$. Then there is a scalar λ such that

$$x + P = \lambda 1_{A/P}. \tag{4.5}$$

(a) Assume that x is a commutator, so that $x = [a, b]$ for some $a, b \in A$. Then by (4.5),

$$[a + P, b + P] = x + P = \lambda 1_{A/P}.$$

As A/P is a unital C^*-algebra, it is well-known that this is only possible if $\lambda = 0$. Thus $x \in P$ as claimed.

If A is non-abelian, then there are $a, b \in A$ such that $x := [a, b] \neq 0$. Then there is $P \in \mathrm{Prim}(A)$ such that $x \notin P$, so by the above arguments $x + P \notin Z(A/P)$. In particular, $x \in \mathrm{CQ}(A) \setminus Z(A)$.

(b) If x is quasi-nilpotent, so is $x + P$, since by the spectral radius formula

$$\nu(x + P) = \lim_n \|x^n + P\|^{\frac{1}{n}} \leq \lim_n \|x^n\|^{\frac{1}{n}} = \nu(x) = 0.$$

This together with (4.5) forces $\lambda = 0$, so $x \in P$.

Now assume that $x = ab$, where $a, b \in A$ and a is quasi-nilpotent. As $a + P$ is quasi-nilpotent, it is a topological divisor of zero (see e.g. [25, Section XXIX.4]). Hence, there is a sequence of elements (x_n) in A such that

$$\|x_n + P\| = 1 \quad \forall n \in \mathbb{N} \qquad \text{and} \qquad \lim_n \|x_n a + P\| = 0.$$

Then, by (4.5), for all $n \in \mathbb{N}$ we have

$$\lambda x_n + P = (x_n + P)(x + P) = (x_n a + P)(b + P),$$

so

$$|\lambda| \leq \|x_n a + P\| \|b + P\|. \tag{4.6}$$

Since the right side in (4.6) tends to zero as n tends to infinity, we conclude that $\lambda = 0$. Therefore $x \in P$ as claimed.

Finally, using the facts that $a \in A$ is quasi-nilpotent if and only if a^* is quasi-nilpotent and that $\mathrm{CQ}(A)$ is a self-adjoint subset of A (Proposition 4.3 (a)), we also conclude that $ba \in \mathrm{CQ}(A)$ for any $b \in A$ and quasi-nilpotent $a \in A$. ■

Remark 4.6. By Proposition 4.5 (a), non-abelian C^*-algebras always contain non-central CQ-elements. On the other hand, in a purely algebraic setting, there are examples of non-abelian algebras in which all centrally stable elements are central [16, Examples 2.5 and 2.6] (where central stability is the algebraic counterpart of the CQ-property). □

If a C^*-algebra A is unital, the following fundamental result gives a necessary and sufficient condition for a central element of A/I to lift to a central element of A. Recall that by $\Psi_A : Z(A) \to C(\mathrm{Prim}(A))$ we denote the Dauns-Hofmann isomorphism.

Theorem 4.7. Let A be a unital C^*-algebra and let I be an ideal of A. Assume that an element $a \in A$ satisfies $a + I \in Z(A/I)$. Then $a \in Z(A) + I$ if and only if

$$\Psi_{A/I}(a + I)(P_1/I) = \Psi_{A/I}(a + I)(P_2/I) \tag{4.7}$$

for all $P_1, P_2 \in \mathrm{Prim}^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$. □

Proof. First assume that $a \in Z(A) + I$, so that $a - z \in I$ for some $z \in Z(A)$. Suppose that $P_1, P_2 \in \mathrm{Prim}^I(A)$ and that $P_1 \cap Z(A) = P_2 \cap Z(A)$. For $i = 1, 2$, there exists $\lambda_i \in \mathbb{C}$ such that

$$a + P_i = z + P_i = \lambda_i 1 + P_i \quad (\text{in } A/P_i).$$

Then $z - \lambda_1 1, z - \lambda_2 1 \in P_1 \cap Z(A)$ and so $(\lambda_1 - \lambda_2)1 \in P_1$. It follows that $\lambda_1 = \lambda_2$ $(= \lambda$, say). Hence

$$(a + I) + P_i/I = \lambda(1 + I) + P_i/I \quad (\text{in } (A/I)/(P_i/I))$$

and therefore

$$\Psi_{A/I}(a + I)(P_1/I) = \lambda = \Psi_{A/I}(a + I)(P_2/I).$$

Conversely, assume that the equality (4.7) holds for all $P_1, P_2 \in \mathrm{Prim}^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$. Since A is unital, $\mathrm{Prim}(A)$ is compact and so the closed subset $\mathrm{Prim}^I(A)$ is also compact. Define a function

$$f \in C(\mathrm{Prim}^I(A)) \qquad \text{by the formula} \qquad f(P) := \Psi_{A/I}(a + I)(P/I).$$

Let $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ be the complete regularization map (see Section 2). Since $\mathrm{Prim}^I(A)$ is a compact subspace of $\mathrm{Prim}(A)$, ϕ_A continuous and $\mathrm{Glimm}(A)$ a compact Hausdorff space, $K := \phi_A(\mathrm{Prim}^I(A))$ is a compact (hence closed) subset of $\mathrm{Glimm}(A)$. Define a function

$$g : K \to \mathbb{C} \qquad \text{by} \qquad g(G) := f(P),$$

where P is any primitive ideal in $\mathrm{Prim}^I(A)$ such that $\phi_A(P) = G$. Since $f(P_1) = f(P_2)$ for any two $P_1, P_2 \in \mathrm{Prim}^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$ (which is equivalent to $\phi_A(P_1) = \phi_A(P_2)$), g is well-defined. We claim that g is continuous on K. Indeed, let (G_α) be an arbitrary net in K that converges to some $G_0 \in K$. By general topology, it suffices to show that for any subnet $(G_{\alpha(\beta)})$ of (G_α) there is a further subnet $(G_{\alpha(\beta(\gamma))})$ such that $(g(G_{\alpha(\beta(\gamma))})$ converges to $g(G_0)$. For each index β choose $P_{\alpha(\beta)} \in \mathrm{Prim}^I(A)$ such that $\phi_A(P_{\alpha(\beta)}) = G_{\alpha(\beta)}$. Then $(P_{\alpha(\beta)})$ is a net in the compact space $\mathrm{Prim}^I(A)$, so it has a subnet $(P_{\alpha(\beta(\gamma))})$ convergent to some $P_0 \in \mathrm{Prim}^I(A)$. Since ϕ_A is continuous and $\mathrm{Glimm}(A)$ Hausdorff, $G_0 = \phi_A(P_0)$. Further, since f is continuous on $\mathrm{Prim}^I(A)$, $(f(P_{\alpha(\beta(\gamma))})$ converges to $f(P_0)$. Therefore

$$\lim_\gamma g(G_{\alpha(\beta(\gamma))}) = \lim_\gamma f(P_{\alpha(\beta(\gamma))}) = f(P_0) = g(G_0).$$

By the Tietze extension theorem, there exists a continuous function $\tilde{g} \in C(\mathrm{Glimm}(A))$ that extends g. Then a function

$$\tilde{f} : \mathrm{Prim}(A) \to \mathbb{C} \qquad \text{defined by} \qquad \tilde{f} := \tilde{g} \circ \phi_A$$

is continuous, so by the Dauns-Hofmann theorem there is $z \in Z(A)$ such that $\Psi_A(z) = \tilde{f}$. Since for any $P \in \mathrm{Prim}^I(A)$ we have $\tilde{f}(P) = f(P)$, we conclude $a - z \in P$. Thus $a - z \in I$, so $a \in Z(A) + I$ as desired. ∎

We now describe the set $\mathrm{CQ}(A)$ for an arbitrary C^*-algebra A. It is somewhat easier to describe its complement V_A. In order to do this, we introduce the following sets:

- V_A^1 as the set of all $a \in A$ for which there exists $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in A/M,
- V_A^2 as the set of all $a \in A$ for which there exist $M_1, M_2 \in \mathrm{Max}(A)$ and scalars $\lambda_1 \neq \lambda_2$ such that $Z(A) \nsubseteq M_i$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ $(i = 1, 2)$.

Theorem 4.8. If A is a C^*-algebra then $V_A = V_A^1 \cup V_A^2$. □

Proof. Assume there exists $a \in V_A^1 \setminus V_A$. Then $a \in \mathrm{CQ}(A)$ and there is $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in A/M. In particular, $a + M \in Z(A/M)$, so the CQ-condition implies $a \in Z(A) + M = M$; a contradiction. This shows $V_A^1 \subseteq V_A$.

Now assume there exists $a \in V_A^2 \setminus V_A$ and let $M_1, M_2 \in \mathrm{Max}(A)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq \lambda_2$, such that $Z(A) \nsubseteq M_i$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ $(i = 1, 2)$. Then by maximality of M_1 and M_2

we have $M_1 + M_2 = A$, so $A/(M_1 \cap M_2) \cong (A/M_1) \oplus (A/M_2)$. Hence $a + (M_1 \cap M_2) \in Z(A/(M_1 \cap M_2))$. Since $a \in \mathrm{CQ}(A)$, this forces $a \in Z(A) + (M_1 \cap M_2)$. Choose a central element $z \in Z(A)$ such that $a - z \in M_1 \cap M_2$. Obviously, $z + M_i = \lambda_i 1_{A/M_i}$ $(i = 1, 2)$. On the other hand, under the canonical isomorphisms

$$\frac{Z(A) + M_1}{M_1} \cong \frac{Z(A)}{M_1 \cap Z(A)} = \frac{Z(A)}{M_2 \cap Z(A)} \cong \frac{Z(A) + M_2}{M_2},$$

$z + M_1$ is mapped to $z + M_2$. This implies $\lambda_1 = \lambda_2$; a contradiction. Therefore $V_A^2 \subseteq V_A$.

Conversely, let $a \in V_A$.

Case 1. A is unital.

In this case $V_A^1 = \emptyset$. Since $a \notin \mathrm{CQ}(A)$, there exists an ideal I of A such that $a + I \in Z(A/I)$ but $a \notin Z(A) + I$. For each $P \in \mathrm{Prim}^I(A)$ set

$$\lambda_P := \Psi_{A/I}(a + I)(P/I),$$

where $\Psi_{A/I} : Z(A/I) \to C(\mathrm{Prim}(A/I))$ is the Dauns-Hofmann isomorphism. By Theorem 4.7 there are $P_1, P_2 \in \mathrm{Prim}^I(A)$ such that $P_1 \cap Z(A) = P_2 \cap Z(A)$ and $\lambda_{P_1} \neq \lambda_{P_2}$. Then $a - \lambda_{P_i} 1 \in P_i$ $(i = 1, 2)$. Choose maximal ideals M_1, M_2 of A such that $P_1 \subseteq M_1$ and $P_2 \subseteq M_2$. Since A/M_i is a quotient of A/P_i, it follows that $\lambda_{M_i} = \lambda_{P_i}$, so $a - \lambda_{P_i} 1 \in M_i$ $(i = 1, 2)$. Therefore, $a + M_1$ and $a + M_2$ are distinct scalars in A/M_1 and A/M_2, which implies $a \in V_A^2$.

Case 2. A is non-unital.

In this case we work inside the unitization $A^\sharp$. By Proposition 4.3 (c) $a \in V_{A^\sharp}$. Then, using Case 1, there are maximal ideals M_1 and M_2 of $A^\sharp$ and scalars $\lambda_1 \neq \lambda_2$ such that $M_1 \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$ and $a - \lambda_i 1 \in M_i$ $(i = 1, 2)$. We have two possibilities.

Case 2.1. One of M_1, M_2 coincides with A. Say $M_1 = A$. Then $\lambda_1 = 0$ (since a belongs to A), so $\lambda_2 \neq 0$. In this case $M_2 \neq A$, since otherwise $\lambda_1 = \lambda_2 = 0$; a contradiction. Then, by Lemma 2.2, $N_2 := M_2 \cap A$ is a modular maximal ideal of A. Since $Z(A) = A \cap Z(A^\sharp) = M_2 \cap Z(A^\sharp)$, N_2 contains $Z(A)$. Under the canonical isomorphism

$$\frac{A}{N_2} \cong \frac{A + M_2}{M_2} = \frac{A^\sharp}{M_2},$$

$a + N_2$ is mapped to $a + M_2 = \lambda_2 1 + M_2$, so $a + N_2 = \lambda_2 1_{A/N_2}$. Since $\lambda_2 \neq 0$, we conclude that a belongs to V_A^1.

Case 2.2. $M_1 \neq A$ and $M_2 \neq A$. Then, by Lemma 2.2, $N_i := M_i \cap A$ $(i = 1, 2)$ are modular maximal ideals of A that have the same intersection with $Z(A)$. Similarly as in Case 2.1, using the canonical isomorphisms $A/N_i \cong A^\sharp/M_i$, we conclude that $a + N_i = \lambda_i 1_{A/N_i}$ $(i = 1, 2)$, which implies that a belongs to V_A^2.

Therefore $a \in V_A^1 \cup V_A^2$, so $V_A = V_A^1 \cup V_A^2$ as claimed. ∎

Remark 4.9. Theorem 4.8 enables us to recapture Corollary 3.9 without using Vesterstrøm's theorem for the unital case (Theorem 1.1). Indeed, if A is weakly central then clearly V_A is empty and so A has the CQ-property. Conversely, suppose that A is not weakly central. If there exists $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$ then, taking any $a \in A$ such that $a + M = 1_{A/M}$, we obtain $a \in V_A^1$. Otherwise, there exist distinct $M_1, M_2 \in \mathrm{Max}(A)$ such that $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$. Since $M_1 + M_2 = A$, there exists $a \in M_1$ such that $a + M_2 = 1_{A/M_2}$ and hence $a \in V_A^2$. Thus V_A is non-empty and so A does not have the CQ-property.

Also, the methods of this section enable us to give a short alternative proof of the fact that $\ker T_A$ is weakly central (Theorem 3.22). By the preceding paragraph $\ker T_A$ is weakly central if and only if $V_{\ker T_A} = \emptyset$. By Theorem 4.8 and Proposition 4.3 (c) it suffices to show that $a \in \ker T_A$ implies $a \notin V_A^1 \cup V_A^2 = V_A$. But this is trivial, since for any $M \in \mathrm{Max}(A)$ that contains $Z(A)$ we have $M \in T_A^1$, so $a \in M$ and therefore $a + M$ is zero in A/M. Similarly, for all $M_1, M_2 \in \mathrm{Max}(A)$ such that $M_1 \neq M_2$ and $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$ we have $M_1, M_2 \in T_A^2$, so $a \in M_1 \cap M_2$ and hence $a + M_i$ is zero in A/M_i $(i = 1, 2)$.

□

If A is a C^*-algebra then by Proposition 4.5 (a) all commutators $[a, b]$ $(a, b \in A)$ belong to $\mathrm{CQ}(A)$. Let us denote by $[A, A]$ the linear span of all commutators of A and by $\overline{[A, A]}$ its norm-closure. We now characterise when $\mathrm{CQ}(A)$ contains $\overline{[A, A]}$.

Theorem 4.10. Let A be a C^*-algebra that is not weakly central.

(a) If for all $M \in T_A$, A/M admits a tracial state then $\overline{[A, A]} \subseteq \mathrm{CQ}(A)$.
(b) If there is $M \in T_A$ such that A/M does not admit a tracial state, then $[A, A] \not\subseteq \mathrm{CQ}(A)$.

□

Proof. (a) Let $x \in \overline{[A, A]}$. In order to show that $x \in \mathrm{CQ}(A)$, it suffices by Theorem 4.8 to prove that for each $M \in T_A$, $x + M \in Z(A/M)$ implies $x \in M$. Therefore, fix some $M \in T_A$ and assume that $x + M \in Z(A/M)$, so that $x + M = \lambda 1_{A/M}$ for some scalar λ. By assumption A/M admits a tracial state τ. As $x \in \overline{[A, A]}$, clearly $x + M \in \overline{[A/M, A/M]}$. Since $\tau(\overline{[A/M, A/M]}) = \{0\}$, we get

$$\lambda = \tau(\lambda 1_{A/M}) = \tau(x + M) = 0.$$

Thus $x \in M$, as claimed.

(b) Assume that A/M does not admit a tracial state for some $M \in T_A$. As $T_A = T_A^1 \cup T_A^2$, we have two possibilities.

Case 1. $M \in T_A^1$, so that $Z(A) \subseteq M$. By [40, Theorem 1] there is an integer $n > 1$, that depends only on A/M, such that any element of A/M can be expressed as a sum of n commutators. In particular, there are

$a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that

$$\sum_{i=1}^{n}[a_i, b_i] + M = \sum_{i=1}^{n}[a_i + M, b_i + M] = 1_{A/M},$$

so by Theorem 4.8

$$\sum_{i=1}^{n}[a_i, b_i] \in V_A^1 \subseteq V_A.$$

Case 2. $M \in T_A^2$. Then $Z(A) \nsubseteq M$ and there exists $N \in \mathrm{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. By Lemma 2.2, $M \cap N$ is a modular maximal ideal of N. As $N/(M \cap N) \cong A/M$, $N/(M \cap N)$ also does not admit a tracial state, so by [40, Theorem 1] there is an integer $n > 1$ and elements $a_1, \dots, a_n, b_1, \dots, b_n \in N$ such that

$$\sum_{i=1}^{n}[a_i, b_i] + M \cap N = 1_{N/(M\cap N)}.$$

Using the canonical isomorphism $N/(M \cap N) \cong A/M$, we get

$$\sum_{i=1}^{n}[a_i, b_i] + M = 1_{A/M}$$

and thus by Theorem 4.8

$$\sum_{i=1}^{n}[a_i, b_i] \in V_A^2 \subseteq V_A.$$

■

Corollary 4.11. If A is a postliminal C^*-algebra or an AF-algebra, then $\overline{[A, A]} \subseteq \mathrm{CQ}(A)$. □

Proof. If A is weakly central, then $\mathrm{CQ}(A) = A$ so we have nothing to prove. Hence assume that A is not weakly central, so that there is $M \in T_A$. If A is postliminal then by Remark 3.13 $A/M \cong \mathrm{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$, so A/M has a (unique) tracial state. If, on the other hand, A is an AF-algebra, then A/M is a unital simple AF-algebra, so it also admits a tracial state (see e.g. [33, Proposition 3.4.11]). Therefore, the assertion follows directly from Theorem 4.10 (a). ■

By Corollary 4.4, for any C^*-algebra A, $\mathrm{CQ}(A)$ always contains $Z(A) + J_{wc}(A)$. The next result in particular demonstrates that $\mathrm{CQ}(A)$ is a C^*-subalgebra of A if and only if $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$. In fact, when this does not hold, $\mathrm{CQ}(A)$ fails dramatically to be a C^*-algebra.

Theorem 4.12. Let A be a C^*-algebra. The following conditions are equivalent:

(i) $A/J_{wc}(A)$ is abelian.

(ii) $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$.

(iii) $\mathrm{CQ}(A)$ is closed under addition.

(iv) $\mathrm{CQ}(A)$ is closed under multiplication.

(v) $\mathrm{CQ}(A)$ is norm-closed.

□

Remark 4.13. Since T_A is dense in $\mathrm{Prim}^{J_{wc}(A)}(A)$, it follows from [21, Proposition 3.6.3] that $A/J_{wc}(A)$ is non-abelian if and only if there is $M \in T_A$ such that $\dim(A/M) > 1$. □

Proof of Theorem 4.12. (i) $\Longrightarrow$ (ii). Assume that $A/J_{wc}(A)$ is abelian. By Corollary 4.4 we already know that $Z(A) + J_{wc}(A) \subseteq \mathrm{CQ}(A)$, so it suffices to show the reverse inclusion. For any $a \in A$ we have $a + J_{wc}(A) \in A/J_{wc}(A) = Z(A/J_{wc}(A))$, so if $a \in \mathrm{CQ}(A)$, this forces $a \in Z(A) + J_{wc}(A)$. Therefore $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$, as claimed.

(ii) $\Longrightarrow$ (iii), (iv), (v) is trivial, since $Z(A) + J_{wc}(A)$ is a C^*-subalgebra of A.

(iii), (iv) or (v) $\Longrightarrow$ (i). Assume that $A/J_{wc}(A)$ is non-abelian. By Remark 4.13 there is $M \in T_A$ such that $\dim(A/M) > 1$. We show that $\mathrm{CQ}(A)$ is not norm-closed and is neither closed under addition nor closed under multiplication. As A/M is non-abelian, by [29, Exercise 4.6.30] A/M contains a nilpotent element $\dot{q}$ of nilpotency index 2. By [1, Proposition 2.8] (see also [38, Theorem 6.7]), we may lift $\dot{q}$ to a nilpotent element $q \in A$ of the same nilpotency index 2. As the norm function $\mathrm{Prim}(A) \ni P \mapsto \|q + P\|$ is lower semi-continuous on $\mathrm{Prim}(A)$ (see e.g. [14, Proposition II.6.5.6 (iii)]) and $q \notin M$, the set

$$U := \{P \in \mathrm{Prim}(A) : \|q + P\| > 0\} \tag{4.8}$$

is an open neighbourhood of M in $\mathrm{Prim}(A)$. As $T_A = T_A^1 \cup T_A^2$ we have two possibilities.

Case 1. $M \in T_A^1$, so that $Z(A) \subseteq M$. Let I be the ideal of A that corresponds to U, so that $U = \mathrm{Prim}_I(A)$. As $\mathrm{CQ}(I) = I \cap \mathrm{CQ}(A)$ (Proposition 4.3 (c)), it suffices to show that $\mathrm{CQ}(I)$ is not norm-closed and is neither closed under addition nor closed under multiplication.

By Lemma 2.2, $M \cap I$ is a modular maximal ideal of I that contains $Z(I) = I \cap Z(A)$, so that $M \cap I \in T_I^1$. Choose a self-adjoint element $a \in I$ such that

$$a + (M \cap I) = 1_{I/(M \cap I)}. \tag{4.9}$$

For each non-zero scalar $\mu \in \mathbb{C}$ consider the element

$$x_\mu := a\,(a + \mu q) \in I.$$

We claim that for any $N' \in \mathrm{Max}(I)$, $x_\mu + N' \in Z(I/N')$ implies $x_\mu \in N'$, so that $x_\mu \in \mathrm{CQ}(I)$ (Theorem 4.8). Indeed, assume there is $N' \in \mathrm{Max}(I)$ such that $x_\mu + N' \in Z(I/N')$. By Lemma 2.2 there exists $N \in \mathrm{Max}_I(A)$ such that $N' = N \cap I$. Then

$$x_\mu + (N \cap I) = \lambda 1_{I/(N \cap I)}$$

for some scalar λ, so using the canonical isomorphism $I/(N\cap I)\cong A/N$ we get

$$(a+N)\left((a+N)+\mu(q+N)\right)=x_\mu+N=\lambda 1_{A/N}. \tag{4.10}$$

Suppose that $\lambda\neq 0$. Then, by (4.10), the element $a+N$ is right invertible in A/N. Since $a+N$ is self-adjoint, it must be invertible in A/N. As $\mu\neq 0$, (4.10) implies

$$q+N=\frac{1}{\mu}\left(\lambda(a+N)^{-1}-(a+N)\right). \tag{4.11}$$

The right side in (4.11) defines a normal element of A/N, as a linear combination of two commuting self-adjoint elements of A/N. Hence, $q+N$ is a normal nilpotent element of A/N which implies $q\in N$. But as $N\in \mathrm{Max}_I(A)$, N belongs to U, which contradicts (4.8). Thus $\lambda=0$ and so $x_\mu\in \mathrm{CQ}(I)$ as claimed.

We claim that

$$x_{-1}+x_1\notin \mathrm{CQ}(I)\qquad\text{and}\qquad x_{-1}x_1\notin\mathrm{CQ}(I).$$

Indeed, by (4.9) we have

$$\begin{aligned} x_{-1}+x_1+(M\cap I) &= (a(a-q)+(M\cap I))+(a(a+q)+(M\cap I))=2a^2+(M\cap I)\\ &= 21_{I/(M\cap I)}. \end{aligned}$$

Further, since $q^2=0$, we have

$$\begin{aligned} x_{-1}x_1+(M\cap I) &= (a(a-q)+(M\cap I))(a(a+q)+(M\cap I))\\ &= (1_{I/(M\cap I)}-(q+(M\cap I)))(1_{I/(I\cap M)}+(q+(M\cap I)))\\ &= 1_{I/(M\cap I)}. \end{aligned}$$

Therefore, both $x_{-1}+x_1$ and $x_{-1}x_1$ belong to $V_I^1\subseteq V_I=I\setminus \mathrm{CQ}(I)$ (Theorem 4.8), which shows that $\mathrm{CQ}(I)$ is neither closed under addition nor closed under multiplication.

It remains to show that $\mathrm{CQ}(I)$ is not norm-closed. In order to do this, consider the sequence

$$y_k:=x_{\frac{1}{k}}=a\left(a+\frac{1}{k}q\right)\qquad (k\in\mathbb{N}).$$

Then (y_k) is a sequence in $\mathrm{CQ}(I)$ that converges to a^2. As $a^2+(M\cap I)=1_{I/(M\cap I)}$ (by (4.9)), we conclude that $a^2\in V_I^1\subseteq V_I$ (Theorem 4.8), so the proof for this case is finished.

Case 2. $M\in T_A^2$. Then $Z(A)\nsubseteq M$ and there exists $N\in\mathrm{Max}(A)$ such that $N\neq M$ and $M\cap Z(A)=N\cap Z(A)$. As singleton subsets of $\mathrm{Max}(A)$ are closed in $\mathrm{Prim}(A)$, $U':=U\setminus\{N\}$ is also an open neighbourhood of M in $\mathrm{Prim}(A)$. Let J be the ideal of A that corresponds to U'. Then, by Lemma 2.2, $M \cap J \in \mathrm{Max}(J)$ and $\dim(J/(M \cap J)) = \dim(A/M) > 1$. Further, since $N \notin U'$, $J \subseteq N$, so

$$Z(J) = (N \cap Z(A)) \cap J = (M \cap Z(A)) \cap J.$$

This implies $Z(J) \subseteq M \cap J$ and therefore $M \cap J \in T_J^1$. Also, by (4.8), we have trivially $\|q + P\| > 0$ for all $P \in U'$. By applying the method of Case 1 to J in place of I, we conclude that $\mathrm{CQ}(J)$ is not norm-closed and is neither closed under addition nor closed under multiplication. As $\mathrm{CQ}(J) = J \cap \mathrm{CQ}(A)$ (Proposition 4.3 (c)), the same is true for $\mathrm{CQ}(A)$. ■

Remark 4.14. Observe that Theorem 4.12 applies to Example 3.28, giving $\mathrm{CQ}(A) = \mathbb{C}1 + \mathrm{K}(\mathcal{H})$. □

Corollary 4.15. If A is a 2-subhomogeneous C^*-algebra, then $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$. □

Proof. Since

$$\{P \in \mathrm{Prim}(A) : \dim(A/P) = 1\}$$

is a closed subset of $\mathrm{Prim}(A)$ (see e.g. [21, Proposition 3.6.3]), A has a 2-homogeneous ideal I such that A/I is abelian. Then I is a central C^*-algebra (Remark 3.18) and so $I \subseteq J_{wc}(A)$. Hence $A/J_{wc}(A)$ is abelian and so the result follows from Theorem 4.12. ■

In the case when a C^*-algebra A is postliminal or an AF-algebra, we also show that the conditions (i)-(v) of Theorem 4.12 are equivalent to one additional condition.

Corollary 4.16. If A is a postliminal C^*-algebra or an AF-algebra, then the conditions (i)-(v) of Theorem 4.12 are also equivalent to:

(vi) For any $x \in \mathrm{CQ}(A)$, $x^n \in \mathrm{CQ}(A)$ for all $n \in \mathbb{N}$.

□

In the proof of Corollary 4.16 we shall use the next two facts. In the sequel we say that a C^*-subalgebra A of a unital C^*-algebra B is *co-unital* if A contains the identity of B.

Lemma 4.17. Let B be a unital simple non-abelian AF-algebra. Then B contains a co-unital finite-dimensional C^*-subalgebra with no abelian summand. □

Proof. Let $(B_k)_{k\in\mathbb{N}}$ be an increasing sequence of finite-dimensional C^*-subalgebras of B such that $1_B \in B_k$ for all $k \in \mathbb{N}$ and

$$B = \overline{\bigcup_{k\in\mathbb{N}} B_k}.$$

We claim that there exists $k \in \mathbb{N}$ such that B_k has no direct summand $*$-isomorphic to $\mathbb{C}$. On a contrary, suppose that for every $k \in \mathbb{N}$, B_k has a direct summand $*$-isomorphic to $\mathbb{C}$ and hence a multiplicative state ω_k. For each

$k \in \mathbb{N}$ let $\psi_k \in \mathcal{S}(B)$ be an extension of ω_k. By the weak*-compactness of $\mathcal{S}(B)$ there exists $\psi \in \mathcal{S}(B)$ and a subnet $(\psi_{k(\alpha)})$ of (ψ_k) such that

$$\psi = w^* - \lim_{\alpha} \psi_{k(\alpha)}.$$

Let $a \in B_{j_0}$ and $b \in B_{k_0}$ for some $j_0, k_0 \in \mathbb{N}$. There exists an index α_0 such that $k(\alpha) \geq \max\{j_0, k_0\}$ for all $\alpha \geq \alpha_0$. Thus

$$\psi_{k(\alpha)}(ab) = \psi_{k(\alpha)}(a)\psi_{k(\alpha)}(b)$$

for all $\alpha \geq \alpha_0$ and so

$$\psi(ab) = \psi(a)\psi(b). \tag{4.12}$$

Now suppose that $a \in B_{j_0}$ for some $j_0 \in \mathbb{N}$, that $b \in B$ and that $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ and $b_0 \in B_{k_0}$ such that

$$\|b - b_0\| < \frac{\varepsilon}{2(1 + \|a\|)}.$$

Then

$$\begin{aligned} |\psi(ab) - \psi(a)\psi(b)| &\leq |\psi(ab) - \psi(ab_0)| + |\psi(a)\psi(b_0) - \psi(a)\psi(b)| \\ &\leq 2\|a\|\|b - b_0\| < \varepsilon. \end{aligned}$$

Thus, again (4.12) holds. A similar approximation in the first variable shows that (4.12) holds for all $a, b \in B$. Thus, B has a multiplicative state, contradicting the fact that B is simple but not $*$-isomorphic to $\mathbb{C}$. ■

Lemma 4.18. Let A be a C^*-algebra such that for some $M \in \text{Max}(A)$, A/M contains a co-unital finite-dimensional C^*-subalgebra with no abelian summand. Then there are $a, b \in A$ and an integer $n > 1$ such that

$$[a, b]^n + M = 1_{A/M}. \tag{4.13}$$

□

Proof. Assume that B is a co-unital finite-dimensional C^*-subalgebra of A/M with no abelian summand. Then there are integers $n_1, \ldots, n_k > 1$ and a $*$-isomorphism $\phi : B \to \text{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \text{M}_{n_k}(\mathbb{C})$. For each $i = 1, \ldots, k$ let $\{\alpha_1^{(i)}, \ldots, \alpha_{n_i}^{(i)}\}$ be the set of all n_i-th roots of unity. It is well-known that the set of all commutators of $\text{M}_{n_i}(\mathbb{C})$ consists precisely of all matrices of trace zero. Hence, as $\alpha_1^{(i)} + \ldots + \alpha_{n_i}^{(i)} = 0$ for all $i = 1, \ldots, k$, there are elements $a, b \in A$ such that $a + M, b + M \in B$ and

$$\phi([a, b] + M) = \text{diag}(\alpha_1^{(1)}, \ldots, \alpha_{n_1}^{(1)}) \oplus \cdots \oplus \text{diag}(\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}).$$

Let n be the least common multiple of $n_1, \ldots, n_k$. Then

$$\phi([a,b]+M)^n = 1_{\mathrm{M}_{n_1}(\mathbb{C})} \oplus \cdots \oplus 1_{\mathrm{M}_{n_k}(\mathbb{C})}.$$

Since B is co-unital in A/M, this is equivalent to (4.13). ■

Proof of Corollary 4.16. By Theorem 4.12 we only have to prove the implication (vi) $\Longrightarrow$ (i). Assume that (i) does not hold. By Remark 4.13 there is $M \in T_A$ such that $\dim(A/M) > 1$. If A is postliminal or AF (respectively), then A/M is a unital simple C^*-algebra that is postliminal or AF (respectively). Hence, by Remark 3.13 and Lemma 4.17 A/M certainly contains a co-unital finite-dimensional C^*-subalgebra with no abelian summand. As $T_A = T_A^1 \cup T_A^2$ we have two possibilities.

Case 1. $M \in T_A^1$, so that $Z(A) \subseteq M$. By Lemma 4.18, there are $a, b \in A$ and an integer $n > 1$ such that for $x := [a,b]$ we have $x^n + M = 1_{A/M}$. In particular $x^n \in V_A^1$ so, by Theorem 4.8, $x^n \notin \mathrm{CQ}(A)$. On the other hand, by Proposition 4.5 (a), $x \in \mathrm{CQ}(A)$.

Case 2. $M \in T_A^2$. Then $Z(A) \nsubseteq M$ and there exists $N \in \mathrm{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. By Lemma 2.2, $M \cap N$ is a modular maximal ideal of N and $N/(M \cap N) \cong A/M$.

By Lemma 4.18 (applied to N) there are $a, b \in N$ and an integer $n > 1$ such that for $x := [a,b]$ we have $x^n + M \cap N = 1_{N/(M\cap N)}$. Then, using the canonical isomorphism $N/(M \cap N) \cong A/M$, we get $x^n + M = 1_{A/M}$. As $x^n \in N$, we have $x^n \in V_A^2$, so $x^n \notin \mathrm{CQ}(A)$ by Theorem 4.8. On the other hand, by Propositions 4.5 (a) and 4.3 (c), $x \in \mathrm{CQ}(N) \subseteq \mathrm{CQ}(A)$. ■

The next example shows that 2-subhomogeneity in Corollary 4.15 cannot be replaced by n-subhomogeneity, where $n > 2$. It also provides an example of a liminal C^*-algebra for which the six equivalent conditions of Corollary 4.16 fail to hold.

Example 4.19. Let A be the C^*-algebra consisting of all functions $a \in C([0,1], \mathrm{M}_3(\mathbb{C}))$ such that

$$a(1) = \begin{pmatrix} \lambda_{11}(a) & \lambda_{12}(a) & 0 \\ \lambda_{21}(a) & \lambda_{22}(a) & 0 \\ 0 & 0 & \mu(a) \end{pmatrix},$$

for some complex numbers $\lambda_{ij}(a), \mu(a)$ $(i,j = 1,2)$. Then A is a unital 3-subhomogeneous C^*-algebra such that

$$Z(A) = \{\mathrm{diag}(f,f,f) : \, f \in C([0,1])\}$$

and

$$T_A = T_A^2 = \{\ker \pi, \ker \mu\},$$

where $\pi : A \to \mathrm{M}_2(\mathbb{C})$ and $\mu : A \to \mathbb{C}$ are irreducible representations of A defined by the assignments $\pi : a \mapsto (\lambda_{ij}(a))$ and $\mu : a \mapsto \mu(a)$. Hence, by Theorem 3.22,

$$J_{wc}(A) = \ker T_A = \{a \in A : \, a(1) = 0\}$$

and so

$$Z(A) + J_{wc}(A) = \{a \in A : \, a(1) \text{ is a scalar matrix}\}.$$

As $A/\ker\pi \cong \mathrm{M}_2(\mathbb{C})$, it follows from Theorem 4.12 and the proofs of Lemma 4.18 and Corollary 4.16 that $\mathrm{CQ}(A)$ is not closed under addition and is not norm-closed, and there is $x \in \mathrm{CQ}(A)$ such that $x^2 \notin \mathrm{CQ}(A)$. To show this explicitly, first by Theorem 4.8 we have

$$V_A = A \setminus \mathrm{CQ}(A) = \{a \in A : \exists\lambda, \mu \in \mathbb{C}, \, \lambda \neq \mu, \text{ such that } a(1) = \mathrm{diag}(\lambda, \lambda, \mu)\}.$$

In particular, $\mathrm{CQ}(A)$ strictly contains $Z(A) + J_{wc}(A)$. Let $b := \mathrm{diag}(1, 0, 0)$ and $c := \mathrm{diag}(0, 1, 0)$ be elements of A, considered as constant functions. Then, $b, c \in \mathrm{CQ}(A)$, but $b + c = \mathrm{diag}(1, 1, 0) \notin \mathrm{CQ}(A)$. Similarly, the constant function $x := \mathrm{diag}(-1, 1, 0)$ belongs to $\mathrm{CQ}(A)$ but $x^2 = \mathrm{diag}(1, 1, 0)$ does not.

We now show that $\mathrm{CQ}(A)$ is not norm-closed in A. In fact, we shall show that $\mathrm{CQ}(A)$ is norm-dense in A, so as A is not weakly central, $\mathrm{CQ}(A)$ cannot be norm-closed (for a more general argument see Proposition 4.24). Choose any $a \in A \setminus \mathrm{CQ}(A)$. Then $a(1) = \mathrm{diag}(\lambda, \lambda, \mu)$ for some distinct scalars λ and μ. For any $\varepsilon > 0$, let $b_\varepsilon := \mathrm{diag}(\varepsilon, 0, 0)$ (as a constant function in A). Then $a + b_\varepsilon \in \mathrm{CQ}(A)$ and $\|(a + b_\varepsilon) - a\| = \|b_\varepsilon\| = \varepsilon$. □

We now demonstrate that Corollary 4.16 can fail when A is not assumed to be postliminal or an AF-algebra. In order to do this, first recall that a C^*-algebra B is said to be *projectionless* if B does not contain non-trivial projections. The first example of a simple projectionless C^*-algebra was given by Blackadar [11] (the non-unital example) and [12] (the unital example). Also, the prominent examples of simple projectionless C^*-algebras include the reduced C^*-algebra $C^*_r(\mathbb{F}_n)$ for the free group $\mathbb{F}_n$ on $n < \infty$ generators [39] and the Jiang-Su algebra $\mathcal{Z}$ [28], which also has the important property that it is KK-equivalent to $\mathbb{C}$.

Lemma 4.20. Let B be a unital projectionless C^*-algebra and let $p \in \mathbb{C}[z]$ be a separable polynomial. An element $b \in B$ satisfies $p(b) = 0$ if and only if $b = \mu 1$, where μ is a root of p. □

Proof. First note that since B is projectionless, all elements of B have connected spectrum. Indeed, otherwise by [29, Corollary 3.3.7] B would contain a non-trivial idempotent e and then by [13, Proposition 4.6.2], e would be similar to a (necessarily non-trivial) projection.

If $p \in \mathbb{C}[z]$ is a separable polynomial of degree n, we can factorize

$$p(z) = \alpha(z - \mu_1) \cdots (z - \mu_n),$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$ are distinct roots of p. If $b \in B$ satisfies $p(b) = 0$, then the spectral mapping theorem implies $\sigma(b) \subseteq \{\mu_1, \ldots, \mu_n\}$. As $\sigma(b)$ is connected, this forces $\sigma(b) = \{\mu_k\}$ for some $1 \le k \le n$. Then for all $i \in \{1, \ldots, n\} \setminus \{k\}$, the element $b - \mu_i 1$ is invertible so

$$0 = p(b) = \alpha(b - \mu_1 1) \cdots (b - \mu_n 1)$$

implies $b = \mu_k 1$ as claimed. The converse is trivial. ■

Example 4.21. Let B be any unital simple projectionless non-abelian C^*-algebra (e.g. $B = \mathcal{Z}$, the Jiang-Su algebra).

Consider the C^*-algebra C of all continuous functions $x : [0,1] \to \mathrm{M}_2(B)$ such that $x(1) = \mathrm{diag}(b(x), 0)$ for some $b(x) \in B$ (note that C can be identified with the tensor product $A \otimes B$, where A is the C^*-algebra from Example 2.9, which is nuclear). As B is unital and simple, $Z(B) = \mathbb{C}1_B$, so

$$Z(C) = \{\mathrm{diag}(f1_B, f1_B) : \ f \in C([0,1]),\ f(1) = 0\},$$

where $(f1_B)(t) = f(t)1_B$, for all $t \in [0,1]$. Consider the ideal M of C defined by

$$M := \{x \in C : \ x(1) = 0\} = C_0([0,1), \mathrm{M}_2(B)).$$

As $C/M \cong B$, M is a modular maximal ideal of C that contains $Z(C)$, so that $M \in T_C^1$. Since $Z(M) \cong C_0([0,1))$ and $\mathrm{Prim}(M)$ is canonically homeomorphic to $[0,1)$, it is easy to check directly that M is a central C^*-algebra (alternatively, $M \cong C_0([0,1)) \otimes \mathrm{M}_2(B)$ is weakly central by Theorem 3.29). Therefore,

$$J_{wc}(C) = M \qquad \text{and} \qquad T_C = T_C^1 = \{M\}.$$

By Theorem 4.8 we have

$$\mathrm{CQ}(C) = \{x \in C : \ b(x) \text{ is not a non-zero scalar}\}.$$

As $C/J_{wc}(C) \cong B$ is non-abelian, by Theorem 4.12 $\mathrm{CQ}(C)$ is not norm-closed and is neither closed under addition nor closed under multiplication.

On the other hand we claim that for any $x \in \mathrm{CQ}(C)$, $x^n \in \mathrm{CQ}(C)$ for all $n \in \mathbb{N}$. On a contrary, assume that there exists $x \in \mathrm{CQ}(C)$ such that $x^n \notin \mathrm{CQ}(C)$ for some $n > 1$. Then, by Theorem 4.8, there is a non-zero $\lambda \in \mathbb{C}$ such that $b(x)^n = \lambda 1_B$. Consider the polynomial $p(z) := z^n - \lambda$. As $\lambda \neq 0$, p is separable. Since B is projectionless and $p(b(x)) = 0$, Lemma 4.20 implies that $b(x) = \mu 1_B$, where μ is some n-th root of λ. But this contradicts the fact that $x \in \mathrm{CQ}(C)$. □

If a unital C^*-algebra A is not weakly central then, even though $\mathrm{CQ}(A)$ might be a C^*-subalgebra of A

(and hence equal to $Z(A) + J_{wc}(A)$ by Theorem 4.12), one may use matrix units to show that $\mathrm{CQ}(\mathrm{M}_2(A))$ is neither closed under addition nor closed under multiplication (for the algebraic counterpart, see the comment following [16, Remark 3.6]). In fact, this is a special case of the following more general result.

Proposition 4.22. Let A be a unital C^*-algebra and let B be a unital simple exact C^*-algebra.

(a) $J_{wc}(A \otimes_{\min} B) = J_{wc}(A) \otimes_{\min} B$.

(b) Suppose that A is not weakly central and that B is not abelian (that is, B is not $*$-isomorphic to $\mathbb{C}$). Then $\mathrm{CQ}(A \otimes_{\min} B)$ is not norm-closed and is neither closed under addition nor closed under multiplication. In particular, $\mathrm{CQ}(\mathrm{M}_n(A))$ is not a C^*-subalgebra of $\mathrm{M}_n(A)$ for any $n > 1$.

□

Proof. (a) If A is weakly central then, since B is weakly central, we have that $A \otimes_{\min} B$ is weakly central (see [3, Theorem 3.1] and Theorem 3.29). So we now assume that A is not weakly central, so that $T_A \neq \emptyset$. Let $M \in T_A$. Then there is $N \in \mathrm{Max}(A)$ such that $N \neq M$ and $M \cap Z(A) = N \cap Z(A)$. Since B is exact,

$$\frac{A \otimes_{\min} B}{M \otimes_{\min} B} \cong \frac{A}{M} \otimes_{\min} B,$$

which is a simple C^*-algebra (see [42, Corollary]). Thus $M \otimes_{\min} B \in \mathrm{Max}(A \otimes_{\min} B)$ and similarly $N \otimes_{\min} B \in \mathrm{Max}(A \otimes_{\min} B)$. Let $x \in (M \otimes_{\min} B) \cap (Z(A) \otimes \mathbb{C}1_B)$. For a state $\omega \in \mathcal{S}(B)$ let $L_\omega : A \otimes_{\min} B \to A$ be the corresponding left slice map (i.e. $L_\omega(a \otimes b) = \omega(b)a$, see [45]). There exists $z \in Z(A)$ such that $x = z \otimes 1_B$ and hence $z = L_\omega(x) \in M$. Thus

$$\begin{aligned}(M \otimes_{\min} B) \cap (Z(A) \otimes \mathbb{C}1_B) &= (M \cap Z(A)) \otimes \mathbb{C}1_B = (N \cap Z(A)) \otimes \mathbb{C}1_B \\ &= (N \otimes_{\min} B) \cap (Z(A) \otimes \mathbb{C}1_B).\end{aligned}$$

Note that also $M \otimes_{\min} B \neq N \otimes_{\min} B$ (for otherwise, by using L_ω, we would obtain $M \subseteq N$ and $N \subseteq M$). Since by [27, Corollary 1], $Z(A) \otimes \mathbb{C}1_B = Z(A \otimes_{\min} B)$, we have shown that $M \otimes_{\min} B \in T_{A \otimes_{\min} B}$. By Theorem 3.22

$$J_{wc}(A \otimes_{\min} B) \subseteq \bigcap_{M \in T_A} (M \otimes_{\min} B) = J_{wc}(A) \otimes_{\min} B. \tag{4.14}$$

For the equality in (4.14), let $y \in \bigcap_{M \in T_A} (M \otimes_{\min} B)$ and $\psi \in B^*$. Then

$$L_\psi(y) \in \bigcap_{M \in T_A} M = J_{wc}(A).$$

Hence

$$0 = q(L_\psi(y)) = \mathcal{L}_\psi((q \otimes \mathrm{id}_B)(y)),$$

where $q : A \to A/J_{wc}(A)$ is the canonical map and $\mathcal{L}_\psi : (A/J_{wc}(A)) \otimes_{\min} B \to A/J_{wc}(A)$ is the left slice map. It follows that

$$y \in \ker(q \otimes \mathrm{id}_B) = J_{wc}(A) \otimes_{\min} B,$$

since B is exact.

On the other hand, it follows from Theorem 3.29 and Corollary 3.9 that $J_{wc}(A) \otimes_{\min} B$ is weakly central. Thus $J_{wc}(A \otimes_{\min} B) = J_{wc}(A) \otimes_{\min} B$, as claimed.

(b) Since B is exact, by (a)

$$\frac{A \otimes_{\min} B}{J_{wc}(A \otimes_{\min} B)} = \frac{A \otimes_{\min} B}{J_{wc}(A) \otimes_{\min} B} \cong \frac{A}{J_{wc}(A)} \otimes_{\min} B,$$

which is non-abelian. The result now follows from Theorem 4.12. ■

In contrast to the second paragraph of Remark 3.26 we now demonstrate there are even separable continuous-trace C^*-algebras A such that $Z(A) = J_{wc}(A) = \{0\}$, while $\mathrm{CQ}(A)$ is norm-dense in A. In order to do this, we shall use the following facts.

Lemma 4.23. Let A be a C^*-algebra such that all primitive ideals of A are maximal and both sets of all modular and non-modular primitive ideals are dense in $\mathrm{Prim}(A)$. Then $Z(A) = J_{wc}(A) = \{0\}$. □

Proof. That $Z(A) = \{0\}$ follows from Remark 2.1. Let I be a non-zero ideal of A. Then $Z(I) = I \cap Z(A) = \{0\}$. On the other hand, the dense set of modular primitive ideals of A meets the open set $\mathrm{Prim}_I(A)$. If P is any modular primitive ideal of A that does not contain I then, by assumption, P is maximal, so by Lemma 2.2 $P \cap I$ is a modular primitive ideal of I such that $\{0\} = Z(I) \subseteq P \cap I$. Therefore, I is not weakly central. ■

Proposition 4.24. Let A be a C^*-algebra.

(a) If either there is $M \in \mathrm{Max}(A)$ of codimension 1 such that $Z(A) \subseteq M$ or there are distinct $M_1, M_2 \in \mathrm{Max}(A)$ of codimension 1 that satisfy $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$, then $\mathrm{CQ}(A)$ is not norm-dense in A.

(b) The converse of (a) is true if T_A is countable.

□

Proof. (a) Assume there is $M \in \mathrm{Max}(A)$ of codimension 1 that contains $Z(A)$. Since $A/M \cong \mathbb{C}$, by Theorem 4.8 for any $a \in \mathrm{CQ}(A)$, $a + M$ is zero in A/M, so $a \in M$. Thus, $\mathrm{CQ}(A) \subseteq M$, so $\mathrm{CQ}(A)$ is clearly not norm-dense in A.

Alternatively, assume there are distinct $M_1, M_2 \in \mathrm{Max}(A)$ of codimension 1 such that $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$. Since $A/(M_1 \cap M_2) \cong (A/M_1) \oplus (A/M_2) \cong \mathbb{C} \oplus \mathbb{C}$, Theorem 4.8 implies $\mathrm{CQ}(A) \subseteq \mathbb{C}1 + (M_1 \cap M_2)$, so $\mathrm{CQ}(A)$ is not norm-dense in A.

(b) Now assume that all $M \in \mathrm{Max}(A)$ that contain $Z(A)$ have codimension greater than 1 and for all distinct $M_1, M_2 \in \mathrm{Max}(A)$ that satisfy $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$, at least one M_i has codimension greater than 1. We may assume that $T_A \neq \emptyset$, for otherwise $\mathrm{CQ}(A) = A$, which is certainly dense in A.

For each $M \in T_A$ such that $\dim(A/M) > 1$, set

$$U_M := \{a \in A : \ a + M \text{ is not a scalar in } A/M\}.$$

Evidently, U_M is an open subset of A. We claim that U_M is norm-dense in A. Let $a \in A \setminus U_M$, so that $a + M$ is a scalar in A/M. Let $\varepsilon > 0$. Since A/M is non-abelian, there is a non-central element $\dot{b}$ of norm one in A/M. Then by [46, Lemma 17.3.3], there is a norm one element $b \in A$ such that $b + M = \dot{b}$. Then the element $a + (\varepsilon/2)b$ lies in U_M and its distance from a is $\varepsilon/2$.

If T_A is countable, then the Baire category theorem implies that

$$U := \bigcap\{U_M : \ M \in T_A, \ \dim(A/M) > 1\}$$

is a dense subset of A. Let $a \in U$. If $M \in T_A^1$ then $a \in U_M$ and so $a + M$ is not a scalar in A/M. Also if $M_1, M_2 \in \mathrm{Max}(A)$, such that $M_1 \neq M_2$ and $M_1 \cap Z(A) = M_2 \cap Z(A) \neq Z(A)$, then for some $i \in \{1, 2\}$ we have $\dim(A/M_i) > 1$ so that $a \in U_{M_i}$ and hence $a + M_i$ is not a scalar in A/M_i. Thus, by Theorem 4.8, $U \subseteq \mathrm{CQ}(A)$, so $\mathrm{CQ}(A)$ is norm-dense in A. ■

The next example is a slight variant of [4, Example 4.4] where we have changed the quotient $A(1)$ in order to avoid an abelian quotient.

Example 4.25. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with orthonormal basis $\{e_n : n \geq 0\}$. For each n let E_n be the projection from $\mathcal{H}$ onto the linear span of the set $\{e_0, e_1, \ldots, e_n\}$. We define A to be the subset of $C([0,1], \mathrm{K}(\mathcal{H}))$ consisting of all elements $a \in C([0,1], \mathrm{K}(\mathcal{H}))$ which satisfy the following requirement: For any dyadic rational $t = p/2^q \in [0, 1)$, where p, q are positive integers such that $2 \not| p$, then

$$a(t) = E_q a(t) = a(t) E_q.$$

Then A is a closed self-adjoint subalgebra of $C([0,1], \mathrm{K}(\mathcal{H}))$ and so is itself a C^*-algebra.

As in [4], standard arguments show that A is a continuous-trace C^*-algebra whose primitive ideal space can be identified with $[0, 1]$, via the homeomorphism

$$[0, 1] \ni t \mapsto P_t := \ker \pi_t \in \mathrm{Prim}(A),$$

where for each $t \in [0,1]$ and $a \in A$, $\pi_t(a) := a(t)$. Moreover, if for each $t \in [0,1]$ we denote the fibre of A at t by $A(t)$ (i.e. $A(t) = \{a(t) : a \in A\}$), then

$$A(t) = \begin{cases} \{K \in \mathrm{K}(\mathcal{H}) : E_q K = K E_q = K\} \cong \mathrm{M}_{q+1}(\mathbb{C}), & \text{if } t = p/2^q \text{ as above} \\ \mathrm{K}(\mathcal{H}), & \text{otherwise.} \end{cases}$$

In particular, all primitive ideals of A are maximal. Further, the sets of modular and non-modular primitive ideals of A are both dense in $\mathrm{Prim}(A)$, and so Lemma 4.23 implies $Z(A) = J_{wc}(A) = \{0\}$. On the other hand, since

$$T_A = T_A^1 = \{P_t : t \in [0,1) \text{ is a dyadic rational}\}$$

is countable and the codimension of each $P_t \in T_A$ is larger than 1, Proposition 4.24 implies that $\mathrm{CQ}(A)$ is norm-dense in A. □

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