Q-W–algebras, Zhelobenko operators and a proof of De Concini–Kac–Procesi conjecture

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Introduction

In this monograph we consider a phenomenon which occurs in the study of certain classes and categories of representations of semisimple Lie algebras, groups of Lie type, and the related quantum groups. This phenomenon is similar to the classical Schur–Weyl duality. However, the relevant classes of representations are quite different from the finite-dimensional irreducible representations of the general linear group or, more generally, of complex semisimple Lie groups which appear in the Schur–Weyl setting.

All examples of the above mentioned type are realizations of the following quite general construction a homological version of which was suggested in [94] (in fact, in the quantum group case the construction presented below requires some technical modifications; we shall not discuss them in the introduction). Let A be an associative algebra over a unital ring $\mathbf{k}$, $B \subset A$ a subalgebra with a character $\chi : B \to \mathbf{k}$. Denote by $\mathbf{k}_\chi$ the corresponding rank one representation of B. Let $Q_\chi = A \otimes_B \mathbf{k}_\chi$ be the induced representation of A.

Let $Hk(A, B, \chi) = \mathrm{End}_A(Q_\chi)^{opp}$ be the algebra of A–endomorphisms of Q_χ with the opposite multiplication. One says that the algebra $Hk(A, B, \chi)$ is obtained from A by a quantum constrained reduction with respect to the subalgebra B. $Hk(A, B, \chi)$ is an algebra of Hecke type. Indeed, if A is the group algebra of a Chevalley group over a finite field, B the group algebra of a Borel subgroup in it, and χ is the trivial complex representation of the Borel subgroup one obtains the Iwahori–Hecke algebra this way (see [49]).

For any representation V of A the algebra $Hk(A, B, \chi)$ naturally acts in the space

$$V_\chi = \mathrm{Hom}_A(Q_\chi, V) = \mathrm{Hom}_B(\mathbf{k}_\chi, V)$$

by compositions of homomorphisms and for any $Hk(A, B, \chi)$–module W $Q_\chi \otimes_{Hk(A,B,\chi)} W$ is a left A–module. Let $Hk(A, B, \chi) - \mathrm{mod}$ be the category of left $Hk(A, B, \chi)$–modules and $A - \mathrm{mod}_B^\chi$ the category of left A–modules of the form $Q_\chi \otimes_{Hk(A,B,\chi)} W$, where $W \in Hk(A, B, \chi) - \mathrm{mod}$, with morphisms induces by morphisms of left $Hk(A, B, \chi)$–modules. Then we have the following theorem.

Theorem 1. *$A - \mathrm{mod}_B^\chi$ is a full subcategory in the category of left A–modules, and the functors $\mathrm{Hom}_A(Q_\chi, \cdot)$ and $Q_\chi \otimes_{Hk(A,B,\chi)} \cdot$ yield mutually inverse equivalences of the categories,*

$$A - \mathrm{mod}_B^\chi \simeq Hk(A, B, \chi) - \mathrm{mod}. \tag{1}$$

Proof. Let $W, W' \in Hk(A, B, \chi) - \mathrm{mod}$. Then by the Frobenius reciprocity and by the definition of the algebra $Hk(A, B, \chi)$

$$\mathrm{Hom}_A(Q_\chi, Q_\chi \otimes_{Hk(A,B,\chi)} W) = \mathrm{Hom}_B(\mathbf{k}_\chi, Q_\chi \otimes_{Hk(A,B,\chi)} W) = \mathrm{Hom}_B(\mathbf{k}_\chi, Q_\chi) \otimes_{Hk(A,B,\chi)} W =$$

$$= \mathrm{Hom}_A(Q_\chi, Q_\chi) \otimes_{Hk(A,B,\chi)} W = Hk(A, B, \chi)^{opp} \otimes_{Hk(A,B,\chi)} W = W.$$

This implies the second claim of this theorem.

By the formula above we also have

$$\mathrm{Hom}_A(Q_\chi \otimes_{Hk(A,B,\chi)} W', Q_\chi \otimes_{Hk(A,B,\chi)} W) = \mathrm{Hom}_{Hk(A,B,\chi)}(W', \mathrm{Hom}_A(Q_\chi, Q_\chi \otimes_{Hk(A,B,\chi)} W)) =$$

$$= \mathrm{Hom}_{Hk(A,B,\chi)}(W', W),$$

and hence $A - \mathrm{mod}_B^\chi$ is a full subcategory in the category of left A–modules.

□

At first sight the category $A-\mathrm{mod}_B^\chi$ looks a bit exotic. But it turns out that in many situations it has alternative descriptions in terms of the algebra A, its subalgebra B and the character χ only, and actually such categories and the related algebras of Hecke type played a very important, if not central, role in representation theory for at least last sixty years.

An important example of equivalences of the type mentioned in Theorem 1 was considered in [60]. In this paper Kostant showed that algebraic analogues of the principal series representations, called irreducible Whittaker modules, for a complex semisimple Lie algebra $\mathfrak{g}$ are in one–to–one correspondence with the one–dimensional representations of the center $Z(U(\mathfrak{g}))$ of the enveloping algebra $U(\mathfrak{g})$

In the situation considered in [60] the algebra $Z(U(\mathfrak{g}))$ is isomorphic to $Hk(A,B,\chi)$ with $A=U(\mathfrak{g})$, $B=U(\mathfrak{n}_-)$, where $\mathfrak{n}_-$ is a nilradical of $\mathfrak{g}$, and χ being a non–singular character of $U(\mathfrak{n}_-)$ which does not vanish on all simple root vectors in $\mathfrak{n}_-$. The category $A-\mathrm{mod}_B^\chi$ in this case can be described as the category of $\mathfrak{g}$–modules on which $x-\chi(x)$ acts locally nilpotently for any $x\in\mathfrak{n}_-$.

This correspondence was generalized in [76] and in the Appendix to [82] to a more general categorical setting and the categorical equivalence established in the Appendix to [82] is called the Skryabin equivalence.

Similar equivalences were obtained in [82] in the case of semisimple Lie algebras over fields of prime characteristic and in [101] in case of quantum groups associated to complex semisimple Lie algebras for generic values of the deformation parameter. Various approaches to the proofs of the above mentioned statements have been developed in [36, 105].

An analogous construction appears also in the case of finite groups of Lie type (see [17], Chapter 10).

Note that the problem of classification of $Hk(A,B,\chi)$–modules is usually very difficult (see e.g. [38] for the case of the Iwahori–Hecke algebra). Sometimes it is easier to classify irreducible objects in the category $A-\mathrm{mod}_B^\chi$ and then to translate the result to the category $Hk(A,B,\chi)-\mathrm{mod}$ (see [64, 65] for the case of algebras $Hk(A,B,\chi)$ considered in [82]).

In all cases considered in [60, 76, 82, 101] the algebras A and B, the characters χ and the appropriate categories $A-\mathrm{mod}_B^\chi$ and $Hk(A,B,\chi)-\mathrm{mod}$ of representations of A and of $Hk(A,B,\chi)$ are relatively easy to define. It is much more difficult to obtain alternative descriptions of the category $A-\mathrm{mod}_B^\chi$. However, one should note that the approach to this problem in all papers mentioned above is slightly different: all those papers start with the description of a category of A–modules in intrinsic terms using the algebra A, its subalgebra B and the character χ. And then one proves that this category is equivalent to the category $Hk(A,B,\chi)-\mathrm{mod}$, the equivalence being established using the functors $\mathrm{Hom}_A(Q_\chi,\cdot)$ and $Q_\chi\otimes_{Hk(A,B,\chi)}\cdot$. Finally one deduces that this category actually coincides with $A-\mathrm{mod}_B^\chi$.

In the Lie algebra case the most simple proofs of statements of this kind were proposed in the Appendix to [82] in the zero characteristic case and in [105] in the prime characteristic case. But the phenomenon behind these proofs is already manifest in [60]. Namely, in the case of Lie algebras over fields of zero characteristic one always has $A=U(\mathfrak{g})$ and $B=U(\mathfrak{m})$ for some reductive Lie algebra $\mathfrak{g}$ and a nilpotent Lie subalgebra $\mathfrak{m}\subset\mathfrak{g}$, and the above mentioned phenomenon amounts to introducing a second $U(\mathfrak{m})$–module structure on Q_χ by tensoring with the one–dimensional representation $\mathbf{k}_\chi$ and to demonstrating that for $\mathbf{k}=\mathbb{C}$ a certain "classical limit" of the $U(\mathfrak{m})$–module $Q_\chi\otimes\mathbf{k}_\chi$ is isomorphic to the algebra of regular functions $\mathbb{C}[\mathcal{C}]$ on a closed algebraic variety $\mathcal{C}$, and the "classical limit" of the $U(\mathfrak{m})$–action on $Q_\chi\otimes\mathbf{k}_\chi$ is induced by a free action of the complex unipotent algebraic group M corresponding to the Lie algebra $\mathfrak{m}$ on $\mathcal{C}$. The "classical limits" here are understood in the sense of taking associate graded objects with respect to suitable filtrations.

The action

$$M\times\mathcal{C}\to\mathcal{C}$$

has a global cross-section $\Sigma\subset\mathcal{C}$, called a Slodowy slice, so that the action map

$$M\times\Sigma\to\mathcal{C} \tag{2}$$

is an isomorphism of varieties, and

$$\mathbb{C}[\mathcal{C}]\simeq\mathbb{C}[\Sigma]\otimes\mathbb{C}[M]. \tag{3}$$

The space $W_0=\mathbb{C}[\Sigma]\simeq\mathbb{C}[\mathcal{C}]^M$ can be regarded as a "classical limit" of $Hk(A,B,\chi)$ which is called a W–algebra in this case. We can also write $\mathbb{C}[\mathcal{C}]\simeq W_0[M]$, where $W_0[M]$ is the algebra of regular functions on M with values in W_0. In fact W_0 carries the natural structure of a Poisson algebra. It is called a Poisson W–algebra.

Let $A-\mathrm{mod}_B^\chi$ be the category of left A–modules V for which the $U(\mathfrak{m})$–action on $V\otimes\mathbf{k}_\chi$ is locally nilpotent. In the Appendix to [82] it is shown that if one equips $V\in A-\mathrm{mod}_B^\chi$ with a second $U(\mathfrak{m})$–module structure by tensoring with $\mathbf{k}_\chi$ then, as a $U(\mathfrak{m})$–module, $V\otimes\mathbf{k}_\chi$ is isomorphic to $\mathrm{hom}_{\mathbf{k}}(U(\mathfrak{m}),V_\chi)$,

$$V\otimes\mathbf{k}_\chi\simeq\mathrm{hom}_{\mathbf{k}}(U(\mathfrak{m}),V_\chi)\simeq V_\chi[M], \tag{4}$$

where $\hom_{\mathbf{k}}$ stands for the space of homomorphisms vanishing on some power of the augmentation ideal of $U(\mathfrak{m})$, $V_\chi = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbf{k}_\chi, V)$ is called the space of Whittaker vectors in V, and, as above, the latter isomorphism holds if $\mathbf{k} = \mathbb{C}$. In the Appendix to [82] it is shown that isomorphisms (4) directly imply an isomorphism between the category of left A–modules V for which the $U(\mathfrak{m})$–action on $V \otimes \mathbf{k}_\chi$ is locally nilpotent and the corresponding category $A - \mathrm{mod}_B^\chi$ introduced before Theorem 1.

Isomorphisms of type (2) occur in the quantum group setting as well (see [98, 99, 101]), and the same idea is applied in [101] to establish similar categorical equivalences in the quantum group case for generic values of the deformation parameter.

In [100, 104] it was observed that an isomorphism of type (2) gives rise to a natural projection operator $\Pi : \mathbb{C}[\Sigma] \to \mathbb{C}[\mathcal{C}]^M \simeq \mathbb{C}[\Sigma] = W_0$. Namely, according to (2) any $x \in \mathcal{C}$ can be uniquely represented in the form

$$x = n(x) \circ \sigma(x), n(x) \in M, \sigma(x) \in \Sigma. \tag{5}$$

If for $f \in \mathbb{C}[\mathcal{C}]$ we define $\Pi f \in \mathbb{C}[\mathcal{C}]$ by

$$(\Pi f)(x) = f(n^{-1}(x) \circ x) = f(\sigma(x)) \tag{6}$$

then Πf is an M–invariant function, and any M–invariant regular function on $\mathcal{C}$ can be obtained this way. Moreover, by the definition $\Pi^2 = \Pi$, i.e. Π is a projection onto $\mathbb{C}[\mathcal{C}]^M$.

In the quantum group setting considered in [98, 99, 101] the "classical limiting" variety $\mathcal{C}$ is always a closed subvariety in a complex semisimple algebraic Lie group G, Σ is an analogue of a Slodowy slice for G introduced in [98, 103], and M is a unipotent subgroup of G, where the "classical limit" simply corresponds now to the $q = 1$ specialization of the deformation parameter q. The peculiarity of the quantum group case is that every element of M can be uniquely represented as an ordered product of elements of some one–parameter subgroups $M_i \subset G$, $i = 1, \ldots, c$ corresponding to roots, i.e. $M = M_1 \ldots M_c$. If we denote by t_i the parameter in M_i and by $X_i(t_i)$ the element of M_i corresponding to the value $t_i \in \mathbb{C}$ of the parameter then factorizing $n(x)$ in (5) as follows

$$n(x) = X_1(t_1(x)) \ldots X_c(t_c(x)) \tag{7}$$

one can express the operator Π as a composition of operators Π_i,

$$(\Pi_i f)(x) = f(X_i(-t_i(x)) \circ x), \tag{8}$$

$$\Pi f = \Pi_1 \ldots \Pi_c f. \tag{9}$$

$t_i(x)$ here can be regarded as regular functions on $\mathcal{C} \subset G$.

The first miracle of the quantum group case is that there are explicit formulas for the functions $t_i(x)$ in (7) expressing them in terms of matrix elements of finite-dimensional irreducible representations of G. These formulas were obtained in [104].

The main objective of this book is to obtain quantum group counterparts of these formulas and to define proper quantum group analogues P_i and Π^q of the operators Π_i and Π. This provides a description of quantum group analogues of W–algebras, called q-W–algebras, as images of operators Π^q. This description implies that q-W–algebras belong to the class of the so-called Mickelsson algebras (see e.g. [115, 118, 119, 120, 122] and [123], Ch. 4).

Magically, the classical formulas for $t_i(x)$ and formulas (8) can be directly extrapolated to the quantum case, so the operator Π^q is given in a factorized form similar to (9). Note that no operators similar to P_i and Π^q can be defined in the Lie algebra setting discussed above.

Using the quantum group analogues B_i of the functions $t_i(x)$ one can also construct natural bases in modules V from the corresponding category $A - \mathrm{mod}_B^\chi$ and establish isomorphisms similar to (4) in the case when the deformation parameter is not a root of unity. Recall that in the Lie algebra case with $\mathbf{k} = \mathbb{C}$ for any $V \in A - \mathrm{mod}_B^\chi$ the Skryabin equivalence provides an isomorphism $V \simeq Q_\chi \otimes_{Hk(A,B,\chi)} V_\chi$. If we denote by V_χ^0 the "classical limit" of V_χ then recalling that the "classical limit" of Q_χ is $\mathbb{C}[\mathcal{C}]$ and the "classical limit" of $Hk(A, B, \chi)$ is $\mathbb{C}[\Sigma]$ we infer from (3) that the "classical limit" of $Q_\chi \otimes_{Hk(A,B,\chi)} V_\chi \simeq V$ is

$$\mathbb{C}[\mathcal{C}] \otimes_{\mathbb{C}[\Sigma]} V_\chi^0 \simeq (\mathbb{C}[\Sigma] \otimes \mathbb{C}[M]) \otimes_{\mathbb{C}[\Sigma]} V_\chi^0 \simeq \mathbb{C}[M] \otimes V_\chi^0.$$

These isomorphisms together with (5) and (7) give a hint how to construct natural bases in modules from the category $V \in A - \mathrm{mod}_B^\chi$ in the quantum group case. Namely, if V is such a module it is natural to expect that if one picks up a linear basis v_j, $j = 1, \ldots$ in the space of Whittaker vectors V_χ then the elements of V given by properly defined ordered monomials in B_i applied to v_j, $j = 1, \dots$ form a linear basis in V. We show that this is indeed the case. These bases are key ingredients for an alternative proof of the Skryabin correspondence for quantum groups.

Operators conceptually similar to Π^q appeared in the literature a long time ago as the projection operators onto subspaces of singular vectors in some modules over a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$ the action of a nilradical $\mathfrak{n}_- \subset \mathfrak{g}$ on which is locally nilpotent. The first example of such operators, called extremal projection operators, for $\mathfrak{g} = \mathfrak{sl}_2$ was explicitly constructed in [66]. In papers [2, 3, 4] the results of [66] were generalized to the case of arbitrary complex semisimple Lie algebras, and explicit formulas for extremal projection operators were obtained. A summary of these results can be found in [111]. Later, using a certain completion of an extension of the universal enveloping algebra of $\mathfrak{g}$, Zhelobenko observed in [115] that the existence of extremal projection operators is an almost trivial fact. In [115] he also introduced a family of operators which are analogues to our operators P_i. These operators are called now Zhelobenko operators. Properties of extremal projection operators and of the Zhelobenko operators have been extensively studied in [115]–[122], and the results obtained in these papers were summarized in book [123].

In out terminology the situation considered in these works corresponds to the case when $A = U(\mathfrak{g})$, $B = U(\mathfrak{n}_-)$ and χ is the trivial character of $U(\mathfrak{n}_-)$. As observed in [100], in this case $\mathcal{C} = \mathfrak{b}_-$, the Borel subalgebra $\mathfrak{b}_- \subset \mathfrak{g}$ containing $\mathfrak{n}_-$, and the action of the unipotent group N_- corresponding to $\mathfrak{n}_-$ on $\mathcal{C}$ is induced by the adjoint action of a Lie group G with the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}$. This action is not free but is gives rise to a birational equivalence

$$N_- \times \mathfrak{h} \to \mathfrak{b}_-,$$

where $\mathfrak{h} = \mathfrak{b}_-/\mathfrak{n}_-$ is a Cartan subalgebra. In [100] it is shown that using this birational equivalence one can still define operators similar to Π_i and Π acting on a certain localization of the algebra of regular functions $\mathbb{C}[\mathfrak{b}_-]$ and these operators are "classical limits" of the Zhelobenko and of the extremal projection operators, respectively.

Remarkably, as observed in [105], the arguments from the Appendix to [82] are applicable to obtain alternative descriptions of the corresponding categories $A - \mathrm{mod}_B^\chi$ from [82] for Lie algebras over fields of prime characteristic. Although no "classical" geometric group action picture is available in this case.

Along the same line, formulas for the quantum group analogues B_i of the functions $t_i(x)$ and for the operators P_i and Π^q can be specialized to the case when q is a primitive odd m-th root of unity ε subject to a few other conditions depending on the Cartan matrix of the corresponding semisimple Lie algebra $\mathfrak{g}$. This provides technical tools for the proof of a root of unity version of the Skryabin correspondence for quantum groups. Similarly to the case of generic q one can construct bases in modules V from the corresponding category $A - \mathrm{mod}_B^\chi$. In case when q is a root of unity all such modules are finite-dimensional, and if one picks up a linear basis v_j, $j = 1, \dots, n$ in the space of Whittaker vectors V_χ then the elements of V given by applied to v_j, $j = 1, \dots, n$ properly defined ordered monomials in B_i, $i = 1, \dots, c$ powers of which are truncated at the degree m form a linear basis in V. In particular, the dimension of V is divisible by m^c. It turns out that any finite-dimensional module over the standard quantum group $U_\varepsilon(\mathfrak{g})$, where ε is a primitive odd m-th root of unity subject to the extra conditions mentioned above, belongs to one of the categories $A - \mathrm{mod}_B^\chi$ with appropriate A, B and χ, so its dimension is divisible by $b = m^c$. Moreover, the number b is equal to the number from the De Concini–Kac–Procesi conjecture on dimensions of irreducible modules over quantum groups at roots of unity suggested in [22]. Thus our result confirms this conjecture. Due to its importance we are going to discuss the De Concini–Kac–Procesi conjecture in more detail.

It is very well known that the number of simple modules for a finite-dimensional algebra over an algebraically closed field is finite. However, often it is very difficult to classify such representations. In some important particular examples even dimensions of simple modules over finite-dimensional algebras are not known.

One of the important examples of that kind is representation theory of semisimple Lie algebras over algebraically closed fields of prime characteristic. Let $\mathfrak{g}'$ be the Lie algebra of a semisimple algebraic group G' over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$. Let $x \mapsto x^{[p]}$ be the p-th power map of $\mathfrak{g}'$ into itself. The structure of the enveloping algebra of $\mathfrak{g}'$ is quite different from the zero characteristic case. Namely, the elements $x^p - x^{[p]}$, $x \in \mathfrak{g}'$ are central. For any linear form θ on $\mathfrak{g}'$, let U_θ be the quotient of the enveloping algebra of $\mathfrak{g}'$ by the ideal generated by the central elements $x^p - x^{[p]} - \theta(x)^p$ with $x \in \mathfrak{g}'$. Then U_θ is a finite-dimensional algebra. Kac and Weisfeiler proved that any simple $\mathfrak{g}'$-module can be regarded as a module over U_θ for a unique θ as above (this explains why all simple $\mathfrak{g}'$–modules are finite-dimensional). The Kac–Weisfeiler conjecture formulated in [54] and proved in [83] says that if the G'–coadjoint orbit of θ has dimension $\dim\, \mathcal{O}_\theta$ then $p^{\frac{\dim\, \mathcal{O}_\theta}{2}}$ divides the dimension of every finite-dimensional U_θ–module.

One can identify θ with an element of $\mathfrak{g}'$ via the Killing form and reduce the proof of the Kac–Weisfeiler conjecture to the case of nilpotent θ. In that case Premet defines in [83] a subalgebra $U_\theta(\mathfrak{m}_\theta) \subset U_\theta$ generated by a Lie subalgebra $\mathfrak{m}_\theta \subset \mathfrak{g}'$ such that $U_\theta(\mathfrak{m}_\theta)$ has dimension $p^{\frac{\dim \mathcal{O}_\theta}{2}}$ and every finite-dimensional U_θ–module is $U_\theta(\mathfrak{m}_\theta)$–free. Verification of the latter fact uses the theory of support varieties (see [33, 34, 35, 84]). Namely, according to the theory of support varieties, in order to prove that a U_θ–module is $U_\theta(\mathfrak{m}_\theta)$–free one should check that it is free over every subalgebra $U_\theta(x)$ generated in $U_\theta(\mathfrak{m}_\theta)$ by a single element $x \in \mathfrak{m}_\theta$.

There is a more elementary and straightforward proof of the Kac–Weisfeiler conjecture given in [81]. The most simple proof of this conjecture follows from the results of [105] on a prime characteristic version of the Skryabin equivalence which we already discussed above. A proof of the conjecture for $p > h$, where h is the Coxeter number of the corresponding root system, using localization of $\mathcal{D}$–modules was presented in [6].

Another important example of finite-dimensional algebras is related to the theory of quantum groups at roots of unity. Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. A remarkable property of the standard Drinfeld-Jimbo quantum group $U_\varepsilon(\mathfrak{g})$ associated to $\mathfrak{g}$, where ε is a primitive m-th root of unity, is that its center contains a huge commutative subalgebra isomorphic to the algebra Z_G of regular functions on (a finite covering of a big cell in) a complex algebraic group G with Lie algebra $\mathfrak{g}$. In this book we consider the simply connected version of $U_\varepsilon(\mathfrak{g})$ and the case when m is odd. In that case G is the connected, simply connected algebraic group corresponding to $\mathfrak{g}$.

Consider finite-dimensional representations of $U_\varepsilon(\mathfrak{g})$, on which Z_G acts according to non–trivial characters η_g given by evaluation of regular functions at various points $g \in G$. Note that all irreducible representations of $U_\varepsilon(\mathfrak{g})$ are of that kind, and every such representation is a representation of the algebra $U_{\eta_g} = U_\varepsilon(\mathfrak{g})/U_\varepsilon(\mathfrak{g})\mathrm{Ker}\ \eta_g$ for some η_g. In [22] De Concini, Kac and Procesi showed that if g_1 and g_2 are two conjugate elements of G then the algebras $U_{\eta_{g_1}}$ and $U_{\eta_{g_2}}$ are isomorphic. Moreover in [22] De Concini, Kac and Procesi formulated the following conjecture.

De Concini–Kac–Procesi conjecture. *The dimension of any finite-dimensional representation of the algebra* U_{η_g} *is divisible by* $b = m^{\frac{1}{2}\dim \mathcal{O}_g}$*, where* $\mathcal{O}_g$ *is the conjugacy class of* g*.*

This conjecture is the quantum group counterpart of the Kac–Weisfeiler conjecture for semisimple Lie algebras over fields of prime characteristic.

As it is shown in [21] it suffices to verify the De Concini–Kac–Procesi conjecture in case of exceptional elements $g \in G$ (an element $g \in G$ is called exceptional if the centralizer in G of its semisimple part has a finite center). However, the De Concini–Kac–Procesi conjecture is related to the geometry of the group G which is much more complicated than the geometry of the linear space $\mathfrak{g}'$ in case of the Kac–Weisfeiler conjecture.

The De Concini–Kac–Procesi conjecture is known to be true for the conjugacy classes of regular elements (see [23]), for the subregular unipotent conjugacy classes in type A_n when m is a power of a prime number (see [12]), for all conjugacy classes in A_n when m is a prime number (see [14]), for the conjugacy classes $\mathcal{O}_g$ of $g \in SL_n$ when the conjugacy class of the unipotent part of g is spherical (see [13]), and for spherical conjugacy classes (see [11]). In [61] a proof of the De Concini–Kac–Procesi conjecture using localization of quantum $\mathcal{D}$–modules was outlined in case of unipotent conjugacy classes. In contract to many papers quoted above the strategy of the proof of the De Concini–Kac–Procesi conjecture developed in this book does not use the reduction to the case of exceptional elements, and all conjugacy classes are treated uniformly.

Namely, following Premet's philosophy we use certain subalgebras $U_{\eta_g}(\mathfrak{m}_-) \subset U_{\eta_g}$ introduced in [102]. These subalgebras have non–trivial characters $\chi : U_{\eta_g}(\mathfrak{m}_-) \to \mathbb{C}$. In terms of the previously introduced notation, we show that for $A = U_{\eta_g}$, $B = U_{\eta_g}(\mathfrak{m}_-)$ and an appropriate χ the category $A - \mathrm{mod}_B^\chi$ can be identified with the category of finite-dimensional representations of U_{η_g} and equivalence (1) holds if $Hk(A, B, \chi) - \mathrm{mod}$ is the category of finite-dimensional representations of the corresponding algebra $Hk(A, B, \chi)$.

As observed in [102] every finite-dimensional U_{η_g}–module is also equipped with an action of the algebra $U_{\eta_1}(\mathfrak{m}_-)$ corresponding to the trivial character η_1 of Z_G given by the evaluation at the identity element of G. In the setting of quantum groups at roots of unity this action is a counterpart of the second $U(\mathfrak{m})$–module structure on objects V of the category $A - \mathrm{mod}_B^\chi$ which appeared in (4) in the case of Lie algebras over fields of zero characteristic.

Since the De Concini–Kac–Procesi conjecture is related to the structure of the set of conjugacy classes in G it is natural to look at transversal slices to the set of conjugacy classes. It turns out that the definition of the subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ is related to the existence of some special transversal slices Σ_s to the set of conjugacy classes in G. These slices Σ_s associated to (conjugacy classes of) elements s in the Weyl group of $\mathfrak{g}$ were introduced by the author in [98]. The slices Σ_s play the role of Slodowy slices in algebraic group theory. In the particular case of elliptic Weyl group elements these slices were also introduced later by He and Lusztig in paper [46] within a different framework.

A remarkable property of a slice Σ_s observed in [102] is that if g is conjugate to an element in Σ_s then the dimension of the corresponding subalgebra $U_{\eta_g}(\mathfrak{m}_-) \subset U_{\eta_g}$ is equal to $m^{\frac{1}{2}\mathrm{codim}\ \Sigma_s}$. The dimension of the algebra

$U_{\eta_1}(\mathfrak{m}_-)$ is also equal to $m^{\frac{1}{2}\mathrm{codim}\ \Sigma_s}$. If $g \in \Sigma_s$ (in fact g may belong to a larger variety) then the corresponding subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ and $U_{\eta_1}(\mathfrak{m}_-)$ can be explicitly described in terms of quantum group analogues of root vectors. Note that one can also define analogues $U_h^s(\mathfrak{m}_-)$ of subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ in the standard Drinfeld–Jimbo quantum group $U_h(\mathfrak{g})$ over the ring of formal power series $\mathbb{C}[[h]]$ (see [99]).

In [99], Theorem 5.2 it is shown that for every conjugacy class $\mathcal{O}$ in G one can find a transversal slice Σ_s such that $\mathcal{O}$ intersects Σ_s and dim $\mathcal{O}$ = codim Σ_s. Using this result we showed in [102] that for every element $g \in G$ one can find a a subalgebra $U_{\eta_g}(\mathfrak{m}_-)$ in U_{η_g} of dimension $m^{\frac{1}{2}\dim\ \mathcal{O}_g}$ with a non–trivial character χ. The dimension of the corresponding algebra $U_{\eta_1}(\mathfrak{m}_-)$ is also equal to $m^{\frac{1}{2}\dim\ \mathcal{O}_g}$.

Following the strategy outlined in the first part of the introduction we show that if m satisfies a certain condition then every finite-dimensional U_{η_g}–module is free over $U_{\eta_1}(\mathfrak{m}_-)$. Thus the dimension of every such module is divisible by $m^{\frac{1}{2}\dim\ \mathcal{O}_g}$. This establishes the De Concini–Kac–Procesi conjecture.

Note that in the case of restricted representations of a small quantum group similar results were obtained in [29]. The situation in [29] is rather similar to the case of the trivial character $\eta = \eta_1$ in our setting.

We also show that the rank of every finite-dimensional U_{η_g}–module V over $U_{\eta_1}(\mathfrak{m}_-)$ is equal to the dimension of the space V_χ and that U_{η_g} is the algebra of matrices of size $m^{\frac{1}{2}\dim\ \mathcal{O}_g}$ over the corresponding q-W–algebra $Hk(A, B, \chi) = Hk(U_{\eta_g}, U_{\eta_g}(\mathfrak{m}_-), \chi)$ which has dimension $m^{\dim\ \Sigma_s}$. In case of Lie algebras over fields of prime characteristic similar results were obtained in [82].

Note that the support variety technique used in [83] to prove the Kac–Weisfeiler conjecture can not be transferred to the case of quantum groups straightforwardly. The notion of the support variety is still available in case of quantum groups (see [29, 41, 78]). But in practical applications it is much less efficient since in the case of quantum groups there is no any underlying linear space.

In conclusion we would like to make a few remarks on the structure of the book. It consists of six chapters. In this introduction we have given a very superficial and incomplete review of the content of the book which rather aims to provide the reader with a general guide outlining the main ideas and the strategy of the main proofs. More technical comments are given in the beginning of each chapter.

In Chapters 1 and 2 we summarize results from [98, 99, 101, 103, 104] on the algebraic group analogues Σ_s of the Slodowy slices and the related results on quantum groups and on the subalgebras $U_h^s(\mathfrak{m}_-) \subset U_h(\mathfrak{g})$. Chapter 1 also contains some results on combinatorics of Weyl groups and on root systems required for the definition of the slices Σ_s, and Chapter 2 contains some advanced results on quantum groups required later for the study of q-W–algebras.

In Chapter 3, following [99, 101], we recall the definition of q-W–algebras and the description of their classical Poisson counterparts given in [104] in terms of the Zhelobenko type operators Π_i and Π. The main purpose of this chapter is to bring this description to a form suitable for quantization. Formulas (3.5.10), (3.5.11) and (3.5.12) obtained in this chapter for Π_i and Π have direct quantum analogues (4.5.5), (4.6.1) and (4.7.3) obtained in Chapter 4 for P_i and Π^q. The main result of Chapter 4 (Theorem 4.7.2) is the description of the q-W–algebra as the image of the operator Π^q.

In Chapter 5 we prove a version of the Skryabin equivalence of type (1) for equivariant modules over quantum groups established in [101]. The new proof of this equivalence in Theorem 5.2.1 is based on Proposition 4.6.7 which allows to construct some nice bases in modules from the category $A - \mathrm{mod}_B^\chi$ (see the discussion in the introduction above). Theorem 5.2.1 also gives precise values of ε of the deformation parameter q for which the categorical equivalence holds while in [101] it was established for generic ε only.

Finally in Chapter 6 we apply the results of Chapter 4 to the study of representations of quantum groups at roots of unity and prove the De Concini–Kac–Procesi conjecture. The strategy of this proof has already been discussed above.

Citations in the main text are reduced to a minimum. References to proofs which are omitted in the body of the text and some historic remarks are given in the bibliographic comments after each chapter.

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Contents

Chapter 1

Algebraic group analogues of Slodowy slices

The q-W–algebras are non-commutative deformations of algebras of regular functions on certain algebraic varieties in algebraic groups transversal to conjugacy classes. In this book these varieties play a role similar to that of the Slodowy slices in the theory of W–algebras and of generalized Gelfand-Geraev representations of semisimple Lie algebras. In this chapter we define these varieties and study their properties. We also develop the relevant Weyl group combinatorics.

1.1 Notation

Fix the notation used throughout the book. Let G be a connected finite-dimensional complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$, Q the corresponding root lattice, and P the weight lattice. Let $\alpha_i,\ i = 1, \ldots, l,\ \ l = rank(\mathfrak{g})$ be a system of simple roots, $\Delta_+ = \{\beta_1, \ldots, \beta_D\}$ the set of positive roots, P_+ the set of the corresponding integral dominant weights, $\omega_1, \ldots, \omega_l$ the fundamental weights. Let $H_1, \ldots, H_l$ be the set of simple root generators of $\mathfrak{h}$.

Denote by a_{ij} the corresponding Cartan matrix, and let $d_1, \ldots, d_l$, $d_i \in \{1, 2, 3\}$, $i = 1, \ldots, l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non–degenerate invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$ such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by $h^\vee$ the element of $\mathfrak{h}$ that corresponds to $h \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let W be the Weyl group of the root system Δ. W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1, \ldots, s_l$,

$$s_i(h) = h - \alpha_i(h) H_i, \quad h \in \mathfrak{h}.$$

The action of W preserves the bilinear form $(,)$ on $\mathfrak{h}$. We denote a representative of $w \in W$ in G by the same letter. For $w \in W, g \in G$ we write $w(g) = wgw^{-1}$. For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection.

For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $l(w)$ of the element w with respect to the system Γ of simple roots in Δ_+. We also write $\Delta_- = -\Delta_+$.

Let $\mathfrak{b}_+$ be the positive Borel subalgebra corresponding to Δ_+ and $\mathfrak{b}_-$ the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nilradicals. Let $H, N_+ = \exp \mathfrak{n}_+, N_- = \exp \mathfrak{n}_-, B_+ = HN_+, B_- = HN_-$ be the maximal torus, the maximal unipotent subgroups and the Borel subgroups of G which correspond to the Lie subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+$ and $\mathfrak{b}_-$, respectively. Note that Δ can also be defined as the root system of the pair (G, H), $\Delta = \Delta(G, H)$.

We identify $\mathfrak{g}$ and its dual by means of the canonical bilinear form. Then the coadjoint action of G on $\mathfrak{g}^*$ is naturally identified with the adjoint one. Using the canonical bilinear form we shall also identify ${\mathfrak{n}_+}^* \simeq \mathfrak{n}_-,\ {\mathfrak{b}_+}^* \simeq \mathfrak{b}_-,\ \mathfrak{h} \simeq \mathfrak{h}^*$.

Let $\mathfrak{g}_\beta$ be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x \text{ for every } h \in \mathfrak{h}\}$. $\mathfrak{g}_\beta \subset \mathfrak{g}$ is a one–dimensional subspace. It is well known that for $\alpha \neq -\beta$ the root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the canonical invariant bilinear form. Moreover $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non–degenerately paired by this form.

Let $X_\alpha \in \mathfrak{g}$ be a non–zero root vector corresponding to a root $\alpha \in \Delta$. Root vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})\alpha^\vee.$$

Note also that in this book we denote by $\mathbb{N}$ the set of non-negative integer numbers, $\mathbb{N} = \{0, 1, \ldots\}$.

1.2 Systems of positive roots associated to Weyl group elements

Algebraic group analogues of the Slodowy slices are associated to (conjugacy classes) in the Weyl group. In this section we recall the relevant combinatorics of the Weyl group and of root systems. We start by defining systems of positive roots associated to Weyl group elements which play the key role in the definition of the algebraic group analogues of the Slodowy slices.

Assume from now on that the group G is simply–connected. Let s be an element of the Weyl group W and denote by $\mathfrak{h}'$ the orthogonal complement in $\mathfrak{h}$, with respect to the canonical bilinear form on $\mathfrak{g}$, to the subspace $\mathfrak{h}'^\perp$ of $\mathfrak{h}$ fixed by the natural action of s on $\mathfrak{h}$. Let $\mathfrak{h}'^*$ be the image of $\mathfrak{h}'$ in $\mathfrak{h}^*$ under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by the canonical bilinear form on $\mathfrak{g}$. By Theorem C in [16] s can be represented as a product of two involutions,

$$s = s^1 s^2, \tag{1.2.1}$$

where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}'^*$.

Let $\mathfrak{h}_\mathbb{R}$ be the real form of $\mathfrak{h}$, the real linear span of simple coroots in $\mathfrak{h}$. The set of roots Δ is a subset of the dual space $\mathfrak{h}^*_\mathbb{R}$.

The Weyl group element s naturally acts on $\mathfrak{h}_\mathbb{R}$ as an orthogonal transformation with respect to the scalar product induced by the symmetric bilinear form of $\mathfrak{g}$.

Let $f_1, \ldots, f_{l'}$ be the vectors of unit length in the directions of $\gamma_1, \ldots \gamma_{l'}$, and $\widehat{f}_1, \ldots, \widehat{f}_{l'}$ the basis of $\mathfrak{h}'_\mathbb{R}$ dual to $f_1, \ldots, f_{l'}$. Let M be the $l' \times l'$ symmetric matrix with real entries $M_{ij} = (f_i, f_j)$. $I - M$ is also a symmetric real matrix, and hence it is diagonalizable and has real eigenvalues.

The following proposition gives a recipe for constructing a spectral decomposition for the action of the orthogonal transformation s on $\mathfrak{h}_\mathbb{R}$.

Proposition 1.2.1. *Let λ be a (real) eigenvalue of the symmetric matrix $I - M$, and $u \in \mathbb{R}^{l'}$ a corresponding non–zero real eigenvector with components u_i, $i = 1, \ldots, l'$. Let $a_u, b_u \in \mathfrak{h}_\mathbb{R}$ be defined by*

$$a_u = \sum_{i=1}^{n} u_i \widehat{f}_i, \; b_u = \sum_{i=n+1}^{l'} u_i \widehat{f}_i. \tag{1.2.2}$$

(i) If $\lambda \neq 0$ then the angle θ between a_u and b_u satisfies $\cos\theta = \lambda$, the plane $\mathfrak{h}_\lambda \subset \mathfrak{h}_\mathbb{R}$ spanned by a_u and b_u is invariant with respect to the involutions $s^{1,2}$, s^1 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by b_u, and s^2 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by a_u. If $\lambda > 0$ the orthogonal transformation $s = s^1 s^2$ acts on $\mathfrak{h}_\lambda$ as a rotation through the angle 2θ.

(ii) If $\lambda \neq 0, \pm 1$ is an eigenvalue of $I - M$ then $-\lambda$ is also an eigenvalue of $I - M$, and if $\lambda \neq \mu$ are two positive eigenvalues of $I - M$, $\lambda, \mu \neq 1$ then the planes $\mathfrak{h}_\lambda$ and $\mathfrak{h}_\mu$ are mutually orthogonal.

(iii) Let $\lambda \neq 0, \pm 1$ be an eigenvalue of $I - M$ of multiplicity greater than 1, and $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult λ a basis of the eigenspace corresponding to λ. If the basis u^k is orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ then the corresponding planes $\mathfrak{h}_\lambda^k$ defined with the help of u^k, $k = 1, \ldots,$ mult λ are mutually orthogonal.

(iv) $\lambda = \pm 1$ are not eigenvalues of $I - M$.

(v) If $\lambda = 0$ is an eigenvalue of $I - M$, then there is a basis $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the corresponding non–zero elements a_{u^k}, b_{u^k} are all mutually orthogonal. Moreover, $s^1 a_{u^k} = -a_{u^k}$, $s^2 a_{u^k} = a_{u^k}$, $s^1 b_{u^k} = b_{u^k}$, $s^2 b_{u^k} = -b_{u^k}$ for non–zero elements a_{u^k}, b_{u^k}. In particular, for non–zero elements a_{u^k}, b_{u^k} we have $s a_{u^k} = -a_{u^k}$, $s b_{u^k} = -b_{u^k}$, and non–zero elements a_{u^k}, b_{u^k} is a basis of the subspace of $\mathfrak{h}_\mathbb{R}$ on which s acts by multiplication by -1.

Proof. By definition the matrix M can be written in a block form,

$$M = \begin{pmatrix} I_n & A \\ A^\top & I_{l'-n} \end{pmatrix}, \tag{1.2.3}$$

where A is an $n\times(l'-n)$ matrix, $A^\top$ is the transpose to A, I_n and $I_{l'-n}$ are the unit matrices of sizes n and $l'-n$. M^{-1} is also symmetric and has a similar block form,

$$M^{-1}=\begin{pmatrix} B & C \\ C^\top & D \end{pmatrix},\ B=B^\top,\ D=D^\top, \tag{1.2.4}$$

with the entries $M^{-1}_{ij}=(\widehat{f}_i,\widehat{f}_j)$.

For any vector $u\in\mathbb{R}^{l'}$ we introduce its $\mathbb{R}^n$ and $\mathbb{R}^{l'-n}$ components $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ in a similar way,

$$u=\begin{pmatrix} \widetilde{u} \\ \widetilde{\widetilde{u}} \end{pmatrix}. \tag{1.2.5}$$

We shall consider both $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ as elements of $\mathbb{R}^{l'}$ using natural embeddings $\mathbb{R}^n,\mathbb{R}^{l'-n}\subset\mathbb{R}^{l'}$ associated to decomposition (1.2.5).

If u is a non–zero eigenvector of $I-M$ corresponding to an eigenvalue λ then the equation $(I-M)u=\lambda u$ gives

$$-A\widetilde{\widetilde{u}}=\lambda\widetilde{u},\ \ -A^\top\widetilde{u}=\lambda\widetilde{\widetilde{u}}. \tag{1.2.6}$$

From these equations we deduce that

$$\begin{pmatrix} -\widetilde{u} \\ \widetilde{\widetilde{u}} \end{pmatrix}$$

is a non–zero eigenvector of $I-M$ corresponding to the eigenvalue $-\lambda$.

Since $M^{-1}M=I$ one has

$$BA+C=0,\ C^\top+DA^\top=0. \tag{1.2.7}$$

Multiplying the first and the second equations in (1.2.6) from the left by B and D, respectively, and using (1.2.7) we obtain that

$$C\widetilde{\widetilde{u}}=\lambda B\widetilde{u},\ C^\top\widetilde{u}=\lambda D\widetilde{\widetilde{u}}. \tag{1.2.8}$$

Now if $u^{1,2}$ are two non–zero eigenvectors of $I-M$ corresponding to an eigenvalue λ then by (1.2.4) we have

$$(a_{u^1},a_{u^2})=\sum_{i,j=1}^n u_i^1u_j^2(\widehat{f}_i,\widehat{f}_j)=\sum_{i,j=1}^n u_i^1u_j^2B_{ij}=\widetilde{u}^1\cdot B\widetilde{u}^2, \tag{1.2.9}$$

where $\cdot$ stands for the standard scalar product in $\mathbb{R}^{l'}$.

Similarly,

$$(b_{u^1},b_{u^2})=D\widetilde{\widetilde{u}}^1\cdot\widetilde{\widetilde{u}}^2,(a_{u^1},b_{u^2})=\widetilde{u}^1\cdot C\widetilde{\widetilde{u}}^2 \tag{1.2.10}$$

From (1.2.8), (1.2.9) and the first identity in (1.2.10) we also obtain that if $\lambda\neq 0$ then

$$(a_{u^1},a_{u^2})=\widetilde{u}^1\cdot B\widetilde{u}^2=\frac{1}{\lambda}\widetilde{u}^1\cdot C\widetilde{\widetilde{u}}^2=\frac{1}{\lambda}C^\top\widetilde{u}^1\cdot\widetilde{\widetilde{u}}^2=D\widetilde{\widetilde{u}}^1\cdot\widetilde{\widetilde{u}}^2=(b_{u^1},b_{u^2}). \tag{1.2.11}$$

Similarly, for any real eigenvalue λ we have

$$(a_{u^1},b_{u^2})=\widetilde{u}^1\cdot C\widetilde{\widetilde{u}}^2=\lambda(a_{u^1},a_{u^2}),\ (b_{u^1},a_{u^2})=\widetilde{\widetilde{u}}^1\cdot C^\top\widetilde{u}^2=\lambda(a_{u^1},a_{u^2}). \tag{1.2.12}$$

Therefore if $\lambda\neq 0$, taking into account (1.2.11), we obtain

$$\lambda=\frac{(a_{u^1},b_{u^2})}{(a_{u^1},a_{u^2})}=\frac{(a_{u^1},b_{u^2})}{\sqrt{(b_{u^1},b_{u^2})}\sqrt{(a_{u^1},a_{u^2})}}=\cos\theta.$$

Now if $\lambda\neq 0,\pm1$ then (1.2.9), (1.2.11), (1.2.12) and the identity $M^{-1}u^2=\frac{1}{1-\lambda}u^2$ yield

$$(a_{u^1}+b_{u^1},a_{u^2}+b_{u^2})=2(a_{u^1},a_{u^2})(\lambda+1)=u^1\cdot M^{-1}u^2=\frac{1}{1-\lambda}u^1\cdot u^2.$$

Thus if $\lambda\neq 0,\pm1$ and $u^{1,2}$ are mutually orthogonal a_{u^1}, a_{u^2} are also mutually orthogonal, and from (1.2.11) and (1.2.12) we obtain that b_{u^1} and b_{u^2}, a_{u^1} and b_{u^2}, a_{u^2} and b_{u^1} are mutually orthogonal. Therefore the planes spanned by a_{u^1},b_{u^1} and by a_{u^2},b_{u^2} are mutually orthogonal.

Let again u be a non–zero eigenvector of $I-M$ corresponding to an eigenvalue λ For $i=1,\ldots,n$ we have

$$(\widehat{f}_i,\lambda a_u-b_u)=\sum_{j=1}^{l'}u_j(\widehat{f}_i,\widehat{f}_j)=\lambda(B\widetilde{u})_i-(C\widetilde{\widetilde{u}})_i=0,$$

where at the last step we used the first identity in (1.2.8). From the last identity we deduce that λa_u-b_u is a linear combination of $f_{n+1},\ldots,f_{l'}$, and hence

$$s^2(\lambda a_u-b_u)=-(\lambda a_u-b_u).$$

However, by the definition of a_u $s^2a_u=a_u$. Therefore

$$s^2b_u=2\lambda a_u-b_u. \qquad (1.2.13)$$

Let $\lambda\neq 0$. Then recalling that by (1.2.11) $(a_u,a_u)=(b_u,b_u)$ we conclude that $\lambda a_u=\cos(\theta)a_u$ is the orthogonal projection of b_u onto the line spanned by a_u and that s^2b_u is obtained from b_u by the reflection in the line spanned by a_u as shown at Figure 1.

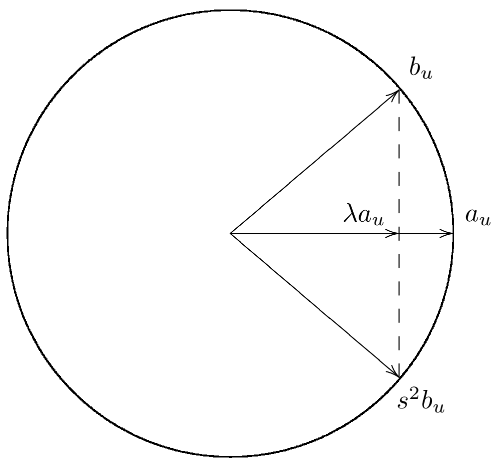


Fig. 1

Similarly, $s^1b_u=b_u$, s^1a_u is obtained from a_u by the reflection in the line spanned by b_u.

Thus the plane $\mathfrak{h}_\lambda\subset\mathfrak{h}_\mathbb{R}$ spanned by a_u and b_u is invariant with respect to the involutions $s^{1,2}$, s^1 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by b_u, and s^2 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by a_u. Since the angle between a_u and b_u is θ, for $\lambda>0$ the orthogonal transformation $s=s^1s^2$ acts on $\mathfrak{h}_\lambda$ as a rotation through the angle 2θ.

$\lambda=1$ is not an eigenvalue of $I-M$ since the matrix M is invertible. $\lambda=-1$ is not an eigenvalue of $I-M$ since otherwise the corresponding elements a_u, b_u would span a non–trivial fixed point subspace for the action of s in $\mathfrak{h}'_\mathbb{R}$ which is impossible as s acts on $\mathfrak{h}'_\mathbb{R}$ without non–trivial fixed points.

From the general theory of orthogonal transformations it follows that if $\lambda\neq\mu$ are two positive eigenvalues of $I-M$, $\lambda,\mu\neq 1$ then the planes $\mathfrak{h}_\lambda$ and $\mathfrak{h}_\mu$ are mutually orthogonal.

If $\lambda=0$ is an eigenvalue of $I-M$ then $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ are the components of an eigenvector u of $I-M$ with eigenvalue 0 if and only if $A\widetilde{\widetilde{u}}=0$ and $A^\top\widetilde{u}=0$. Therefore using the usual orthogonalization procedure one can construct a basis $u^k\in\mathbb{R}^{l'}$, $k=1,\ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the components $\widetilde{u}^k$ and $\widetilde{\widetilde{u}}^k$ $k=1,\ldots,$ mult 0 are all mutually orthogonal.

By (1.2.13) $s^2b_{u^k}=-b_{u^k}$. Also by the definition of a_{u^k} $s^2a_{u^k}=a_{u^k}$. Similarly, $s^1a_{u^k}=-a_{u^k}$ and $s^1b_{u^k}=b_{u^k}$.

Now using the definition of the eigenvector we deduce that for the basis u^k the following relations hold: $B\widetilde{u}^k=\widetilde{u}^k$, $D\widetilde{\widetilde{u}}^k=\widetilde{\widetilde{u}}^k$. For $k\neq l$ by (1.2.9)

$$(a_{u^k},a_{u^l})=\widetilde{u}^k\cdot B\widetilde{u}^l=\widetilde{u}^k\cdot\widetilde{u}^l=0$$

and by (1.2.10)

$$(b_{u^k},b_{u^l})=D\widetilde{\widetilde{u}}^k\cdot\widetilde{\widetilde{u}}^l=\widetilde{\widetilde{u}}^k\cdot\widetilde{\widetilde{u}}^l=0.$$

By (1.2.12) we always have

$$(a_{u^k}, b_{u^l}) = \lambda(a_{u^k}, a_{u^l}) = 0.$$

This completes the proof.

□

Using the previous proposition we can decompose $\mathfrak{h}_{\mathbb{R}}$ into a direct orthogonal sum of s–invariant subspaces,

$$\mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^{K} \mathfrak{h}_i, \tag{1.2.14}$$

where each of the subspaces $\mathfrak{h}_i \subset \mathfrak{h}_{\mathbb{R}}$, $i = 1, \ldots, K$ is invariant with respect to both involutions $s^{1,2}$ in the decomposition $s = s^1 s^2$, and there are the following three possibilities for each $\mathfrak{h}_i$: $\mathfrak{h}_i$ is two–dimensional ($\mathfrak{h}_i = \mathfrak{h}_\lambda^k$ for an eigenvalue $0 < \lambda < 1$ of the matrix $I - M$, and $k = 1, \ldots, \text{mult } \lambda$) and the Weyl group element s acts on it as rotation with angle θ_i, $0 < \theta_i < \pi$ or $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\lambda = 0$, $k = 1, \ldots, \text{mult } \lambda$ has dimension 1 and s acts on it by multiplication by -1 or $\mathfrak{h}_i$ coincides with the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. Note that since s has finite order $\theta_i = \frac{2\pi n_i}{m_i}$, $n_i, m_i \in \{1, 2, \ldots\}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \ldots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}_i$ with respect to the natural pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$.

Now we consider certain s–invariant subsets of roots Δ_i, $i = 0, \ldots, K$, defined as follows

$$\Delta_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i,\ h_i(\alpha) \neq 0\}, \tag{1.2.15}$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets Δ_i are empty, and that the definition of these subsets depends on the order of the terms in direct sum (1.2.14).

Now consider the nonempty s–invariant subsets of roots Δ_{i_k}, $k = 0, \ldots, M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$.

Observe also that the root system Δ is the disjoint union of the subsets Δ_{i_k},

$$\Delta = \bigcup_{k=0}^{M} \Delta_{i_k}.$$

Now assume that

$$|h_{i_k}(\alpha)| > |\sum_{l \leq j < k} h_{i_j}(\alpha)|, \text{ for any } \alpha \in \Delta_{i_k},\ k = 0, \ldots, M,\ l < k. \tag{1.2.16}$$

Condition (1.2.16) can be always fulfilled by suitable rescalings of the elements h_{i_k}.

Consider the element

$$\bar{h} = \sum_{k=0}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}. \tag{1.2.17}$$

From definition (1.2.15) of the sets Δ_i we obtain that for $\alpha \in \Delta_{i_k}$

$$\bar{h}(\alpha) = \sum_{j \leq k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j<k} h_{i_j}(\alpha) \tag{1.2.18}$$

Now condition (1.2.16), the previous identity and the inequality $|x + y| \geq ||x| - |y||$ imply that for $\alpha \in \Delta_{i_k}$ we have

$$|\bar{h}(\alpha)| \geq ||h_{i_k}(\alpha)| - |\sum_{j<k} h_{i_j}(\alpha)|| > 0.$$

Since Δ is the disjoint union of the subsets Δ_{i_k}, $\Delta = \bigcup_{k=0}^{M} \Delta_{i_k}$, the last inequality ensures that $\bar{h}$ belongs to a Weyl chamber of the root system Δ, and one can define the subset of positive roots Δ_+^s with respect to that chamber. From condition (1.2.16) and formula (1.2.18) we also obtain that a root $\alpha \in \Delta_{i_k}$ is positive if and only if

$$h_{i_k}(\alpha) > 0. \tag{1.2.19}$$

We denote by $(\Delta_{i_k})_+^s$ the set of positive roots contained in Δ_{i_k}, $(\Delta_{i_k})_+^s = \Delta_+^s \cap \Delta_{i_k}$.

We also define other s–invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0, \ldots, M$,

$$\overline{\Delta}_{i_k} = \bigcup_{i_j \leq i_k} \Delta_{i_j}. \tag{1.2.20}$$

According to this definition we have a chain of strict inclusions

$$\overline{\Delta}_{i_M} \supset \overline{\Delta}_{i_{M-1}} \supset \ldots \supset \overline{\Delta}_{i_0}, \tag{1.2.21}$$

such that $\overline{\Delta}_{i_M} = \Delta$, $\overline{\Delta}_0 = \Delta_0$, and $\overline{\Delta}_{i_k} \setminus \overline{\Delta}_{i_{k-1}} = \Delta_{i_k}$.

The following lemma shows that the subsets of roots $\overline{\Delta}_{i_k} \subset \Delta$ are root systems of some standard Levi subalgebras in $\mathfrak{g}$.

Lemma 1.2.2. *Let Γ^s be the set of simple roots in Δ_+^s. Then $\Gamma^s \cap \overline{\Delta}_{i_k}$ is a set of simple roots in $\overline{\Delta}_{i_k}$.*

Proof. Indeed, let $\alpha \in \overline{\Delta}_{i_k} \cap \Delta_+^s$, $\alpha = \sum_{i=1}^l n_i \alpha_i$, where $n_i \in \{0, 1, 2, \ldots\}$ and $\Gamma^s = \{\alpha_1, \ldots, \alpha_l\}$. Assume that α does not belong to the linear span of roots from $\Gamma^s \cap \overline{\Delta}_{i_k}$ and $t > i_k$ is maximal possible such that for some $\alpha_q \in \Delta_t$ one has $n_q > 0$. Then by (1.2.15) and (1.2.19) $h_t(\alpha) = \sum_{i=1}^l n_i h_t(\alpha_i) = \sum_{\alpha_i \in \Delta_t} n_i h_t(\alpha_i) > 0$, and by the choice of t $h_r(\alpha) = 0$ for $r > t$. Therefore $\alpha \in \Delta_t$, and hence $\alpha \notin \overline{\Delta}_{i_k}$. Thus we arrive at a contradiction.

□

1.3 Algebraic group analogues of Slodowy slices

In this section we define analogues of the Slodowy slices for algebraic groups.

Let $s \in W$ be a Weyl group element, Δ_+^s a system of positive roots associated to (the conjugacy class of) s in the previous section, Γ^s the set of simple roots in Δ_+^s. We shall assume that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. According to this convention $\Delta_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s.

We shall need the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing the Borel subalgebra corresponding to $\Delta_-^s = -\Delta_+^s$ and associated to the subset $-\Gamma_0^s$ of the set of simple roots $-\Gamma^s$, where $\Gamma_0^s = \Gamma^s \cap \Delta_0$. Denote by P the corresponding parabolic subgroup of G. Let $\mathfrak{n}$ and $\mathfrak{l}$ be the nilradical and the Levi factor of $\mathfrak{p}$, N and L the unipotent radical and the Levi factor of P, respectively.

Note that we have natural inclusions of Lie algebras $\mathfrak{p} \supset \mathfrak{n}$, and by Lemma 1.2.2 Δ_0 is the root system of the reductive Lie algebra $\mathfrak{l}$. We also denote by $\overline{\mathfrak{n}}$ the nilpotent subalgebra opposite to $\mathfrak{n}$ and by $\overline{N}$ the subgroup in G corresponding to $\overline{\mathfrak{n}}$.

Denote a representative for the Weyl group element s in G by the same letter. Let Z be the connected subgroup of G generated by the semisimple part of the Levi subgroup L and by the identity component H_0 of centralizer of s in H. H_0 is the connected Lie subgroup of H corresponding to the Lie subalgebra $\mathfrak{h}'^{\perp} \subset \mathfrak{h}$. We shall also denote by $\mathfrak{z}$ the Lie algebra of Z. Let $H' \subset H$ be the subgroup corresponding to the Lie subalgebra $\mathfrak{h}' \subset \mathfrak{h}$. We obviously have $L = ZH'$, and $\mathfrak{l} = \mathfrak{z} + \mathfrak{h}'$, where the sum is not direct.

Recall that G is in fact an algebraic group and all its subgroups introduced above are algebraic subgroups (see e.g. see §104, Theorem 12 in [114]).

Proposition 1.3.1. *Let $N_s = \{v \in N | svs^{-1} \in \overline{N}\}$. Then* dim $N_s = l(s)$*, where $l(s)$ is the length of the Weyl group element $s \in W$ with respect to the system of simple roots of Δ_+^s, the subvarieties $NZsN$ and sZN_s of G are closed and the conjugation map*

$$N \times sZN_s \to NZsN \tag{1.3.1}$$

is an isomorphism of varieties. Moreover, the variety $\Sigma_s = sZN_s$ is a transversal slice to the set of conjugacy classes in G.

Proof. dim $N_s = l(s)$ since $N_s \subset N$ is a closed subgroup generated by the one–parameter subgroups corresponding to the roots from the set $\{\alpha \in -\Delta_+^s : s\alpha \in \Delta_+^s\}$ the cardinality of which is equal to $l(s)$ (see e.g. [15], §2.2, 8.4).

Next we show that $NZsN$ is closed in G. Using a decomposition of N as a product of one–dimensional subgroups corresponding to roots one can write $N = N_s' N_s$, where $N_s' = N \cap s^{-1}Ns$, and hence

$$NsZN = NsZN_s'N_s = NsZN_s, \tag{1.3.2}$$

as Z normalizes N_s'.

Observe that an element $g \in G$ belongs to $NZsN = NsZN_s = NZsN_s$ if and only if $gs^{-1} \in NZsN_s s^{-1}$. The variety $NZsN_s s^{-1}$ is a subvariety of $NZ\overline{N}$. We prove that $NZ\overline{N}$ is closed in G.

Let $\mathfrak{h}_{\mathbb{R}}' = \mathfrak{h}' \cap \mathfrak{h}_{\mathbb{R}}$. $\mathfrak{h}_{\mathbb{R}}'$ and $\mathfrak{h}_0$ are annihilators of each other with respect to the restriction of the bilinear form on $\mathfrak{g}$ to $\mathfrak{h}_{\mathbb{R}}$. Let $\mathfrak{h}_{\mathbb{R}}'^*$ and $\mathfrak{h}_0^*$ be the images of $\mathfrak{h}_{\mathbb{R}}'$ and $\mathfrak{h}_0$, respectively, under the isomorphism $\mathfrak{h}_{\mathbb{R}} \simeq \mathfrak{h}_{\mathbb{R}}^*$ induced by the bilinear form on $\mathfrak{g}$.

Introduce the element $\bar{h}_0 = \sum_{k=1}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}$. By the definition of Δ_+^s for any $x \in \mathfrak{h}_0^*$ one has $\bar{h}_0(x) = 0$ and a root $\alpha \in \Delta \setminus \Delta_0$ belongs to Δ_+^s if and only if $\bar{h}_0(\alpha) > 0$. Let $\bar{h}_0^* \in \mathfrak{h}_{\mathbb{R}}^*$ be the image in $\mathfrak{h}_{\mathbb{R}}$ of the element $\bar{h}_0 \in \mathfrak{h}_{\mathbb{R}}'$. Since $\bar{h}_0 \in \mathfrak{h}_{\mathbb{R}}'$ we actually have $\bar{h}_0^* \in \mathfrak{h}_{\mathbb{R}}'^*$.

Let $\alpha_1, \ldots, \alpha_p$ be the simple roots in Γ^s which do not belong to Δ_0, $\omega_1, \ldots, \omega_p$ the corresponding fundamental weights. $\mathfrak{h}_{\mathbb{R}}'^*$ is a linear subspace in the real linear span Π of $\omega_1, \ldots, \omega_p$ as Π is the annihilator of the subspace of $\mathfrak{h}_{\mathbb{R}}^*$ spanned by the roots from Δ_0 which is contained in $\mathfrak{h}_0^*$. The subset Π_+ of Π which consists of x satisfying the condition $(x, \alpha) > 0$, $\alpha \in \Delta_+^s \setminus \Delta_0$ is open in Π and by definition $\bar{h}_0^* \in \Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$. Therefore the intersection $\Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$ is not empty and open in $\mathfrak{h}_{\mathbb{R}}'^*$.

The roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}_{\mathbb{R}}'^*$. They also span a $\mathbb{Z}$–sublattice Q' in the $\mathbb{Z}$–lattice generated by $\omega_1, \ldots, \omega_p$ as every root is a linear combination of fundamental weights with integer coefficients and $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}_{\mathbb{R}}'^* \subset \Pi$. Linear combinations of elements of Q' with rational coefficients are dense in $\mathfrak{h}_{\mathbb{R}}'^*$, and, in particular, in the open set $\Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$. Since the subset Π_+ of Π consists of x satisfying the condition $(x, \alpha) > 0$, $\alpha \in \Delta_+^s \setminus \Delta_0$ there is a linear basis of $\mathfrak{h}_{\mathbb{R}}'^*$ which consists linear combinations of $\omega_1, \ldots, \omega_p$ with positive rational coefficients. Multiplying the elements of this basis by appropriate positive integer numbers we obtain a linear basis Ω_i, $i = 1, \ldots, l'$ of $\mathfrak{h}_{\mathbb{R}}'^*$ which consists of integral dominant weights of the form $\Omega_i = \sum_{j=1}^{p} g_{ij}\omega_j$, $g_{ij} \in \mathbb{Z}, g_{ij} > 0$.

Let $\mathfrak{h}'^{\perp}$ be the orthogonal compliment to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the restriction of the symmetric bilinear form on $\mathfrak{g}$ to $\mathfrak{h}$. $\mathfrak{h}'^{\perp}$ is the complexification of $\mathfrak{h}_0$, and hence we deduce that an element $x \in \mathfrak{h}$ belongs to $\mathfrak{h}'^{\perp}$ if and only if $\Omega_i(x) = 0$, $i = 1, \ldots, l'$.

Let B_+^s be the Borel subgroup of G corresponding to the system Δ_+^s of positive roots, B_-^s the opposite Borel subgroup , $N_\pm^s$, $\mathfrak{b}_\pm^s$ their unipotent radicals and Lie algebras, respectively. Let V_{Ω_i}, $i = 1, \ldots, l'$ be the irreducible finite–dimensional representation of $\mathfrak{g}$ with highest weight Ω_i with respect to the system Δ_+^s of positive roots. Denote by v_{Ω_i} a nonzero highest weight vector in V_{Ω_i} and by $< \cdot, \cdot >$ the contravariant bilinear form on V_{Ω_i} normalized in such a way that $< v_{\Omega_i}, v_{\Omega_i} >= 1$. The matrix element $< v_{\Omega_i}, \cdot\, v_{\Omega_i} >$ can be regarded as a regular function on G whose restriction to the big dense cell $N_-^s H N_+^s$ is given by the character Ω_i of H, $< v_{\Omega_i}, n_- h n_+ v_{\Omega_i} >=< v_{\Omega_i}, h v_{\Omega_i} >= \Omega_i(h)$, $n_- \in N_-^s, h \in H, n_+ \in N_+^s$.

Each fundamental weight ω_i can be regarded as a regular function $< v_{\omega_j}, \cdot\, v_{\omega_j} >$ on G defined as above with V_{Ω_i} replaced by the irreducible finite–dimensional representation V_{ω_i} with highest weight ω_i. By the definition of Ω_i the function $< v_{\Omega_i}, \cdot\, v_{\Omega_i} >$ can be expressed as a product of functions $< v_{\omega_j}, \cdot\, v_{\omega_j} >$, $< v_{\Omega_i}, g v_{\Omega_i} >= \prod_{j=1}^{p} < v_{\omega_j}, g v_{\omega_j} >^{g_{ij}}$, $g \in G$.

Consider the closed subvariety in G defined by the equations $< v_{\Omega_i}, g v_{\Omega_i} >= 1$, $i = 1, \ldots, l'$, $g \in G$. According to the Bruhat decomposition every element $g \in G$ belongs to $g \in B_-^s w B_+^s$ for some $w \in W$. In this case g can be written in the form $g = n_- w h n_+$ for some $n_\pm \in N_\pm^s$, $h \in H$. Now $< v_{\Omega_i}, g v_{\Omega_i} >= \Omega_i(h) < v_{\Omega_i}, w v_{\Omega_i} >= \Omega_i(h) \prod_{j=1}^{p} < v_{\omega_j}, w v_{\omega_j} >^{g_{ij}}$. As different weight spaces of V_{ω_j} are orthogonal with respect to the contravariant form, the right hand side of the last identity is not zero for all $i = 1, \ldots, l'$ if and only if w fixes all weights ω_i, $i = 1, \ldots, p$, i.e. if and only if w belongs to the Weyl group of the root subsystem Δ_0. Since Δ_0 is the root system of the Levi factor $L = ZH'$, and $< v_{\omega_i}, v_{\omega_i} >= 1$, one has $< v_{\Omega_i}, w v_{\Omega_i} >\neq 0$, $i = 1, \ldots, l'$ if and only if $g \in \overline{N} Z H' N$, and in that case $< v_{\Omega_i}, g v_{\Omega_i} >= \Omega_i(h)$, where $g = n_- w h n_+$ for some $n_\pm \in N_\pm^s$, $h \in H$, and w is an element of the Weyl group of the root subsystem Δ_0.

As we observed above an element $x \in \mathfrak{h}$ belongs to $\mathfrak{h}'^{\perp}$ if and only if $\Omega_i(x) = 0$, $i = 1, \ldots, l'$. Therefore the conditions $< v_{\Omega_i}, g v_{\Omega_i} >= \Omega_i(h) = 1$, $i = 1, \ldots, l'$ are equivalent to the fact that h belongs to a subgroup H_0' of H with Lie algebra $\mathfrak{h}'^{\perp}$. Hence the equations $< v_{\Omega_i}, g v_{\Omega_i} >= 1$, $i = 1, \ldots, l'$ hold if and only if $g \in NZ'\overline{N}$, where $Z' \subset L$ is a subgroup of L with the same Lie algebra as Z. Thus the variety $NZ'\overline{N}$ is closed in G. Its closed connected component containing the identity element of G is obviously $NZ\overline{N}$. Thus the variety $NZ\overline{N}$ is closed in G.

The variety $NZsN_s s^{-1}$ is a closed subvariety of $NZ\overline{N}$ as $sN_s s^{-1}$ is the closed algebraic subgroup in $\overline{N}$ generated by the one–parameter subgroups corresponding to the roots from the set $\{\alpha \in \Delta_+^s : s^{-1}\alpha \in -\Delta_+^s\}$. So finally $NZsN_s s^{-1}$ is closed in G, and hence $NsZN = NZsN_s$ is also closed in G.

$sZN_s \subset NsZN_s = NsZN$ is a closed subvariety as $N_s \subset N$ is a closed subgroup generated by the one–parameter subgroups corresponding to the roots from the set $\{\alpha \in -\Delta_+^s : s\alpha \in \Delta_+^s\}$, and N is a closed algebraic subgroup of G. Since $NsZN$ is closed in G its closed subvariety sZN_s is also closed in G.

Next we show that map (1.3.1) is an isomorphism of varieties. It suffices to show that this map is bijective. Then by Zariski's main theorem this map is an isomorphism of varieties.

Fix a system of root vectors $X_\alpha \in \mathfrak{g}, \alpha \in \Delta$ such that if $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$ for any pair $\alpha, \beta \in \Gamma^s$ of simple positive roots then $[X_{-\alpha}, X_{-\beta}] = N_{\alpha,\beta}X_{-\alpha-\beta}$.

Recall that by Theorem 5.4.2. in [43] one can uniquely choose a representative $w \in G$ for any Weyl group element $w \in W$ in such a way that the operator Adw sends root vectors $X_{\pm\alpha}$ to $X_{\pm w\alpha}$ for any simple positive root $\alpha \in \Gamma^s$. We denote this representative by the same letter, $w \in G$. The representative $w \in G$ is called the normal representative of the Weyl group element $w \in W$. If the order of the Weyl group element $w \in W$ is equal to p then the inner automorphism Adw of the Lie algebra $\mathfrak{g}$ has order at most $2p$, $\mathrm{Ad}w^{2p} = \mathrm{id}$.

Since any representative of $s \in W$ in the normalizer of $\mathfrak{h}$ in G is H–conjugate to an element from Zs, where $s \in G$ is the normal representative of $s \in W$, we can assume without loss of generality that $s \in G$ is the normal representative for the Weyl group element $s \in W$.

Note that, since the operator Ads sends root vectors $X_{\pm\alpha}$ to $X_{\pm s\alpha}$ for any simple positive root $\alpha \in \Gamma^s$ and the root system of the reductive Lie algebra $\mathfrak{l}$ is fixed by the action of s, the semisimple part of the Levi subalgebra $\mathfrak{l}$ is fixed by the action of Ads. Therefore Z is the connected component of the centralizer of s in L containing the identity element,

$$Z = \{z \in L \mid s^{-1}zs = z\}^\circ, \tag{1.3.3}$$

Observe that map (1.3.1) is bijective if and only if for any given $k_s \in N_s, u \in N$ and $z \in Z$ the equation

$$uszk_s = nsz'n_sn^{-1} \tag{1.3.4}$$

has a unique solution $n \in N, n_s \in N_s, z' \in Z$. First observe that any element $uzsk_s$ is uniquely conjugated by $k_s \in N$ to $vsz, v = k_su$, and hence we can assume that $k_s = 1$ in (1.3.4),

$$vsz = nsz'n_sn^{-1}. \tag{1.3.5}$$

Now we show that for any given $v \in N$ and $z \in Z$ equation (1.3.5) has a unique solution $n \in N, n_s \in N_s, z' \in Z$. We shall use induction over certain s–invariant reductive subgroups in G that we are going to define. Consider the reductive Lie subalgebras $\mathfrak{g}_{i_k}$, $k = 0, \ldots, M$ defined by induction as follows: $\mathfrak{g}_{i_M} = \mathfrak{g}$, $\mathfrak{g}_{i_{k-1}} = \mathfrak{z}_{\mathfrak{g}_{i_k}}(h_{i_k})$, where $\mathfrak{z}_{\mathfrak{g}_{i_k}}(h_{i_k})$ is the centralizer of h_{i_k} in $\mathfrak{g}_{i_k}$. We denote by G_{i_k} the corresponding subgroups in G.

By construction $\overline{\Delta}_{i_k}$ is the root system of $\mathfrak{g}_{i_k}$, and we have chains of strict inclusions

$$\mathfrak{g} = \mathfrak{g}_{i_M} \supset \mathfrak{g}_{i_{M-1}} \supset \ldots \supset \mathfrak{g}_0 = \mathfrak{l}, \tag{1.3.6}$$

$$G = G_{i_M} \supset G_{i_{M-1}} \supset \ldots \supset G_0 = L \tag{1.3.7}$$

corresponding to inclusions (1.2.21). Note that by Lemma 1.2.2 all subalgebras $\mathfrak{g}_{i_k}$ are standard levi subalgebras in $\mathfrak{g}$ and $\mathfrak{g}_{i_{k-1}}$ is the Levi factor of the parabolic subalgebra $\mathfrak{p}_{i_{k-1}} \subset \mathfrak{g}_{i_k}$ containing the Borel subalgebra $\mathfrak{b}_-^s \cap \mathfrak{g}_{i_k} \subset \mathfrak{g}_{i_k}$ and associated to the set of simple roots $-\Gamma^s \cap \overline{\Delta}_{i_{k-1}}$. Let $\mathfrak{n}_{i_{k-1}}$ be the nilradical of $\mathfrak{p}_{i_{k-1}}$. We also denote by $\overline{\mathfrak{n}}_{i_{k-1}}$ the nilradical of the opposite parabolic subalgebra. Let $P_{i_{k-1}}$, $N_{i_{k-1}}$, and $\overline{N}_{i_{k-1}}$ be the corresponding subgroups of G_{i_k}. Below we shall need the following direct vector space decompositions of linear spaces

$$\mathfrak{g}_{i_k} = \mathfrak{p}_{i_{k-1}} + \overline{\mathfrak{n}}_{i_{k-1}} = \mathfrak{n}_{i_{k-1}} + \mathfrak{g}_{i_{k-1}} + \overline{\mathfrak{n}}_{i_{k-1}}, \tag{1.3.8}$$

$$\mathfrak{n} = \sum_{k=0}^{M-1} \mathfrak{n}_{i_k} \tag{1.3.9}$$

following straightforwardly from the definitions of the subalgebras $\mathfrak{p}_{i_{k-1}}$, $\mathfrak{n}_{i_k}$ and $\overline{\mathfrak{n}}_{i_{k-1}}$. Decompositions (1.3.8) imply decompositions of dense subsets $\overline{G}_{i_k} \subset G_{i_k}$,

$$\overline{G}_{i_k} = P_{i_{k-1}}\overline{N}_{i_{k-1}} = N_{i_{k-1}}G_{i_{k-1}}\overline{N}_{i_{k-1}}, \tag{1.3.10}$$

and decomposition (1.3.9) implies two decompositions

$$N = N_{i_{M-1}}N_{i_{M-2}}\ldots N_{i_0}, N = N_{i_0}N_{i_1}\ldots N_{i_{M-1}}. \tag{1.3.11}$$

Note that, since the subsets of roots $\overline{\Delta}_{i_k}$ are s–invariant, the subalgebras $\mathfrak{g}_{i_k}$ are Ads–invariant and the subgroups G_{i_k} are invariant with respect to the action of s on G by conjugations.

Applying decomposition (1.3.10) successively we also obtain decompositions of dense subsets $\overline{G}_k \subset G$,

$$\overline{G}_k = N_k G_{i_k} \overline{N}_k,\ N_k = N_{i_{M-1}} N_{i_{M-2}} \ldots N_{i_k}, \overline{N}_k = \overline{N}_{i_{M-1}} \overline{N}_{i_{M-2}} \ldots \overline{N}_{i_k}. \tag{1.3.12}$$

Due to the inclusions

$$[\mathfrak{g}_{i_{k-1}}, \mathfrak{n}_{i_{k-1}}] \subset \mathfrak{n}_{i_{k-1}}, [\mathfrak{g}_{i_{k-1}}, \overline{\mathfrak{n}}_{i_{k-1}}] \subset \overline{\mathfrak{n}}_{i_{k-1}}, \mathfrak{n}_{i_{k-2}} \subset \mathfrak{g}_{i_{k-1}}, \overline{\mathfrak{n}}_{i_{k-2}} \subset \mathfrak{g}_{i_{k-1}} \tag{1.3.13}$$

N_k and $\overline{N}_k$ are Lie subgroups in G with Lie algebras $\sum_{m=k}^{M-1} \mathfrak{n}_{i_m}$ and $\sum_{m=k}^{M-1} \overline{\mathfrak{n}}_{i_m}$, respectively.

We shall prove that equation (1.3.5) has a unique solution by induction over the reductive subgroups G_{i_k} starting with $k = 0$. First we rewrite equation (1.3.5) in a slightly different form,

$$vzsns^{-1} = nz'sn_s s^{-1}. \tag{1.3.14}$$

To establish the base of induction we first observe that both the l.h.s. and the r.h.s. of equation (1.3.14) belong to the dense subset $\overline{G}_0 \subset G$ and that the $G_0 = L$–component of equation (1.3.14) with respect to decomposition (1.3.12) for $k = 0$ is reduced to

$$z = z'. \tag{1.3.15}$$

Indeed, using a decomposition of N as a product of one–dimensional subgroups corresponding to roots one can write $N = N_s' N_s$, and hence

$$sNs^{-1} = sN_s's^{-1}sN_ss^{-1} \subset N\overline{N}. \tag{1.3.16}$$

If $n = mm_s$ is the decomposition of n corresponding to the decomposition $N = N_s'N_s$ then recalling that Z normalizes both N and $\overline{N}$ we deduce that the decompositions of the r.h.s. and of the l.h.s. of equation (1.3.14) corresponding to the decomposition $\overline{G}_0 = NL\overline{N}$ take the form

$$vzsms^{-1}z^{-1}zsm_ss^{-1} = nz'sn_ss^{-1},$$

where $vzsms^{-1}z^{-1}, n \in N$ and $sm_ss^{-1}, sn_ss^{-1} \in \overline{N}$, $z, z' \in Z \subset L$. This implies (1.3.15) and establishes the base of induction.

Now let

$$n = n_{i_0} \ldots n_{i_{M-2}} n_{i_{M-1}}, v = v_{i_{M-1}} \ldots v_{i_1} v_{i_0},\ n_s = n_{s i_0} \ldots n_{s i_{M-2}} n_{s i_{M-1}} \tag{1.3.17}$$

be the decompositions of the elements n, v, n_s corresponding to decompositions (1.3.11) and assume that n_{i_j} and $n_{s i_j}$ have already been uniquely defined for $j < k-1$. We shall show that using equation (1.3.14) one can find $n_{i_{k-1}}$ and $n_{s i_{k-1}}$ in a unique way.

Observe that both the l.h.s. and the r.h.s. of equation (1.3.14) belong to the dense subset $\overline{G}_k \subset G$ and that the G_{i_k}–component of equation (1.3.14) with respect to decomposition (1.3.12) is reduced to

$$v_{i_{k-1}}(v)_{k-1} zs(n)_{k-1} n_{i_{k-1}} s^{-1} = (n)_{k-1} n_{i_{k-1}} zs(n_s)_{k-1} n_{s i_{k-1}} s^{-1}, \tag{1.3.18}$$

where $(n_s)_{k-1} = n_{s i_0} \ldots n_{s i_{k-2}} \in G_{i_{k-1}}$, $(n)_{k-1} = n_{i_0} \ldots n_{i_{k-2}} \in G_{i_{k-1}}$, $(v)_{k-1} = v_{i_{k-2}} \ldots v_{i_0} \in G_{i_{k-1}}$ and $(n)_{k-1}, (n_s)_{k-1}, (u)_{k-1}, v_{i_{k-1}}, z$ are already known. This follows, similarly to the case $k = 0$, from decompositions (1.3.17), inclusions (1.3.13), which also imply that $G_{i_{k-1}}$ normalizes both $N_{i_{k-1}}$ and $\overline{N}_{i_{k-1}}$, and the fact that the subgroups G_{i_k} are invariant with respect to the action of s on G by conjugations.

The same properties imply that after multiplying by $(n)_{k-1}^{-1}$ from the left, equation (1.3.18) takes the form

$$wz_0 sn_{i_{k-1}} s^{-1} = n_{i_{k-1}} z_0' sn_{s i_{k-1}} s^{-1}, \tag{1.3.19}$$

with some known $w \in N_{i_{k-1}}$, $z_0, z_0' \in G_{i_{k-1}}$, and the compatibility of the equation of type (1.3.18) with k replaced by $k-1$ implies that $z_0 = z_0'$. Therefore (1.3.19) takes the form

$$wz_0 s\bar{n}s^{-1} = \bar{n} z_0 s\bar{n}_s s^{-1} \tag{1.3.20}$$

where we renamed the unknowns $\bar{n} = n_{i_{k-1}}, \bar{n}_s = n_{s i_{k-1}}$ to simplify the notation.

Let $\bar{n} = \bar{m}\bar{m}_s$ be the decomposition of the element $\bar{n}$ corresponding to the factorization

$$N_{i_{k-1}} = (N_{i_{k-1}} \cap N_s')(N_{i_{k-1}} \cap N_s).$$

In terms of this factorization equation (1.3.20) can be rewritten as follows

$$wz_0 s\bar{m}s^{-1}s\bar{m}_s s^{-1} = \bar{n}z_0 s\bar{n}_s s^{-1}, \qquad (1.3.21)$$

and the $\overline{N}_{i_{k-1}}$–component of the last equation with respect to factorization (1.3.10) is

$$s\bar{m}_s s^{-1} = s\bar{n}_s s^{-1}.$$

From this relation we obtain that

$$\bar{m}_s = \bar{n}_s, \qquad (1.3.22)$$

and hence (1.3.21) yields

$$wz_0 s\bar{m}s^{-1}z_0^{-1} = \bar{n}. \qquad (1.3.23)$$

Now we show that the last equation defines $\bar{n}$ in a unique way.

First observe that $\bar{n} \in N_{i_{k-1}}$, and $N_{i_{k-1}}$ is generated by one–parametric subgroups corresponding to roots from the set $(\Delta_{i_k})_+ = \Delta_{i_k} \cap \Delta_+$.

By the definition of the set Δ_s^s each s–orbit in the s–invariant set Δ_{i_k} contains a unique element from $\Delta_s^s \cap \Delta_{i_k}$. This observation implies that the set $(\Delta_{i_k})_+$ is the disjoint union of the subsets $\Delta_{i_k}^p = \{\alpha \in -(\Delta_{i_k})_+ : s^{-1}\alpha, \ldots, s^{-(p-1)}\alpha \in -(\Delta_{i_k})_+, s^{-p}\alpha \in (\Delta_{i_k})_+\}$, $p = 1, \ldots, D_k + 1$. Here D_k is chosen in such a way that $\Delta_{i_k}^{D_k+1} \subset -\Delta_s^s \cap \Delta_{i_k}$, and $\Delta_{i_k}^{D_k} = {\Delta'}_{i_k}^{D_k} \cup {\Delta''}_{i_k}^{D_k}$ (disjoint union), where ${\Delta'}_{i_k}^{D_k} = \Delta_{i_k}^{D_k} \cap -\Delta_s^s$, and ${\Delta''}_{i_k}^{D_k} = \Delta_{i_k}^{D_k} \setminus {\Delta'}_{i_k}^{D_k}$. The set $\Delta_{i_k}^{D_k+1}$ may be empty.

The orthogonal projections of the roots from the subsets $\Delta_{i_k}^p$ onto $\mathfrak{h}_{i_k}$ are contained in the interior of the sectors labeled $\Delta_{i_k}^p$ at Figure 2. All those sectors belong to the lower half plane and have the same central angles equal to θ_{i_k}, except for the last sector labeled by $\Delta_{i_k}^{D_k+1}$, which can possibly have a smaller angle.

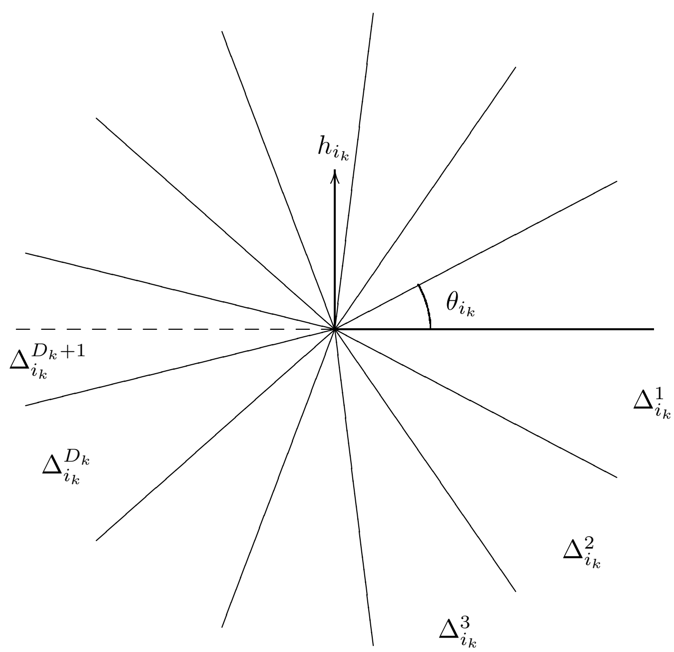


Fig. 2

The vector h_{i_k} is directed upwards at the picture, and the orthogonal projections of elements from $-(\Delta_{i_k})_+$ onto $\mathfrak{h}_{i_k}$ are contained in the lower half plane. The element $s \in W$ acts on the plane $\mathfrak{h}_{i_k}$ by clockwise rotation by the angle θ_{i_k}.

Now consider the unipotent subgroups $N_{i_{k-1}}^p$, $p = 1, \ldots, D_k + 1$, ${N'}_{i_{k-1}}$, ${N''}_{i_{k-1}}$ generated by the one–dimensional subgroups corresponding to the roots from the sets $\Delta_{i_k}^p$, ${\Delta'}_{i_k}^{D_k}$, ${\Delta''}_{i_k}^{D_k}$, respectively.

Obviously we have decompositions

$$N_{i_{k-1}} = N_{i_{k-1}}^1 N_{i_{k-1}}^2 \ldots N_{i_{k-1}}^{D_k+1}, N_{i_{k-1}}^{D_k} = {N''}_{i_{k-1}} {N'}_{i_{k-1}}, N_{i_{k-1}} \cap N_s = {N'}_{i_{k-1}} N_{i_{k-1}}^{D_k+1}. \qquad (1.3.24)$$

Let

$$\bar{n} = \bar{n}^1 \ldots \bar{n}^{D_k+1},\ \bar{n}^{D_k} = \bar{n}''\bar{n}',\ \bar{m} = \bar{n}^1 \ldots \bar{n}^{D_k-1}\bar{n}'',\ w = w^1 w^2 \ldots w^{D_k+1}, \bar{m}_s = \bar{n}'\bar{n}^{D_k+1} \quad (1.3.25)$$

be the corresponding decomposition of elements $\bar{n}$, $\bar{m}$, w and $\bar{m}_s$, respectively.

We claim that the components $\bar{n}^p$, $p = 1, \ldots, D_k + 1$ can be uniquely calculated by induction starting with $\bar{n}^1$. Indeed, substituting decompositions (1.3.25) into (1.3.23) we obtain

$$w^1 w^2 \ldots w^{D_k+1} z_0 s \bar{n}^1 \ldots \bar{n}^{D_k-1}\bar{n}'' s^{-1} z_0^{-1} = \bar{n}^1 \ldots \bar{n}^{D_k+1}.$$

Now comparing the $N^p_{i_{k-1}}$–components of the last equation, with respect to the first factorization in (1.3.24), and using the fact that $sN^p_{i_{k-1}}s^{-1} \subset N^{p+1}_{i_{k-1}}$, $p = 1, \ldots, D_k - 1$, $sN''_{i_{k-1}}s^{-1} \subset N^{D_k+1}_{i_{k-1}}$, and that z_0 normalizes the subgroups $N^p_{i_{k-1}}$, $p = 1, \ldots, D_k + 1$, $N'_{i_{k-1}}$, $N''_{i_{k-1}}$, we obtain

$$\bar{n}^1 = w^1, \bar{n}^p = (w^p \ldots w^{D_k+1} z_0 s \bar{n}^1 \ldots \bar{n}^{p-1} s^{-1} z_0^{-1})_p, p = 2, \ldots, D_k,$$
$$\bar{n}^{D_k+1} = (w^{D_k+1} z_0 s \bar{n}^1 \ldots \bar{n}^{D_k-1}\bar{n}'' s^{-1} z_0^{-1})_{D_k+1},$$

where $\bar{n}''$ is defined from the factorization $\bar{n}^{D_k} = \bar{n}''\bar{n}'$, and the subscript $(\ldots)_p$ stands for the $N^p_{i_{k-1}}$–component with respect to the first factorization in (1.3.24). From the formulas above one can recursively find the components $\bar{n}^p$ starting from $\bar{n}^1 = w^1$, and finally one can find $\bar{n}_s$ using (1.3.22). This proves the induction step and establishes isomorphism (1.3.1).

Finally we have to show that the variety $sZN_s \subset G$ is a transversal slice to the set of conjugacy classes in G, i.e. that the differential of the conjugation map

$$\gamma : G \times sZN_s \to G \quad (1.3.26)$$

is surjective.

Note that the set of smooth points of map (1.3.26) is stable under the G–action by left translations on the first factor of $G \times sZN_s$. Therefore it suffices to show that the differential of map (1.3.26) is surjective at points $(1, szn_s)$, $n_s \in N_s$, $z \in Z$.

In terms of the left trivialization of the tangent bundle TG and the induced trivialization of $T(sZN_s)$ the differential of map (1.3.26) at points $(1, szn_s)$ takes the form

$$d\gamma_{(1,szn_s)} : (x, (n, w)) \to -(Id - \mathrm{Ad}(szn_s)^{-1})x + n + w, \quad (1.3.27)$$
$$x \in \mathfrak{g} \simeq T_1(G), (n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T_{szn_s}(sZN_s),$$

where $\mathfrak{n}_s \subset \mathfrak{g}$ is the Lie algebra of $\mathfrak{n}_s$.

In order to show that the image of map (1.3.27) coincides with $T_{szn_s}G \simeq \mathfrak{g}$ we shall need a direct orthogonal, with respect to the bilinear form, vector space decomposition of the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{z} + \overline{\mathfrak{n}} + \mathfrak{h}'. \quad (1.3.28)$$

We shall use isomorphism (1.3.1), $\alpha : N \times sZN_s \to NsZN_s = NsZN$. By definition α is the restriction of the map γ to the subset $N \times sZN_s \subset G \times sZN_s$. Observe that in terms of the left trivialization of the tangent bundle TG the differential of the map α at points $(1, szn_s) \in N \times sZN_s$, $n_s \in N_s, z \in Z$ is given by

$$d\alpha_{(1,szn_s)} : (x, (n, w)) \to -(Id - \mathrm{Ad}(szn_s)^{-1})x + n + w, \quad (1.3.29)$$
$$x \in \mathfrak{n} \simeq T_1(N), (n, w) \in \mathfrak{n}_s + \mathfrak{z} \simeq T_{szn_s}(sZN_s).$$

Recall that the conjugation map $\alpha : N \times sZN_s \to NsZN_s$ is an isomorphism, and hence its differential is an isomorphism of the corresponding tangent spaces at all points. Using the left trivialization of the tangent bundle TG the tangent space $T_{szn_s}(NsZN_s)$ can be identified with $\mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(szn_s)^{-1}\mathfrak{n}$, $T_{szn_s z}(NsZN_s) = \mathfrak{n}_s + \mathfrak{z} + \mathrm{Ad}(szn_s)^{-1}\mathfrak{n}$. Therefore using (1.3.29) and the fcat that $d\alpha_{(1,szn_s)}$ is a linear space isomorphism we deduce that

$$(Id - \mathrm{Ad}(szn_s)^{-1})\mathfrak{n} + \mathfrak{n}_s + \mathfrak{z} = \mathrm{Ad}(szn_s)^{-1}\mathfrak{n} + \mathfrak{n}_s + \mathfrak{z}. \quad (1.3.30)$$

Now observe that by definition the subset $(Id - \mathrm{Ad}(szn_s)^{-1})\mathfrak{n} \subset \mathfrak{g}$ is contained in the image of $d\alpha_{(1,szn_s)}$, and by (1.3.30) the subset $\mathrm{Ad}(szn_s)^{-1}\mathfrak{n} \subset \mathfrak{g}$ is also contained in the image of $d\alpha_{(1,szn_s)}$. Since $\mathfrak{n} = (Id - \mathrm{Ad}(szn_s)^{-1})\mathfrak{n} + \mathrm{Ad}(szn_s)^{-1}\mathfrak{n}$, we deduce that $\mathfrak{n}$ is contained in the image of $d\alpha_{(1,szn_s)}$, and hence in the image of $d\gamma_{(1,szn_s)}$,

$$\mathfrak{n} \subset \mathrm{Im}\ d\gamma_{(1,szn_s)}. \quad (1.3.31)$$

Next observe that similarly to (1.3.1) one can show that the conjugation map

$$N \times N_{s^{-1}}Zs \to NZsN = N_{s^{-1}}ZsN, N_{s^{-1}} = \{n \in N : s^{-1}ns \in \overline{N}\} \tag{1.3.32}$$

is an isomorphism of varieties.

Interchanging the roles of N and $\overline{N}$ in (1.3.32) we immediately obtain that the conjugation map $\overline{\alpha} : \overline{N} \times \overline{N}_{s^{-1}}Zs \to \overline{N}_{s^{-1}}Zs\overline{N}$, $\overline{N}_{s^{-1}} = sN_s s^{-1}$ is an isomorphism. Observe also that by definition the map $\overline{\alpha}$ is the restriction of γ to the subset $\overline{N} \times \overline{N}_{s^{-1}}Zs = \overline{N} \times sZN_s \subset G \times sZN_s$.

Using the differential of the map $\overline{\alpha}$ we immediately infer, similarly to inclusion (1.3.31), that

$$\overline{\mathfrak{n}} \subset \mathrm{Im}\; d\gamma_{(1,szn_s)}. \tag{1.3.33}$$

Now observe that since $\mathfrak{h}$ normalizes $\mathfrak{n}$, and Z is a subgroup of L the adjoint action of which fixes $\mathfrak{h}'$, we have for any $x' \in \mathfrak{h}'$

$$\left((Id - \mathrm{Ad}(szn_s)^{-1})x'\right)_{\mathfrak{h}'} = (Id - \mathrm{Ad}s^{-1})x', \tag{1.3.34}$$

where $\left((Id - \mathrm{Ad}(szn_s)^{-1})x'\right)_{\mathfrak{h}'}$ stands for the $\mathfrak{h}'$–component of $(Id - \mathrm{Ad}(szn_s)^{-1})x'$ with respect to decomposition (1.3.28).

Since by definition the operator $\mathrm{Ad}s^{-1}$ has no fixed points in the invariant subspace $\mathfrak{h}'$, from formula (1.3.34) it follows that $\mathfrak{h}'$ is contained in the image of $(Id - \mathrm{Ad}(szn_s)^{-1})$, and hence, by formula (1.3.27), in the image of $d\gamma_{(1,szn_s)}$. Recalling also inclusions (1.3.31) and (1.3.33) and taking into account the obvious inclusion $\mathfrak{z} \subset \mathrm{Im}\; d\gamma_{(1,szn_s)}$ and decomposition (1.3.28) we deduce that the image of the map $d\gamma_{(1,szn_s)}$ coincides with $\mathfrak{g} \simeq T_{szn_s}G$. Therefore the differential of the map γ is surjective at all points. This completes the proof.

□

In the course of the proof of the previous proposition we obtained that the subvariety $NZ\overline{N} \subset G$ is closed in G. Similarly one can show that $\overline{N}ZN \subset G$ is closed in G. For later references we state this result as a corollary

Corollary 1.3.2. *The subvarieties $NZ\overline{N}, \overline{N}ZN \subset G$ are closed in G.*

The subvarieties $\Sigma_s \subset G$ are analogues of the Slodowy slices in algebraic group theory.

Remark 1.3.3. *In the proof of isomorphism (1.3.1) we only used commutation relations between one–parameter subgroups of G. Therefore isomorphisms similar to (1.3.1) hold in case when G is replaced with an algebraic group of the same type as G over an algebraically closed field or with a Chevalley group of the same type as G over an arbitrary field.*

1.4 The Lusztig partition

In this section $G_{\mathbf{k}}$ is a connected finite–dimensional semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for $G_{\mathbf{k}}$ and of the same type as G. Let G_p be a connected finite–dimensional semisimple algebraic group of the same type as G over an algebraically closed field of characteristic exponent p.

In the next section we shall show that for every conjugacy class $\mathcal{O}$ in $G_{\mathbf{k}}$ one can find a subvariety $\Sigma_{s,\mathbf{k}} \subset G_{\mathbf{k}}$ defined similarly to $\Sigma_s \subset G$ such that $\mathcal{O}$ intersects $\Sigma_{s,\mathbf{k}}$ and $\dim\, \mathcal{O} = \mathrm{codim}\; \Sigma_{s,\mathbf{k}}$. It turns out that there is a remarkable partition of the group $G_{\mathbf{k}}$ the strata of which are unions of conjugacy classes of the same dimension. For each stratum of this partition there is a Weyl group element s such that all conjugacy classes $\mathcal{O}$ from that stratum intersect $\Sigma_{s,\mathbf{k}}$, and $\dim\, \mathcal{O} = \mathrm{codim}\; \Sigma_{s,\mathbf{k}}$. In this section, which is rather descriptive, we define this partition called the Lusztig partition. The exposition in this section mainly follows paper [70] to which we refer the reader for technical details.

For any Weyl group W let $\widehat{W}$ be the set of isomorphism classes of irreducible representations of W over $\mathbb{Q}$. For any $E \in \widehat{W}$ let b_E be the smallest nonnegative integer such that E appears with non–zero multiplicity in the b_E-th symmetric power of the reflection representation of W. If this multiplicity is equal to 1 then one says that E is good. If $W' \subset W$ are two Weyl groups, and $E \in \widehat{W'}$ is good then there is a unique $\widetilde{E} \in \widehat{W}$ such that $\widetilde{E}$ appears in the decomposition of the induced representation $\mathrm{Ind}_{W'}^{W}E$, $b_{\widetilde{E}} = b_E$, and $\widetilde{E}$ is good. The representation $\widetilde{E}$ is called j-induced from E, $\widetilde{E} = j_{W'}^{W}E$.

Let $g \in G_p$, and $g = g_s g_u$ its decomposition as a product of the semisimple part g_s and the unipotent part g_u. Let $C = Z_{G_p}(g_s)^0$ be the identity component of the centralizer of g_s in G_p. C is a reductive subgroup of G_p of the same rank as G_p. Let H_p be a maximal torus of C. H_p is also a maximal torus in G_p, and hence one has a natural imbedding

$$W' = N_C(H_p)/H_p \to N_{G_p}(H_p)/H_p = W,$$

where $N_C(H_p), N_{G_p}(H_p)$ stand for the normalizers of H_p in C and in G_p, respectively, W' is the Weyl group of C and W is the Weyl group of G_p.

Let E be the irreducible representation of W' associated with the help of the Springer correspondence to the conjugacy class of g_u and the trivial local system on it. Then E is good, and let $\widetilde{E}$ be the j-induced representation of W. This gives a well-defined map $\phi_{G_p} : G_p \to \widehat{W}$. The fibers of this map are called the strata of G_p. By definition the map ϕ_{G_p} is constant on each conjugacy class in G_p. Therefore the strata are unions of conjugacy classes.

Moreover, by 1.3 in [70] we have the following formula for the dimension of the centralizer $Z_{G_p}(g)$ of any element $g \in G_p$ in G_p:

$$\dim Z_{G_p}(g) = \text{rank } G_p + 2b_{\phi_{G_p}(g)}, \tag{1.4.1}$$

where rank G_p is the rank of G_p.

It turns out that the image $\mathcal{R}(W)$ of ϕ_{G_p} only depends on W. It can be described as follows. Let $\mathcal{N}(G_p)$ be the unipotent variety of G_p and $\underline{\mathcal{N}}(G_p)$ the set of unipotent conjugacy classes in G_p. Let $\mathcal{X}^p(W)$ be the set of irreducible representations of W associated by the Springer correspondence to unipotent classes in $\underline{\mathcal{N}}(G_p)$ and the trivial local systems on them. We shall identify $\mathcal{X}^p(W)$ and $\underline{\mathcal{N}}(G_p)$. Let $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ be the corresponding bijective map.

Proposition 1.4.1. ([70], Proposition 1.4) *We have*

$$\mathcal{R}(W) = \mathcal{X}^1(W) \cup_{r \text{ prime}} \mathcal{X}^r(W).$$

If G_p is of type A_n, $(n \geq 1)$ or E_6 then $\mathcal{R}(W) = \mathcal{X}^1(W)$.

If G_p is of type B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, F_4 or E_7 then $\mathcal{R}(W) = \mathcal{X}^2(W)$.

If G_p is of type G_2 then $\mathcal{R}(W) = \mathcal{X}^3(W)$.

If G_p is of type E_8 then $\mathcal{R}(W) = \mathcal{X}^2(W) \cup \mathcal{X}^3(W)$, and $\mathcal{X}^2(W) \cap \mathcal{X}^3(W) = \mathcal{X}^1(W)$.

The above description of the set $\mathcal{R}(W)$ and the bijections $\underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ yield certain maps between sets $\underline{\mathcal{N}}(G_p)$ which preserve dimensions of conjugacy classes by (1.4.1). For instance, one always has an inclusion $\mathcal{X}^1(W) \subset \mathcal{X}^r(W)$ for any $r \geq 2$. The corresponding inclusion $\underline{\mathcal{N}}(G_1) \subset \underline{\mathcal{N}}(G_p)$ coincides with the Spaltenstein map $\pi_p^{G_{\mathbf{k}}} : \underline{\mathcal{N}}(G_1) \to \underline{\mathcal{N}}(G_p)$ (see [107], Théorème III.5.2).

Fix a system of positive roots in Δ. Note that Δ can be regarded as the root system of the pair (G_p, H_p), $\Delta = \Delta(G_p, H_p)$. Let B_p be the Borel subgroup in G_p associated to the corresponding system of negative roots, $H_p \subset B_p$ the maximal torus, and l the corresponding length function on W. Denote by $\underline{W}$ the set of conjugacy classes in W. For each $w \in W = N_{G_p}(H_p)/H_p$ one can pick up a representative $\dot{w} \in G_p$. If p is good for G_p, we write $G_p = G_{\mathbf{k}}$, $B_p = B_{\mathbf{k}}$, $\underline{\mathcal{N}}(G_p) = \underline{\mathcal{N}}(G_{\mathbf{k}})$.

Let $\mathcal{C}$ be a conjugacy class in W. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to l. By Theorem 0.4 in [73] there is a unique conjugacy class $\mathcal{O} \in \underline{\mathcal{N}}(G_{\mathbf{k}})$ of minimal possible dimension which intersects the Bruhat cell $B_{\mathbf{k}}\dot{w}B_{\mathbf{k}}$ and does not depend on the choice of the minimal possible length representative w in $\mathcal{C}$. We denote this class by $\Phi_1^{G_{\mathbf{k}}}(\mathcal{C})$.

As shown in Section 1.1 in [73], one can always find a representative $w \in \mathcal{C}$ of minimal possible length with respect to l which is elliptic in a parabolic Weyl subgroup $W' \subset W$, i.e. w acts without fixed points in the reflection representation of W'. Indeed, by Theorem 3.2.12 in [38] there is a parabolic subgroup $W' \subset W$ such that $\mathcal{C} \cap W'$ is a cuspidal conjugacy class in W', i.e. every element in it is elliptic in W'. By Lemma 3.1.14 in [38] if $w \in \mathcal{C} \cap W'$ is of minimal possible length with respect to the restriction of l to W' then it is also of minimal possible length with respect to l.

Let $P'_{\mathbf{k}} \subset G_{\mathbf{k}}$ be the parabolic subgroup which contains $B_{\mathbf{k}}$ and corresponds to W', and $M'_{\mathbf{k}}$ the semi-simple part of the Levi factor of $P'_{\mathbf{k}}$, so that W' is the Weyl group of $M'_{\mathbf{k}}$. Let $\Phi_p^{G_{\mathbf{k}}}(\mathcal{C})$ be the unipotent class in G_p containing the class $\pi_p^{M'_{\mathbf{k}}}\Phi_1^{M'_{\mathbf{k}}}(\mathcal{C})$. This class only depends on the conjugacy class $\mathcal{C}$, and hence one has a map $\Phi_p^{G_{\mathbf{k}}} : \underline{W} \to \underline{\mathcal{N}}(G_p)$ which is in fact surjective by 4.5(a) in [73].

Let $\mathcal{C} \in \underline{W}$, and $m_{\mathcal{C}}$ the dimension of the fixed point space for the action of any $w \in \mathcal{C}$ in the reflection representation. Then by Theorem 0.2 in [72] for any $\gamma \in \underline{\mathcal{N}}(G_p)$ there is a unique $\mathcal{C}_0 \in (\Phi_p^{G_{\mathbf{k}}})^{-1}(\gamma)$ such that the function $m_{\mathcal{C}} : (\Phi_p^{G_{\mathbf{k}}})^{-1}(\gamma) \to \mathbb{N}$ reaches its minimum at $\mathcal{C}_0$. We denote $\mathcal{C}_0$ by $\Psi_p^{G_{\mathbf{k}}}(\gamma)$. Thus one obtains an injective map $\Psi_p^{G_{\mathbf{k}}} : \underline{\mathcal{N}}(G_p) \to \underline{W}$.

Now recall that using identifications $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ one can define a bijection

$$F : \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}}) = \underline{\mathcal{N}}(G_1) \cup_{r \text{ prime}} \underline{\mathcal{N}}(G_r) \to \mathcal{X}^1(W) \cup_{r \text{ prime}} \mathcal{X}^r(W) = \mathcal{R}(W).$$

Using maps $\Phi_p^{G_{\mathbf{k}}}$ one can also define a surjective map $\Phi^W : \underline{W} \to \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$ as follows. If $\Phi_r^G(\mathcal{C}) \in \underline{\mathcal{N}}(G_1)$ for all $r > 1$ then $\Phi_r^{G_{\mathbf{k}}}(\mathcal{C})$ is independent of r, and one puts $\Phi^W(\mathcal{C}) = \Phi_r^{G_{\mathbf{k}}}(\mathcal{C})$ for any $r > 1$. If $\Phi_r^{G_{\mathbf{k}}}(\mathcal{C}) \not\in \underline{\mathcal{N}}(G_1)$ for some $r > 1$ then r is unique, one defines $\Phi^W(\mathcal{C}) = \Phi_r^{G_{\mathbf{k}}}(\mathcal{C})$.

By definition there is a right-sided injective inverse Ψ^W to Φ^W such that if $\gamma \in \underline{\mathcal{N}}(G_1)$ then $\Psi^W(\gamma) = \Psi_1^{G_{\mathbf{k}}}(\gamma)$, and if $\gamma \not\in \underline{\mathcal{N}}(G_1)$, and $\gamma \in \underline{\mathcal{N}}(G_r)$ then $\Psi^W(\gamma) = \Psi_r^{G_{\mathbf{k}}}(\gamma)$.

Denote by $C(W)$ the image of $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$ in $\underline{W}$ under the map Ψ^W, $C(W) = \Psi^W(\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}}))$. We shall identify $C(W)$, $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$ and $\mathcal{R}(W)$.

Now assume that p is not a bad prime for G_p. In this case the strata of the Lusztig partition can be described geometrically as follows. Let $\mathcal{C} \in C(W)$. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to l. Denote by $\underline{G}_p$ the set of conjugacy classes in G_p, and by $G'_{\mathcal{C}}$ the set of all conjugacy classes in G_p which intersect the Bruhat cell $B_p \dot{w} B_p$. This definition does not depend on the choice of the the minimal possible length representative w. Let

$$d_{\mathcal{C}} = \min_{\gamma \in G'_{\mathcal{C}}} \dim \gamma.$$

Then the stratum $G_{\mathcal{C}} = \phi_{G_p}^{-1}(F(\Phi^W(\mathcal{C})))$ can be described as follows (see Theorem 2.2, [70]),

$$G_{\mathcal{C}} = \bigcup_{\gamma \in G'_{\mathcal{C}},\ \dim\gamma = d_{\mathcal{C}}} \gamma.$$

Thus we have a disjoint union

$$G_p = \bigcup_{\mathcal{C} \in C(W)} G_{\mathcal{C}}.$$

Note that by the definition of the stratum , if $\mathcal{C} \in \mathrm{Im}(\Psi_1^{G_{\mathbf{k}}})$ then $G_{\mathcal{C}}$ contains a unique unipotent class, and if $\mathcal{C} \not\in \mathrm{Im}(\Psi_1^{G_{\mathbf{k}}})$ then $G_{\mathcal{C}}$ does not contain unipotent classes.

For good p the maps introduced above are summarized in the following diagram

$$\begin{array}{ccccccc} & & \mathcal{X}^1(W) & \stackrel{f_1}{\longleftarrow} & \underline{\mathcal{N}}(G_{\mathbf{k}}) & & \\ & & \downarrow \iota & & \downarrow \pi^{G_{\mathbf{k}}} & & \\ G_{\mathbf{k}} & \stackrel{\phi_{G_{\mathbf{k}}}}{\longrightarrow} & \mathcal{R}(W) & \stackrel{F}{\longleftarrow} & \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}}) & \begin{array}{c} \stackrel{\Phi^W}{\longleftarrow} \\ \underset{\Psi^W}{\longrightarrow} \end{array} & \underline{W}, \end{array} \tag{1.4.2}$$

where ι is an inclusion, bijections f_1 and F are induced by the Springer correspondence with the trivial local data, and the inclusion $\pi^{G_{\mathbf{k}}}$ is induced by the Spaltenstein map.

For exceptional groups the maps f_1 and F can be described explicitly using tables in [108], the maps Φ^W and Ψ^W can be described using the tables in Section 2 in [72], and the maps ι and π^G can be described explicitly using the tables of unipotent classes in [63], Chapter 22 or [108] (note that the labeling for unipotent classes in bad characteristics in [63] differs from that in [108]). The dimensions of the conjugacy classes in the strata in $G_{\mathbf{k}}$ can be obtained using dimension tables of centralizers of unipotent elements in case when a stratum contains a unipotent class (see [17, 63]), the tables for dimensions of the centralizers of unipotent elements in bad characteristic when a stratum does not contain a unipotent class (see [63]) or formula (1.4.1) and the tables of the values of the b–invariant b_E for representations of Weyl groups (see [17, 38]). Note that formula (1.4.1) implies that if $\mathcal{O}$ is any conjugacy class in $G_{\mathcal{C}}$, $\mathcal{O} \in G_{\mathcal{C}}$ then

$$\dim \mathcal{O} = \dim \Phi^W(\mathcal{C}). \tag{1.4.3}$$

In case of classical groups all those maps and dimensions are described in terms of partitions (see [17, 39, 63, 71, 72, 73, 107]). In case of classical matrix groups the strata can also be described explicitly (see [70]). We recall this description below. By (1.4.3) the dimensions of the conjugacy classes in every stratum of $G_{\mathbf{k}}$ are equal to the dimension of the corresponding conjugacy class in $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$. The dimensions of centralizers of unipotent elements in arbitrary characteristic can be found in [48, 63].

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ is a partition we denote by $\lambda^* = (\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_m^*)$ the corresponding dual partition. It is defined by the property that $\lambda_1^* = m$ and $\lambda_i^* - \lambda_{i+1}^* = l_i(\lambda)$, where $l_i(\lambda)$ is the number of times i appears in the partition λ. We also denote by $\tau(\lambda)$ the length of λ, $\tau(\lambda) = m$. If a partition μ is obtained from λ by adding a number of zeroes, we shall identify λ and μ.

$\mathbf{A_n}$

$G_{\mathbf{k}}$ is of type $\mathrm{SL}(V)$ where V is a vector space of dimension $n+1 \geq 1$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \geq 1$. W is the group of permutations of $n+1$ elements. All sets in (1.4.2), except for $G_{\mathbf{k}}$, are isomorphic to the set of partitions of $n+1$, and all maps, except for $\phi_{G_{\mathbf{k}}}$, are the identity maps.

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{SL}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$ and let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_m^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. Then $\phi_{G_{\mathbf{k}}}(g)$ is the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_m$ given by $\lambda(g)_j = \sum_{x \in \mathbf{k}^*} \lambda_j^x$.

If g is any element in the stratum G_λ corresponding to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$, $\lambda_m \geq 1$, then

$$\dim Z_{G_{\mathbf{k}}}(g) = n + 2\sum_{i=1}^{m}(i-1)\lambda_i. \tag{1.4.4}$$

The element of $\underline{W}$ which corresponds to λ is the Coxeter class in the Weyl subgroup of the type

$$A_{\lambda_1 - 1} + A_{\lambda_2 - 1} + \ldots + A_{\lambda_m - 1}. \tag{1.4.5}$$

The summands in the diagram above are called blocks. Blocks of type A_0 are called trivial.

$\mathbf{C_n}$

$G_{\mathbf{k}}$ is of type $\mathrm{Sp}(V)$ where V is a symplectic space of dimension $2n$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$. W is the group of permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. Each element $s \in W$ can be expressed as a product of disjoint cycles of the form

$$\varepsilon_{k_1} \to \pm\varepsilon_{k_2} \to \pm\varepsilon_{k_3} \to \ldots \to \pm\varepsilon_{k_r} \to \pm\varepsilon_{k_1}.$$

The cycle above is of length r; it is called positive if $s^r(\varepsilon_{k_1}) = \varepsilon_{k_1}$ and negative if $s^r(\varepsilon_{k_1}) = -\varepsilon_{k_1}$. The lengths of the cycles together with their signs give a set of positive or negative integers called the signed cycle-type of s. To each positive cycle of s of length r there corresponds a pair of positive orbits $X, -X$, $|X| = r$, for the action of the group $\langle s \rangle$ generated by s on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$, and to each negative cycle of s of length r there corresponds a negative orbit X, $|X| = 2r$, for the action of $\langle s \rangle$ on E. A positive cycle of length 1 is called trivial. It corresponds to a pair of fixed points for the action of $\langle s \rangle$ on E.

Elements of $\underline{W}$ are parametrized by pairs of partitions (λ, μ), where the parts of λ are even (for any $w \in \mathcal{C} \in \underline{W}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w), μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X$, $X \neq -X$), and $\sum \lambda_i + \sum \mu_j = 2n$. We denote this set of pairs of partitions by $\mathcal{A}^1_{2n}$. An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \leq \ldots \leq \mu_{2k-1} = \mu_{2k})$ is the Coxeter class in the Weyl subgroup of the type

$$C_{\frac{\lambda_1}{2}} + C_{\frac{\lambda_2}{2}} + \ldots + C_{\frac{\lambda_m}{2}} + A_{\mu_1 - 1} + A_{\mu_3 - 1} + \ldots + A_{\mu_{2k-1} - 1}. \tag{1.4.6}$$

Elements of $\underline{\mathcal{N}}(G_{\mathbf{k}})$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for odd j. We denote this set of partitions by $\mathcal{T}_{2n}$. In case of $G_{\mathbf{k}} = \mathrm{Sp}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}}) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{Sp}(V_2)$ where V_2 is a symplectic space of dimension $2n$ over an algebraically closed field of characteristic 2. Elements of $\underline{\mathcal{N}}(G_2)$ are parametrized by pairs (λ, ε), where $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m) \in \mathcal{T}_{2n}$, and $\varepsilon : \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \to \{0, 1, \omega\}$ is a function such that

$$\varepsilon(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 1 & \text{if } k = 0; \\ 1 & \text{if } k > 0 \text{ is even, } l_k(\lambda) \text{ is odd;} \\ 0 \text{ or } 1 & \text{if } k > 0 \text{ is even, } l_k(\lambda) \text{ is even.} \end{cases} \tag{1.4.7}$$

We denote the set of such pairs (λ, ε) by $\mathcal{T}^2_{2n}$.

Elements of $\widehat{W}$ are parametrized by pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length $\tau(\alpha)$ of α is related to the length of β by $\tau(\alpha) = \tau(\beta) + 1$. The set of such pairs is denoted by $X_{n,1}$.

The maps f_1, F can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{T}_{2n}$, and assume that $\lambda_1 = 0$. If $f_1(\lambda) = ((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m}))$ then the parts c'_i are defined by induction starting from $c'_1 = 0$,

$$\begin{array}{ll} c'_i = \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even and } c'_{i-1} \text{ is already defined;} \\ c'_i = \frac{\lambda_i+1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_{i-1} \text{ is already defined;} \\ c'_{i+1} = \frac{\lambda_i-1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_i \text{ is already defined.} \end{array}$$

The image of f_1 consists of all pairs $((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m})) \in X_{n,1}$ such that $c'_i \leq c'_{i+1} + 1$ for all i.

If $F(\lambda, \varepsilon) = ((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m}))$ then the parts c_i are defined by induction starting from $c_1 = 0$,

$$\begin{array}{ll} c_i = \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even, } \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined;} \\ c_i = \frac{\lambda_i+1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined;} \\ c_{i+1} = \frac{\lambda_i-1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined;} \\ c_i = \frac{\lambda_i+2}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_{i-1} \text{ is already defined;} \\ c_{i+1} = \frac{\lambda_i-2}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_i \text{ is already defined.} \end{array}$$

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The map Φ^W is defined by $\Phi^W(\lambda, \mu) = (\nu, \varepsilon)$, where the set of parts of ν is just the union of the sets of parts of λ and μ, and

$$\varepsilon(k) = \begin{cases} 1 & \text{if } k \in 2\mathbb{N} \text{ is a part of } \lambda; \\ 0 & \text{if } k \in 2\mathbb{N} \text{ is not a part of } \lambda; \\ \omega & \text{if } k \text{ is odd.} \end{cases}$$

The map Ψ^W associates to each pair (ν, ε) a unique point (λ, μ) in the preimage $(\Phi^W)^{-1}(\nu, \varepsilon)$ such that the number of parts of μ is minimal possible. This point is defined by the conditions

$$l_k(\lambda) = \begin{cases} 0 & \text{if } k \text{ is odd or } k \text{ is even, } l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\ l_k(\nu) & \text{otherwise,} \end{cases}$$

$$l_k(\mu) = \begin{cases} l_k(\nu) & \text{if } k \text{ is odd or } k \text{ is even, } l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The map $\pi^{G_{\mathbf{k}}}$ is given by $\pi^{G_{\mathbf{k}}}(\lambda) = (\lambda, \varepsilon')$, where

$$\varepsilon'(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

The map $\pi^{G_{\mathbf{k}}}$ is injective and its image consists of pairs $(\lambda, \varepsilon) \in \mathcal{T}^2_{2n}$, where ε satisfies the conditions above.

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{Sp}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m+1}$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m+1}$ be the sequence in $\mathbb{N}$, where $((\lambda^x_1 \geq \lambda^x_3 \geq \ldots \geq \lambda^x_{2m+1}), (\lambda^x_2 \geq \lambda^x_4 \geq \ldots \geq \lambda^x_{2m}))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{\dim V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda^x_j$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_G(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}^2_{2n}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then

$$\dim Z_{G_{\mathbf{k}}}(g) = n + \sum_{i=1}^{m}(i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|. \tag{1.4.8}$$

$\mathbf{B_n}$

$G_{\mathbf{k}}$ is of type $\mathrm{SO}(V)$ where V is a vector space of dimension $2n+1$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$ equipped with a non–degenerate symmetric bilinear form. W is the same as in case of C_n.

An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [16], Section 7)

$$\begin{aligned} &A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1} + \\ &\quad + D_{\frac{\lambda_1+\lambda_2}{2}}(a_{\frac{\lambda_2}{2}-1}) + D_{\frac{\lambda_3+\lambda_4}{2}}(a_{\frac{\lambda_4}{2}-1}) + \ldots + D_{\frac{\lambda_{m-2}+\lambda_{m-1}}{2}}(a_{\frac{\lambda_{m-1}}{2}-1}) + B_{\frac{\lambda_m}{2}} \ (m \text{ is odd}), \\ &A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1} + \\ &\qquad + D_{\frac{\lambda_1+\lambda_2}{2}}(a_{\frac{\lambda_2}{2}-1}) + D_{\frac{\lambda_3+\lambda_4}{2}}(a_{\frac{\lambda_4}{2}-1}) + \ldots + D_{\frac{\lambda_{m-1}+\lambda_m}{2}}(a_{\frac{\lambda_m}{2}-1}) \ (m \text{ is even}), \end{aligned} \tag{1.4.9}$$

where it is assumed that $D_k(a_0) = D_k$.

The elements of $\underline{\mathcal{N}}(G_{\mathbf{k}})$ are parametrized by partitions λ of $2n+1$ for which $l_j(\lambda)$ is even for even j. We denote this set of partitions by $\mathcal{Q}_{2n+1}$. In case of $G_{\mathbf{k}} = \mathrm{SO}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}}) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{SO}(V_2)$ where V_2 is a vector space of dimension $2n+1$ over an algebraically closed field of characteristic 2 equipped with a bilinear form $(\cdot,\cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x+y) - Q(x) - Q(y), \ x, y \in V_2,$$

and the restriction of Q to the null space $V_2^{\perp} = \{x \in V_2 : (x, y) = 0 \ \forall \ y \in V_2\}$ of $(\cdot,\cdot)$ has zero kernel. In fact G_2 is isomorphic to a group of type $\mathrm{Sp}(V_2)$, $\dim V_2 = 2n$, and hence $\underline{\mathcal{N}}(G_2) \simeq \mathcal{T}_{2n}^2$.

We also have $\widehat{W} \simeq X_{n,1}$, and the map F is the same as in case of C_n.

The map f_1 can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$. If

$$f_1(\lambda) = ((c_1', c_3', \ldots, c_{2m+1}'), (c_2', c_4', \ldots, c_{2m}'))$$

then the parts c_i' are defined by induction starting from c_1',

$$\begin{array}{ll} c_i' = \frac{\lambda_i - 1}{2} + i - 1 - 2\left[\frac{i-1}{2}\right] & \text{if } \lambda_i \text{ is odd and } c_{i-1}' \text{ is already defined;} \\ c_i' = \frac{\lambda_i}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c_{i-1}' \text{ is already defined;} \\ c_{i+1}' = \frac{\lambda_i}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c_i' \text{ is already defined.} \end{array}$$

The image of f_1 consists of all pairs $((c_1', c_3', \ldots, c_{2m+1}'), (c_2', c_4', \ldots, c_{2m}')) \in X_{n,1}$ such that $c_i' \leq c_{i+1}'$ for all odd i and $c_i' \leq c_{i+1}' + 2$ for all even i.

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The maps Φ^W and Ψ^W are the same as in case of C_n.

The map π^G is given by $\pi^G(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$, where

$$\nu_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i \text{ and } i \text{ are odd and } \lambda_{i-1} < \lambda_i; \\ \lambda_i + 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_i < \lambda_{i+1}; \\ \lambda_i & \text{otherwise,} \end{cases}$$

and

$$\varepsilon'(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with even } i \text{ such that } \lambda_{i-1} < \lambda_i; \\ 1 & \text{otherwise.} \end{cases}$$

The map $\pi^{G_{\mathbf{k}}}$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2$ such that $\varepsilon(k) \neq 0$ if ν_k^* is odd and for each even i such that ν_i^* is even we have $\nu_{i-1}^* = \nu_i^*$, i.e. $i-1$ does not appear in the partition ν. Here $\nu_1^* \geq \nu_2^* \geq \ldots \geq \nu_m^*$ is the partition dual to ν.

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{SO}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$, where $((\lambda_1^x \geq \lambda_3^x \geq \ldots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \ldots \geq \lambda_{2m}^x))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{(\dim V_x - 1)/2}$ (if $x \neq -1$) or $D_{\dim V_x/2}$ (if $x = -1$) is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda_j^x$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_G(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then the dimension of the centralizer of g in $G_{\mathbf{k}}$ is given by formula (1.4.8),

$$\dim Z_{G_{\mathbf{k}}}(g) = n + \sum_{i=1}^{m}(i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|. \tag{1.4.10}$$

$\mathbf{D_n}$

$G_{\mathbf{k}}$ is of type $\mathrm{SO}(V)$ where V is a vector space of dimension $2n$, $n \geq 3$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$ equipped with a non–degenerate symmetric bilinear form. W is the group of even permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. W can be regarded as a subgroup in the Weyl group W' of type C_n.

Let $\underline{\widetilde{W}}$ be the set of W'–conjugacy classes in W. Elements of $\underline{\widetilde{W}}$ are parametrized by pairs of partitions (λ, μ), where the parts of λ are even (for any $w \in \mathcal{C} \in \underline{\widetilde{W}}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w), the number of parts of λ is even, μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X$, $X \neq -X$), and $\sum \lambda_i + \sum \mu_j = 2n$. We denote this set of pairs of partitions by $\mathcal{A}_{2n}^0$. To each pair $(-, \mu)$, where all parts of μ are even, there correspond two conjugacy classes in W. To all other elements of $\mathcal{A}_{2n}^0$ there corresponds a unique conjugacy class in W.

An element of $\underline{\widetilde{W}}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [16], Section 7)

$$A_{\mu_1 - 1} + A_{\mu_3 - 1} + \ldots + A_{\mu_{2k-1} - 1} + D_{\frac{\lambda_1 + \lambda_2}{2}}(a_{\frac{\lambda_2}{2} - 1}) + D_{\frac{\lambda_3 + \lambda_4}{2}}(a_{\frac{\lambda_4}{2} - 1}) + \ldots + D_{\frac{\lambda_{m-1} + \lambda_m}{2}}(a_{\frac{\lambda_m}{2} - 1}). \tag{1.4.11}$$

Let $G'_{\mathbf{k}}$ be the extension of $G_{\mathbf{k}}$ by the Dynkin graph automorphism of order 2. Then $G'_{\mathbf{k}}$ is of type $\mathrm{O}(V)$. Denote by $\underline{\widetilde{\mathcal{N}}}(G_{\mathbf{k}})$ the set of unipotent classes of $G'_{\mathbf{k}}$. Note that they are all contained in $G_{\mathbf{k}}$. The elements of $\underline{\widetilde{\mathcal{N}}}(G_{\mathbf{k}})$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for even j. Note that the number of parts of such partitions is even. We denote this set of partitions by $\mathcal{Q}_{2n}$. In case when $G_{\mathbf{k}} = \mathrm{SO}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ. If λ has only even parts then λ corresponds to two unipotent classes in $G_{\mathbf{k}}$ of the same dimension. In all other cases there is a unique unipotent class in G which corresponds to λ.

Let V_2 be a vector space of dimension $2n$ over an algebraically closed field of characteristic 2 equipped with a non–degenerate bilinear form $(\cdot, \cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x + y) - Q(x) - Q(y),\ x, y \in V_2.$$

We remind that $\mathrm{SO}(V_2)$ is the connected component containing the identity of the group of linear automorphisms of V_2 preserving the quadratic, and hence the bilinear, form.

One has $\underline{\widehat{\mathcal{N}}}(G_{\mathbf{k}}) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{SO}(V_2)$.

Let G'_2 be the extension of G_2 by the Dynkin graph automorphism of order 2. Then G'_2 is of type $\mathrm{O}(V_2)$. Denote by $\underline{\widetilde{\mathcal{N}}}(G_2)$ the set of unipotent classes of G'_2 contained in G_2. Since the bilinear form $(\cdot, \cdot)$ is also alternating in characteristic 2 there is a natural injective homomorphism from $\mathrm{O}(V_2)$ to $\mathrm{Sp}(V_2)$, $\dim V_2 = 2n$, and hence $\underline{\widetilde{\mathcal{N}}}(G_2) \simeq \widetilde{\mathcal{T}}_{2n}^2$, where $\widetilde{\mathcal{T}}_{2n}^2$ is the set of elements $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$ such that λ has an even number of parts.

Let $\widehat{\widehat{W}}$ be the set of orbits of irreducible characters of W under the action of W'. Elements of $\widehat{\widehat{W}}$ are parametrized by unordered pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length of α is equal to the length of β. The set of such pairs is denoted by $Y_{n,0}$.

Instead of the maps in (1.4.2) we shall describe the following maps

$$\begin{array}{ccccccc} & & \widetilde{\mathcal{X}}^1(W) & \xleftarrow{\widetilde{f}_1} & \underline{\widetilde{\mathcal{N}}}(G_{\mathbf{k}}) & & \\ & & \downarrow\ \iota & & \downarrow\ \widetilde{\pi}^{G_{\mathbf{k}}} & & \\ G_{\mathbf{k}} & \xrightarrow{\widetilde{\phi}_{G_{\mathbf{k}}}} & \widetilde{\mathcal{R}}(W) & \xleftarrow{\widetilde{F}} & \underline{\widetilde{\mathcal{N}}}(G_2) & \overset{\widetilde{\Phi}^W}{\underset{\widetilde{\Psi}^W}{\rightleftarrows}} & \underline{\widetilde{W}}, \end{array} \qquad (1.4.12)$$

where $\widetilde{f}_1$ and $\widetilde{F}$ are induced by the restrictions of the maps f_1 and F for $G'_{\mathbf{k}}$, G'_2 to $\underline{\widetilde{\mathcal{N}}}(G_{\mathbf{k}})$, $\underline{\widetilde{\mathcal{N}}}(G_2)$, respectively, $\widetilde{\mathcal{X}}^1(W)$ and $\widetilde{\mathcal{R}}(W)$ are their images, $\widetilde{\phi}_{G_{\mathbf{k}}}, \widetilde{\Psi}^W, \widetilde{\Phi}^W$ and $\widetilde{\pi}^G$ are also induced by the corresponding maps for $G'_{\mathbf{k}}$, G'_2 and W'.

The map $\widetilde{f}_1$ is defined as in case of B_n. The image of $\widetilde{f}_1$ consists of all pairs $((c'_1, c'_3, \dots, c'_{2m+1}), (c'_2, c'_4, \dots, c'_{2m})) \in Y_{n,0}$ such that $c'_i \le c'_{i+1}$ for all odd i and $c'_i \le c'_{i+1} + 2$ for all even i.

If $(\lambda, \varepsilon) \in \widetilde{\mathcal{T}}^2_{2n}$, $\lambda = (\lambda_1 \le \lambda_2 \le \dots \le \lambda_{2m})$ and $\widetilde{F}(\lambda, \varepsilon) = ((c_1, c_3, \dots, c_{2m-1}), (c_2, c_4, \dots, c_{2m}))$ then the parts c_i are defined by induction starting from c_1,

$$\begin{array}{ll} c_i = \frac{\lambda_i - 2}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i \text{ is even, } \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined;} \\ c_i = \frac{\lambda_i - 1}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined;} \\ c_{i+1} = \frac{\lambda_i - 3}{2} + 2i - 4\left[\frac{i}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined;} \\ c_i = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_{i-1} \text{ is already defined;} \\ c_{i+1} = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_i \text{ is already defined.} \end{array}$$

The image $\widetilde{\mathcal{R}}(W)$ of $\widetilde{F}$ consists of all pairs $((c_1, c_3, \dots, c_{2m+1}), (c_2, c_4, \dots, c_{2m})) \in Y_{n,0}$ such that $c_i \le c_{i+1}$ for all odd i and $c_i \le c_{i+1} + 4$ for all even i.

The maps $\widetilde{\Phi}^W$ and $\widetilde{\Psi}^W$ are defined by the same formulas as in case of C_n.

The map $\widetilde{\pi}^{G_{\mathbf{k}}}$ is given by $\widetilde{\pi}^{G_{\mathbf{k}}}(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \le \lambda_2 \le \dots \le \lambda_{2m}) \in \mathcal{Q}_{2n}$, where

$$\nu_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_{i-1} < \lambda_i; \\ \lambda_i + 1 & \text{if } \lambda_i \text{ and } i \text{ are odd, and } \lambda_i < \lambda_{i+1}; \\ \lambda_i & \text{otherwise,} \end{cases}$$

and

$$\varepsilon'(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with odd } i \text{ such that } \lambda_{i-1} < \lambda_i; \\ 1 & \text{otherwise.} \end{cases}$$

The map $\widetilde{\pi}^{G_{\mathbf{k}}}$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \widetilde{\mathcal{T}}^2_{2n}$ such that $\varepsilon(k) \ne 0$ if ν^*_k is odd and for each even i such that ν^*_i is even we have $\nu^*_{i-1} = \nu^*_i$, i.e. $i-1$ does not appear in the partition ν. Here $\nu^*_1 \ge \nu^*_2 \ge \dots \ge \nu^*_m$ is the partition dual to ν.

To describe $\widetilde{\phi}_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{SO}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \ne 1$ let $\lambda^x_1 \ge \lambda^x_2 \ge \dots \ge \lambda^x_{2m}$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda^x_1 \ge \lambda^x_2 \ge \dots \ge \lambda^x_{2m}$ be the sequence in $\mathbb{N}$, where $((\lambda^x_1 \ge \lambda^x_3 \ge \dots \ge \lambda^x_{2m-1}), (\lambda^x_2 \ge \lambda^x_4 \ge \dots \ge \lambda^x_{2m}))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $D_{\dim\, V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \ge \lambda(g)_2 \ge \dots \ge \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda^x_j$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\widetilde{\phi}_G(g)$ is the pair of partitions $((\lambda(g)_1 \ge \lambda(g)_3 \ge \dots \ge \lambda(g)_{2m-1}), (\lambda(g)_2 \ge \lambda(g)_4 \ge \dots \ge \lambda(g)_{2m}))$.

The preimage $\widetilde{\phi}^{-1}_{G_{\mathbf{k}}}(\lambda, \mu)$ is a stratum in $G_{\mathbf{k}}$ in all cases except for the one when the pair (λ, μ) is of the form $((\lambda_1 \ge \lambda_3 \ge \dots \ge \lambda_{2m-1}), (\lambda_1 \ge \lambda_3 \ge \dots \ge \lambda_{2m-1}))$. In that case $\widetilde{\phi}^{-1}_{G_{\mathbf{k}}}(\lambda, \mu)$ is a union of two strata, and the conjugacy classes in each of them have the same dimension.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \widetilde{\mathcal{T}}^2_{2n}$, $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_m)$ then the dimension of the centralizer of g in $G_{\mathbf{k}}$ is given by the following formula

$$\dim\, Z_{G_{\mathbf{k}}}(g) = n + \sum_{i=1}^{m} (i-1)\lambda_i - \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| - |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 1\}|. \qquad (1.4.13)$$

1.5 The strict transversality condition

Let Δ_+^s be a set of positive roots associated to $s \in W = W(G_{\mathbf{k}}, H_{\mathbf{k}})$ in Section 1.2, where we again assume that that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. Denote by $P_{\mathbf{k}}$ the parabolic subgroup of $G_{\mathbf{k}}$ containing the Borel subgroup corresponding to $-\Delta_+^s$ and assocuated to the subset $-\Gamma_0^s$ of the set of simple roots in $-\Gamma^s$, where $\Gamma_0^s = \Gamma^s \cap \Delta_0$. Let $N_{\mathbf{k}}$ and $L_{\mathbf{k}}$ the unipotent radical and the Levi factor of $P_{\mathbf{k}}$, respectively, and $\overline{N}_{\mathbf{k}}$ the opposite unipotent radical.

Denote a representative for the Weyl group element s in $G_{\mathbf{k}}$ by $\dot{s}$. Let $Z_{\mathbf{k}}$ be the connected subgroup of $G_{\mathbf{k}}$ generated by the semisimple part of the Levi subgroup $L_{\mathbf{k}}$ and by the identity component $H_{\mathbf{k}}^0$ of centralizer of $\dot{s}$ in $H_{\mathbf{k}}$. Let $N_{s,\mathbf{k}} = \{v \in N_{\mathbf{k}} | \dot{s}v\dot{s}^{-1} \in \overline{N}_{\mathbf{k}}\}$.

Following the definition of transversal slices Σ_s we define

$$\Sigma_{s,\mathbf{k}} = sZ_{\mathbf{k}}N_{s,\mathbf{k}}.$$

Recall that the definition of Δ_+^s, and hence of $\Sigma_{s,\mathbf{k}}$, depends on the choice of ordering of terms in decomposition (1.2.14). In this section for every conjugacy class $\mathcal{C} \in C(W)$ we define a variety $\Sigma_{s,\mathbf{k}}$, $s \in \mathcal{C}$ such that every conjugacy class $\mathcal{O} \in G_{\mathcal{C}}$ intersects $\Sigma_{s,\mathbf{k}}$ and

$$\dim \mathcal{O} = \text{codim } \Sigma_{s,\mathbf{k}}. \tag{1.5.1}$$

It turns out that in order to fulfill condition (1.5.1) the subspaces $\mathfrak{h}_i$ in (1.2.14) should be ordered in such a way that $\mathfrak{h}_0 \subset \mathfrak{h}_{\mathbb{R}}$ is the subspace fixed by the action of s, and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$, where λ and μ are eigenvalues of the corresponding matrix $I - M$ for s. In the case of exceptional root systems this is verified using a computer program, and in the case of classical root systems this is confirmed by explicit computation based on a technical lemma. In order to formulate this lemma we recall realizations of classical irreducible root systems.

Let V be a real Euclidean n–dimensional vector space with an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. The root systems of types A_{n-1}, B_n, C_n and D_n can be realized in V as follows.

$\mathbf{A_n}$

The roots are $\varepsilon_i - \varepsilon_j$, $1 \leq i, j \leq n$, $i \neq j$, $\mathfrak{h}_{\mathbb{R}}$ is the hyperplane in V consisting of the points the sum of whose coordinates is zero.

$\mathbf{B_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

$\mathbf{C_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm 2\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

$\mathbf{D_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

In all cases listed above the corresponding Weyl group W is a subgroup of the Weyl group of type C_n acting on the elements of the basis $\varepsilon_1, \ldots, \varepsilon_n$ by permuting the basis vectors and changing the sign of an arbitrary subset of them.

Now we formulate the main lemma.

Lemma 1.5.1. *Let s be an element of the Weyl group of type C_n operating on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ as indicated in Section 1.4, where $\varepsilon_1, \ldots, \varepsilon_n$ is the basis of V introduced above. Assume that s has either only one nontrivial cycle of length $k/2$ (k is even), which is negative, or only one nontrivial cycle of length k, which is positive, $1 < k \leq n$. Let Δ be a root system of type A_{n-1}, B_n, C_n or D_n realized in V as above.*

(i) If s has only one nontrivial cycle of length $k/2$, which is negative, then k is even, the spectrum of s in the complexification $V_{\mathbb{C}}$ of V is $\epsilon_r = \exp(\frac{2\pi i(k-2r+1)}{k})$, $r = 1, \ldots, k/2$, and possibly $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

(ii) If s only has one nontrivial cycle of length k, which is positive, then the spectrum of s in the complexification of V is $\epsilon_r = \exp(\frac{2\pi i(k-r)}{k})$, $r = 1, \ldots, k-1$, and $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

In both cases we denote by V_r the invariant subspace in V which corresponds to $\epsilon_r = \exp(\frac{2\pi i([k/2]+1-r)}{k})$, $r = 1, \dots, \left[\frac{k}{2}\right]$ or $\epsilon_0 = 1$ in case of a positive nontrivial cycle and to $\epsilon_r = \exp(\frac{2\pi i(2\left[\frac{k/2+1}{2}\right]+1-2r)}{k})$, $r = 1, \dots, \left[\frac{k/2+1}{2}\right]$ or $\epsilon_0 = 1$ in case of a negative cycle. For $r \neq 0$ the space V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_{\mathbb{C}}$ corresponding to ϵ_r, and V_0 is the subspace of fixed points of s in V.

V_r is two–dimensional if $\epsilon_r \neq \pm 1$, one–dimensional if $\epsilon_r = -1$ or may have arbitrary dimension if $\epsilon_r = 1$.

Let Δ_+^s be a system of positive roots associated to s and defined as in Section 1.2, where we use the decomposition

$$V = \bigoplus_i V_i \tag{1.5.2}$$

as (1.2.14) in the definition of Δ_+^s. Denote by $\Delta_i \subset \Delta$ the corresponding subsets of roots defined as in (1.2.15).

Let Δ_0^s be the root subsystem fixed by the action of s and $\underline{l}(s)$ the number of positive roots which become negative under the action of s.

(iii) If s has only one nontrivial cycle of length k, which is positive, we have

1. *if $\Delta = A_{n-1}$ then $\Delta_0^s = A_{n-k-1}$, $\underline{l}(s) = 2n - k - 1$;*
2. *if $\Delta = B_n(C_n)$ then $\Delta_0^s = B_{n-k}(C_{n-k})$, $\underline{l}(s) = 4n - 2k$ for odd k and $\underline{l}(s) = 4n - 2k + 1$ for even k;*
3. *if $\Delta = D_n$ then $\Delta_0^s = D_{n-k}$, $\underline{l}(s) = 4n - 2k - 2$ for odd k and $\underline{l}(s) = 4n - 2k - 1$ for even k.*

(iv) If s has only one nontrivial cycle of length $\frac{k}{2}$, which is negative, we have

1. *if $\Delta = B_n(C_n)$ then $\Delta_0^s = B_{n-k/2}(C_{n-k/2})$, $\underline{l}(s) = 2n - k/2$;*
2. *if $\Delta = D_n$ then $\Delta_0^s = D_{n-k/2}$, $\underline{l}(s) = 2n - k/2 - 1$.*

(v) If s has only one nontrivial cycle of length k, which is positive, Δ is of type B_n, C_n or D_n, and k is even then $\Delta = \Delta_{k/2} \cup \Delta_{k/2-1} \cup \Delta_0^s$ (disjoint union), and all roots in $\Delta_{k/2-1}$ are orthogonal to the fixed point subspace for the action of s on V.

(vi) In all other cases $\Delta = \Delta_{i_{\max}} \cup \Delta_0^s$ (disjoint union), where $i_{\max}$ is the maximal possible index i which appears in decomposition (1.5.2).

Proof. The proof is similar in all cases. We only give details in the most complicated case when s has only one nontrivial cycle, which is positive, Δ is of type $B_n(C_n)$, and k is even. Without loss of generality one can assume that s corresponds to the cycle of the form

$$\varepsilon_1 \to \varepsilon_2 \to \varepsilon_4 \to \varepsilon_6 \to \dots \to \varepsilon_{k-2} \to \varepsilon_k \to \varepsilon_{k-1} \to \varepsilon_{k-3} \to \dots \to \varepsilon_3 \to \varepsilon_1 \ (k > 2),\ \varepsilon_1 \to \varepsilon_2 \to \varepsilon_1 \ (k = 2).$$

From this definition one easily sees that $\Delta_0^s = B_{n-k}(C_{n-k}) = \Delta \cap V'$, where $V' \subset V$ is the subspace generated by $\varepsilon_{k+1}, \dots, \varepsilon_n$. Computing the eigenvalues of s in $V_{\mathbb{C}}$ is a standard exercise in linear algebra. The eigenvalues are expressed in terms of the exponents of the root system of type A_{k-1} (see [15], Ch. 10).

The invariant subspace V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_{\mathbb{C}}$ corresponding to the eigenvalue ϵ_r. If $\epsilon_r \neq \pm 1$ then V_r is two–dimensional, and for $\epsilon_r = -1$ V_r is one–dimensional. In the former case V_r will be regarded as the real form of a complex plane with the orthonormal basis $1, i$. Under this convention the orthogonal projection operator onto V_r acts on the basic vectors ε_j as follows

$$\varepsilon_{2j+1} \mapsto c\epsilon_r^j,\ j = 0, \dots, \frac{k}{2} - 1, \varepsilon_{2j} \mapsto c\epsilon_r^{-j},\ j = 1, \dots, \frac{k}{2}, \tag{1.5.3}$$

where $c = \sqrt{\frac{2}{k}}$. Consider the case when $k > 2$; the case $k = 2$ can be analyzed in a similar way.

To compute $\underline{l}(s)$ using the definition of Δ_+^s given in Section 1.2 one should first look at all roots which have nonzero projections onto $V_{k/2}$ on which s acts by rotation with the angle $\frac{2\pi}{k}$.

From (1.5.3) we deduce that the roots which are not fixed by s and have zero orthogonal projections onto $V_{k/2}$ are $\pm(\varepsilon_j + \varepsilon_{k-j+1})$, $j = 1, \dots \frac{k}{2}$. The number of those roots is equal to k, and they all have nonzero orthogonal projections onto $V_{k/2-1}$. From (1.5.3) we also obtain that all the other roots which are not fixed by s have nonzero orthogonal projections onto $V_{k/2}$, hence $|\Delta_{k/2-1}| = k$. The number of roots fixed by s is $2(n-k)^2$ since it is equal to the number of roots in $\Delta_0 = \Delta_0^s = B_{n-k}(C_{n-k})$. Hence $\Delta = \Delta_{k/2} \cup \Delta_{k/2-1} \cup \Delta_0$ (disjoint union), the number of roots in $\Delta_{k/2}$ is $|\Delta| - |\Delta_0| - |\Delta_{k/2-1}| = 2n^2 - 2(n-k)^2 - k = 4nk - 2k^2 - k$, $|\Delta_{k/2}| = 4nk - 2k^2 - k$.

Now using the symmetry of the root system Δ as a subset of V and the fact that s acts as rotation by the angles $\frac{2\pi}{k}$ and $\frac{4\pi}{k}$ in $V_{k/2}$ and $V_{k/2-1}$, respectively, we deduce that the number of positive roots in $\Delta_{k/2}$ ($\Delta_{k/2-1}$) which become negative under the action of s is equal to the number of roots in $\Delta_{k/2}$ ($\Delta_{k/2-1}$) divided by the order of s in $V_{k/2}(V_{k/2-1})$. Therefore

$$\underline{l}(s) = \frac{|\Delta_{k/2}|}{k} + \frac{|\Delta_{k/2-1}|}{k/2} = \frac{4nk-2k^2-k}{k} + \frac{k}{k/2} = 4n-2k+1.$$

This completes the proof in the considered case. □

Now we are in a position to prove the main statement of this section.

Theorem 1.5.2. *Let $G_{\mathbf{k}}$ be a connected semisimple algebraic group over an algebraically closed filed $\mathbf{k}$ of characteristic good for $G_{\mathbf{k}}$, and $\mathcal{O} \in \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$. Let Δ be the root system of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$ and Δ_+^s a system of positive roots in Δ associated to s and defined in Section 1.2 with the help of decomposition (1.2.14), where the subspaces $\mathfrak{h}_i$ are ordered in such a way that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s, and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$. In the case of exceptional root systems we assume, in addition, that Δ_+^s is chosen as in the tables in Appendix 2, so that $s = s^1 s^2$ is defined by the data from columns three and four in the tables in Appendix 2. Then all conjugacy classes in the stratum $G_{\mathcal{O}} = \phi_G^{-1}(F(\mathcal{O}))$ intersect the corresponding variety $\Sigma_{s,\mathbf{k}}$ at some points of the subvariety $\dot{s}H_{0,\mathbf{k}}N_{s,\mathbf{k}}$, where $H_{0,\mathbf{k}} \subset H_{\mathbf{k}}$ is the identity component of the centralizer of $\dot{s}$ in $H_{\mathbf{k}}$. Moreover, if $\mathcal{O} \in \underline{\mathcal{N}}(G_p) \subset \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$ for some p, then for any $g \in G_{\mathcal{O}}$*

$$\dim Z_{G_{\mathbf{k}}}(g) = \dim \Sigma_{s,\mathbf{k}} = \mathrm{codim}_{G_p}\, \mathcal{O}. \tag{1.5.4}$$

Proof. We shall divide the proof into several lemmas. First we compute the dimension of the slice $\Sigma_{s,\mathbf{k}}$, $s \in \Psi^W(\mathcal{O})$ and justify that for any $g \in G_{\mathcal{O}}$ equality (1.5.4) holds.

Lemma 1.5.3. *Assume that the conditions of Theorem 1.5.2 are satisfied. Then for any $g \in G_{\mathcal{O}}$, where $\mathcal{O} \in \underline{\mathcal{N}}(G_p) \subset \widehat{\underline{\mathcal{N}}}(G)$ for some p, equality (1.5.4) holds, i.e.*

$$\dim Z_{G_{\mathbf{k}}}(g) = \dim \Sigma_{s,\mathbf{k}} = \mathrm{codim}_{G_p}\, \mathcal{O}.$$

Proof. Observe that by the definition of the slice $\Sigma_{s,\mathbf{k}}$

$$\dim \Sigma_{s,\mathbf{k}} = l(s) + |\Delta_0| + \dim \mathfrak{h}_0,$$

where $l(s)$ is the length of s with respect to the system of simple roots in Δ_+^s. Hence to compute $\dim \Sigma_{s,\mathbf{k}}$ we have to find all numbers in the right hand side of the last equality.

Consider the case of classical Lie algebras when each Weyl group element is a product of cycles in a permutation group. In this case identity (1.5.4) is proved by a straightforward calculation using Lemma 1.5.1.

If G is of type A_n let s be a representative in the conjugacy class of the Weyl group which corresponds to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$. The particular ordering of the invariant subspaces $\mathfrak{h}_i$ in the formulation of this theorem implies that the length $l(s)$ should be computed by successive application of Lemma 1.5.1 to the cycles s_i of s, which correspond to λ_i placed in a non–increasing order.

We should first apply Lemma 1.5.1 to the cycle s_1 of s which corresponds to the maximal part λ_1. In this case $\underline{l}(s_1) = 2n - \lambda_1 + 1$ and $\Delta_0^{s_1} = A_{n-\lambda_1} = \Delta \setminus \Delta_{i_M}$ in the notation of Section 1.2. The remaining cycles $s_2, \ldots, s_m$ of s corresponding to $\lambda_2 \geq \ldots \geq \lambda_m$ act on $\Delta_0^{s_1}$, and we can apply Lemma 1.5.1 to s_2 acting on $\Delta_0^{s_1}$ to get $\underline{l}(s_2) = 2(n-\lambda_1) - \lambda_2 + 1$ and $\Delta_0^{s_2} = A_{n-\lambda_1-\lambda_2} = \Delta \setminus \left(\Delta_{i_M} \cup \Delta_{i_{M-1}}\right)$. Iterating this procedure and observing that $l(s)$ is equal to the number of positive roots which become negative under the action of s we obtain

$$l(s) = \sum \underline{l}(s_k),\ \underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1, \tag{1.5.5}$$

where the first sum in (1.5.5) is taken over k for which $\lambda_k > 1$.

The number of roots fixed by s can be represented in a similar form,

$$|\Delta_0| = \sum \underline{l}(s_k),\ \underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1, \tag{1.5.6}$$

where the sum in (1.5.6) is taken over k for which $\lambda_k = 1$.

Finally the dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}$ is $m-1$, dim $\mathfrak{h}_0 = m-1$.

Observe now that

$$\dim \Sigma_{s,\mathbf{k}} = l(s) + |\Delta_0| + \dim \mathfrak{h}_0, \tag{1.5.7}$$

and hence

$$\dim \Sigma_{s,\mathbf{k}} = \sum_{k=1}^{m} \underline{l}(s_k) + m - 1 = \sum_{k=1}^{m}\left(2(n - \sum_{i=1}^{k-1}\lambda_i) - \lambda_k + 1\right) + m - 1.$$

Exchanging the order of summation and simplifying this expression we obtain that

$$\dim \Sigma_{s,\mathbf{k}} = n + 2\sum_{i=1}^{m}(i-1)\lambda_i$$

which coincides with (1.4.4).

The computations of dim $\Sigma_{s,\mathbf{k}}$ in case of B_n and of C_n are similar. If $(\nu,\varepsilon) \in \mathcal{T}_{2n}^2$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, corresponds to $\mathcal{O} \in \widehat{\underline{N}}(G_{\mathbf{k}}) = \underline{N}(G_2)$ then $\Psi^W(\nu,\varepsilon) = (\lambda,\mu) \in \mathcal{A}_{2n}^1 \simeq \underline{W}$ is defined in Section 1.4, part $\mathbf{C_n}$. λ consists of even parts ν_i of ν for which $\varepsilon(\nu_i) = 1$, and μ consists of all odd parts of ν and of even parts ν_i of ν for which $\varepsilon(\nu_i) = 0$, the last two types of parts appear in pairs of equal parts. Let s be a representative in the conjugacy class $\Psi^W(\nu,\varepsilon)$. Then each part λ_i corresponds to a negative cycle of s of length $\frac{\lambda_i}{2}$, and each pair $\mu_i = \mu_{i+1}$ of equal parts of μ corresponds to a positive cycle of s of length μ_i. We order the cycles s_k of s associated to the (pairs of equal) parts of the partition ν in a way compatible with a non–increasing ordering of the parts of the partition $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, i.e. if we denote by s_k the cycle that corresponds to an even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ or to a pair $\nu_k = \nu_{k+1}$ of odd parts of ν or of even parts of ν for which $\varepsilon(\nu_k) = 0$ then $s_k \geq s_l$ if $\nu_k \geq \nu_l$.

Similarly to the case of A_n, by the definition of Δ_+^s and by Lemma 1.5.1 applied iteratively to the cycles s_k in the order defined above, the length $l(s)$ of s is the sum of the following terms $\underline{l}(s_k)$.

To each even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ we associate the term

$$\underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2};$$

to each pair of odd parts $\nu_k = \nu_{k+1} > 1$ we associate the term

$$\underline{l}(s_k) = 4(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - 2\nu_k = \left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \left(2(n - \sum_{i=1}^{k}\frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2}\right);$$

note that the sum of these terms over all pairs $\nu_k = \nu_{k+1} = 1$ gives the number $|\Delta_0|$ of the roots fixed by s;

to each pair of even parts $\nu_k = \nu_{k+1}$ for which $\varepsilon(\nu_k) = 0$ we associate the term

$$\underline{l}(s_k) = 4(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - 2\nu_k + 1 = \left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2} + \frac{1}{2}\right) + \left(2(n - \sum_{i=1}^{k}\frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2} + \frac{1}{2}\right).$$

The dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}_{\mathbb{R}}$ is equal to a half of the sum of the number of all even parts ν_k for which $\varepsilon(\nu_k) = 0$ and of the number of all odd parts ν_k,

$$\dim \mathfrak{h}_0 = \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|. \tag{1.5.8}$$

Finally substituting all the computed contributions into formula (1.5.7) we obtain

$$\dim \Sigma_{s,\mathbf{k}} = \sum_{k=1}^{m}\left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| +$$
$$+\frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|.$$

Exchanging the order of summation and simplifying this expression we obtain that

$$\dim \Sigma_{s,\mathbf{k}} = n + \sum_{i=1}^{m}(i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| \tag{1.5.9}$$

which coincides with (1.4.8) or (1.4.10).

In case of D_n the number $\dim \Sigma_{s,\mathbf{k}}$ can be easily obtained if we observe that the map $\widetilde{\Psi}^W$ is defined by the same formula as Ψ^W in case of C_n. In case when $\widetilde{\Psi}^W(\nu,\varepsilon) = (-,\mu)$, where all parts of μ are even, there are two conjugacy classes in W which correspond to $\widetilde{\Psi}^W(\nu,\varepsilon)$. However, the numbers $l(s)$, $|\Delta_0|$ and $\dim \mathfrak{h}_0$ are the same in both cases. They only depend on $\widetilde{\Psi}^W(\nu,\varepsilon)$ in all cases. Let $s \in W$ be a representative from the conjugacy class $\widetilde{\Psi}^W(\nu,\varepsilon)$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$.

From Lemma 1.5.1 we deduce that in the case of D_n the contributions of the cycles s_k of s to the formula for $\dim \Sigma_{s,\mathbf{k}}$ can be obtained from the corresponding contributions in case of C_n in the following way: for each pair of odd parts $\nu_k = \nu_{k+1}$ and for each pair of even parts $\nu_k = \nu_{k+1}$ with $\varepsilon(\nu_k) = 0$ the corresponding contribution $\underline{l}(s_k)$ to $l(s)$ should be reduced by 2 and for each even part ν_k of ν with $\varepsilon(\nu_k) = 1$ the corresponding contribution $\underline{l}(s_k)$ to $l(s)$ should be reduced by 1. This observation and formula (1.5.9) yield

$$\begin{aligned}\dim \Sigma_{s,\mathbf{k}} = n + \sum_{i=1}^{m}(i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| - \\ -|\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even}\}| = \\ = n + \sum_{i=1}^{m}(i-1)\nu_i - \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 1\}|\end{aligned}$$

which coincides with (1.4.13).

In case of root systems of exceptional types $\dim \Sigma_{s,\mathbf{k}}$ can be found in the tables in Appendix 2. According to those tables equality (1.5.4) holds in all cases.

□

Now we show that all conjugacy classes in the stratum $G_{\mathcal{O}} = \phi_G^{-1}(F(\mathcal{O}))$ intersect the corresponding variety $\Sigma_{s,\mathbf{k}}$, $s \in \Psi^W(\mathcal{O})$. Let Δ_+^s be the system of the positive roots introduced in the statement of Theorem 1.5.2,

$$\mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^{K} \mathfrak{h}_i$$

the corresponding decomposition of $\mathfrak{h}_{\mathbb{R}}$ and $h_i \in \mathfrak{h}_i$ the corresponding elements of the subspaces $\mathfrak{h}_i$.

Recall that $\mathfrak{h}_0$ is the subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. If $\mathfrak{h}_0 = 0$, let $\mathfrak{h}'_i = \mathfrak{h}_i$ and $h'_i = h_i$, $i = 0, \ldots, K$. Otherwise let $\mathfrak{h}'_K = \mathfrak{h}_0$, $\mathfrak{h}'_i = \mathfrak{h}_{i+1}$, $h'_i = h_{i+1}$, $i = 0, \ldots, K-1$ and choose an element $h'_K \in h'_K$ such that $h'_K(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}'_K$ with respect to the natural pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$.

By a suitable rescaling of h'_K we can assume that conditions (1.2.16) are satisfied for the elements h'_i and roots α from sets Δ'_i defined as in (1.2.20) with h_j, h_i replaced by h'_j, h'_i. Indeed, observe that

$$\Delta'_i = \{\alpha \in \Delta : h'_j(\alpha) = 0, j > i,\ h'_i(\alpha) \neq 0\} \subset \{\alpha \in \Delta : h_j(\alpha) = 0, j > i+1,\ h_{i+1}(\alpha) \neq 0\} = \Delta_{i+1}, i = 0, \ldots, K-1$$

by the definition of the elements h'_i. Thus, since (1.2.16) is satisfied for h_i, $i = 0, \ldots, K$, it is also satisfied for $h'_i = h_i$ if $i < K$. By a suitable rescaling of h'_K we can assume that (1.2.16) is satisfied for h'_K as well.

Let Δ_+^1 be a system of positive roots in $\Delta = \Delta(G_{\mathbf{k}}, H_{\mathbf{k}})$ which corresponds to the Weyl chamber containing the element $\bar{h}' = \sum_{i=0}^{K} h'_i$. By Lemma 1.2.2 the set of roots $\Delta(L_1, H_{\mathbf{k}})$ with zero orthogonal projections onto $\mathfrak{h}_0$ is the root system of a standard Levi subgroup $L_1 \subset G_{\mathbf{k}}$ with respect to the system of simple roots in Δ_+^1.

Using formula (1.2.1) and recalling that the roots $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$, i.e. for $i = 1, \ldots, l'$ γ_i has zero orthogonal projection onto ${\mathfrak{h}'}^{\perp}$, we deduce that for $i = 1, \ldots, l'$ $\gamma_i \in \Delta(L_1, H_{\mathbf{k}})$, and hence s belongs to the Weyl group $W_1 \subset W$ of the root system $\Delta(L_1, H_{\mathbf{k}})$. Note that, as L_1 is a standard Levi subgroup in $G_{\mathbf{k}}$, W_1 is a parabolic subgroup in W with respect to the system of simple roots in Δ_+^1. Since $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$, the linear span of roots from $\Delta(L_1, H_{\mathbf{k}})$ coincides with $\mathfrak{h}'$, and hence the element s is elliptic in W_1 as s acts without fixed points on $\mathfrak{h}'$.

Let w be a minimal length representative in the conjugacy class of s in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta^1_+ \cap \Delta(L_1, H_{\mathbf{k}})$. By Lemma 3.1.14 in [38] if $w \in \Psi^W(\mathcal{O}) \cap W_1$ is of minimal possible length with respect to the system of simple reflections in W_1 then it is also of minimal possible length with respect to the system of simple reflections in W, where in both cases the simple reflections are the reflections with respect to the simple roots in Δ^1_+. Note that w is elliptic in W_1 as well.

Let B_1 be the Borel subgroup in G corresponding to $-\Delta^1_+$, $P_1 \supset B_1$ the parabolic subgroup of G corresponding to W_1. Thus L_1 is the Levi factor of P_1.

Denote by $B_2 = B_1 \cap L_1$ the Borel subgroup in L_1. One can always find a representative $\dot{w} \in L_1$ of w.

Lemma 1.5.4. *Any conjugacy class in $G_{\mathcal{O}}$ intersects $B_2\dot{w}B_2 \subset B_1\dot{w}B_1$.*

Proof. By the definition the stratum $G_{\mathcal{O}}$ consists of all conjugacy classes of minimal possible dimension which intersect the Bruhat cell $B_1\dot{w}B_1$. Denote by U_1 the unipotent radical of P_1. Then by the definition of parabolic subgroups one can always find a one parameter subgroup $\rho : \mathbf{k}^* \to Z_{G_{\mathbf{k}}}(L_1)$ such that

$$\lim_{t\to 0} \rho(t)n\rho(t^{-1}) = 1 \tag{1.5.10}$$

for any $n \in U_1$.

Let $\gamma \in G_{\mathcal{O}}$ be a conjugacy class which intersects $B_1\dot{w}B_1$ at point $b\dot{w}b'$, $b, b' \in B_1$ such that $b\dot{w}b' \notin B_2\dot{w}B_2$. Since by definitions of B_2 and U_1 we have $B_1 = B_2U_1$ there are unique factorizations $b = un$, $b' = u'n'$, $u, u' \in B_2$, $n, n' \in U_1$. By (1.5.10) we have

$$\lim_{t\to 0} \rho(t)b\dot{w}b'\rho(t^{-1}) = \lim_{t\to 0} u\rho(t)n\rho(t^{-1})\dot{w}u'\rho(t)n'\rho(t^{-1}) = u\dot{w}u' \in B_2\dot{w}B_2,$$

and hence the closure of γ contains a conjugacy class γ' which intersects $B_1\dot{w}B_1$ at some point of $B_2\dot{w}B_2 \subset B_1\dot{w}B_1$. In particular, $\dim \gamma > \dim \gamma'$. This is impossible by the definition of $G_{\mathcal{O}}$, and hence γ intersects $B_1\dot{w}B_1$ at some point of $B_2\dot{w}B_2 \subset B_1\dot{w}B_1$.

□

Lemma 1.5.5. *Let $G_{\mathbf{k}}$ be a connected semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for $G_{\mathbf{k}}$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an elliptic element. Denote by $\mathcal{O}_s$ the conjugacy class of s in W. Then $\Phi^W(\mathcal{O}_s) \subset \underline{\mathcal{N}}(G_{\mathbf{k}})$.*

Proof. The statement of this lemma is a consequence of the fact that s is elliptic. Indeed, it suffices to consider the case when $G_{\mathbf{k}}$ is simple.

In case when $G_{\mathbf{k}}$ is of type A_n this is obvious since $\widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$ contains only unipotent classes. In fact in this case $\mathcal{O}_s$ is the Coxeter class, and $\Phi^W(\mathcal{O}_s)$ is the class of regular unipotent elements.

If $G_{\mathbf{k}}$ is of type B_n, C_n or D_n, formula (1.5.8) implies that if $\Phi^W(\mathcal{O}_s)$ corresponds to $(\nu, \varepsilon) \in \mathcal{T}^2_{2n}(\widetilde{\mathcal{T}}^2_{2n})$ then ν has no odd parts and no even parts ν_i with $\varepsilon(\nu_i) = 0$. According to the description given in the previous section the map $\pi^{G_{\mathbf{k}}}(\widetilde{\pi}^{G_{\mathbf{k}}})$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}^2_{2n}(\widetilde{\mathcal{T}}^2_{2n})$ such that $\varepsilon(k) \neq 0$ if ν^*_k is odd and for each even i such that ν^*_i is even we have $\nu^*_{i-1} = \nu^*_i$, i.e. $i-1$ does not appear in the partition ν. We deduce that $\Phi^W(\mathcal{O}_s)$ is contained in the image of $\pi^{G_{\mathbf{k}}}(\widetilde{\pi}^{G_{\mathbf{k}}})$, i.e. $\Phi^W(\mathcal{O}_s) \in \underline{\mathcal{N}}(G_{\mathbf{k}})$ is a unipotent class in $G_{\mathbf{k}}$.

In case when $G_{\mathbf{k}}$ is of exceptional type this can be checked by examining the tables in Appendix 2.

□

Now we show that in fact one can always take $w = s$.

Lemma 1.5.6. *The element s is of minimal length representative in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta^1_+ \cap \Delta(L_1, H_{\mathbf{k}})$.*

Proof. Let M_1 be the semisimple part of L_1 and $\mathcal{O}_n = \Phi^{W_1}(\mathcal{O}_w) \subset M_1$, where $\mathcal{O}_w$ is the conjugacy class of w in the Weyl group $W_1 = W(L_1, H)$. By the previous lemma applied to the group M_1 and the elliptic element $w \in W_1$ we have $\mathcal{O}_n \in \underline{\mathcal{N}}(M_1)$.

Therefore $\mathcal{O}_n$ is the unipotent class of minimal possible dimension which intersects $B_2\dot{w}B_2$. By Theorem 0.7 in [73] the codimension of $\mathcal{O}_n$ in M_1 is equal to $l_1(w)$, where l_1 is the length function in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta^1_+ \cap \Delta(L_1, H_{\mathbf{k}})$,

$$\mathrm{codim}_{M_1}\ \mathcal{O}_n = l_1(w). \tag{1.5.11}$$

Now we show that s has minimal length in the Weyl group W_1 with respect to the system of simple roots in the set of positive roots $\Delta(L_1, H_{\mathbf{k}})_+$.

Indeed, let $\Sigma'_{s,\mathbf{k}}$ be the variety in M_1 associated to $s \in W_1$ in the beginning of this section, where we use $\Delta(L_1, H_{\mathbf{k}})_+$ as the system of positive roots in the definition of $\Sigma'_{s,\mathbf{k}}$.

Formula (1.5.4) confirmed in Lemma 1.5.3 is applicable to the slice $\Sigma'_{s,\mathbf{k}}$ and yields

$$\mathrm{codim}_{M_1}\ \mathcal{O}_n = \dim\ \Sigma'_{s,\mathbf{k}}.$$

Formula (1.5.7) and the fact that s is elliptic in W_1 imply that

$$\dim\ \Sigma'_{s,\mathbf{k}} = l_1(s),$$

From the last two formulas we infer

$$\mathrm{codim}_{M_1}\ \mathcal{O}_n = \dim\ \Sigma'_{s,\mathbf{k}} = l_1(s).$$

The last formula and (1.5.11) yield $l_1(w) = l_1(s)$, and hence s has minimal possible length in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+$.

□

Now we can assume that $s = w$.

Lemma 1.5.7. *Any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $\Sigma_{s,\mathbf{k}}$ at some point of $\dot{s}H_{0,\mathbf{k}}N_{s,\mathbf{k}}$.*

Proof. Let $\alpha \in \Delta \cap \mathfrak{h}'$. Then $\alpha \in \Delta_i \cap \mathfrak{h}'$ for some $i > 0$. Observe that by the definition of the subspaces $\mathfrak{h}'_k$ and by the choice of the elements h_k, $k = 0, \ldots, K$

$$\Delta_i \cap \mathfrak{h}' = \{\beta \in \Delta : h_j(\beta) = 0, j > i, h_i(\beta) \neq 0, h_0(\beta) = 0\} = \{\beta \in \Delta : h'_j(\beta) = 0, j > i-1, h'_{i-1}(\beta) \neq 0, \} = \Delta'_{i-1},$$

and hence by (1.2.19) $\alpha \in (\Delta_i)_+ \cap \mathfrak{h}'$ if and only if $h'_{i-1}(\alpha) = h_i(\alpha) > 0$, i.e. $\alpha \in (\Delta_i)_+ \cap \mathfrak{h}'$ if and only if $\alpha \in \Delta^1_+ \cap \Delta'_i$. Therefore if we denote $B_3 = B_{\mathbf{k}} \cap L_1$, where $B_{\mathbf{k}}$ is the Borel subgroup corresponding to $-\Delta^s_+$, then $B_3 = B_1 \cap L_1 = B_2$.

By Lemma 1.5.4 applied to the element $s \in W$ any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $B_2\dot{s}B_2$, and hence it also intersects $B_3\dot{s}B_3 \subset B\dot{s}B$ as $B_3 = B_2$. But by the definition of L_1 s acts on the root system of the pair $(L_1, H_{\mathbf{k}})$ without fixed points. Since $B_3 = B \cap L_1$ and s fixes all the roots of the pair $(Z_{\mathbf{k}}H_{\mathbf{k}}, H_{\mathbf{k}})$ we have an inclusion $B_3 \subset H_{\mathbf{k}}N_{\mathbf{k}}$, where $N_{\mathbf{k}}$ is the unipotent radical of the parabolic subgroup $P_{\mathbf{k}}$ associated to s in the beginning of this section. Hence $B_3\dot{s}B_3 \subset N_{\mathbf{k}}\dot{s}H_{\mathbf{k}}N_{\mathbf{k}}$.

Let $H_{0,\mathbf{k}}$ be the identity component of the centralizer of $\dot{s}$ in $H_{\mathbf{k}}$. Let $\mathfrak{h}_{\mathbf{k}}$ be the Lie algebra of $H_{\mathbf{k}}$ and $\mathfrak{h}'^{\perp}_{\mathbf{k}}$ the Lie algebra of $H_{0,\mathbf{k}}$. Let $\mathfrak{h}'_{\mathbf{k}}$ be the s–invariant complementary subspace to $\mathfrak{h}'^{\perp}_{\mathbf{k}}$ in $\mathfrak{h}_{\mathbf{k}}$. Since $\mathfrak{h}_{\mathbf{k}}$ is abelian $\mathfrak{h}'_{\mathbf{k}} \subset \mathfrak{h}_{\mathbf{k}}$ is a Lie algebra. Let $H'_{\mathbf{k}} \subset H_{\mathbf{k}}$ be the subgroup which corresponds to $\mathfrak{h}'_{\mathbf{k}}$ in H. If $h, h' \in H'_{\mathbf{k}}$, $h = e^x, h' = e^y$, $x, y \in \mathfrak{h}'_{\mathbf{k}}$ then

$$h'\dot{s}h(h')^{-1} = \dot{s}e^{(s^{-1}-1)x+y},$$

and for any y one can find a unique $x = \frac{1}{1-s^{-1}}y$ such that $h'\dot{s}h(h')^{-1} = \dot{s}$. Note also that $H_{\mathbf{k}}$ normalizes $N_{\mathbf{k}}$. Therefore the factorization $H_{\mathbf{k}} = H_{0,\mathbf{k}}H'_{\mathbf{k}}$ implies that any element of $N_{\mathbf{k}}\dot{s}H_{\mathbf{k}}N_{\mathbf{k}}$ can be conjugated by an element of $H'_{\mathbf{k}}$ to an element from $N_{\mathbf{k}}\dot{s}H_{0,\mathbf{k}}N_{\mathbf{k}}$.

Finally observe that $N_{\mathbf{k}}\dot{s}H_{0,\mathbf{k}}N_{\mathbf{k}} \subset N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k}}$, and hence any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $N_{\mathbf{k}}\dot{s}H_{0,\mathbf{k}}N_{\mathbf{k}} \subset N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k}}$. By Remark 1.3.3 γ also intersects $\Sigma_{s,\mathbf{k}}$. The proof of isomorphism (1.3.1) in Proposition 1.3.1 implies that the $Z_{\mathbf{k}}$–component of any element from $N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k}}$ is equal to the $Z_{\mathbf{k}}$–component in $\Sigma_{s,\mathbf{k}} = \dot{s}Z_{\mathbf{k}}N_{s,\mathbf{k}}$ of its image under the isomorphism $N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k}} \simeq N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{s,\mathbf{k}}$. Therefore any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $\Sigma_{s,\mathbf{k}}$ at some point of $\dot{s}H_{0,\mathbf{k}}N_{s,\mathbf{k}}$. This completes the proof.

□

By the previous lemma the statement of this theorem holds.

□

1.6 Some normal orderings of positive roots systems associated to Weyl group elements

For the purpose of quantization we shall need a certain normal ordering on the root system Δ_+^s.

An ordering of a set of positive roots Δ_+ is called normal if for any three roots $\alpha,\ \beta,\ \gamma$ such that $\gamma=\alpha+\beta$ we have either $\alpha<\gamma<\beta$ or $\beta<\gamma<\alpha$. Let $\alpha_1,\ldots,\alpha_l$ be the simple roots in Δ_+, $s_1,\ldots,s_l$ the corresponding simple reflections. Let $\overline{w}$ be the element of W of maximal length with respect to the system $s_1,\ldots,s_l$ of simple reflections. For any reduced decomposition $\overline{w}=s_{i_1}\ldots s_{i_D}$ of $\overline{w}$ the ordering

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_D=s_{i_1}\ldots s_{i_{D-1}}\alpha_{i_D}$$

is a normal ordering in Δ_+, and there is a one–to–one correspondence between normal orderings of Δ_+ and reduced decompositions of $\overline{w}$.

From this fact and from properties of Coxeter groups it follows that any two normal orderings in Δ_+ can be reduced to each other by the so–called elementary transpositions. The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

$$\begin{array}{ll}\alpha,\ \beta & A_1+A_1\\ \alpha,\ \alpha+\beta,\ \beta & A_2\\ \alpha,\ \alpha+\beta,\ \alpha+2\beta,\ \beta & B_2\\ \alpha,\ \alpha+\beta,\ 2\alpha+3\beta,\ \alpha+2\beta,\ \alpha+3\beta,\ \beta & G_2\end{array} \tag{1.6.1}$$

where it is assumed that $(\alpha,\alpha)\geq(\beta,\beta)$. Moreover, any normal ordering in a rank 2 root system is one of orderings (1.6.1) or one of the inverse orderings.

In general an elementary inversion of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (1.6.1) (or of a segment with the inverse ordering) in the ordered set Δ_+, where $\alpha-\beta\not\in\Delta$.

From now on we shall always assume that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the action of s and that the one–dimensional subspaces $\mathfrak{h}_i$ on which s^1 acts by multiplication by -1 are immediately preceding $\mathfrak{h}_0$ in (1.2.14). According to this convention $\Delta_0=\{\alpha\in\Delta: s\alpha=\alpha\}$ is the set of roots fixed by the action of s.

Suppose that the direct sum $\bigoplus_{k=0,i_k>0}^r\mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 is not trivial. Since the one–dimensional subspaces $\mathfrak{h}_i$ on which s^1 acts by multiplication by -1 are immediately preceding $\mathfrak{h}_0$ in sum (1.2.14), the roots from the union $\bigcup_{k=0,i_k>0}^r\Delta_{i_k}$ must be orthogonal to all subspaces $\mathfrak{h}_{i_k}, i_k>0$ on which s^1 does not act by multiplication by -1 and to all roots from the set $\gamma_{n+1},\ldots\gamma_{l'}$ as s^2 acts trivially on $\bigoplus_{k=0}^r\mathfrak{h}_{i_k}$. Pick up a root $\gamma\in\bigcup_{k=0,i_k>0}^r\Delta_{i_k}$. Then γ is orthogonal to the roots $\gamma_{n+1},\ldots\gamma_{l'}$. Therefore, by the choice of γ, $s_0^1=s^1s_\gamma$ is an involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^1 plus one, and $s_0^2=s_\gamma s^2$ is another involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^2 minus one. We also have a decomposition $s=s_0^1s_0^2$.

Now we can apply the above construction of the system of positive roots to the new decomposition of s. Iterating this procedure we shall eventually arrive at the situation when the direct sum $\bigoplus_{k=0,i_k>0}^r\mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 is trivial. From now on we shall always consider decompositions $s=s^1s^2$ which satisfy this property. This implies

$$s^2\alpha=\alpha\Rightarrow\alpha\in\Delta_0. \tag{1.6.2}$$

Proposition 1.6.1. *Let $s\in W$ be an element of the Weyl group W of the pair $(\mathfrak{g},\mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g},\mathfrak{h})$, Δ_+^s a system of positive roots associated to s. Then the following statements are true.*

(i) The decomposition $s=s^1s^2$ is reduced in the sense that $l(s)=l(s^2)+l(s^1)$, where $l(\cdot)$ is the length function in W with respect to the system of simple roots in Δ_+^s, and $\Delta_s^s=\Delta_{s^2}^s\cup s^2(\Delta_{s^1}^s)$, $\Delta_{s^{-1}}^s=\Delta_{s^1}^s\cup s^1(\Delta_{s^2}^s)$ (disjoint unions), where $\Delta_s^s=\{\alpha\in\Delta_+^s: s\alpha\in-\Delta_+^s\}$, $\Delta_{s^{-1}}^s=\{\alpha\in\Delta_+^s: s^{-1}\alpha\in-\Delta_+^s\}$, $\Delta_{s^{1,2}}^s=\{\alpha\in\Delta_+^s: s^{1,2}\alpha\in-\Delta_+^s\}$. Here s^1,s^2 are the involutions in decomposition (1.2.1), $s^1=s_{\gamma_1}\ldots s_{\gamma_n}$, $s^2=s_{\gamma_{n+1}}\ldots s_{\gamma_{l'}}$ satisfying (1.6.2), the roots in each of the sets $\gamma_1,\ldots,\gamma_n$ and $\gamma_{n+1},\ldots,\gamma_{l'}$ are positive and mutually orthogonal.

(ii) For any root $\alpha\in\Delta_{s^1}^s$ one has $s^2\alpha\in\Delta_+^s\setminus\left(\Delta_{s^1}^s\cup\Delta_{s^2}^s\cup\Delta_0\right)$.

(iii) There is a normal ordering of the root system Δ_+^s of the following form

$$\begin{gathered}\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta^1_{t+\frac{p-n}{2}},\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\\ \beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},\gamma_3,\ldots,\gamma_n,\beta^1_{t+p+1},\ldots,\beta^1_{l(s^1)},\ldots,\\ \beta_1^2,\ldots,\beta_q^2,\gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta^2_{q+m_{l(s^2)}+1},\ldots,\beta^2_{2q+2m_{l(s^2)}-(l'-n)},\beta^2_{2q+2m_{l(s^2)}-(l'-n)+1},\ldots,\beta^2_{l(s^2)},\\ \beta_1^0,\ldots,\beta^0_{D_0},\end{gathered} \tag{1.6.3}$$

where

$$\begin{gathered}\{\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta^1_{t+\frac{p-n}{2}},\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\\ \beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},\gamma_3,\ldots,\gamma_n,\beta^1_{t+p+1},\ldots,\beta^1_{l(s^1)}\}=\Delta^s_{s^1},\end{gathered}$$

$$\begin{gathered}\{\beta_{t+1}^1,\ldots,\beta^1_{t+\frac{p-n}{2}},\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\\ \beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},\gamma_3,\ldots,\gamma_n\}=\{\alpha\in\Delta_+^s|s^1\alpha=-\alpha\}=\Delta^{-1}_{s^1},\end{gathered}$$

$$\begin{gathered}\{\beta_1^2,\ldots,\beta_q^2,\gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta^2_{q+m_{l(s^2)}+1},\ldots,\beta^2_{2q+2m_{l(s^2)}-(l'-n)},\beta^2_{2q+2m_{l(s^2)}-(l'-n)+1},\ldots,\beta^2_{l(s^2)}\}=\Delta^s_{s^2},\end{gathered}$$

$$\begin{gathered}\{\gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta^2_{q+m_{l(s^2)}+1},\ldots,\beta^2_{2q+2m_{l(s^2)}-(l'-n)}\}=\{\alpha\in\Delta_+^s|s^2\alpha=-\alpha\}=\Delta^{-1}_{s^2},\end{gathered}$$

$$\{\beta_1^0,\ldots,\beta^0_{D_0}\}=\{\alpha\in\Delta_+^s|s(\alpha)=\alpha\}.$$

(iv) The length of the ordered segment $\Delta_{\mathfrak{m}_+}\subset\Delta$ in normal ordering (1.6.3),

$$\begin{gathered}\Delta_{\mathfrak{m}_+}=\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},\\ \gamma_3,\ldots,\gamma_n,\beta^1_{t+p+1},\ldots,\beta^1_{l(s^1)},\ldots,\beta_1^2,\ldots,\beta_q^2,\\ \gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\gamma_{l'},\end{gathered} \tag{1.6.4}$$

is equal to

$$D-\left(\frac{l(s)-l'}{2}+D_0\right), \tag{1.6.5}$$

where D is the number of roots in Δ_+^s, and D_0 is the number of positive roots fixed by the action of s.

(v) For any two roots $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$ such that $\alpha<\beta$ the sum $\alpha+\beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k\gamma_{i_k}$, where $c_k\in\mathbb{N}$ and $\alpha<\gamma_{i_1}<\ldots<\gamma_{i_t}<\beta$.

(vi) The roots from the set Δ_s^s form a minimal segment in Δ_+^s of the form $\gamma,\ldots,\beta^2_{l(s^2)}$ which contains $\Delta^s_{s^2}$, and the roots from the set $\Delta^s_{s^{-1}}$ form a minimal segment in Δ_+^s of the form $\beta_1^1,\ldots,\delta$ which contains $\Delta^s_{s^1}$.

(vii) For any $\alpha\in(\Delta_{i_k})_+$, $i_k>0$ such that $s\alpha\in(\Delta_{i_k})_+$ one has $s\alpha>\alpha$, and if $\beta,\gamma\in\Delta_{i_j}\cup\{0\}$, $j<k$ and $s\alpha+\beta,\alpha+\gamma\in\Delta$ then $s\alpha+\beta,\alpha+\gamma\in\Delta_+^s$ and $s\alpha+\beta>\alpha+\gamma$.

In particular, for any $\alpha\in\Delta_+^s$, $\alpha\notin\Delta_0$ and any $\alpha_0\in\Delta_0$ such that $s\alpha\in\Delta_+^s$ one has $s\alpha>\alpha$ and if $s\alpha+\alpha_0\in\Delta$ then $s\alpha+\alpha_0>\alpha$.

(viii) If $\alpha,\beta\in\Delta_{i_k}\cap\Delta_+^s$, $i_k>0$, $\alpha<\beta$ and $s\beta\in\Delta_+^s$ then the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

Proof. Firstly we describe the set $(\Delta_{i_k})_+ = \Delta_{i_k} \cap \Delta_+^s$ for $i_k > 0$. Suppose that the corresponding s–invariant subspace $\mathfrak{h}_{i_k}$ is a two–dimensional plane. The case when $\mathfrak{h}_{i_k}$ is an invariant line on which s^2 acts by reflection and s^1 acts trivially can be treated in a similar way. The plane $\mathfrak{h}_{i_k}$ is shown at Figure 3.

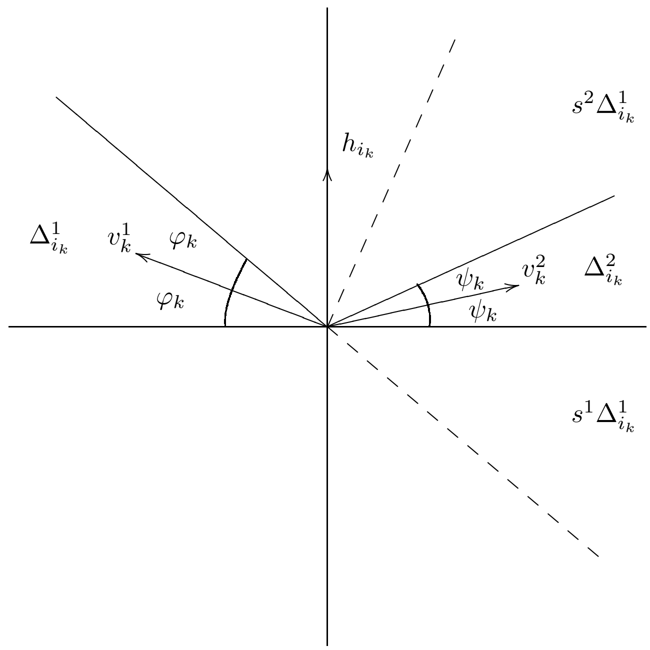


Fig. 3

The vector h_{i_k} is directed upwards at the picture. By (1.2.19) a root $\alpha \in \Delta_{i_k}$ belongs to the set $(\Delta_{i_k})_+$ if and only if $h_{i_k}(\alpha) > 0$. Identifying $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}^*_\mathbb{R}$ with the help of the bilinear form one can deduce that $\alpha \in \Delta_{i_k}$ is in Δ_+^s if and only if its orthogonal, with respect to the bilinear form, projection onto $\mathfrak{h}_{i_k}$ is contained in the upper–half plane shown at Figure 3.

The involutions s^1 and s^2 act in $\mathfrak{h}_{i_k}$ as reflections with respect to the lines orthogonal to the vectors labeled by v_k^1 and v_k^2, respectively, at Figure 3, the angle between v_k^1 and v_k^2 being equal to $\pi - \theta_{i_k}/2$. The nonzero projections of the roots from the set $\{\gamma_1, \ldots \gamma_n\} \cap \Delta_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ have the same (or the opposite) direction as the vector v_k^1, and the nonzero projections of the roots from the set $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \Delta_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ have the same (or the opposite) direction as the vector v_k^2.

The element s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$. Therefore the set $\Delta_s^s \cap \Delta_{i_k}$ consists of the roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the union of the sectors labeled $s^2\Delta^1_{i_k}$ and $\Delta^2_{i_k}$ at Figure 3.

Note that by Lemma 1.2.2 each $\overline{\Delta}_{i_k}$ is the root system of a standard Levi subalgebra in $\mathfrak{g}$.

Let $i_k > 0$ and $\Delta_{i_k}^{(r)}$ the subset of roots in $(\Delta_{i_k})_+$ orthogonal projections of which onto $\mathfrak{h}_{i_k}$ are directed along a ray $r \subset \mathfrak{h}_{i_k}$ starting at the origin. We call $\Delta_{i_k}^{(r)}$ the family corresponding to the ray r. Below we shall only consider rays which correspond to sets of the form $\Delta_{i_k}^{(r)}$.

Lemma 1.6.2. *Suppose that $i_k > 0$. Then each $\Delta_{i_k}^{(r)}$ is an additively closed set of roots.*

Let $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ be two families corresponding to rays r_1 and r_2, and $\delta_1 \in \Delta_{i_k}^{(r_1)}$, $\delta_2 \in \Delta_{i_k}^{(r_2)}$ two roots such that $\delta_1 + \delta_2 = \delta_3 \in \Delta$. Then $\delta_3 \in \Delta_{i_k}^{(r_3)}$, where $\Delta_{i_k}^{(r_3)}$ is the family corresponding to a ray r_3 such that r_3 lies inside of the angle formed by r_1 and r_2.

Proof. All statements are simple consequences of the fact that the sum of the orthogonal projections of any two roots onto $\mathfrak{h}_{i_k}$ is equal to the orthogonal projection of the sum.

In the first case the orthogonal projections of any two roots α, β from $\Delta_{i_k}^{(r)}$ onto $\mathfrak{h}_{i_k}$ have the same direction therefore the orthogonal projection of the sum $\alpha + \beta$ onto $\mathfrak{h}_{i_k}$ has the same direction as the orthogonal projections of α and β, and hence $\alpha + \beta \in \Delta_{i_k}^{(r)}$.

In the second case it suffices to observe that the sum of the orthogonal projections of δ_1 and δ_2 onto $\mathfrak{h}_{i_k}$ is equal to the orthogonal projection of the sum, and the sum of the orthogonal projections of δ_1 and δ_2 onto $\mathfrak{h}_{i_k}$ lies inside of the angle formed by r_1 and r_2.

□

Now we construct an auxiliary normal ordering on Δ_+^s by induction starting from the set $(\Delta_{i_0})_+$ as follows.

If $i_0 = 0$ or $\mathfrak{h}_{i_0}$ is one–dimensional then we fix an arbitrary normal order on $(\Delta_{i_0})_+$.

If $\mathfrak{h}_{i_0}$ is two–dimensional then we choose a normal ordering in $(\Delta_{i_0})_+$ in the following way. First fix an initial arbitrary normal ordering on $(\Delta_{i_0})_+$. Since by Lemma 1.6.2 each set $\Delta_{i_0}^{(r)}$ is additively closed we obtain an induced ordering for $\Delta_{i_0}^{(r)}$ which satisfies the defining property for the normal ordering.

Now using these induced orderings on the sets $\Delta_{i_0}^{(r)}$ we define an auxiliary normal ordering on $(\Delta_{i_0})_+$ such that on the sets $\Delta_{i_0}^{(r)}$ it coincides with the induced normal ordering defined above, and if $\Delta_{i_0}^{(r_1)}$ and $\Delta_{i_0}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_0}$ then for any $\alpha \in \Delta_{i_0}^{(r_1)}$ and $\beta \in \Delta_{i_0}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 1.6.2 the two conditions imposed on the auxiliary normal ordering in $(\Delta_{i_0})_+$ are compatible and define it in a unique way for the given initial normal ordering on $(\Delta_{i_0})_+$. Since s acts by a clockwise rotation on $\mathfrak{h}_{i_0}$ we have $s(\Delta_{i_0}^{(r)}) = \Delta_{i_0}^{(s(r))}$ for $s(r)$ in the upper–half plane, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_0})_+$ such that $s\alpha \in (\Delta_{i_0})_+$ one has $s\alpha > \alpha$.

Now assume that an auxiliary normal ordering has already been constructed for the set $\overline{\Delta}_{i_{k-1}}$ and define it for the set $\overline{\Delta}_{i_k}$.

By Lemma 1.2.2 $\overline{\Delta}_{i_{k-1}} \subset \overline{\Delta}_{i_k}$ is the root system of a standard Levi subalgebra $\mathfrak{g}_{i_{k-1}}$ inside of the standard Levi subalgebra $\mathfrak{g}_{i_k}$ of $\mathfrak{g}$ with the root system $\overline{\Delta}_{i_k}$. In particular, $\overline{\Delta}_{i_{k-1}}$ is generated by some subset of simple roots of the set of simple roots of $(\overline{\Delta}_{i_k})_+$. Therefore there exists an initial normal ordering on $(\overline{\Delta}_{i_k})_+$ in which the roots from the set $(\overline{\Delta}_{i_k})_+ \setminus (\overline{\Delta}_{i_{k-1}})_+ = (\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. As in case of the induction base this initial normal ordering gives rise to an induced ordering on each set $\Delta_{i_k}^{(r)}$.

Now using these induced orderings on the sets $\Delta_{i_k}^{(r)}$ we define an auxiliary normal ordering on $(\overline{\Delta}_{i_k})_+$. We impose the following conditions on it. Firstly we require that the roots from the set $(\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. Secondly, on the sets $\Delta_{i_k}^{(r)}$ the auxiliary normal ordering coincides with the induced normal ordering defined above, and if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_k}$ then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 1.6.2 the conditions imposed on the auxiliary normal ordering in $(\Delta_{i_k})_+$ are compatible and define it in a unique way. Since s acts by a clockwise rotation on $\mathfrak{h}_{i_k}$ we have $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper–half plane, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$.

Note also that the roots from $\overline{\Delta}_{i_{k-1}}$ have zero orthogonal projections onto $\mathfrak{h}_{i_k}$. Therefore if $\alpha \in (\Delta_{i_k})_+$, $\beta, \gamma \in \overline{\Delta}_{i_{k-1}}$ are such that $s\alpha \in (\Delta_{i_k})_+$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then by (1.2.15) and (1.2.19) $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$ as $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper–half plane.

These properties of the new normal ordering are summarized in the following lemma.

Lemma 1.6.3. *Suppose that $i_k > 0$. Then for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$ and if $\beta, \gamma \in \overline{\Delta}_{i_{k-1}} \cup \{0\}$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$.*

Now we proceed by induction and obtain an auxiliary normal ordering on Δ_+^s.

Observe that, according to the definition of the auxiliary normal ordering of Δ_+^s constructed above we have the following properties of this normal ordering.

Lemma 1.6.4. *Let $i_k > 0$. Then for any $\alpha \in \Delta_{i_k}$ and $\beta \in \Delta_{i_{k+1}}$ we have $\alpha > \beta$, and if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_k}$ then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$. Moreover, the roots from the sets $\Delta_{i_k}^{(r)}$ form minimal segments, and the roots from the set $(\Delta_0)_+$ form a final minimal segment.*

For each of the involutions s^1 and s^2 we obviously have decompositions $\Delta_{s^{1,2}}^s = \bigcup_{k=0}^M \Delta_{i_k}^{1,2}$, $\Delta_s^s = \bigcup_{k=0}^M \Delta_{i_k}^s$, where $\Delta_{i_k}^{1,2} = \Delta_{i_k} \cap \Delta_{s^{1,2}}^s$, $\Delta_{s^{1,2}}^s = \{\alpha \in \Delta_+^s : s^{1,2}\alpha \in -\Delta_+^s\}$, $\Delta_{i_k}^s = \Delta_{i_k} \cap \Delta_s^s$, $\Delta_{i_k}^s = \Delta_{i_k}^2 \cup s^2\Delta_{i_k}^1$. In the plane

$\mathfrak{h}_{i_k}$, the elements from the sets $\Delta_{i_k}^{1,2}$ are projected onto the interiors of the sectors labeled by $\Delta_{i_k}^{1,2}$ and the elements from the set $\Delta_{i_k}^s$ are projected onto the interior of the union of the sectors labeled by $\Delta_{i_k}^2$ and $s^2\Delta_{i_k}^1$. Therefore the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sector $\Delta_{i_k}^{1,2}$, and the sets $\Delta_{i_k}^s$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the union of the sectors labeled by $\Delta_{i_k}^2$ and $s^2\Delta_{i_k}^1$.

Let $\alpha \in \Delta_{s^1}^s$. By Theorem C in [16] the roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis in the annihilator ${\mathfrak{h}'_{\mathbb{R}}}^*$ of $\mathfrak{h}_0$ with respect to the pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$. Therefore $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ fixes all roots from Δ_0, and hence $\alpha \notin \Delta_0$. Also one obviously has $\alpha \in \Delta_{i_k}^1$, where $\mathfrak{h}_{i_k}$ is a two–dimensional plane, as by the assumption imposed before (1.6.2) there are no one–dimensional subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1. Thus in case if $\mathfrak{h}_{i_k}$ is an invariant line on which s acts by multiplication by -1 the set $\Delta_{i_k}^1$ is empty and hence $\Delta_{i_k}^s = \Delta_{i_k}^2$. This set is the set $\Delta_{i_k}^{(r)} = (\Delta_{i_k})_+$, where r is the positive semi-axis in $\mathfrak{h}_{i_k}$. From the observations made in the last two paragraphs we deduce that the sets $\Delta_{s^1}^s$ and $\Delta_{s^2}^s$ have always empty intersection. In particular, by the results of §3 in [115] the decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^1) + l(s^2)$, and $\Delta_s^s = \Delta_{s^2}^s \cup s^2(\Delta_{s^1}^s)$ (disjoint union). Similarly, $\Delta_{s^{-1}}^s = \Delta_{s^1}^s \cup s^1(\Delta_{s^2}^s)$ (disjoint union). This proves (i).

Note also that for any root α fixed by the action of s^2 one has $\alpha \in \Delta_0$ by (1.6.2). As we observed above for any root $\alpha \in \Delta_{s^1}^s$ one obviously has $\alpha \in \Delta_{i_k}^1$, where $\mathfrak{h}_{i_k}$ is a two–dimensional plane. Hence using Figure 3 we deduce that $s^2\alpha \in \Delta_+^s \setminus \left(\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0\right)$. This proves (ii).

Lemma 1.6.5. *Assume that Δ_+^s is equipped with an arbitrary normal ordering such that for some ray $r \subset \mathfrak{h}_{i_k}$ the roots from the set $\Delta_{i_k}^{(r)} = \{\delta_1, \ldots, \delta_a\}$ form a minimal segment $\delta_1, \ldots, \delta_a$. Suppose also that for some natural p such that $0 \leq p < k$, $i_p \neq 0$ and some ray $t \subset \mathfrak{h}_{i_p}$ the roots from the set $\Delta_{i_p}^{(t)} = \{\xi_1, \ldots, \xi_b\}$ form a minimal segment $\xi_1, \ldots, \xi_b$ and that the segment $\delta_1, \ldots, \delta_a, \xi_1, \ldots, \xi_b$ is also minimal. Then applying elementary transpositions one can reduce the last segment to the form $\xi_{i_1}, \ldots, \xi_{i_b}, \delta_{j_1}, \ldots, \delta_{j_a}$.*

Proof. The proof is by induction. First consider the minimal segment $\delta_1, \ldots, \delta_a, \xi_1$.

Since the orthogonal projection of the roots from the set Δ_{i_p} onto $\mathfrak{h}_{i_k}$ are equal to zero, for any $\alpha \in \Delta_{i_p}^{(t)}$ and $\beta \in \Delta_{i_k}^{(r)}$ such that $\alpha + \beta \in \Delta$ we have $\alpha + \beta \in \Delta_{i_k}^{(r)}$. Assume now that α and β are contained in an ordered segment of form (1.6.1) or in a segment with the inverse ordering. By the above observation this segment contains no other roots from $\Delta_{i_p}^{(t)}$, and α is the first or the last element in that segment. For the same reason the other roots in that segment must also belong to $\Delta_{i_k}^{(r)}$. Therefore applying an elementary transposition, if necessarily, one can move α to the first position in that segment.

Applying this procedure iteratively to the segment $\delta_1, \ldots, \delta_a, \xi_1$ we can reduce it to the form $\xi_1, \delta_{k_1}, \ldots, \delta_{k_a}$.

Now we can apply the same procedure to the segment $\delta_{k_1}, \ldots, \delta_{k_a}, \xi_2$ to reduce the segment $\xi_1, \delta_{k_1}, \ldots, \delta_{k_a}, \xi_2$ to the form $\xi_1, \xi_2, \delta_{l_1}, \ldots, \delta_{l_a}$.

Iterating this procedure we obtain the statement of the lemma.

□

Now observe that according to Lemma 1.6.4 the roots from each of the sets $(\Delta_{i_k})_+$ form a minimal segment in the auxiliary normal ordering of Δ_+^s, and the roots from the sets $\Delta_{i_k}^{(r)}$ form minimal segments inside $(\Delta_{i_k})_+$. As we observed above the sets $\Delta_{i_k}^{1,2}$ are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sectors $\Delta_{i_k}^{1,2}$ and hence by Lemma 1.6.4 the roots from the sets $\Delta_{i_k}^{1,2}$ form an initial and a final segment inside $(\Delta_{i_k})_+$.

Therefore we can apply Lemmas 1.6.4 and 1.6.5 to move all roots from the segments $\Delta_{i_k}^1$, $k = 0, \ldots, M$ to the left and to move all roots from the segments $\Delta_{i_k}^2$, $k = 0, \ldots, M$ to the right to positions preceding the final segment formed by the roots from $(\Delta_0)_+$.

Now using similar arguments the roots from the sets $s^2\Delta_{i_k}^1$, $k = 0, \ldots, M$ forming minimal segments by Lemma 1.6.4 as well can be moved to the right to positions preceding the final segment formed by the roots from the set $\Delta_{s^2}^s \cup (\Delta_0)_+$. After this modification the roots from the set $\Delta_s^s = \Delta_{s^2}^s \cap s^2(\Delta_{s^1}^s)$ will form a minimal segment in Δ_+^s of the form $\gamma, \ldots, \beta_{l(s^2)}^2$ which contains $\Delta_{s^2}^s$.

Similarly, the roots from the sets $s^1\Delta_{i_k}^2$, $k = 0, \ldots, M$ forming minimal segments by Lemma 1.6.4 can be moved to the left to positions after the initial segment formed by the roots from the set $\Delta_{s^1}^s$. After this modification the roots from the set $\Delta_{s^{-1}}^s = \Delta_{s^1}^s \cap s^1(\Delta_{s^2}^s)$ will form a minimal segment in Δ_+^s of the form $\beta_1^1, \ldots, \delta$ which contains $\Delta_{s^1}^s$. This proves (vi).

Note that according to the algorithm given in Lemma 1.6.5 for each fixed k the order of the minimal segments formed by the roots from the sets $\Delta_{i_k}^{(r)}$ is preserved after applying that lemma. Therefore the new normal ordering obtained this way still satisfies the second property mentioned in Lemma 1.6.4, i.e. if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_k}$ then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$.

Now we can apply elementary transpositions to bring the initial segment formed by the roots from $\Delta_{s^1}^s = \{\beta_1^1, \ldots, \beta_{l(s^1)}^1\}$ and the segment formed by the roots from $\Delta_{s^2}^s = \{\beta_1^2, \ldots, \beta_{l(s^2)}^2\}$ and preceding the final segment $(\Delta_0)_+$ to the form described in (1.6.3).

Recall that by Theorem A in [86] every involution w in the Weyl group W is the longest element of the Weyl group of a Levi subalgebra in $\mathfrak{g}$, and w acts by multiplication by -1 on the Cartan subalgebra $\mathfrak{h}_w \subset \mathfrak{h}$ of the semisimple part $\mathfrak{m}_w$ of that Levi subalgebra. By Lemma 5 in [16] the involution w can also be expressed as a product of $\dim \mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal roots. In case of the involution s^1, $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ is such an expression, and the roots $\gamma_1, \ldots, \gamma_n$ span the Cartan subalgebra $\mathfrak{h}_{s^1}$.

Since $\mathfrak{m}_{s^1}$ is the semisimple part of a Levi subalgebra, using elementary transpositions one can reduce the normal ordering of the segment $\beta_1^1, \ldots, \beta_{l(s^1)}^1$ to the form

$$\beta_1^1, \ldots, \beta_t^1, \beta_{t+1}^1, \ldots, \beta_{t+p}^1, \beta_{t+p+1}^1, \ldots, \beta_{l(s^1)}^1, \tag{1.6.6}$$

where $\beta_{t+1}^1, \ldots, \beta_{t+p}^1$ is a normal ordering of the system $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_+^s$ of positive roots in the root system $\Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ of the pair $(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$. Now applying elementary transpositions we can reduce the ordering $\beta_{t+1}^1, \ldots, \beta_{t+p}^1$ to the form compatible with the decomposition $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ (see Appendix 1).

Applying similar arguments to the involution s^2 and using the normal ordering of the system of positive roots $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_+^s$ in the root system $\Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ of the pair $(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$ we finally obtain the following normal ordering of the set Δ_+^s

$$\begin{gathered}
\beta_1^1, \ldots, \beta_t^1, \beta_{t+1}^1, \ldots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \ldots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\
\beta_{t+\frac{p-n}{2}+n_1+2}^1 \ldots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \ldots, \gamma_n, \beta_{t+p+1}^1, \ldots, \beta_{l(s^1)}^1, \ldots, \\
\beta_1^2, \ldots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \ldots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \ldots, \beta_{q+m_2}^2, \gamma_{n+3}, \ldots, \\
\gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \ldots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \ldots, \beta_{l(s^2)}^2, \\
\beta_1^0, \ldots, \beta_{D_0}^0,
\end{gathered} \tag{1.6.7}$$

where

$$\begin{gathered}
\gamma_{n+1}, \beta_{q+2}^2, \ldots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \ldots, \beta_{q+m_2}^2, \gamma_{n+3}, \ldots, \\
\gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \ldots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2
\end{gathered}$$

is the normal ordering of the system of positive roots $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$. We caim that normal ordering (1.6.7) has all the properties listed in the statement of this proposition, i.e. it is the required ordering (1.6.3). This establishes (iii).

The elementary transpositions used above to bring the segments $\Delta_{s^1}^s$ and $\Delta_{s^2}^s$ to the form required in (1.6.3) do not affect the positions of the roots which do not belong to $\Delta_{s^1}^s$ and $\Delta_{s^2}^s$. We claim that for $\alpha \in \Delta_{i_k}^{(r)}$, $i_k > 0$ we still have $s\alpha > \alpha$ if $s\alpha \in \Delta_+^s$.

Indeed, if $\alpha \in (\Delta_{i_k})_+$, $\alpha \notin \Delta_{i_k}^1$, $s\alpha \notin \Delta_{i_k}^2$, $s\alpha \in (\Delta_{i_k})_+$ this follows from the second property mentioned in Lemma 1.6.4.

If $\alpha \in \Delta_{i_k}^1$ and $s\alpha \in \Delta_+^s$ then $s\alpha \notin \Delta_{s^1}^s$ as $s^1(s\Delta_{s^1}^s) = s^2(\Delta_{s^1}^s) \subset \Delta_+^s$ since the decomposition $s = s^1 s^2$ is reduced. Therefore $s\alpha > \alpha$ as the roots from the set $\Delta_{s^1}^s$ form an initial segment in the normal ordering of Δ_+^s.

If $\alpha \in (\Delta_{i_k})_+$, $\alpha \notin \Delta_{i_k}^1$, $s\alpha \in \Delta_{i_k}^2$ then $\alpha \notin \Delta_s^s$ and $s\alpha \in \Delta_s^s$ as $\Delta_{i_k}^2 \subset \Delta_s^s$. Therefore $s\alpha > \alpha$ as the roots from the set $\Delta_s^s \cup (\Delta_0)_+$ form a final segment in the normal ordering of Δ_+^s and α does not belong to that segment.

Finally if $\alpha \in \Delta_{i_k}^2$ then $s\alpha \in -\Delta_+^s$.

Moreover, similar arguments together with the fact that all roots from $\overline{\Delta}_{i_{k-1}}$ have zero orthogonal projections onto $\mathfrak{h}_{i_k}$ show that the new normal ordering still satisfies the property stated in Lemma 1.6.3. Note that for one–dimensional $\mathfrak{h}_{i_k}$ this property is void.

The fact that for any $\alpha \in \Delta_+^s$, $\alpha \notin \Delta_0$ and any $\alpha_0 \in \Delta_0$ such that $s\alpha \in \Delta_+^s$ one has $s\alpha > \alpha$ and if $s\alpha + \alpha_0 \in \Delta$ then $s\alpha + \alpha_0 > \alpha$ is a particular case of the property stated in Lemma 1.6.3 because Δ_+^s is the disjoint union of the sets $(\Delta_{i_k})_+$. This proves (vii).

Now we show that if $\alpha,\beta\in\Delta_{i_k}\cap\Delta_+^s$, $\alpha<\beta$ and $s\beta\in\Delta_+^s$ then the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

Indeed, observe that the elementary transpositions which we used above to bring the segments formed by the roots from the sets $\Delta_{s^1}^s$ and $\Delta_{s^2}^s$ to the form required in normal ordering (1.6.3) do not affect the positions of other roots and after this rearrangement the orthogonal projections of roots from $\Delta_{i_k}^{1,2}$ onto $\mathfrak{h}_{i_k}$ still belong to the sectors labeled $\Delta_{i_k}^{1,2}$ at Figure 3.

Therefore by Lemma 1.6.4 if $\alpha,\beta\in\Delta_{i_k}\cap\Delta_+^s$, $\beta\in\Delta_{i_k}^1$ and $\alpha<\beta$ then the orthogonal projection of α onto $\mathfrak{h}_{i_k}$ belongs to the sector labeled $\Delta_{i_k}^1$ at Figure 3. On the other hand since $s\beta\in\Delta_+^s$ the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ belongs to the upper half plane and does not belong to the sector labeled $\Delta_{i_k}^1$ at Figure 3 as s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation by the angle $\theta_{i_k}=2(\varphi_k+\psi_k)>2\varphi_k$. Thus the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

The case when $\beta\in(\Delta_{i_k})_+$, $\beta\notin\Delta_{i_k}^1$, $s\beta\in(\Delta_{i_k})_+$ is treated in a similar way with the help of Lemma 1.6.4. This proves (viii).

Next we claim that $t=l(s^1)-(t+p)$, i.e. there are equal numbers of roots on the left and on the right from the segment $\beta_{t+1}^1,\ldots,\beta_{t+p}^1$ in the segment

$$\beta_1^1,\ldots,\beta_t^1,\beta_{t+1}^1,\ldots,\beta_{t+p}^1,\beta_{t+p+1}^1,\ldots,\beta_{l(s^1)}^1,$$

and

$$t=\frac{l(s^1)-p}{2}. \tag{1.6.8}$$

Recall that by formula (3.5) in [115], given a reduced decomposition $w=s_{i_1}\ldots s_{i_m}$ of a Weyl group element w, one can represent w as a product of reflections with respect to the roots from the set

$$\begin{gathered}\Delta_{w^{-1}}=\{\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}\},\\ w=s_{i_1}\ldots s_{i_m}=s_{\beta_m}\ldots s_{\beta_1}\end{gathered} \tag{1.6.9}$$

Note that

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}$$

is the initial segment of a normal ordering of the corresponding system of positive roots.

Applying this observation to the segment of the normal ordering (1.6.6) consisting of elements from the set $\Delta_{s^1}^s$ one can represent $(s^1)^{-1}=s^1$ as follows

$$s^1=s_{\beta_{l(s^1)}^1}\ldots s_{\beta_{t+p+1}^1}s_{\beta_{t+p}^1}\ldots s_{\beta_{t+1}^1}s_{\beta_t^1}\ldots s_{\beta_1^1}, \tag{1.6.10}$$

where if $s^1=s_{i_1}\ldots s_{i_{l(s^1)}}$ is the corresponding reduced decomposition of s^1 then

$$\beta_1^1=\alpha_{i_1},\beta_2^1=s_{i_1}\alpha_{i_2},\ldots,\beta_{l(s^1)}^1=s_{i_1}\ldots s_{i_{l(s^1)-1}}\alpha_{i_{l(s^1)}}. \tag{1.6.11}$$

Since $\beta_{t+1}^1,\ldots,\beta_{t+p}^1$ is a normal ordering of the system of positive roots of the pair $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$ and s^1 is the longest element in the Weyl group of the pair $(\mathfrak{m}_{s^1},\mathfrak{h}_{s^1})$ we also have

$$s^1=s_{\beta_{t+p}^1}\ldots s_{\beta_{t+1}^1},$$

and hence by (1.6.10)

$$s^1=s_{\beta_{l(s^1)}^1}\ldots s_{\beta_{t+p+1}^1}s^1s_{\beta_t^1}\ldots s_{\beta_1^1}. \tag{1.6.12}$$

From the last formula we deduce that

$$s_{\beta_1^1}\ldots s_{\beta_t^1}=(s_{\beta_t^1}\ldots s_{\beta_1^1})^{-1}(s^1)^{-1}s_{\beta_{l(s^1)}^1}\ldots s_{\beta_{t+p+1}^1}s^1s_{\beta_t^1}\ldots s_{\beta_1^1}. \tag{1.6.13}$$

Now formula (1.6.9) implies that $s_{\beta_1^1}\ldots s_{\beta_t^1}=s_{i_t}\ldots s_{i_1}$, and relations (1.6.11) combined with (1.6.9) yield

$$\begin{aligned}u(s^1)^{-1}s_{\beta_{l(s^1)}^1}\ldots s_{\beta_{t+p+1}^1}s^1u^{-1}=s_{u(s^1)^{-1}(\beta_{l(s^1)}^1)}\ldots s_{u(s^1)^{-1}(\beta_{t+p+1}^1)}=\\ =s_{s_{i_{t+p+1}}\ldots s_{i_{l(s^1)-1}}\alpha_{i_{l(s^1)}}}\ldots s_{i_{t+p+1}}=s_{i_{t+p+1}}\ldots s_{i_{l(s^1)}},\end{aligned}$$

where $u = s_{\beta_1^1} \dots s_{\beta_t^1} = s_{i_t} \dots s_{i_1}$. Therefore from formula (1.6.13) we deduce that

$$s_{i_t} \dots s_{i_1} = s_{i_{t+p+1}} \dots s_{i_{l(s^1)}}. \tag{1.6.14}$$

Since the decompositions in both sides of (1.6.14) are parts of reduced decompositions they are reduced as well, and we have $t = l(s^1) - (t+p)$. This is equivalent to formula (1.6.8).

Using a similar formula for the involution s^2 and recalling the definition of the orderings of positive roots of the pairs $(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$, $(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ compatible with decompositions $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$ and $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$ (see Appendix 1) we deduce that the number of roots in the segment $\Delta_{\mathfrak{m}_+}$ of normal ordering (1.6.7),

$$\begin{aligned}\Delta_{\mathfrak{m}_+} = \gamma_1, \beta^1_{t+\frac{p-n}{2}+2}, \dots, \beta^1_{t+\frac{p-n}{2}+n_1}, \gamma_2, \beta^1_{t+\frac{p-n}{2}+n_1+2} \dots, \beta^1_{t+\frac{p-n}{2}+n_2},\\ \gamma_3, \dots, \gamma_n, \beta^1_{t+p+1}, \dots, \beta^1_{l(s^1)}, \dots, \beta^2_1, \dots, \beta^2_q,\\ \gamma_{n+1}, \beta^2_{q+2}, \dots, \beta^2_{q+m_1}, \gamma_{n+2}, \beta^2_{q+m_1+2}, \dots, \beta^2_{q+m_2}, \gamma_{n+3}, \dots, \gamma_{l'}\end{aligned}$$

is equal to $D - (\frac{l(s)-l'}{2} + D_0)$, where $l(s) = l(s^1) + l(s^2)$ is the length of s and D_0 is the number of positive roots fixed by the action of s. This establishes (iv).

Now let $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$, be any two roots such that $\alpha < \beta$. We shall show that the sum $\alpha+\beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_k} < \beta$.

Suppose that such a decomposition exists, $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k}$. Obviously at least one of the roots α, β must belong to the set $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_{\mathfrak{m}_+}$ or to the set $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ for otherwise the set of roots γ_{i_k} such that $\alpha < \gamma_{i_k} < \beta$ is empty.

Suppose that $\alpha \in \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_{\mathfrak{m}_+}$. The other cases are considered in a similar way.

If $\beta \notin \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k}$, and $\gamma_{i_k} \le \gamma_n$. In particular, since $\alpha \in \mathfrak{h}_{s^1}$ and $\gamma_{i_k} \in \mathfrak{h}_{s^1}$ if $\gamma_{i_k} \le \gamma_n$, we have $\beta = \sum_{k=1}^q c_k \gamma_{i_k} - \alpha \in \mathfrak{h}_{s^1}$. This is impossible by the definition of the ordering of the set $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ compatible with the decomposition $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$.

If $\beta \in \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k} = \sum_{i_k \le n} c_k \gamma_{i_k} + \sum_{i_k > n} c_k \gamma_{i_k}$. This implies

$$\alpha - \sum_{i_k \le n} c_k \gamma_{i_k} = \sum_{i_k > n} c_k \gamma_{i_k} - \beta.$$

The l.h.s. of the last formula is an element of $\mathfrak{h}_{s^1}$ and the r.h.s. is an element $\mathfrak{h}_{s^2}$. Since $\mathfrak{h}' = \mathfrak{h}_{s^1} + \mathfrak{h}_{s^2}$ is a direct vector space decomposition we infer that

$$\alpha = \sum_{i_k \le n, \alpha < \gamma_{i_k}} c_k \gamma_{i_k}$$

and

$$\beta = \sum_{i_k > n, \gamma_{i_k} < \beta} c_k \gamma_{i_k}.$$

This is impossible by the definition of the orderings of the sets $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ and $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ compatible with the decompositions $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$ and $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, respectively.

Therefore the sum $\alpha + \beta$, $\alpha < \beta$, $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_k} < \beta$. This confirms (v) and completes the proof of the proposition.

□

We shall also need another system of positive roots associated to (the conjugacy class of) the Weyl group element s. In order to define it we need to recall the definition of a circular normal ordering of the root system Δ.

Let $\beta_1, \beta_2, \dots, \beta_D$ be a normal ordering of a positive root system Δ_+. Then one can introduce the corresponding circular normal ordering of the root system Δ where the roots in Δ are located on a circle in the following way

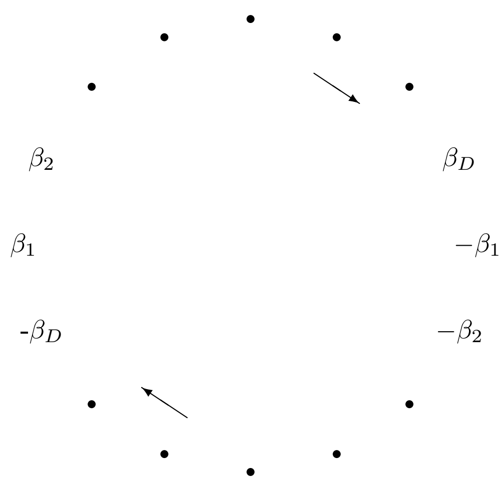


Fig. 4

Let $\alpha, \beta \in \Delta$. One says that the segment $[\alpha, \beta]$ of the circle is minimal if it does not contain the opposite roots $-\alpha$ and $-\beta$ and the root β follows after α on the circle above, the circle being oriented clockwise. In that case one also says that $\alpha < \beta$ in the sense of the circular normal ordering,

$$\alpha < \beta \Leftrightarrow \text{the segment } [\alpha, \beta] \text{ of the circle is minimal.} \tag{1.6.15}$$

Later we shall need the following property of minimal segments which is a direct consequence of Proposition 3.3 in [59].

Lemma 1.6.6. *Let $[\alpha, \beta]$ be a minimal segment in a circular normal ordering of a root system Δ. Then if $\alpha + \beta$ is a root we have*

$$\alpha < \alpha + \beta < \beta.$$

Note that any segment in a circular normal ordering of Δ of length equal to the number of positive roots is a system of positive roots.

Now consider the circular normal ordering of Δ corresponding to the system of positive roots Δ_+^s and to its normal ordering introduced in Proposition 1.6.1. The segment which consists of the roots α satisfying $\gamma_1 \leq \alpha < -\gamma_1$ is a system of positive roots in Δ as its length is equal to the number of positive roots and it is closed under addition of roots by Lemma 1.6.6.

The system of positive roots Δ_+ introduced this way and equipped with the normal ordering induced by the circular normal ordering is called the normally ordered system of positive roots associated to the (conjugacy class of) the Weyl group element $s \in W$.

We have the following property of the length of s with respect to the sets of simple roots in Δ_+^s.

Proposition 1.6.7. *For all systems of positive roots Δ_+^s the lengths $l(s)$ of s with respect to the sets of simple roots in Δ_+^s are the same, and they are equal to the length of s with respect to the set of simple roots in any system of positive roots Δ_+ associated to s.*

Proof. The first statement is a consequence of the definition of Δ_+^s.

To prove the second assertion we recall that a root $\alpha \in \Delta_{i_k}$ belongs to the set $(\Delta_{i_k})_+ = (\Delta_{i_k})_+ \cap \Delta_+^s$ if and only if $h_{i_k}(\alpha) > 0$. Identifying $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$ with the help of the bilinear form one can deduce that $\alpha \in \Delta_{i_k}$ is in Δ_+^s if and only if its orthogonal projection onto $\mathfrak{h}_{i_k}$ is contained in the upper–half plane shown at Figure 3.

According to the definition of the set Δ_+,

$$\Delta_{i_k} \cap \Delta_+ = ((\Delta_{i_k})_+ \setminus \{\alpha \in (\Delta_{i_k})_+ \cap \Delta_{s^1}^s : \alpha < \gamma_1\}) \cup \{-\alpha : \alpha \in (\Delta_{i_k})_+ \cap \Delta_{s^1}^s, \alpha < \gamma_1\},$$

i.e $\Delta_{i_k} \cap \Delta_+$ is obtained from $(\Delta_{i_k})_+$ by removing some roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the sector labeled $\Delta_{i_k}^1$ at Figure 3 and by adding the the opposite negative roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the sector labeled $s^1\Delta_{i_k}^1$ at Figure 3.

The element s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$. Therefore the set $\Delta_s^s \cap \Delta_{i_k}$ consists of the roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the union of the sectors labeled $s^2\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ at Figure 3. Together with the description of the set $\Delta_{i_k} \cap \Delta_+$ given above it implies that the number of roots in the set $\Delta_s \cap \Delta_{i_k} = \{\alpha \in \Delta_{i_k} \cap \Delta_+ : s\alpha \in \Delta_-\}$, where $\Delta_s = \{\alpha \in \Delta_+ : s\alpha \in \Delta_-\}$, is equal to the number of roots in the set $\Delta_s^s \cap \Delta_{i_k}$. From this observation we deduce that the length $l(s)$ of s with respect to the system of simple roots in Δ_+^s is the same as the length of s with respect to the system of simple roots in Δ_+, as both of them are equal to the cardinality to the set $\bigcup_{k=0}^M \Delta_s^s \cap \Delta_{i_k}$ (disjoint union) which is the same as the cardinality of the set $\bigcup_{k=0}^M \Delta_s \cap \Delta_{i_k}$ (disjoint union).

□

The relative positions of the systems of positive rots Δ_+^s, Δ_+ and of the minimal segments introduced in Proposition 1.6.1 are shown at the following picture where all the segments are placed on a circle according to the circular normal ordering of roots corresponding to normal ordering (1.6.3) of Δ_+^s.

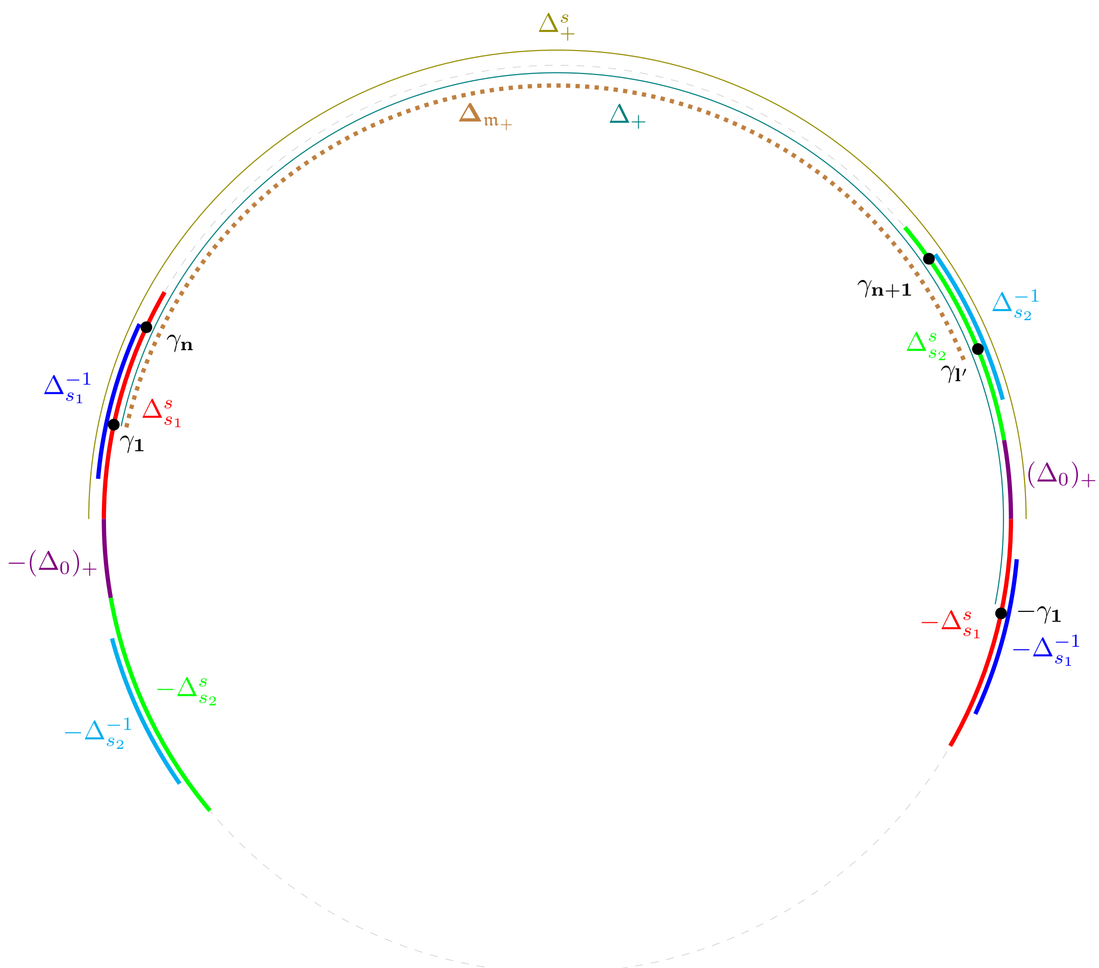


Fig. 5

The reader may find this picture useful in combination with Lemma 1.6.6 when adding roots or commuting roots vectors. This picture can be also useful for deriving some formulas containing q-commutators of quantum root vectors as explained in the next chapter.

1.7 Bibliographic comments

A uniform classification of conjugacy classes of Weyl group elements from which one can obtain presentation (1.2.1) was suggested in [16].

The definition of systems of positive roots Δ_+^s associated to (conjugacy classes of) Weyl group elements was suggested in [98]. It is based on a deep generalization of the results by Coxeter and Steinberg on the properties of the Coxeter elements. In our notation this corresponds to the case when $\gamma_1, \ldots, \gamma_{l'}$ is a set of simple roots in Δ, so that according to (1.2.1) s is a product of simple reflections, i.e. a Coxeter element. In this case there is a unique plane in $\mathfrak{h}_\mathbb{R}$, called a Coxeter plane, on which s acts by rotation by the angle $2\pi/h$, where h is the Coxeter number of $\mathfrak{g}$. This plane was introduced by Coxeter in book [19], and the pictures of root systems of Lie algebras of high ranks which one can find in many textbooks are obtained using orthogonal projections of roots onto these planes. The key observation is that all these projections are non–zero. Coxeter originally applied the above mentioned procedure to construct regular polytopes.

Later in paper [109] Steinberg proved interesting properties of Coxeter elements using the properties of the action of Coxeter elements on Coxeter planes.

The construction of the spectral decomposition for Weyl group elements in Proposition 1.2.1 suggested in [103] is a generalization of similar results on the properties of the Coxeter plane which can be found in [15], Section 10.4.

The slices Σ_s introduced in [98] are generalizations of the Steinberg cross-sections to the set of conjugacy classes of regular elements in G suggested in [110]. Σ_s reduces to a Steinberg cross-section when $\gamma_1, \ldots, \gamma_{l'}$ is a set of simple roots, i.e. when s is a Coxeter element. In this case isomorphism (1.3.1) is stated in [110] without proof. The first proof of this result appeared in [92]. Although that proof is not applicable for the root system of type E_6. The proof of isomorphism (1.3.1) given in Proposition 1.3.1 is a refined version of the proof of this result presented in [98].

Another construction of the slices Σ_s in the case when s are elliptic can be found in [46].

The closedness of the varieties $NZsN$ was justified in [101], Proposition 6.2.

In book [106] Slodowy proved Brieskorn's conjecture on the realization of simple singularities using the adjoint quotient of complex semisimple Lie algebras. Although a significant part of Slodowy's book is devoted to the study of the conjugation quotient for semisimple algebraic groups and to constructing some its resolutions, he ended up with a Lie algebra version of the construction of simple singularities and introduced transversal slices for the adjoint action for this purpose. These slices are called now the Slodowy slices. The slices Σ_s can be regarded as algebraic group analogues of the Slodowy slices.

The Lusztig partition was introduced in [70]. Its definition is related to the study of the properties of intersections of conjugacy classes in G with Bruhat cells established in [72, 73] where the map Φ^W from the set of Weyl group conjugacy classes to the set of unipotent classes and its one sided inverse Ψ^W, which we use in Section 1.4, are defined using these properties. These properties are also related to the generalized Springer correspondence. We only briefly discussed the relevant results in this book.

The study of intersections of conjugacy classes in G with Bruhat cells was initiated in [110]. Some results on these intersections were obtained in [30], and another map from nilpotent orbits in a complex semisimple Lie algebra to conjugacy classes in the Weyl group was defined in [56]. In [72] it is mentioned that this map is likely to coincide with the map Ψ^W introduced in [72].

The main result of Theorem 1.5.2 on the dimensions of the slices Σ_s is an experimental observation made in [103], Theorem 5.2. Other results of Section 1.5 can also be found in [103].

Note that the the slices Σ_s listed in Appendix 2 are slightly different from those from Appendix B to [103]. The corresponding slices in both sets have the same dimensions. But in this book the roots $\gamma_1, \ldots, \gamma_{l'}$ in the tables in Appendix 2 are chosen in such a way that the corresponding root systems Δ_+^s satisfy condition (1.6.2). The algorithm for constructing the slices Σ_s listed in the tables in Appendix B to [103] was modified to fulfill this condition. The description of the original algorithm can be found in [103].

The ordering of the s–invariant planes in $\mathfrak{h}_\mathbb{R}$ according to the angles of rotations by which s acts in the planes as in Theorem 1.5.2 was used in [47] to prove properties of minimal length elements in finite Coxeter groups.

Normal orderings of positive root systems of the form Δ_+^s described in Proposition 1.6.1 were firstly introduced in [99] where one can also find the construction of normal orderings of positive root systems compatible with Weyl group involutions from Appendix 1. Later the original definition was refined in [104]. Proposition 1.6.1 is a modified version of Proposition 5.1 in [99] and Proposition 2.2 in [104].

Circular orderings of root systems were defined in [58] to describe commutation relations between quantum group analogues of root vectors. In [101] this construction was used to modify the positive root systems Δ_+^s in order to construct positive root systems associated to (conjugacy classes of) Weyl group elements which appear in the end of Section 1.6.

Chapter 2

Quantum groups

In this chapter we recall some definitions and results on quantum groups required for the study of q-W–algebras. Besides the standard definitions and results related to quantum groups we shall need some rather non–standard realizations of the Drinfeld–Jimbo quantum group in terms of which q-W–algebras are defined. These realizations are related to the definition of the algebraic group analogues of the Slodowy slices in the previous section. We shall consider the Drinfeld–Jimbo quantum group $U_h(\mathfrak{g})$ defined over the ring of formal power series $\mathbb{C}[[h]]$, where h is an indeterminate, and some its specializations defined over smaller rings.

2.1 The definition of quantum groups

In this section we remind the definition of the standard Drinfeld-Jimbo quantum group $U_h(\mathfrak{g})$.

Let V be a $\mathbb{C}[[h]]$–module equipped with the h–adic topology. This topology is characterized by requiring that $\{h^nV \mid n \geq 0\}$ is a base of the neighborhoods of 0 in V, and that translations in V are continuous. In this book all $\mathbb{C}[[h]]$–modules are supposed to be complete with respect to this topology.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$–module equipped with a structure of $\mathbb{C}[[h]]$–Hopf algebra, the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the h–adic topology.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by elements $H_i,\ X_i^+,\ X_i^-,\ i = 1,\ldots,l$, subject to the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \quad X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} (X_i^{\pm})^{1-a_{ij}-r} X_j^{\pm} (X_i^{\pm})^r = 0, \ i \neq j,$$

where

$$K_i = e^{d_i h H_i}, \quad e^h = q, \quad q_i = q^{d_i} = e^{d_i h},$$

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![n-m]_q!}, \ [n]_q! = [n]_q \ldots [1]_q, \ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta_h(X_i^+) = X_i^+ \otimes K_i^{-1} + 1 \otimes X_i^+, \quad \Delta_h(X_i^-) = X_i^- \otimes 1 + K_i \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ K_i, \quad S_h(X_i^-) = -K_i^{-1} X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^{\pm}) = 0.$$

We shall also use the weight–type generators

$$Y_i = \sum_{j=1}^{l} d_i (a^{-1})_{ij} H_j.$$

Let $L_i^{\pm 1} = e^{\pm h Y_i}$. These elements commute with the quantum simple root vectors $X_i^\pm$ as follows:

$$L_i X_j^\pm L_i^{-1} = q_i^{\pm \delta_{ij}} X_j^\pm. \quad (2.1.1)$$

We also obviously have

$$L_i L_j = L_j L_i. \quad (2.1.2)$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on $\mathfrak{g}$ in the sense that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$, $\Delta_h = \Delta \pmod h$, where Δ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod h = -\delta.$$

Here $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on $\mathfrak{g}$, and $\Delta_h^{opp} = \sigma \Delta_h$, σ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$. Recall that

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2 r_\pm, \quad r_\pm \in \mathfrak{g} \otimes \mathfrak{g},$$

$$r_\pm = \pm \frac{1}{2} \sum_{i=1}^{l} Y_i \otimes H_i \pm \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_{\pm\beta} \otimes X_{\mp\beta}. \quad (2.1.3)$$

Here $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ are non–zero root vectors of $\mathfrak{g}$. The element $r_\pm \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r–matrix.

2.2 The braid group action

One can define a quantum group analogue of the braid group action on $\mathfrak{g}$. Let m_{ij}, $i \neq j$ be equal to $2, 3, 4, 6$ if $a_{ij} a_{ji}$ is equal to $0, 1, 2, 3$, respectively. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators T_i, $i = 1, \ldots, l$, and defining relations

$$T_i T_j T_i T_j \ldots = T_j T_i T_j T_i \ldots$$

for all $i \neq j$, where there are m_{ij} T's on each side of the equation.

Recall that if $X_{\pm\alpha_i}$ are non–zero simple root vectors of $\mathfrak{g}$ then one can introduce an action of the braid group $\mathcal{B}_\mathfrak{g}$ by algebra automorphisms of $\mathfrak{g}$ defined on the standard generators as follows:

$$T_i(X_{\pm\alpha_i}) = -X_{\mp\alpha_i}, \; T_i(H_j) = H_j - a_{ji} H_i,$$

$$T_i(X_{\alpha_j}) = \frac{1}{(-a_{ij})!} \mathrm{ad}_{X_{\alpha_i}}^{-a_{ij}} X_{\alpha_j}, \; i \neq j, \quad (2.2.1)$$

$$T_i(X_{-\alpha_j}) = \frac{(-1)^{a_{ij}}}{(-a_{ij})!} \mathrm{ad}_{X_{-\alpha_i}}^{-a_{ij}} X_{-\alpha_j}, \; i \neq j.$$

Similarly, $\mathcal{B}_\mathfrak{g}$ acts by algebra automorphisms of $U_h(\mathfrak{g})$ as follows:

$$T_i(X_i^+) = -X_i^- e^{h d_i H_i}, \; T_i(X_i^-) = -e^{-h d_i H_i} X_i^+, \; T_i(H_j) = H_j - a_{ji} H_i,$$

$$T_i(X_j^+) = \sum_{r=0}^{-a_{ij}} (-1)^{r - a_{ij}} q_i^{-r} (X_i^+)^{(-a_{ij} - r)} X_j^+ (X_i^+)^{(r)}, \; i \neq j, \quad (2.2.2)$$

$$T_i(X_j^-) = \sum_{r=0}^{-a_{ij}} (-1)^{r - a_{ij}} q_i^{r} (X_i^-)^{(r)} X_j^- (X_i^-)^{(-a_{ij} - r)}, \; i \neq j,$$

where

$$(X_i^+)^{(r)} = \frac{(X_i^+)^r}{[r]_{q_i}!},\ (X_i^-)^{(r)} = \frac{(X_i^-)^r}{[r]_{q_i}!},\ r \geq 0,\ i = 1, \dots, l.$$

Recall that action (2.2.1) of the generators T_i is induced by the adjoint action of certain representatives of the Weyl group elements s_i in G. Similarly, action (2.2.2) is induced by conjugation by certain elements of a completion of $U_h(\mathfrak{g})$.

More precisely, define a q-exponential by

$$exp'_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{x^k}{[k]_q!}.$$

Then the automorphism T_i in (2.2.2) is given by conjugation by the invertible element (see [88])

$$\begin{aligned} T_i = exp'_{q_i^{-1}}(-q_i^{-1}X_i^- K_i) exp'_{q_i^{-1}}(X_i^+) exp'_{q_i^{-1}}(-q_i X_i^- K_i^{-1}) q_i^{\frac{H_i(H_i+1)}{2}} = \\ = exp'_{q_i^{-1}}(q_i^{-1}X_i^+ K_i^{-1}) exp'_{q_i^{-1}}(-X_i^-) exp'_{q_i^{-1}}(q_i X_i^+ K_i) q_i^{\frac{H_i(H_i+1)}{2}} \end{aligned} \tag{2.2.3}$$

which belongs to the completion $\mathbb{C}_h[G]^*$ of $U_h(\mathfrak{g})$, where $\mathbb{C}_h[G]$ is the restricted dual of $U_h(\mathfrak{g})$, i.e. the Hopf algebra generated by the matrix elements of finite rank representations of $U_h(\mathfrak{g})$. The inverse of T_i in (2.2.3) can be found using the identity

$$exp'_q(x) exp'_{q^{-1}}(-x) = 1$$

which implies

$$\begin{aligned} T_i^{-1} = q_i^{-\frac{H_i(H_i+1)}{2}} exp'_{q_i}(q_i X_i^- K_i^{-1}) exp'_{q_i}(-X_i^+) exp'_{q_i}(q_i^{-1} X_i^- K_i) = \\ = q_i^{-\frac{H_i(H_i+1)}{2}} exp'_{q_i}(-q_i X_i^+ K_i) exp'_{q_i}(X_i^-) exp'_{q_i}(-q_i^{-1} X_i^+ K_i^{-1}). \end{aligned} \tag{2.2.4}$$

From formula (2.2.4) we obtain the following relations in $\mathbb{C}_h[G]^*$

$$\begin{aligned} exp'_{q_i}(-X_i^+) = exp'_{q_i^{-1}}(-q_i X_i^- K_i^{-1}) q_i^{\frac{H_i(H_i+1)}{2}} T_i^{-1} exp'_{q_i^{-1}}(-q_i^{-1} X_i^- K_i) = \\ = exp'_{q_i^{-1}}(-q_i X_i^- K_i^{-1}) q_i^{\frac{H_i(H_i+1)}{2}} exp'_{q_i^{-1}}(q_i^{-1} X_i^+) T_i^{-1}. \end{aligned} \tag{2.2.5}$$

The comultiplication in $U_h(\mathfrak{g})$ induces a comultiplication in $\mathbb{C}_h[G]^*$ with respect to which we have

$$\Delta_h(T_i) = \theta_i T_i \otimes T_i = T_i \otimes T_i \overline{\theta}_i, \tag{2.2.6}$$

$$\theta_i = exp_{q_i}[(1 - q_i^{-2}) X_i^+ \otimes X_i^-], \overline{\theta}_i = exp_{q_i}[(1 - q_i^{-2}) K_i^{-1} X_i^- \otimes X_i^+ K_i],$$

$$\Delta_h(T_i^{-1}) = \overline{\theta}_i^{-1} T_i \otimes T_i = T_i \otimes T_i \theta_i^{-1}, \tag{2.2.7}$$

$$\theta_i^{-1} = exp_{q_i^{-1}}[(1 - q_i^2) X_i^+ \otimes X_i^-], \overline{\theta}_i^{-1} = exp_{q_i^{-1}}[(1 - q_i^2) K_i^{-1} X_i^- \otimes X_i^+ K_i].$$

For a reduced decomposition $w = s_{i_1} \dots s_{i_k}$, $T_w = T_{i_1} \dots T_{i_k}$ only depends on w and (2.2.6) implies

$$\Delta_h(T_w) = \prod_{p=1}^{k} \theta_{\beta_p} T_w \otimes T_w = T_w \otimes T_w \prod_{p=1}^{k} \overline{\theta}_{\beta'_p}, \tag{2.2.8}$$

where for $p = 1, \dots, k$

$$X_{\beta_p}^{\pm} = T_{i_1} \dots T_{i_{p-1}} X_{i_p}^{\pm},\ \overline{X}_{\beta'_p}^{\pm} = T_{i_k}^{-1} \dots T_{i_{p+1}}^{-1} X_{i_p}^{\pm},\ K_{\beta'_p} = T_{i_k}^{-1} \dots T_{i_{p+1}}^{-1} K_{i_p},$$

$$\theta_{\beta_p} = exp_{q_{\beta_p}}[(1 - q_{\beta_p}^{-2}) X_{\beta_p}^+ \otimes X_{\beta_p}^-],\ \overline{\theta}_{\beta'_p} = exp_{q_{\beta'_p}}[(1 - q_{\beta'_p}^{-2}) K_{\beta'_p}^{-1} \overline{X}_{\beta'_p}^- \otimes \overline{X}_{\beta'_p}^+ K_{\beta'_p}].$$

Similarly, for a reduced decomposition $w = s_{i_1} \dots s_{i_k}$, $\overline{T}_w = T_{i_1}^{-1} \dots T_{i_k}^{-1}$ only depends on w and (2.2.7) yields

$$\Delta_h(\overline{T}_w) = \prod_{p=1}^{k} \overline{\theta}'_{\beta_p} \overline{T}_w \otimes \overline{T}_w = \overline{T}_w \otimes \overline{T}_w \prod_{p=1}^{k} \theta'_{\beta'_p}, \tag{2.2.9}$$

where for $p=1,\ldots,k$

$$\overline{X}^{\pm}_{\beta_p}=T_{i_1}^{-1}\ldots T_{i_{p-1}}^{-1}X_{i_p}^{\pm},\ X^{\pm}_{\beta'_p}{}'=T_{i_k}\ldots T_{i_{p+1}}X_{i_p}^{\pm},\ \overline{K}_{\beta_p}=T_{i_1}^{-1}\ldots T_{i_{p-1}}^{-1}K_{i_p},$$

$$\theta'_{\beta'_p}=exp_{q^{-1}_{\beta'_p}}[(1-q^2_{\beta'_p})X^{+}_{\beta'_p}{}'\otimes X^{-}_{\beta'_p}{}'],\ \overline{\theta}'_{\beta_p}=exp_{q^{-1}_{\beta_p}}[(1-q^2_{\beta_p})\overline{K}^{-1}_{\beta_p}\overline{X}^{-}_{\beta_p}\otimes\overline{X}^{+}_{\beta_p}\overline{K}_{\beta_p}].$$

If $w\alpha_i=s_{i_1}\ldots s_{i_k}\alpha_i=\alpha_j$ for some i and j then

$$T_wX_i^{\pm}=X_j^{\pm},\ \overline{T}_wX_i^{\pm}=X_j^{\pm}. \tag{2.2.10}$$

2.3 Quantum root vectors

In this section we recall the construction of analogues of root vectors for $U_h(\mathfrak{g})$ in terms of the braid group action on $U_h(\mathfrak{g})$. Recall that for any reduced decomposition $\overline{w}=s_{i_1}\ldots s_{i_D}$ of the longest element $\overline{w}$ of the Weyl group W of $\mathfrak{g}$ the ordering

$$\beta_1=\alpha_{i_1},\beta_2=s_{i_1}\alpha_{i_2},\ldots,\beta_D=s_{i_1}\ldots s_{i_{D-1}}\alpha_{i_D}$$

is a normal ordering in Δ_+, and there is a one–to–one correspondence between normal orderings of Δ_+ and reduced decompositions of $\overline{w}$.

Fix a reduced decomposition $\overline{w}=s_{i_1}\ldots s_{i_D}$ of $\overline{w}$ and define the corresponding quantum root vectors in $U_h(\mathfrak{g})$ by

$$X^{\pm}_{\beta_k}=T_{i_1}\ldots T_{i_{k-1}}X^{\pm}_{i_k}. \tag{2.3.1}$$

Note that one can construct root vectors in the Lie algebra $\mathfrak{g}$ in a similar way. Namely, the root vectors $X_{\pm\beta_k}\in\mathfrak{g}_{\pm\beta_k}$ of $\mathfrak{g}$ can be defined by

$$X_{\pm\beta_k}=T_{i_1}\ldots T_{i_{k-1}}X_{\pm\alpha_{i_k}}, \tag{2.3.2}$$

where $X_{\pm\alpha_{i_k}}$ are as in (2.2.1).

The root vectors $X^{\pm}_{\beta}$ satisfy the following relations:

$$\begin{aligned}X^{\pm}_{\alpha}X^{\pm}_{\beta}-q^{(\alpha,\beta)}X^{\pm}_{\beta}X^{\pm}_{\alpha}&=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C(k_1,\ldots,k_n)(X^{\pm}_{\delta_1})^{k_1}(X^{\pm}_{\delta_2})^{k_2}\ldots(X^{\pm}_{\delta_n})^{k_n}=\\&=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}C'(k_1,\ldots,k_n)(X^{\pm}_{\delta_1})^{(k_1)}(X^{\pm}_{\delta_2})^{(k_2)}\ldots(X^{\pm}_{\delta_n})^{(k_n)},\ \alpha<\beta,\end{aligned} \tag{2.3.3}$$

where for $\alpha\in\Delta_+$ we put $(X^{\pm}_{\alpha})^{(k)}=\frac{(X^{\pm}_{\alpha})^k}{[k]_{q_\alpha}!}$, $k\geq 0$, $q_\alpha=q^{d_i}$ if the positive root α is Weyl group conjugate to the simple root α_i, $C'(k_1,\ldots,k_n)\in\mathbb{C}[q,q^{-1}]$, $C(k_1,\ldots,k_n)\in\mathcal{P}$, and $\mathcal{P}=\mathbb{C}[q,q^{-1}]$ if $\mathfrak{g}$ is simply-laced, $\mathcal{P}=\mathbb{C}[q,q^{-1},\frac{1}{[2]_q}]$ if $\mathfrak{g}$ is of type B_l,C_l or F_4, and $\mathcal{P}=\mathbb{C}[q,q^{-1},\frac{1}{[2]_q},\frac{1}{[3]_q}]$ if $\mathfrak{g}$ is of type G_2.

Note that by construction

$$\begin{aligned}X^{+}_{\beta}\ (\mathrm{mod}\ h)&=X_\beta\in\mathfrak{g}_\beta,\\X^{-}_{\beta}\ (\mathrm{mod}\ h)&=X_{-\beta}\in\mathfrak{g}_{-\beta}\end{aligned} \tag{2.3.4}$$

are root vectors of $\mathfrak{g}$.

Define an algebra antiautomorphism ψ of $U_h(\mathfrak{g})$ by

$$\psi(X_i^{\pm})=X_i^{\pm},\psi(H_i)=-H_i,\psi(h)=h.$$

It satisfies the relations $T_i^{-1}=\psi T_i\psi$ and hence for any $\alpha\in\Delta_+$

$$\psi(X^{\pm}_{\alpha})=\overline{X}^{\pm}_{\alpha},$$

where

$$\overline{X}^{\pm}_{\beta_k}=T_{i_1}^{-1}\ldots T_{i_{k-1}}^{-1}X^{\pm}_{i_k}.$$

By applying ψ to relations (2.3.3) one can obtain the following relations for the root vectors $\overline{X}_\beta^{\pm}$

$$\overline{X}_\alpha^{\pm}\overline{X}_\beta^{\pm} - q^{-(\alpha,\beta)}\overline{X}_\beta^{\pm}\overline{X}_\alpha^{\pm} = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} D(k_1,\ldots,k_n)(\overline{X}_{\delta_1}^{\pm})^{k_1}(\overline{X}_{\delta_2}^{\pm})^{k_2}\ldots(\overline{X}_{\delta_n}^{\pm})^{k_n} =$$

$$= \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} D'(k_1,\ldots,k_n)(\overline{X}_{\delta_1}^{\pm})^{(k_1)}(\overline{X}_{\delta_2}^{\pm})^{(k_2)}\ldots(\overline{X}_{\delta_n}^{\pm})^{(k_n)},\ \alpha<\beta, \quad (2.3.5)$$

where for $\alpha \in \Delta_+$ we put $(\overline{X}_\alpha^{\pm})^{(k)} = \frac{(\overline{X}_\alpha^{\pm})^k}{[k]_{q_\alpha}!}$, $k \geq 0$, $D(k_1,\ldots,k_n) \in \mathcal{P}$, $D'(k_1,\ldots,k_n) \in \mathbb{C}[q,q^{-1}]$.

One can also obtain commutation relations between positive and negative root vectors. These relations are known in some form. For completeness we give a proof of them using (2.3.3) and (2.3.5) only.

Lemma 2.3.1. *Let $[-\beta,\alpha]$, $\alpha,\beta \in \Delta_+$ be a minimal segment with respect to the circular normal ordering of Δ corresponding to a normal ordering $\beta_1,\ldots,\beta_D$ of Δ_+. Then*

$$X_\alpha^+X_\beta^- - X_\beta^-X_\alpha^+ = \sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha} C(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n} =$$

$$= \sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha} C'(k_1,\ldots,k_n)(X_{\delta_1})^{(k_1)}(X_{\delta_2})^{(k_2)}\ldots(X_{\delta_n})^{(k_n)}, \quad (2.3.6)$$

where the inequalities for roots in the sum are with respect to the circular normal ordering in Δ corresponding to the normal ordering $\beta_1,\ldots,\beta_D$ of Δ_+, $X_\delta = X_\delta^+$ for $\delta \in \Delta_+$ and $X_\delta = X_\delta^-$ for $\delta \in \Delta_-$, $C'(k_1,\ldots,k_n) \in U_q(H)$, $U_q(H)$ is the $\mathbb{C}[q,q^{-1}]$–subalgebra of $U_h(\mathfrak{g})$ generated by $K_i^{\pm1}$, $i=1,\ldots,l$, $C(k_1,\ldots,k_n) \in \mathcal{P}'$, and $\mathcal{P}' = U_q(H)$ if $\mathfrak{g}$ is simply-laced, $\mathcal{P}' = U_q(H)[\frac{1}{[2]_q}]$ if $\mathfrak{g}$ is of type B_l, C_l or F_4, and $\mathcal{P}' = U_q(H)[\frac{1}{[2]_q},\frac{1}{[3]_q}]$ if $\mathfrak{g}$ is of type G_2.

Also

$$\overline{X}_\alpha^{+}\overline{X}_\beta^{-} - \overline{X}_\beta^{-}\overline{X}_\alpha^{+} = \sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha} D(k_1,\ldots,k_n)(\overline{X}_{\delta_1})^{k_1}(\overline{X}_{\delta_2})^{k_2}\ldots(\overline{X}_{\delta_n})^{k_n} =$$

$$= \sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha} D'(k_1,\ldots,k_n)(\overline{X}_{\delta_1})^{(k_1)}(\overline{X}_{\delta_2})^{(k_2)}\ldots(\overline{X}_{\delta_n})^{(k_n)},\ \alpha<\beta, \quad (2.3.7)$$

where the inequalities for roots in the sum are with respect to the circular normal ordering in Δ corresponding to the normal ordering $\beta_1,\ldots,\beta_D$ of Δ_+, $\overline{X}_\delta = \overline{X}_\delta^+$ for $\delta \in \Delta_+$ and $\overline{X}_\delta = \overline{X}_\delta^-$ for $\delta \in \Delta_-$, $D'(k_1,\ldots,k_n) \in U_q(H)$, $D(k_1,\ldots,k_n) \in \mathcal{P}'$.

Proof. The proof is by induction. We shall consider the first identity in the case when $\alpha < \beta$. The others are proved in a similar way.

Let $\overline{w} = s_{i_1}\ldots s_{i_D}$ be the reduced decomposition of the longest element of the Weyl group corresponding to the normal ordering $\beta_1,\ldots,\beta_D$ of Δ_+.

First assume that $\alpha = \beta_1 = \alpha_{i_1}$. Let $\beta = \beta_m = s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}$. Then $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ is another normal ordering of Δ_+, and by (2.3.3) for this normal ordering

$$T_{i_1}^{-1}(X_\alpha^+X_\beta^- - X_\beta^-X_\alpha^+) = T_{i_1}^{-1}(X_{i_1}^+X_{\beta_m}^- - X_{\beta_m}^-X_{i_1}^+) =$$
$$= K_{i_1}^{-1}(-X_{i_1}^-X_{s_{i_1}^{-1}\beta_m}^- + q^{-(\alpha_{i_1},s_{i_1}^{-1}\beta_m)}X_{s_{i_1}^{-1}\beta_m}^-X_{i_1}^-) =$$
$$= \sum_{-s_{i_1}^{-1}\beta_m<\delta_1<\ldots<\delta_n<-\alpha_{i_1}} K_{i_1}^{-1}C(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n},$$

where $C(k_1,\ldots,k_n) \in \mathcal{P}$, the inequalities for the roots in the sum are with respect to the circular normal ordering of Δ associated to the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ of Δ_+, and and the quantum root vectors are defined using the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ of Δ_+.

Now applying T_{i_1} to the last identity we get

$$X_\alpha^+X_\beta^- - X_\beta^-X_\alpha^+ = \sum_{-\beta_m<\delta_1<\ldots<\delta_n\leq-\beta_D} K_{i_1}C(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n} =$$
$$= \sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha} K_{i_1}C(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n},$$

where the inequalities for the roots in the sum are with respect to the circular normal ordering associated to the original ordering $\beta_1,\ldots,\beta_D$ of Δ_+ and the quantum root vectors are defined using the ordering $\beta_1,\ldots,\beta_D$ of Δ_+. This establishes the base of the induction.

Now assume that the identity in question is proved for all normal orderings of Δ_+ and for all $\alpha=\beta_k$ with $k<n$ for some $n>0$ and for all possible β such that $[-\beta,\alpha]$, $\alpha,\beta\in\Delta_+$ is a minimal segment.

Let $\alpha=\beta_n=s_{i_1}\ldots s_{i_{n-1}}\alpha_{i_n}$, $\beta=\beta_m=s_{i_1}\ldots s_{i_{m-1}}\alpha_{i_m}$, $n<m$. Then $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ is another normal ordering of Δ_+, and by the induction hypothesis for this normal ordering with $s_{i_1}^{-1}\alpha=s_{i_2}\ldots s_{i_{n-1}}\alpha_{i_n}$, $s_{i_1}^{-1}\beta=\beta_m=s_{i_2}\ldots s_{i_{m-1}}\alpha_{i_m}$ we have

$$X^+_{s_{i_1}^{-1}\alpha}X^-_{s_{i_1}^{-1}\beta}-X^-_{s_{i_1}^{-1}\beta}X^+_{s_{i_1}^{-1}\alpha}=$$

$$=\sum_{-s_{i_1}^{-1}\beta_m<\delta_1'<\ldots<\delta_n'<s_{i_1}^{-1}\alpha}C(k_1',\ldots,k_n')(X_{\delta_1'})^{k_1'}(X_{\delta_2'})^{k_2'}\ldots(X_{\delta_n'})^{k_n'},$$

where $C(k_1',\ldots,k_n')\in\mathcal{P}'$, the inequalities for the roots in the sum are with respect to the circular normal ordering in Δ associated to the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ of Δ_+ and the quantum root vectors are defined using the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D,\alpha_{i_1}$ of Δ_+.

Now applying T_{i_1} to the last identity we get

$$X^+_\alpha X^-_\beta-X^-_\beta X^+_\alpha=\sum_{-\beta<\delta_1<\ldots<\delta_n<\alpha}K_{i_1}^{k_q'}\overline{C}(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n},$$

where k_q' is such that $\delta_q'=-\alpha_{i_1}$ in the previous formula, the inequalities for the roots in the sum are with respect to the circular normal ordering associated to the original ordering $\beta_1,\ldots,\beta_D$ of Δ_+, the quantum root vectors are defined using the ordering $\beta_1,\ldots,\beta_D$ of Δ_+, and $\overline{C}(k_1,\ldots,k_n)\in\mathcal{P}'$. This establishes the induction step and completes the proof.

□

2.4 Some subalgebras in quantum groups and their Poincaré–Birkhoff–Witt bases

Now we shall explicitly describe a topological $\mathbb{C}[[h]]$–basis for $U_h(\mathfrak{g})$. We shall also recall the definition of some rational forms of $U_h(\mathfrak{g})$ and of their bases.

Denote by $U_h(\mathfrak{n}_+),U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$ the $\mathbb{C}[[h]]$–subalgebras of $U_h(\mathfrak{g})$ topologically generated by the X_i^+, by the X_i^- and by the H_i, respectively. For any $\alpha\in\Delta_+$ one has $X_\alpha^\pm\in U_h(\mathfrak{n}_\pm)$. From the definition of the quantum root vectors it also follows that $[H_i,X_\alpha^\pm]=\pm\alpha(H_i)X_\alpha^\pm$, $i=1,\ldots,l$. Therefore using the uniqueness of the presentation of any positive root as a sum of simple roots we immediately deduce the following property of the quantum root vectors.

Proposition 2.4.1. *For $\beta=\sum_{i=1}^l m_i\alpha_i,\ m_i\in\mathbb{N}$ $X_\beta^\pm$ is a polynomial in the noncommutative variables $X_i^\pm$ homogeneous in each $X_i^\pm$ of degree m_i.*

Denote by $U_q^{res}(\mathfrak{g})$ the subalgebra in $U_h(\mathfrak{g})$ generated over $\mathbb{C}[q,q^{-1}]$ by the elements

$$K_i^{\pm1},\ (X_i^\pm)^{(k)},\ i=1,\ldots,l,k\geq 1.$$

The elements

$$\left[\begin{array}{c}K_i;c\\ r\end{array}\right]_{q_i}=\prod_{s=1}^r\frac{K_iq_i^{c+1-s}-K_i^{-1}q_i^{s-1-c}}{q_i^s-q_i^{-s}}\ ,\ i=1,\ldots,l,\ c\in\mathbb{Z},\ r\in\mathbb{N}$$

belong to $U_q^{res}(\mathfrak{g})$. Denote by $U_q^{res}(H)$ the subalgebra of $U_q^{res}(\mathfrak{g})$ generated by those elements and by $K_i^{\pm1}$, $i=1,\ldots,l$.

Let $U_{\mathcal{P}}(\mathfrak{n}_+),U_{\mathcal{P}}(\mathfrak{n}_-)$ ($U_q^{res}(\mathfrak{n}_+),U_q^{res}(\mathfrak{n}_-)$) be the $\mathcal{P}(\mathbb{C}[q,q^{-1}])$–subalgebras of $U_h(\mathfrak{g})$ generated by the X_i^+ and by the X_i^-, $i=1,\ldots,l$ (by the $(X_i^+)^{(r)}$ and by the $(X_i^-)^{(r)}$, $i=1,\ldots,l$, $r\geq 0$), respectively. Using the root vectors $X_\beta^\pm$, $(X_\beta^\pm)^{(r)}$ we can construct bases for these subalgebras Namely, let $(X^\pm)^{\mathbf{r}}=(X_{\beta_1}^\pm)^{r_1}\ldots(X_{\beta_D}^\pm)^{r_D}$, $(X^\pm)^{(\mathbf{r})}=(X_{\beta_1}^\pm)^{(r_1)}\ldots(X_{\beta_D}^\pm)^{(r_D)}$, $\mathbf{r}=(r_1,\ldots r_D)\in\mathbb{N}^D$, $H^{\mathbf{k}}=H_1^{k_1}\ldots H_l^{k_l}$, $\mathbf{k}=(k_1,\ldots,k_l)\in\mathbb{N}^l$.

Commutation relations (2.3.3), (2.3.5), (2.3.6) and (2.3.7) between quantum root vectors imply the following lemma.

Lemma 2.4.2. *The elements* $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ *(*$(X^+)^{(\mathbf{r})}$, $(X^-)^{(\mathbf{t})}$*) for* $\mathbf{r},\ \mathbf{t}\in\mathbb{N}^D$ *form bases of* $U_{\mathcal{P}}(\mathfrak{n}_+)$, $U_{\mathcal{P}}(\mathfrak{n}_-)$ *(*$U_q^{res}(\mathfrak{n}_+)$, $U_q^{res}(\mathfrak{n}_-)$*), respectively.*

The elements $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ *and* $H^{\mathbf{k}}$ *form topological bases of* $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ *and* $U_h(\mathfrak{h})$*, respectively.*

The multiplication defines an isomorphisms of $\mathbb{C}[q,q^{-1}]$*–modules:*

$$U_q^{res}(\mathfrak{n}_-)\otimes U_q^{res}(H)\otimes U_q^{res}(\mathfrak{n}_+)\to U_q^{res}(\mathfrak{g}), \tag{2.4.1}$$

and of complete $\mathbb{C}[[h]]$*–modules*

$$U_h(\mathfrak{n}_-)\otimes U_h(\mathfrak{h})\otimes U_h(\mathfrak{n}_+)\to U_h(\mathfrak{g}),$$

where the tensor products in the leftt hand side are completed in the h–adic topology.

Let $[\alpha,\beta]=\{\beta_p,\ldots,\beta_q\}$ *be a minimal segment in* Δ_+, $U_{\mathcal{P}}([\alpha,\beta])$, $U_{\mathcal{P}}([-\alpha,-\beta])$ *(*$U_q^{res}([\alpha,\beta])$, $U_q^{res}([-\alpha,-\beta])$*) the* $\mathcal{P}(\mathbb{C}[q,q^{-1}])$*–subalgebras of* $U_h(\mathfrak{g})$ *generated by the* X_γ^+ *and by the* X_γ^-, $\gamma\in[\alpha,\beta]$ *(by the* $(X_\gamma^+)^{(r)}$ *and by the* $(X_\gamma^-)^{(r)}$, $\gamma\in[\alpha,\beta]$, $r\geq 0$*), respectively. Then the elements* $(X_{\beta_p}^\pm)^{r_p}\ldots(X_{\beta_q}^\pm)^{r_q}$ *(*$(X_{\beta_p}^\pm)^{(r_p)}\ldots(X_{\beta_q}^\pm)^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $U_{\mathcal{P}}([\alpha,\beta])$, $U_{\mathcal{P}}([-\alpha,-\beta])$ *(*$U_q^{res}([\alpha,\beta])$, $U_q^{res}([-\alpha,-\beta])$*), respectively.*

Let $\overline{U}_{\mathcal{P}}([\alpha,\beta])$, $\overline{U}_{\mathcal{P}}([-\alpha,-\beta])$ *(*$\overline{U}_q^{res}([\alpha,\beta])$, $\overline{U}_q^{res}([-\alpha,-\beta])$*) be the* $\mathcal{P}(\mathbb{C}[q,q^{-1}])$*–subalgebras of* $U_h(\mathfrak{g})$ *generated by the* $\overline{X}_\gamma^+$ *and by the* $\overline{X}_\gamma^-$, $\gamma\in[\alpha,\beta]$ *(by the* $(\overline{X}_\gamma^+)^{(r)}$ *and by the* $(\overline{X}_\gamma^-)^{(r)}$, $\gamma\in[\alpha,\beta]$, $r\geq 0$*), respectively. Then the elements* $(\overline{X}_{\beta_p}^\pm)^{r_p}\ldots(\overline{X}_{\beta_q}^\pm)^{r_q}$ *(*$(\overline{X}_{\beta_p}^\pm)^{(r_p)}\ldots(\overline{X}_{\beta_q}^\pm)^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $\overline{U}_{\mathcal{P}}([\alpha,\beta])$, $\overline{U}_{\mathcal{P}}([-\alpha,-\beta])$ *(*$\overline{U}_q^{res}([\alpha,\beta])$, $\overline{U}_q^{res}([-\alpha,-\beta])$*), respectively.*

Let $[\alpha,-\beta]=\{\beta_p,\ldots,\beta_q\}$, $\alpha,\beta\in\Delta_+$ *be a minimal segment in* Δ, $U_{\mathcal{P}'}([\alpha,-\beta])$, $U_{\mathcal{P}'}([-\alpha,\beta])$ *(*$U_{U_q^{res}(H)}^{res}([\alpha,-\beta])$, $U_{U_q^{res}(H)}^{res}([-\alpha,\beta])$*) the* $\mathcal{P}'(U_q^{res}(H))$*–subalgebras of* $U_h(\mathfrak{g})$ *generated by the* X_γ *(*$(X_\gamma)^{(r)}$*), where* $\gamma\in[\alpha,-\beta]$ *or* $\gamma\in[-\alpha,\beta]$*, respectively, and* $X_\gamma=X_\gamma^+$ *if* $\gamma\in\Delta_+$, $X_\gamma=X_\gamma^-$ *if* $\gamma\in\Delta_-$*. Then the elements* $(X_{\beta_p})^{r_p}\ldots(X_{\beta_q})^{r_q}$ *(*$(X_{\beta_p})^{(r_p)}\ldots(X_{\beta_q})^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $U_{\mathcal{P}'}([\alpha,-\beta])$ *(*$U_{U_q^{res}(H)}^{res}([\alpha,-\beta])$*), and the elements* $(X_{-\beta_p})^{r_p}\ldots(X_{-\beta_q})^{r_q}$ *(*$(X_{-\beta_p})^{(r_p)}\ldots(X_{-\beta_q})^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $U_{\mathcal{P}'}([-\alpha,\beta])$ *(*$U_{U_q^{res}(H)}^{res}([-\alpha,\beta])$*), respectively.*

Let $\overline{U}_{\mathcal{P}'}([\alpha,-\beta])$, $\overline{U}_{\mathcal{P}'}([-\alpha,\beta])$ *(*$\overline{U}_{U_q^{res}(H)}^{res}([\alpha,-\beta])$, $\overline{U}_{U_q^{res}(H)}^{res}([-\alpha,\beta])$*) be the* $\mathcal{P}'(U_q^{res}(H))$*–subalgebras of* $U_h(\mathfrak{g})$ *generated by the* $\overline{X}_\gamma$ *(*$(\overline{X}_\gamma)^{(r)}$*), where* $\gamma\in[\alpha,-\beta]$ *or* $\gamma\in[-\alpha,\beta]$*, respectively, and* $\overline{X}_\gamma=\overline{X}_\gamma^+$ *if* $\gamma\in\Delta_+$, $\overline{X}_\gamma=\overline{X}_\gamma^-$ *if* $\gamma\in\Delta_-$*. Then the elements* $(\overline{X}_{\beta_p})^{r_p}\ldots(\overline{X}_{\beta_q})^{r_q}$ *(*$(\overline{X}_{\beta_p})^{(r_p)}\ldots(\overline{X}_{\beta_q})^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $\overline{U}_{\mathcal{P}'}([\alpha,-\beta])$ *(*$\overline{U}_{U_q^{res}(H)}^{res}([\alpha,-\beta])$*), and the elements* $(\overline{X}_{-\beta_p})^{r_p}\ldots(\overline{X}_{-\beta_q})^{r_q}$ *(*$(\overline{X}_{-\beta_p})^{(r_p)}\ldots(\overline{X}_{-\beta_q})^{(r_q)}$*),* $r_i\in\mathbb{N}$ *form bases of* $\overline{U}_{\mathcal{P}'}([-\alpha,\beta])$ *(*$\overline{U}_{U_q^{res}(H)}^{res}([-\alpha,\beta])$*), respectively.*

Proof. The first four statements of this lemma are just Propositions 8.1.7, 9.1.3 and 9.3.3 in [18]. The proofs of the other claims are similar to each other. Consider, for instance, the case of the algebra $U_{U_q^{res}(H)}^{res}([\alpha,-\beta])$.

Let $[\alpha,-\beta]=\{\beta_p,\ldots,\beta_q\}$, $\alpha,\beta\in\Delta_+$ be a minimal segment in Δ, $U_{U_q^{res}(H)}^{res}([\alpha,-\beta])$ the $U_q^{res}(H)$–subalgebra of $U_h(\mathfrak{g})$ generated by the $(X_\gamma)^{(r)}$, where $\gamma\in[\alpha,-\beta]$, and $X_\gamma=X_\gamma^+$ if $\gamma\in\Delta_+$, $X_\gamma=X_\gamma^-$ if $\gamma\in\Delta_-$. We show that the elements $(X_{\beta_p})^{(r_p)}\ldots(X_{\beta_q})^{(r_q)}$, $r_i\in\mathbb{N}$ form a basis of $U_{U_q^{res}(H)}^{res}([\alpha,-\beta])$.

Consider the algebra $U_q(\mathfrak{g})=U_q^{res}(\mathfrak{g})\otimes_{\mathbb{C}[q,q^{-1}]}\mathbb{C}(q)$. If $x\in U_{U_q^{res}(H)}^{res}([\alpha,-\beta])\subset U_q(\mathfrak{g})$ then using commutation relations (2.3.3) and (2.3.6) one can represent x as a $U_q^{res}(H)\otimes_{\mathbb{C}[q,q^{-1}]}\mathbb{C}(q)$–linear combination of the elements $(X_{\beta_p})^{(r_p)}\ldots(X_{\beta_q})^{(r_q)}$, $r_i\in\mathbb{N}$. We can also consider x as an element of $U_q(\mathfrak{g})$ and by the Poincaré–Birkhoff–Witt theorem for $U_q(\mathfrak{g})$ (see Proposition 9.1.3 in [18]) the above mentioned presentation of x is unique. Now by the Poincaré–Birkhoff–Witt theorem for $U_q^{res}(\mathfrak{g})$ (see Proposition 9.3.3 in [18] or the third claim of this lemma) the coefficients in this presentation must belong to $U_q^{res}(H)$. This completes the proof in the considered case.

□

A basis for $U_q^{res}(H)$ is a little bit more difficult to describe. We do not need its explicit description.

2.5 The universal R–matrix

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R}\in U_h(\mathfrak{g})\otimes U_h(\mathfrak{g})$ (completed tensor product), called a universal R–matrix, such that

$$\Delta_h^{opp}(a)=\mathcal{R}\Delta_h(a)\mathcal{R}^{-1} \text{ for all } a\in U_h(\mathfrak{g}), \tag{2.5.1}$$

and

$$(\Delta_h \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},$$
$$(id \otimes \Delta_h)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.5.2}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1,\ \mathcal{R}_{23} = 1 \otimes \mathcal{R},\ \mathcal{R}_{13} = (\sigma \otimes id)\mathcal{R}_{23}$.

From (2.5.1) and (2.5.2) it follows that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.5.3}$$

For every quasitriangular Hopf algebra we also have

$$(S \otimes id)\mathcal{R} = (id \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1},$$

and

$$(S \otimes S)\mathcal{R} = \mathcal{R}. \tag{2.5.4}$$

An explicit expression for $\mathcal{R}$ may be written by making use of the q–exponential

$$exp_q(x) = exp'_q(qx) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$$

in terms of which the element $\mathcal{R}$ takes the form

$$\mathcal{R} = \prod_{\beta} exp_{q_\beta}[(1 - q_\beta^{-2})X_\beta^- \otimes X_\beta^+] exp\left[h \sum_{i=1}^{l} (Y_i \otimes H_i)\right], \tag{2.5.5}$$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the α–term appears to the left of the β–term if $\alpha < \beta$ with respect to the normal ordering

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D}$$

of Δ_+ which is used in the definition of the quantum root vectors $X_\beta^\pm$.

One can calculate the action of the comultiplication on the root vectors $X_{\beta_k}^\pm$ in terms of the universal R–matrix. For instance for $\Delta_h(X_{\beta_k}^-)$ one has

$$\Delta_h(X_{\beta_k}^-) = \widetilde{\mathcal{R}}_{<\beta_k}(X_{\beta_k}^- \otimes 1 + e^{h\beta^\vee} \otimes X_{\beta_k}^-)\widetilde{\mathcal{R}}_{<\beta_k}^{-1}, \tag{2.5.6}$$

where

$$\widetilde{\mathcal{R}}_{<\beta_k} = \widetilde{\mathcal{R}}_{\beta_1} \ldots \widetilde{\mathcal{R}}_{\beta_{k-1}},\ \widetilde{\mathcal{R}}_{\beta_r} = exp_{q_{\beta_r}}[(1 - q_{\beta_r}^{-2})X_{\beta_r}^+ \otimes X_{\beta_r}^-].$$

The r–matrix $r_- = -\frac{1}{2}h^{-1}(\mathcal{R} - 1 \otimes 1) \pmod h$, which is the classical limit of $\mathcal{R}$, coincides with the classical r–matrix (2.1.3).

2.6 Realizations of quantum groups associated to Weyl group elements

q-W–algebras will be defined in terms of certain integral forms of non–standard realizations of quantum groups associated to Weyl group elements.

Let s be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{h}'$ the orthogonal complement, with respect to the symmetric bilinear form, to the subspace of $\mathfrak{h}$ fixed by the natural action of s on $\mathfrak{h}$. Let $\mathfrak{h}'^*$ be the image of $\mathfrak{h}'$ in $\mathfrak{h}^*$ under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by the symmetric bilinear form on $\mathfrak{g}$. The restriction of the natural action of s on $\mathfrak{h}^*$ to the subspace $\mathfrak{h}'^*$ has no fixed points. Therefore one can define the Cayley transform $\frac{1+s}{1-s}$ of the restriction of s to $\mathfrak{h}'^*$. Denote by $P_{\mathfrak{h}'^*}$ the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to the bilinear form.

Let $\kappa \in \mathbb{Z}$ be an integer number and $U_h^s(\mathfrak{g})$ the topological algebra over $\mathbb{C}[[h]]$ topologically generated by elements $e_i, f_i, H_i,\ i = 1, \dots l$ subject to the relations:

$$\begin{aligned}
&[H_i, H_j] = 0, \quad [H_i, e_j] = a_{ij} e_j, \quad [H_i, f_j] = -a_{ij} f_j, \quad e_i f_j - q^{c_{ij}} f_j e_i = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
&c_{ij} = \kappa \left(\tfrac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j \right), \quad K_i = e^{d_i h H_i}, \\
&\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r q^{r c_{ij}} \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r = 0, \ i \neq j, \\
&\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r q^{r c_{ij}} \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (f_i)^{1-a_{ij}-r} f_j (f_i)^r = 0, \ i \neq j.
\end{aligned} \tag{2.6.1}$$

Proposition 2.6.1. *For every solution $n_{ij} \in \mathbb{C},\ i, j = 1, \dots, l$ of equations*

$$d_j n_{ij} - d_i n_{ji} = c_{ij} \tag{2.6.2}$$

there exists an algebra isomorphism $\psi_{\{n_{ij}\}} : U_h^s(\mathfrak{g}) \to U_h(\mathfrak{g})$ defined by the formulas:

$$\psi_{\{n_{ij}\}}(e_i) = X_i^+ \prod_{p=1}^{l} L_p^{n_{ip}}, \quad \psi_{\{n_{ij}\}}(f_i) = \prod_{p=1}^{l} L_p^{-n_{ip}} X_i^-, \quad \psi_{\{n_{ij}\}}(H_i) = H_i.$$

Proof. The proof of this proposition is by direct verification of defining relations (2.6.1). The most nontrivial part is to verify the deformed quantum Serre relations, i.e. the last two relations in (2.6.1). For instance, the defining relations of $U_h(\mathfrak{g})$ imply the following relations for $\psi_{\{n_{ij}\}}(e_i)$,

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q^{k(d_j n_{ij} - d_i n_{ji})} \psi_{\{n_{ij}\}}(e_i)^{1-a_{ij}-k} \psi_{\{n_{ij}\}}(e_j) \psi_{\{n_{ij}\}}(e_i)^k = 0,$$

for any $i \neq j$. Now using equation (2.6.2) we arrive to the quantum Serre relations for e_i in (2.6.1).

□

The general solution of equation (2.6.2) is given by

$$n_{ij} = \frac{1}{2d_j}(c_{ij} + s_{ij}), \tag{2.6.3}$$

where $s_{ij} = s_{ji}$.

We shall only use the solution for which $s_{ij} = 0$ for all $i, j = 1, \dots l$. Then

$$n_{ij} = \frac{1}{2d_j} c_{ij} \tag{2.6.4}$$

From now on we assume that solution (2.6.4) is used to identify $U_h^s(\mathfrak{g})$ and $U_h(\mathfrak{g})$.

The algebra $U_h^s(\mathfrak{g})$ is called the realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the element $s \in W$. Denote by $U_h^s(\mathfrak{n}_\pm)$ the subalgebra in $U_h^s(\mathfrak{g})$ generated by $e_i\ (f_i), i = 1, \dots, l$. Let $U_h^s(\mathfrak{h})$ be the subalgebra in $U_h^s(\mathfrak{g})$ generated by $H_i,\ i = 1, \dots, l$.

We shall construct analogues of quantum root vectors for $U_h^s(\mathfrak{g})$. It is convenient to introduce an operator $K \in \mathrm{End}\, \mathfrak{h}$ defined by

$$K H_i = \sum_{j=1}^{l} \frac{n_{ij}}{d_i} Y_j. \tag{2.6.5}$$

From (2.6.4) we obtain that

$$K h = \frac{\kappa}{2} \frac{1+s}{1-s} P_{\mathfrak{h}'} h, h \in \mathfrak{h}.$$

Proposition 2.6.2. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Let $U_h^s(\mathfrak{g})$ be the realization of the quantum group $U_h(\mathfrak{g})$ associated to s.*

For any normal ordering of the root system Δ_+ the elements

$$e_\beta = \psi^{-1}_{\{n_{ij}\}}(X^+_\beta e^{hK\beta^\vee}) \text{ and } f_\beta = \psi^{-1}_{\{n_{ij}\}}(e^{-hK\beta^\vee} X^-_\beta),\ \beta \in \Delta_+$$

lie in the subalgebras $U_h^s(\mathfrak{n}_+)$ and $U_h^s(\mathfrak{n}_-)$, respectively.

The elements $f_\beta \in U_h^s(\mathfrak{n}_-)$, $\beta \in \Delta_{\mathfrak{m}_+}$ generate a subalgebra $U_h^s(\mathfrak{m}_-) \subset U_h^s(\mathfrak{g})$ such that

$$U_h^s(\mathfrak{m}_-)/hU_h^s(\mathfrak{m}_-) \simeq U(\mathfrak{m}_-),$$

where $\mathfrak{m}_-$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors $X_{-\alpha}$, $\alpha \in \Delta_{\mathfrak{m}_+}$.

Proof. Fix a normal ordering of the root system Δ_+. Let $\beta = \sum_{i=1}^l m_i\alpha_i \in \Delta_+$ be a positive root, $X^+_\beta \in U_h(\mathfrak{g})$ the corresponding quantum root vector constructed with the help of the fixed normal ordering of Δ_+. Then $\beta^\vee = \sum_{i=1}^l m_i d_i H_i$, and so $K\beta^\vee = \sum_{i,j=1}^l m_i n_{ij} Y_j$. Now the proof of the first statement follows immediately from Proposition 2.4.1, commutation relations (2.1.1) and the definition of the isomorphism $\psi_{\{n_{ij}\}}$.

The second assertion is a consequence of (2.3.4).

□

The realizations $U_h^s(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are related to quantizations of some nonstandard bialgebra structures on $\mathfrak{g}$. At the quantum level changing bialgebra structure corresponds to the so–called Drinfeld twist. The relevant class of such twists described in the following proposition.

Proposition 2.6.3. *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra over a commutative ring with multiplication μ, unit $\imath$, comultiplication Δ, counit ε and antipode S. Let $\mathcal{F}$ be an invertible element of $A \otimes A$ such that*

$$\begin{gathered}\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}), \\ (\varepsilon \otimes id)(\mathcal{F}) = (id \otimes \varepsilon)(\mathcal{F}) = 1.\end{gathered} \tag{2.6.6}$$

Then, $v = \mu(id \otimes S)(\mathcal{F})$ is an invertible element of A with

$$v^{-1} = \mu(S \otimes id)(\mathcal{F}^{-1}).$$

Moreover, if we define $\Delta^{\mathcal{F}} : A \to A \otimes A$ and $S^{\mathcal{F}} : A \to A$ by

$$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1},\quad S^{\mathcal{F}}(a) = vS(a)v^{-1},$$

then $(A, \mu, \imath, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a Hopf algebra denoted by $A^{\mathcal{F}}$ and called the twist of A by $\mathcal{F}$.

Corollary 2.6.4. *Suppose that A and $\mathcal{F}$ are as in Proposition 2.6.3, but assume in addition that A is quasitriangular with universal R–matrix $\mathcal{R}$. Then $A^{\mathcal{F}}$ is quasitriangular with universal R–matrix*

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{2.6.7}$$

where $\mathcal{F}_{21} = \sigma\mathcal{F}$.

Let $P_{\mathfrak{h}'}$ be the orthogonal projection operator onto $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the bilinear form on $\mathfrak{h}$. Equip $U_h^s(\mathfrak{g})$ with the comultiplication Δ_s given by

$$\Delta_s(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_s(e_i) = e_i \otimes e^{-hd_iH_i} + e^{h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i} \otimes e_i,\quad \Delta_s(f_i) = f_i \otimes 1 + e^{-h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i + hd_iH_i} \otimes f_i,$$

the antipode $S_s(x)$ given by

$$S_s(e_i) = -e^{-h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i} e_i e^{hd_iH_i},\ S_s(f_i) = -e^{h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i - hd_iH_i} f_i,\ S_s(H_i) = -H_i,$$

and counit defined by

$$\varepsilon_s(H_i) = \varepsilon_s(X_i^\pm) = 0.$$

The comultiplication Δ_s is obtained from the standard comultiplication by a Drinfeld twist. Namely, let

$$\mathcal{F} = exp(-h\sum_{i,j=1}^{l}\frac{n_{ij}}{d_i}Y_i \otimes Y_j) \in U_h(\mathfrak{h}) \otimes U_h(\mathfrak{h}). \tag{2.6.8}$$

Then

$$\Delta_s(a) = (\psi^{-1}_{\{n_{ij}\}} \otimes \psi^{-1}_{\{n_{ij}\}})\mathcal{F}\Delta_h(\psi_{\{n_{ij}\}}(a))\mathcal{F}^{-1}. \tag{2.6.9}$$

Note that the Hopf algebra $U_h^s(\mathfrak{g})$ is a quantization of the bialgebra structure on $\mathfrak{g}$ defined by the cocycle

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)2r^s_{\pm}, \quad r^s_{\pm} \in \mathfrak{g} \otimes \mathfrak{g}, \tag{2.6.10}$$

where $r^s_{\pm} = r_{\pm} + \frac{1}{2}\sum_{i=1}^{l}\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i$, and $r_{\pm}$ is given by (2.1.3).

$U_h^s(\mathfrak{g})$ is a quasitriangular topological Hopf algebra with the universal R–matrix $\mathcal{R}^s = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$,

$$\begin{array}{l}\mathcal{R}^s = \prod_{\beta} exp_{q_\beta}[(1-q_\beta^{-2})f_\beta \otimes e_\beta e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee}]\times \\ exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) - \sum_{i=1}^{l}\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i)\right],\end{array} \tag{2.6.11}$$

where the order of the terms is such that the α–term appears to the left of the β–term if $\alpha < \beta$ with respect to the normal ordering of Δ_+ with the help of which the quantum root vectors e_β, f_β are defined in Proposition 2.6.2.

Similarly to (2.2.8) one obtains that for a reduced decomposition $w = s_{i_1}\ldots s_{i_k}$ and $T_w = T_{i_1}\ldots T_{i_k}$ only depending on w one has from (2.2.8) and (2.6.9)

$$\Delta_s(T_w) = \prod_{p=1}^{k}\theta^s_{\beta_p}q^{\sum_{i=1}^{l}(-Y_i\otimes KH_i + T_wY_i\otimes T_wKH_i)}T_w \otimes T_w = \tag{2.6.12}$$

$$= T_w \otimes T_w q^{\sum_{i=1}^{l}(-T_{w^{-1}}Y_i\otimes T_{w^{-1}}KH_i + Y_i\otimes KH_i)}\prod_{p=1}^{k}\overline{\theta}^s_{\beta'_p},$$

where for $p = 1, \ldots, k$

$$e_{\beta_p} = \psi^{-1}_{\{n_{ij}\}}(X^+_{\beta_p}e^{hK\beta_p^\vee}), f_\beta = \psi^{-1}_{\{n_{ij}\}}(e^{-hK\beta_p^\vee}X^-_{\beta_p}), \beta_p = s_{i_1}\ldots s_{i_{p-1}}\alpha_{i_p},$$

$$X^\pm_{\beta_p} = T_{i_1}\ldots T_{i_{p-1}}X^\pm_{i_p},\ \overline{X}^\pm_{\beta'_p} = T^{-1}_{i_k}\ldots T^{-1}_{i_{p+1}}X^\pm_{i_p},$$

$$\overline{e}_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}(\overline{X}^+_{\beta'_p}e^{hK{\beta'_p}^\vee}), \overline{f}_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}(e^{-hK{\beta'_p}^\vee}\overline{X}^-_{\beta'_p}),\ K_{\beta'_p} = T^{-1}_{i_k}\ldots T^{-1}_{i_{p+1}}K_{i_p}, \beta'_p = s_{i_k}\ldots s_{i_{p+1}}\alpha_{i_p}$$

$$\theta^s_{\beta_p} = exp_{q_{\beta_p}}[(1-q^{-2}_{\beta_p})e_{\beta_p}e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_p^\vee} \otimes f_{\beta_p}],\ \overline{\theta}^s_{\beta'_p} = exp_{q_{\beta'_p}}[(1-q^{-2}_{\beta'_p})K^{-1}_{\beta'_p}e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}{\beta'_p}^\vee}\overline{f}_{\beta'_p} \otimes \overline{e}_{\beta'_p}K_{\beta'_p}].$$

In the same way, for $\overline{T}_w = T^{-1}_{i_1}\ldots T^{-1}_{i_k}$ only depending on w one has from (2.2.9) and (2.6.9)

$$\Delta_s(\overline{T}_w) = \prod_{p=1}^{k}{\overline{\theta}^s_{\beta_p}}'q^{\sum_{i=1}^{l}(-Y_i\otimes KH_i + T_wY_i\otimes T_wKH_i)}\overline{T}_w \otimes \overline{T}_w = \tag{2.6.13}$$

$$= \overline{T}_w \otimes \overline{T}_w q^{\sum_{i=1}^{l}(-T_{w^{-1}}Y_i\otimes T_{w^{-1}}KH_i + Y_i\otimes KH_i)}\prod_{p=1}^{k}{\theta^s_{\beta'_p}}',$$

where for $p = 1, \ldots, k$

$$\overline{e}_{\beta_p} = \psi^{-1}_{\{n_{ij}\}}(\overline{X}^+_{\beta_p}e^{hK\beta_p^\vee}), \overline{f}_{\beta_p} = \psi^{-1}_{\{n_{ij}\}}(e^{-hK\beta_p^\vee}\overline{X}^-_{\beta_p}), \overline{K}_{\beta_p} = T^{-1}_{i_1}\ldots T^{-1}_{i_{p-1}}K_{i_p},$$

$$\overline{X}^\pm_{\beta_p} = T^{-1}_{i_1}\ldots T^{-1}_{i_{p-1}}X^\pm_{i_p}, \beta_p = s_{i_1}\ldots s_{i_{p-1}}\alpha_{i_p},$$

$$e'_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}({X^+_{\beta'_p}}'e^{hK{\beta'_p}^\vee}), f'_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}(e^{-hK{\beta'_p}^\vee}{X^-_{\beta'_p}}'), \beta'_p = s_{i_k}\ldots s_{i_{p+1}}\alpha_{i_p}, {X^\pm_{\beta'_p}}' = T_{i_k}\ldots T_{i_{p+1}}X^\pm_{i_p},$$

$${\theta^s_{\beta'_p}}' = exp_{q^{-1}_{\beta'_p}}[(1-q^2_{\beta'_p})e'_{\beta'_p}e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}{\beta'_p}^\vee} \otimes f'_{\beta'_p}],\ {\overline{\theta}^s_{\beta_p}}' = exp_{q^{-1}_{\beta_p}}[(1-q^2_{\beta_p})\overline{K}^{-1}_{\beta_p}e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_p^\vee}\overline{f}_{\beta_p} \otimes \overline{e}_{\beta_p}\overline{K}_{\beta_p}].$$

2.7 Some forms and specializations of quantum groups

In order to define q-W–algebras we shall actually need not the algebras $U_h^s(\mathfrak{g})$ themselves but some their forms defined over certain rings. They are similar to the rational form and the restricted integral form for the standard quantum group $U_h(\mathfrak{g})$. The motivations of the definitions given below will be clear in Section 3.2. The results below are slight modifications of similar statements for $U_h(\mathfrak{g})$.

We start with a very important technical lemma which will play the key role in the definition of q-W–algebras. Below we keep the notation introduced in Section 1.2.

Let $s \in W$ be an element of the Weyl group. Recall that s can be represented as a product of two involutions, $s = s^1 s^2$, where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, and the roots $\gamma_1, \ldots, \gamma_{l'}$ form a basis of a subspace $\mathfrak{h}'^* \subset \mathfrak{h}^*$ on which s acts without fixed points. We shall study the matrix elements of the Cayley transform of the restriction of s to $\mathfrak{h}'^*$ with respect to this basis.

Lemma 2.7.1. *Let $P_{\mathfrak{h}'^*}$ be the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to the bilinear form. Then the matrix elements of the operator $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ in the basis $\gamma_1, \ldots, \gamma_{l'}$ are of the form*

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i, \gamma_j\right) = \varepsilon_{ij}(\gamma_i, \gamma_j), \tag{2.7.1}$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

Proof. First we calculate the matrix of the element s with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$. We obtain this matrix in the form of the Gauss decomposition of the operator $s : \mathfrak{h}'^* \to \mathfrak{h}'^*$.

Let $z_i = s\gamma_i$. Recall that $s_{\gamma_i}(\gamma_j) = \gamma_j - A_{ij}\gamma_i$, $A_{ij} = (\gamma_i^\vee, \gamma_j)$. Using this definition the elements z_i may be represented as

$$z_i = y_i - \sum_{k \geq i} A_{ki} y_k,$$

where

$$y_i = s_{\gamma_1} \ldots s_{\gamma_{i-1}} \gamma_i. \tag{2.7.2}$$

Using the matrix notation we can rewrite the last formula as follows

$$z_i = (I+V)_{ki} y_k, \qquad \text{where } V_{ki} = \begin{cases} A_{ki} & k \geq i \\ 0 & k < i \end{cases} \tag{2.7.3}$$

To calculate the matrix of the operator $s : \mathfrak{h}'^* \to \mathfrak{h}'^*$ with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$ we have to express the elements y_i via $\gamma_1, \ldots, \gamma_{l'}$. Applying the definition of reflections to (2.7.2) we can pull out the element γ_i to the right,

$$y_i = \gamma_i - \sum_{k<i} A_{ki} y_k.$$

Therefore

$$\gamma_i = (I+U)_{ki} y_k \ , \ \text{where } U_{ki} = \begin{cases} A_{ki} & k < i \\ 0 & k \geq i \end{cases}$$

Thus

$$y_k = (I+U)^{-1}_{jk} \gamma_j. \tag{2.7.4}$$

Summarizing (2.7.4) and (2.7.3) we obtain

$$s\gamma_i = \left((I+U)^{-1}(I-V)\right)_{ki} \gamma_k. \tag{2.7.5}$$

This implies

$$\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i = \left(\frac{2I+U-V}{U+V}\right)_{ki} \gamma_k. \tag{2.7.6}$$

Observe that $(U+V)_{ki}=A_{ki}$ and $(2I+U-V)_{ij}=-A_{ij}\varepsilon_{ij}$. Substituting these expressions into (2.7.6) we get

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j\right)=-(A^{-1})_{kp}\varepsilon_{pi}A_{pi}(\gamma_j,\gamma_k)=\varepsilon_{ij}(\gamma_i,\gamma_j). \quad (2.7.7)$$

This completes the proof of the lemma.

□

Let γ_i^*, $i=1,\ldots,l'$ be the basis of $\mathfrak{h}'^*$ dual to γ_i, $i=1,\ldots,l'$ with respect to the restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{h}'^*$. Since the numbers (γ_i,γ_j) are integer each element γ_i^* has the form $\gamma_i^*=\sum_{j=1}^{l'}m_{ij}\gamma_j$, where $m_{ij}\in\mathbb{Q}$. Therefore by Lemma 2.7.1 and using for simple roots α_i the decomposition of the form $P_{\mathfrak{h}'^*}\alpha_i=\sum_{p=1}^{l'}(\alpha_i,\gamma_p)\gamma_p^*=\sum_{p,q=1}^{l'}(\alpha_i,\gamma_p)m_{pq}\gamma_q$ we deduce that the numbers

$$p_{ij}=\frac{1}{2d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)= \quad (2.7.8)$$

$$=\frac{1}{2d_j}\sum_{k,l,p,q=1}^{l'}(\gamma_k,\alpha_i)(\gamma_l,\alpha_j)\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_p,\gamma_q\right)m_{kp}m_{lq},\ i,j=1,\ldots,l$$

are rational, $p_{ij}\in\mathbb{Q}$, as all factors in the products in the sum in the right hand side are rational. Denote by d an integer number divisible by all the denominators of the rational numbers p_{ij}, $i,j=1,\ldots,l$.

Let $r\in\mathbb{N}$ be such that $a_{ij}^{-1}\in\frac{1}{r}\mathbb{Z}$, $i,j=1,\ldots,l$. Let $U_q^s(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{dr^2}})$-algebra generated by the elements $e_i,f_i,L_i^{\pm1},t_i^{\pm1}$, $i=1,\ldots,l$ with the same relations as the relations in $U_h^s(\mathfrak{g})$ for the generators denoted by the same symbols, where we assume that $t_i^{\pm1}=\exp(\pm\frac{h\kappa}{d}Y_i)$. The coefficients of these relations indeed belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$, where $q^{\frac{1}{dr^2}}=e^{h\frac{1}{dr^2}}$.

Let $U_q(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{dr^2}})$-algebra generated by the elements $X_i^{\pm},L_i^{\pm1},t_i^{\pm1}$, $i=1,\ldots,l$ subject to the same relations as the relations in $U_h(\mathfrak{g})$ for the generators denoted by the same symbols, where we assume that $t_i^{\pm1}=\exp(\pm\frac{h\kappa}{d}Y_i)$. The coefficients of these relations indeed belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$, where $q^{\frac{1}{dr^2}}=e^{h\frac{1}{dr^2}}$.

Note that by the choice of d we have $q^{c_{ij}}\in\mathbb{C}[q^{\frac{\kappa}{d}},q^{-\frac{\kappa}{d}}]$.

The second form of $U_h^s(\mathfrak{g})$ is a subalgebra $U_{\mathcal{A}}^s(\mathfrak{g})$ in $U_q^s(\mathfrak{g})$ over the ring $\mathcal{A}=\mathcal{P}[q^{\frac{1}{dr^2}},q^{-\frac{1}{dr^2}}]$. $U_{\mathcal{A}}^s(\mathfrak{g})$ is the subalgebra in $U_q^s(\mathfrak{g})$ generated over $\mathcal{A}$ by the elements

$$L_i^{\pm1},t_i^{\pm1},\ \frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\ e_i,\ f_i,\ i=1,\ldots,l.$$

Denote also by $U_{\mathcal{A}}(\mathfrak{g})$ the subalgebra in $U_q(\mathfrak{g})$ generated over $\mathcal{A}$ by the elements

$$L_i^{\pm1},t_i^{\pm1},\ \frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\ X_i^{\pm},\ i=1,\ldots,l.$$

For the solution $n_{ij}=\frac{1}{2d_j}c_{ij}$ to equations (2.6.2) the root vectors e_β,f_β belong to all the above introduced specializations of $U_h(\mathfrak{g})$.

For any normal ordering of Δ_+ we denote $e_\beta^{(k)}=\frac{e_\beta^k}{[k]_{q_\alpha}!}$, $f_\beta^{(k)}=\frac{f_\beta^k}{[k]_{q_\alpha}!}$, where e_β,f_β are the corresponding quantum root vectors from Proposition 2.6.2. Let $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ be the subalgebra in $U_q^s(\mathfrak{g})$ generated over $\mathcal{B}=\mathbb{C}[q^{\frac{1}{dr^2}},q^{-\frac{1}{dr^2}}]$ by the elements

$$L_i^{\pm1},t_i^{\pm1},\ e_i^{(k)},\ f_i^{(k)},\ i=1,\ldots,l,k\geq1.$$

Denote also by $U_{\mathcal{B}}^{res}(\mathfrak{g})$ the subalgebra in $U_q(\mathfrak{g})$ generated over $\mathcal{B}$ by the elements

$$L_i^{\pm1},t_i^{\pm1},\ (X_i^{\pm})^{(k)},\ i=1,\ldots,l,k\geq1.$$

Let $\varepsilon\in\mathbb{C}^*$. Fix a root of ε of degree r^2d, $\varepsilon^{\frac{1}{dr^2}}$ and if $\varepsilon=1$ put $\varepsilon^{\frac{1}{r^2d}}=1$. Then we define the specialization $U_\varepsilon^s(\mathfrak{g})$ of $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_\varepsilon^s(\mathfrak{g})=U_{\mathcal{A}}^s(\mathfrak{g})/(q^{\frac{1}{dr^2}}-\varepsilon^{\frac{1}{dr^2}})U_{\mathcal{A}}^s(\mathfrak{g})$, and

$$U_\varepsilon^{s,res}(\mathfrak{g})=U_{\mathcal{B}}^{s,res}(\mathfrak{g})/(q^{\frac{1}{dr^2}}-\varepsilon^{\frac{1}{dr^2}})U_{\mathcal{B}}^{s,res}(\mathfrak{g})$$

$U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$, $U_{\varepsilon}^{s,res}(\mathfrak{g})$ and $U_{\varepsilon}^s(\mathfrak{g})$ are Hopf algebras with the comultiplication given on generators by the same formulas as in $U_h^s(\mathfrak{g})$ with $q^{\frac{1}{dr^2}} = e^{h\frac{1}{dr^2}}$.

The elements t_i and L_i are central in the algebra $U_1^s(\mathfrak{g})$, and the quotient of $U_1^s(\mathfrak{g})$ by the two–sided ideal generated by $t_i - 1$ and $L_i - 1$ is isomorphic to $U(\mathfrak{g})$. Note that none of the specializations of $U_h^s(\mathfrak{g})$ introduced above is quasitriangular.

The algebra isomorphism $\psi_{\{n_{ij}\}}$ with $n_{ij} = \frac{1}{2d_j}c_{ij}$ induces isomorphisms of $U_h(\mathfrak{g})$ and $U_h^s(\mathfrak{g})$ and of the forms and specializations of $U_h(\mathfrak{g})$ and $U_h^s(\mathfrak{g})$ with the superscript s defined above and of their counterparts with the superscript s dropped. We shall always identify them using these isomorphisms.

Note that the structure constants in the commutation relations for $U_{\mathcal{A}}^s(\mathfrak{g}) = U_{\mathcal{A}}(\mathfrak{g})$ actually belong to $\mathcal{P}[q^{\frac{\kappa}{d}}, q^{-\frac{\kappa}{d}}]$, so if κ is divisible by d the specialization $U_{\varepsilon}^s(\mathfrak{g})$ actually depends on $\varepsilon \in \mathbb{C}^*$ but not on its root $\varepsilon^{\frac{1}{dr^2}}$. As we shall see below one can define an action of the universal R-matrix $\mathcal{R}^s$ on tensor products of finite rank $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–modules. This action will play a crucial role in subsequent considerations.

$U_{\mathcal{A}}(\mathfrak{g})$, $U_{\mathcal{B}}^{res}(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ can be also regarded as subalgebras of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$, and $U_q^{res}(\mathfrak{g})$ can be regarded as a $\mathbb{C}[q, q^{-1}]$–subalgebra in $U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \simeq U_{\mathcal{B}}^{res}(\mathfrak{g})$.

Denote by $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$ the subalgebras of $U_q^s(\mathfrak{g})$ generated by the e_i, f_i and by the t_i, L_i, respectively, and let $U_q^s(\mathfrak{b}_\pm)$ be the subalgebra in $U_q^s(\mathfrak{g})$ generated by $U_q^s(\mathfrak{n}_\pm)$ and by $U_q^s(\mathfrak{h})$, $U_q^s(\mathfrak{b}_\pm) = U_q^s(\mathfrak{n}_+)U_q^s(\mathfrak{h})$.

Let $U_{\mathcal{A}}^s(\mathfrak{n}_+)$, $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ ($U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+)$, $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$) be the subalgebras of $U_{\mathcal{A}}^s(\mathfrak{g})$ ($U_{\mathcal{B}}^{s,res}(\mathfrak{g})$) generated by the e_i and by the f_i, $i = 1, \ldots, l$ (by the $e_i^{(r)}$ and by the $f_i^{(r)}$, $i = 1, \ldots, l$, $r \geq 0$), respectively, and $U_{\mathcal{A}}(\mathfrak{h}) = U_{\mathcal{A}}^s(\mathfrak{h})$ the subalgebra in $U_{\mathcal{A}}(\mathfrak{g}) = U_{\mathcal{A}}^s(\mathfrak{g})$ generated by t_i, L_i, $i = 1, \ldots, l$.

The elements

$$\left[\begin{array}{c} K_i; c \\ r \end{array} \right]_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}}, \ i = 1, \ldots, l, \ c \in \mathbb{Z}, \ r \in \mathbb{N}$$

belong to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Denote by $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$ the subalgebra of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ generated by those elements and by $t_i^{\pm 1}, L_i^{\pm 1}$, $i = 1, \ldots, l$.

Define an algebra antiautomorphism ω of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$ by

$$\omega(X_i^{\pm}) = X_i^{\mp}, \omega(H_i) = H_i, \omega(h) = -h.$$

It commutes with the braid group action and for any $\alpha \in \Delta_+$ satisfies

$$\omega(X_\alpha^+) = X_\alpha^-, \omega(f_\alpha) = e_\alpha, \omega(e_\alpha) = f_\alpha.$$

Define also an algebra antiautomorphism ω_0 of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$ by

$$\omega_0(e_i) = e_i, \omega_0(f_i) = f_i, \omega_0(H_i) = -H_i, \omega_0(h) = -h.$$

It satisfies

$$\omega_0(X_i^{\pm}) = q_i^{\mp n_{ii}} X_i^{\pm} = X_i^{\pm}, \omega\omega_0 = \omega_0\omega,$$

$$\omega_0(T_i X_j^{\pm}) = (-1)^{a_{ij}} q_i^{\pm a_{ij}(n_{ii}-1)} T_i(\omega_0 X_j^{\pm}) = (-1)^{a_{ij}} q_i^{\mp a_{ij}} T_i(\omega_0 X_j^{\pm}), i \neq j,$$

$$\omega_0(T_i X_i^+) = q_i^{-2} q_i^{2n_{ii}} T_i(\omega_0 X_i^+) = q_i^{-2} T_i(\omega_0 X_i^+), \omega_0(T_i X_i^-) = q_i^2 T_i(\omega_0 X_i^-).$$

As a consequence we obtain that if X is a homogeneous polynomial in quantum simple root vectors then

$$T_i(\omega_0 X) = c_X \omega_0(T_i X),$$

where $c_X = \epsilon p$, $\epsilon = \pm 1$, $p \in q^{\mathbb{Z}}$, and hence

$$\omega_0(e_\alpha) = c_\alpha e_\alpha, \tag{2.7.9}$$

where $c_\alpha = \epsilon_\alpha p_\alpha$, $\epsilon_\alpha = \pm 1$, $p_\alpha \in q^{\mathbb{Z}}$. We also have

$$\omega_0(f_\alpha) = \omega_0\omega(e_\alpha) = \omega\omega_0(e_\alpha) = \omega(c_\alpha e_\alpha) = c_\alpha^{-1}\omega(e_\alpha) = c_\alpha^{-1} f_\alpha \tag{2.7.10}$$

Note also that ω_0 is a coalgebra homomorphism.

The antiautomorphisms ω and ω_0 give rise to antiautomorphisms of $U_q^s(\mathfrak{g})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and $U_{\mathcal{A}}^s(\mathfrak{g})$ which we denote by the same letters.

Using the root vectors e_β and f_β we can construct bases for the algebras introduced above.

Lemma 2.7.2. *Fix a normal ordering of the system of positive roots Δ_+ and let e_β, f_β be the corresponding quantum root vectors defined in Proposition 2.6.2. Then the elements f_β satisfy the following commutation relations*

$$f_\alpha f_\beta - q^{(\alpha,\beta)+\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)} f_\beta f_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(p_1,\ldots,p_n) f_{\delta_1}^{(p_1)} f_{\delta_2}^{(p_2)} \ldots f_{\delta_n}^{(p_n)} = \qquad (2.7.11)$$

$$= \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C'(p_1,\ldots,p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n},\ \alpha<\beta,$$

where $C(p_1,\ldots,p_n)\in\mathcal{B}$, $C'(p_1,\ldots,p_n)\in\mathcal{A}$.

The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1}\ldots e_{\beta_D}^{r_D}$, $f^{\mathbf{t}} = f_{\beta_D}^{t_D}\ldots f_{\beta_1}^{t_1}$, for $\mathbf{r}=(r_1,\ldots r_D)$, $\mathbf{t}=(t_1,\ldots t_D)\in\mathbb{N}^D$, form bases of $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$, respectively, and the multiplication defines an isomorphism of $\mathbb{C}(q^{\frac{1}{dr^2}})$–modules:

$$U_q^s(\mathfrak{n}_-)\otimes U_q^s(\mathfrak{h})\otimes U_q^s(\mathfrak{n}_+)\rightarrow U_q^s(\mathfrak{g}).$$

The elements $e^{\mathbf{r}}$, $f^{\mathbf{t}}$ ($e^{(\mathbf{r})} = e_{\beta_1}^{(r_1)}\ldots e_{\beta_D}^{(r_D)}$, $f^{(\mathbf{t})} = f_{\beta_D}^{(t_D)}\ldots f_{\beta_1}^{(t_1)}$) for $\mathbf{r}$, $\mathbf{t}\in\mathbb{N}^D$ form bases of $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$ ($U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+), U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$), respectively.

The multiplication defines an isomorphisms of $\mathcal{B}$–modules:

$$U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)\otimes U_{\mathcal{B}}^{s,res}(\mathfrak{h})\otimes U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+)\rightarrow U_{\mathcal{B}}^{s,res}(\mathfrak{g}).$$

Let $[\alpha,\beta]=\{\beta_p,\ldots,\beta_q\}$ be a minimal segment in Δ_+, $U_{\mathcal{A}}^s([\alpha,\beta])$, $U_{\mathcal{A}}^s([-\alpha,-\beta])$ ($U_{\mathcal{B}}^{s,res}([\alpha,\beta])$, $U_{\mathcal{B}}^{s,res}([-\alpha,-\beta])$) the $\mathcal{A}(\mathcal{B})$–subalgebras of $U_h^s(\mathfrak{g})$ generated by the e_γ and by the f_γ, $\gamma\in[\alpha,\beta]$ (by the $(e_\gamma)^{(r)}$ and by the $(f_\gamma)^{(r)}$, $\gamma\in[\alpha,\beta]$, $r\in\mathbb{N}$), respectively. Then the elements $(e_{\beta_p})^{r_p}\ldots(e_{\beta_q})^{r_q}$, $(f_{\beta_q})^{r_q}\ldots(f_{\beta_p})^{r_p}$ ($(e_{\beta_p})^{(r_p)}\ldots(e_{\beta_q})^{(r_q)}$, $(f_{\beta_q})^{(r_q)}\ldots(f_{\beta_p})^{(r_p)}$), $r_i\in\mathbb{N}$ form bases of $U_{\mathcal{A}}^s([\alpha,\beta])$, $U_{\mathcal{A}}^s([-\alpha,-\beta])$ ($U_{\mathcal{B}}^{s,res}([\alpha,\beta])$, $U_{\mathcal{B}}^{s,res}([-\alpha,-\beta])$), respectively.

The elements $f^{\mathbf{t}}$ ($f^{(\mathbf{t})} = f_{\beta_D}^{(t_D)}\ldots f_{\beta_1}^{(t_1)}$) for $\mathbf{t}\in\mathbb{N}^D$ with $t_i>0$ for at least one $i\geq p$ form a basis in the right ideal of $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ ($U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$) generated by f_γ, $\gamma\in[\beta_p,\beta_D]$ (by $(f_\gamma)^{(r)}$, $\gamma\in[\beta_p,\beta_D]$, $r>0$).

Proof. Commutation relations (2.7.11) follow from commutation relations (2.3.3), (2.1.1), (2.1.2), Proposition 2.4.1, the definition of the elements e_β, f_β and the definition of the isomorphism $\psi_{\{n_{ij}\}}$.

The other statements of this lemma, except for the last one, follow straightforwardly from Lemma 2.4.2 and Propositions 2.6.1 and 2.6.2.

For the last statement, using commutation relations (2.7.11) we can represent any element of the right ideal of $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ generated by f_γ, $\gamma\in[\beta_p,\beta_D]$ as an $\mathcal{A}$–linear combination of the elements $f^{\mathbf{t}}$ for $\mathbf{t}\in\mathbb{N}^D$ with $t_i>0$ for at least one $i\geq p$. This presentation is unique by the Poincaré–Birkhoff–Witt decomposition for $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ stated above.

Note that a similar result holds for the algebra $U_q^s(\mathfrak{n}_-)=U_{\mathcal{A}}^s(\mathfrak{n}_-)\otimes_{\mathcal{A}}\mathbb{C}(q^{\frac{1}{dr^2}})$ for the same reasons.

We can apply it to represent any element of the right ideal of $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)\subset U_q^s(\mathfrak{n}_-)$ generated by $(f_\gamma)^{(r)}$, $\gamma\in[\beta_p,\beta_D]$, $r>0$ as a $\mathbb{C}(q^{\frac{1}{dr^2}})$–linear combination of the elements $f^{(\mathbf{t})}$ for $\mathbf{t}\in\mathbb{N}^D$ with $t_i>0$ for at least one $i\geq p$. This presentation is unique and by the uniqueness of the Poincaré–Birkhoff–Witt decomposition for $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ stated above the coefficients in this decomposition belong to $\mathcal{B}$. This completes the proof.

□

A basis for $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$ is a little bit more difficult to describe. We do not need its explicit description.

Remark 2.7.3. *Applying the antiautomorphism ω_0 to the elements of the bases constructed in Lemma 2.7.2 and using (2.7.9) and (2.7.10) we obtain other bases of similar types where the order of the quantum root vectors in the products defining the elements of the bases is reversed.*

By specializing the above constructed bases for $q^{\frac{1}{rd^2}}=\varepsilon^{\frac{1}{rd^2}}$ one can obtain similar bases and similar subalgebras for $U_\varepsilon^s(\mathfrak{g})$ and $U_\varepsilon^{s,res}(\mathfrak{g})$.

Using formulas (2.5.6) and (2.6.9) one can also find that

$$\Delta_s(f_{\beta_k}) = \widetilde{\mathcal{R}}^s_{<\beta_k}(e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_k^\vee+h\beta_k^\vee}\otimes f_{\beta_k}+f_{\beta_k}\otimes 1)(\widetilde{\mathcal{R}}^s_{<\beta_k})^{-1} = \qquad (2.7.12)$$

$$= G_{\beta_k}^{-1}\otimes f_{\beta_k}+f_{\beta_k}\otimes 1+\sum_i y_i\otimes x_i,$$

where

$$G_\beta = e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee - h\beta^\vee}, y_i = e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\gamma^\vee_{x_i}+h\gamma^\vee_{x_i}}\overline{y}_i,$$
$$\overline{y}_i \in U_{\mathcal{A}}^s([-\beta_{k+1},-\beta_D]) \cap U_{\mathcal{B}}^{s,res}([-\beta_{k+1},-\beta_D]),$$
$$x_i \in U_{\mathcal{A}}^s([-\beta_1,-\beta_{k-1}]) \cap U_{\mathcal{B}}^{s,res}([-\beta_1,-\beta_{k-1}]),$$

$\overline{y}_i, x_i$ belong to weight components and have non-zero weights, γ_{x_i} is the weight of x_i,

$$\widetilde{\mathcal{R}}^s_{<\beta_k} = \widetilde{\mathcal{R}}^s_{\beta_1}\ldots\widetilde{\mathcal{R}}^s_{\beta_{k-1}},\ \widetilde{\mathcal{R}}^s_{\beta_r} = exp_{q_{\beta_r}}[(1-q_{\beta_r}^{-2})e_{\beta_r}e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} \otimes f_{\beta_r}],$$

and

$$(\widetilde{\mathcal{R}}^s_{<\beta_k})^{-1} = (\widetilde{\mathcal{R}}^s_{\beta_{k-1}})^{-1}\ldots(\widetilde{\mathcal{R}}^s_{\beta_1})^{-1},\ (\widetilde{\mathcal{R}}^s_{\beta_r})^{-1} = exp_{q_{\beta_r}^{-1}}[(1-q_{\beta_r}^{2})e_{\beta_r}e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} \otimes f_{\beta_r}].$$

From (2.7.12) we also obtain

$$\Delta_s(f_{\beta_k}^{(n)}) = \frac{1}{[n]_{q_{\beta_k}}!}\widetilde{\mathcal{R}}^s_{<\beta_k}(G_{\beta_k}^{-1}\otimes f_{\beta_k} + f_{\beta_k}\otimes 1)^n(\widetilde{\mathcal{R}}^s_{<\beta_k})^{-1} = \tag{2.7.13}$$

$$= \widetilde{\mathcal{R}}^s_{<\beta_k}(\sum_{k=0}^n q_{\beta_k}^{k(n-k)}G_{\beta_k}^{-k}f_{\beta_k}^{(n-k)}\otimes f_{\beta_k}^{(k)})(\widetilde{\mathcal{R}}^s_{<\beta_k})^{-1} =$$

$$= \sum_{k=0}^n q_{\beta_k}^{k(n-k)}G_{\beta_k}^{-k}f_{\beta_k}^{(n-k)}\otimes f_{\beta_k}^{(k)} + \sum_i y_i^{(n)}\otimes x_i^{(n)},$$

where

$$y_i^{(n)} = e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\gamma^\vee_{x_i^{(n)}}+h\gamma^\vee_{x_i^{(n)}}}\overline{y}_i^{(n)},$$

$\overline{y}_i^{(n)} \in I_k^>$, $x_i^{(n)} \in I_k^<$ belong to weight components and have non-zero weights, $\gamma_{x_i^{(n)}}$ is the weight of $x_i^{(n)}$, $I_k^>$ is the ideal in $U_{\mathcal{B}}^{s,res}([-\beta_k,-\beta_D])$ generated by $f_{\beta_i}^{(p)}$, $i=k+1,\ldots,D$, $p>0$, and $I_k^<$ is the ideal in $U_{\mathcal{B}}^{s,res}([-\beta_1,-\beta_k])$ generated by $f_{\beta_i}^{(p)}$, $i=1,\ldots,k-1$, $p>0$.

Similarly

$$\Delta_s(e_{\beta_k}) = \widetilde{\mathcal{R}}^s_{<\beta_k}(e_{\beta_k}\otimes e^{-h\beta_k^\vee} + e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_k^\vee}\otimes e_{\beta_k})(\widetilde{\mathcal{R}}^s_{<\beta_k})^{-1} = \tag{2.7.14}$$
$$= e_{\beta_k}\otimes e^{-h\beta_k^\vee} + e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_k^\vee}\otimes e_{\beta_k} + \sum_i x_i'\otimes y_i',$$

where

$$y_i' \in U_{\mathcal{A}}^s([\beta_{k+1},\beta_D])U_{\mathcal{A}}^s(\mathfrak{h}) \cap U_{\mathcal{B}}^{s,res}([\beta_{k+1},\beta_D])U_{\mathcal{B}}^{s,res}(\mathfrak{h}),$$
$$x_i' \in U_{\mathcal{A}}^s([\beta_1,\beta_{k-1}])U_{\mathcal{A}}^s(\mathfrak{h}) \cap U_{\mathcal{B}}^{s,res}([\beta_1,\beta_{k-1}])U_{\mathcal{B}}^{s,res}(\mathfrak{h}),$$

and y_i', x_i' belong to weight components and have non-zero weights.

From these formulas we deduce

$$S_s(f_\beta) = -G_\beta f_\beta - \sum_i S_s(y_i)x_i = -G_\beta(f_\beta + \sum_i y_iS_s(x_i)), \tag{2.7.15}$$

$$S_s^{-1}(f_\beta) = -(f_\beta + \sum_i S_s^{-1}(x_i)y_i)G_\beta = -f_\beta G_\beta - \sum_i x_iS_s^{-1}(y_i). \tag{2.7.16}$$

We also have

$$\omega_0S_s^{-1}(f_\beta) = -G_\beta(\omega_0 f_\beta + \sum_i \omega_0(y_i)\omega_0S_s^{-1}(x_i)) =$$

$$= -G_\beta\omega_0 f_\beta - \sum_i \omega_0S_s^{-1}(y_i)\omega_0(x_i),$$

$$\omega_0 f_\beta = -G_\beta^{-1}(\omega_0S_s^{-1}(f_\beta) + \sum_i \omega_0S_s^{-1}(y_i)).$$

As usual, one can define highest weight, Verma and finite-dimensional modules for all forms and specializations of the quantum group $U_h^s(\mathfrak{g})$ introduced above. We recall that by Propositions 6.5.5 and 6.5.7 in [18] $U_h(\mathfrak{g}) \simeq U(\mathfrak{g})[[h]]$, and this isomorphism restricts to the identity map on $U(\mathfrak{h})$ and induces a canonical isomorphism of the center of $U_h(\mathfrak{g})$ and of $Z(U(\mathfrak{g}))[[h]]$, where $Z(U(\mathfrak{g}))$ is the center of $U(\mathfrak{g})$. Therefore if V is a $U_h(\mathfrak{g})$–module topologically free and of finite rank over $\mathbb{C}[[h]]$ then $V_1 = V/hV$ is a finite-dimensional $U(\mathfrak{g})$–module which is completely reducible, and its irreducible components are highest weight irreducible finite-dimensional representations of $U(\mathfrak{g})$. Since, as observed in Section 6.5 B in [18], such modules have no non–trivial deformations one has $V \simeq V_1[[h]]$.

If V_1 is a highest weight irreducible representation of $U(\mathfrak{g})$ with highest weight $\lambda \in P_+$ we call the corresponding representation $V = V_\lambda$ a highest weight indecomposable representation of highest weight λ. V_λ is generated by a highest weight vector with respect to the action of $\mathfrak{h} \subset U_h(\mathfrak{h})$. All indecomposable $U_h(\mathfrak{g})$–modules topologically free and of finite rank over $\mathbb{C}[[h]]$ can be obtained this way.

Recall that $U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \simeq U_{\mathcal{B}}^{res}(\mathfrak{g})$ can be regarded as a subalgebra of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$. Let V be a $U_h(\mathfrak{g})$–module topologically free and of finite rank over $\mathbb{C}[[h]]$ (for brevity we shall call such modules finite rank $U_h(\mathfrak{g})$–modules). Then a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–module V^{res} is called a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in V if $V^{res} \otimes_{\mathcal{B}} \mathbb{C}[[h]] \simeq V$.

For any $U_h(\mathfrak{g})$–module V topologically free and of finite rank over $\mathbb{C}[[h]]$ a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice V^{res} exists. Indeed, by the above discussion it suffices to consider the case of indecomposable $V = V_\lambda$ generated by a highest weight vector v. In this case, similarly to the proof of Proposition 4.2 in [75] (see also Proposition 10.1.4 in [18]) one can show that $V_\lambda^{res} = U_{\mathcal{B}}^{s,res}(\mathfrak{g})v$ is a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice, V_λ^{res} is the direct sum of its intersections with the weight spaces of V_λ and each such intersection is a finitely generated free $\mathcal{B}$–module of finite rank. Moreover, using the arguments from the proof of Proposition 4.2 in Section 4.9 of [75] one can see that the last two properties hold for any $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in any topologically free $U_h(\mathfrak{g})$–module V of finite rank over $\mathbb{C}[[h]]$.

The specialization of V_λ^{res} at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ is a highest weight $U_\varepsilon^{s,res}(\mathfrak{g})$–module. In general this module is not irreducible even if V is indecomposable.

The R-matrix $\mathcal{R}^s$ acts in tensor products of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattices in $U_h(\mathfrak{g})$–modules topologically free and of finite rank over $\mathbb{C}[[h]]$. Namely, recalling (2.6.11) we can represent $\mathcal{R}^s$ in the form $\mathcal{R}^s = \mathcal{E}^s\mathcal{R}_0^s$, where

$$\mathcal{R}_0^s = exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) - \sum_{i=1}^{l}\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i)\right], \tag{2.7.17}$$

$$\mathcal{E}^s = \prod_{\beta}\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}}(1-q_\beta^{-2})^k f_\beta^{(k)} \otimes e_\beta^k e^{-kh\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} =$$

$$= \prod_{\beta}\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}}(1-q_\beta^{-2})^k f_\beta^{k} \otimes e_\beta^{(k)} e^{-kh\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee},$$

and for $\beta = \sum_{i=1}^{l} c_i\alpha_i$

$$e^{-kh\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} = e^{-kh\sum_{i,j=1}^{l}\frac{c_{ij}}{d_j}c_iY_j} \in U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \cap U_{\mathcal{A}}^s(\mathfrak{h}).$$

We can define the action of $\mathcal{R}_0^s$ on tensor products of modules of the form V^{res}, where V is a highest weight finite rank $U_h(\mathfrak{g})$–module as follows. If V^{res}, W^{res} are two such modules and $v_\lambda \in V^{res}$, $w_\mu \in W^{res}$ are vectors of weights λ and μ then we define

$$\mathcal{R}_0^s v_\lambda \otimes w_\mu = q^{(\lambda,\mu)-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\mu,\lambda)} v_\lambda \otimes w_\mu,$$

and $q^{(\lambda,\mu)-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\mu,\lambda)} \in \mathcal{B}$.

Since for any $\beta \in \Delta_+$ and any module of the form V^{res}, where V is a finite rank $U_h(\mathfrak{g})$–module, $f_\beta^{(k)}$ and e_β^k belong to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and act as zero operators on $V_{\mathcal{B}}^{res}$ for k large enough, and for any k

$$e^{-kh\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} \in U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \cap U_{\mathcal{A}}^s(\mathfrak{h}),$$

$\mathcal{E}^s$ naturally acts on tensor products of such modules.

For two modules of the form V^{res}, W^{res}, where V and W are highest weight finite rank $U_h(\mathfrak{g})$–modules, we denote by R^{VW} the operator corresponding to the action of $\mathcal{R}^s$ in $V^{res} \otimes W^{res}$.

With this definition the identity

$$\Delta_s^{opp}(x) = \mathcal{R}^s\Delta_s(x)\mathcal{R}^{s\,-1}, x \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$$

still holds being evaluated in tensor products of modules of the form V^{res}, where V is a finite rank $U_h(\mathfrak{g})$–module.

From the explicit formulas and the results in [68], Section 5.2 it follows that the braid group elements given by (2.2.3) and (2.2.4) act in $U_h(\mathfrak{g})$–modules topologically free and of finite rank over $\mathbb{C}[[h]]$ and in $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattices in them.

We shall also need the following technical lemma regarding the action of the elements T_w and $\overline{T}_w$ on finite rank indecomposable modules.

Lemma 2.7.4. *Let V be a finite rank $U_h(\mathfrak{g})$–module. If T, T' are two elements of the braid group $\mathcal{B}_{\mathfrak{g}}$ which act as the same transformation on $\mathfrak{h} \subset U_h(\mathfrak{h})$ and v is a highest weight vector in V then $Tv = tT'v$, where t is a non–zero multiple of a power of q.*

Proof. Since v generates a highest weight indecomposable submodule $V_\lambda \subset V$ of highest weight λ equal to the weight of v, and V_λ is invariant under the braid group action, we can assume without loss of generality that $V = V_\lambda$.

First observe that similarly to the proof of Proposition 4.2 in [75] (see also Proposition 10.1.4 in [18]) one can show that $V' = U_q^{res}(\mathfrak{g})v$ is a $U_q^{res}(\mathfrak{g})$–lattice in V_λ in the sense that $V' \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}[[h]] \simeq V_\lambda$. This module coincides with the one defined in Section 4.1 in [70] (see also Proposition 10.1.4 in [18]).

It is well known that any element T of the braid group acts as an invertible linear automorphism of V' which can be specialized to any non–zero numeric value of q in the sense that for any $\varepsilon \in \mathbb{C}^*$ T gives rise to a linear automorphism of $U_q^{res}(\mathfrak{g})/(q-\varepsilon)U_q^{res}(\mathfrak{g})$–module $V'_\varepsilon = V'/(q-\varepsilon)V'$. It suffices to verify this statement when $T = T_i$ for $i = 1, \ldots, l$, and in this case it follows from the explicit formulas and the results in [68], Section 5.2.

Recall that elements of the braid group act as Weyl group elements on $\mathfrak{h} \subset U_h(\mathfrak{h})$. Assume that the action of T and T' on $\mathfrak{h} \subset U_h(\mathfrak{h})$ coincides with the action of a Weyl group element w. Since the $\mathbb{C}[q, q^{-1}]$–submodule of V' which consists of elements of weight $w\mu$ has rank one and Tv and $T'v$ must belong to this submodule, the relation $Tv = t(q)T'v$ must hold for some rational function $t(q)$ of q with poles or zeroes only at zero and infinity. Indeed, if $t(q_0) = 0$, $q_0 \neq 0, \infty$ then in V'_{q_0} we have $Tv = 0$, i.e. T does not induce an automorphism of V'_{q_0}, and if $t^{-1}(q'_0) = 0$, $q'_0 \neq 0, \infty$ then in $V'_{q'_0}$ we have $T'v = 0$, i.e. T' does not induce an automorphism of $V'_{q'_0}$. In both cases we arrive at a contradiction. Thus $t(q)$ must be a non–zero multiple of a power of q.

□

2.8 Bibliographic comments

The material presented in Sections 2.1, 2.2, 2.3, 2.4 and 2.5 is mostly standard and we refer to books [18, 51, 68] for more details and omitted proofs. Formula (2.2.3) can be found in [88].

Realizations of quantum groups associated to Weyl group elements were introduces in [93] in the case of Coxeter elements and in [99] in general.

Lemma 2.7.1 is a generalization of the result of Exercise 3 in Chapter V, §6, [9].

Specializations of quantum groups similar to those which appear in this book were considered in [101]. In this book we introduce slightly different specializations of quantum groups in order to use restricted specializations as well.

Chapter 3

q-W–algebras

In this chapter we introduce q–W–algebras and study the structure of their the quasi–classical versions, Poisson q-W–algebras. In the next chapter similar results will be obtained for q-W–algebras.

As we briefly mentioned in the introduction the naive definition of q-W–algebras as Hecke type algebras $Hk(A, B, \chi)$ requires some modification. In fact the main ingredient of the definition of q-W–algebras is the adjoint action of the quantum group on itself, and they are defined using a $\mathcal{B}$–subalgebra $\mathbb{C}_{\mathcal{B}}[G_*]$ of the quantum group the restriction of the adjoint action to which is locally finite. When q is specialized to $\varepsilon \in \mathbb{C}^*$ which is not a root of unity the algebra $\mathbb{C}_{\mathcal{B}}[G_*]$ becomes the locally finite part of the quantum group with respect to the adjoint action which was introduced and studied by Joseph.

The algebra $\mathbb{C}_{\mathcal{B}}[G_*]$ is a quantization of the algebra of regular functions on a Poisson manifold G_* which is isomorphic to G as a manifold and the Poisson structure of which is closely related to that of the Poisson–Lie group G^* dual to a quasitriangular Poisson–Lie group G.

After recalling basic facts on Poisson–Lie groups in Section 3.1 we introduce an algebra $\mathbb{C}[G^*]$ of functions on G^* in Section 3.2, its quantization $\mathbb{C}_{\mathcal{B}}[G^*] \subset U_h^s(\mathfrak{g})$ and the subalgebra $\mathbb{C}_{\mathcal{B}}[G_*] \subset \mathbb{C}_{\mathcal{B}}[G^*]$.

A special choice of the bialgebra structure entering the definitions of $\mathbb{C}[G^*]$, $\mathbb{C}_{\mathcal{B}}[G^*] \subset U_h^s(\mathfrak{g})$ and $\mathbb{C}_{\mathcal{B}}[G_*]$ is crucial for the definition of q-W–algebras. It depends on the choice of a Weyl group element $s \in W$ and ensures that one can define a subalgebra $\mathbb{C}_{\mathcal{B}}[M_+] \subset \mathbb{C}_{\mathcal{B}}[G^*]$ equipped with a non–trivial character, so that the q-W–algebra $W_{\mathcal{B}}^s(G)$ can be defined as the result of a quantum constrained reduction with respect to the subalgebra $\mathbb{C}_{\mathcal{B}}[M_+]$.

Next, in Section 3.4 we proceed with the study of the specialization $W^s(G)$ of the algebra $W_{\mathcal{B}}^s(G)$ at $q^{\frac{1}{rd^2}} = 1$. We recall that $W^s(G)$ is naturally a Poisson algebra which can be regarded as the algebra of regular functions on a reduced Poisson manifold which is also an algebraic variety. Poisson reduction works well for differential Poisson manifolds. Therefore it is easier firstly to describe the reduced Poisson structure on the algebra of C^∞–functions on the reduced Poisson manifold and then to recover the structure of the algebraic variety on it. This is done in Proposition 3.4.3 and Theorem 3.4.5.

In Section 3.5 we define a projection operator Π into the algebra $W^s(G)$. In Theorem 3.5.6, which is central in this chapter, we obtain a formula for the operator Π suitable for quantization. This formula plays the key role in the proof of Theorem 4.7.2 describing a localization of the algebra $W_{\mathcal{B}}^s(G)$ in terms of a quantum counterpart of the operator Π. Miraculously the formula for Π from Theorem 3.5.6 can be directly extrapolated to the quantum case.

3.1 Some facts on Poisson–Lie groups

In this section we recall some notions related to Poisson–Lie groups. These facts will be needed for the study of Poisson q-W–algebras.

Let G be a finite-dimensional Lie group equipped with a Poisson bracket, $\mathfrak{g}$ its Lie algebra. G is called a Poisson–Lie group if the multiplication $G \times G \to G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T_e^*G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of G.

Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy the following compatibility condition:

Let $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the dual of the commutator map $[,]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. Then δ is a 1-cocycle on $\mathfrak{g}$ (with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$).

Let c_{ij}^k, f_c^{ab} be the structure constants of $\mathfrak{g}, \mathfrak{g}^*$ with respect to the dual bases $\{e_i\}, \{e^i\}$ in $\mathfrak{g}, \mathfrak{g}^*$. The compatibility condition means that

$$c_{ab}^s f_s^{ik} - c_{as}^i f_b^{sk} + c_{as}^k f_b^{si} - c_{bs}^k f_a^{si} + c_{bs}^i f_a^{sk} = 0.$$

This condition is symmetric with respect to exchange of c and f. Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite-dimensional Lie bialgebras is isomorphic to the category of finite-dimensional connected simply connected Poisson–Lie groups.

Proposition 3.1.1. *If G is a connected simply connected finite-dimensional Lie group, every bialgebra structure on $\mathfrak{g}$ is the tangent bialgebra of a unique Poisson structure on G which makes G into a Poisson–Lie group.*

Let G be a finite-dimensional Poisson–Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of G. The connected simply connected finite-dimensional Poisson–Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson–Lie group and denoted by G^*.

$(\mathfrak{g}, \mathfrak{g}^*)$ is called a factorizable Lie bialgebra if the following conditions are satisfied:

1. *$\mathfrak{g}$ is equipped with a non–degenerate invariant scalar product $(\cdot, \cdot)$.*

   We shall always identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by means of this scalar product.

2. *The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by*

$$[X, Y]_* = \frac{1}{2}\left([rX, Y] + [X, rY]\right), X, Y \in \mathfrak{g}, \tag{3.1.1}$$

   where $r \in \mathrm{End}\ \mathfrak{g}$ is a skew symmetric linear operator (classical r-matrix).

3. *r satisfies the modified classical Yang-Baxter identity:*

$$[rX, rY] - r\left([rX, Y] + [X, rY]\right) = -\left[X, Y\right],\ X, Y \in \mathfrak{g}. \tag{3.1.2}$$

Define operators $r_\pm \in \mathrm{End}\ \mathfrak{g}$ by

$$r_\pm = \frac{1}{2}\left(r \pm id\right).$$

We shall need some properties of the operators $r_\pm$. Denote by $\mathfrak{b}_\pm$ and $\mathfrak{n}_\mp$ the image and the kernel of the operator $r_\pm$:

$$\mathfrak{i}_\pm = Im\ r_\pm,\quad \mathfrak{k}_\mp = Ker\ r_\pm. \tag{3.1.3}$$

Proposition 3.1.2. *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*

(i) $\mathfrak{i}_\pm \subset \mathfrak{g}$ is a Lie subalgebra, the subspace $\mathfrak{k}_\pm$ is a Lie ideal in $\mathfrak{i}_\pm$, $\mathfrak{i}_\pm^\perp = \mathfrak{k}_\pm$.

(ii) $\mathfrak{k}_\pm$ is an ideal in $\mathfrak{g}^*$.

(iii) $\mathfrak{i}_\pm$ is a Lie subalgebra in $\mathfrak{g}^*$. Moreover $\mathfrak{i}_\pm = \mathfrak{g}^*/\mathfrak{k}_\pm$.

(iv) $(\mathfrak{i}_\pm, \mathfrak{i}_\pm^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{i}_\pm, \mathfrak{i}_\pm^*) \simeq (\mathfrak{i}_\pm, \mathfrak{i}_\mp)$. The canonical paring between $\mathfrak{i}_\mp$ and $\mathfrak{i}_\pm$ is given by

$$(X_\mp, Y_\pm)_\pm = (X_\mp, r_\pm^{-1} Y_\pm),\ X_\mp \in \mathfrak{i}_\mp;\ Y_\pm \in \mathfrak{i}_\pm. \tag{3.1.4}$$

The classical Yang–Baxter equation implies that $r_\pm$, regarded as a mapping from $\mathfrak{g}^*$ into $\mathfrak{g}$, is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

Put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ (direct sum of two copies). The mapping

$$\mathfrak{g}^* \to \mathfrak{d} \quad : X \mapsto (X_+,\ X_-), \quad X_\pm \ = \ r_\pm X \tag{3.1.5}$$

is a Lie algebra embedding. Thus we may identify $\mathfrak{g}^*$ with a Lie subalgebra in $\mathfrak{d}$.

Naturally, embedding (3.1.5) extends to a homomorphism

$$G^* \to G \times G,\quad L \mapsto (L_+, L_-).$$

We shall identify G^* with the corresponding subgroup in $G \times G$.

3.2 Quantization of algebraic Poisson–Lie groups and the definition of q-W–algebras

In this section we introduce the main object of this book, q-W–algebras. We start by defining the relevant Poisson–Lie groups and their quantizations. We consider algebras defined over the ring $\mathcal{B}$ since later in our construction the restricted specialization of the quantum group $U_h^s(\mathfrak{g})$ defined over $\mathcal{B}$ will play the key role.

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots associated to s. Observe that cocycle (2.6.10) equips $\mathfrak{g}$ with the structure of a factorizable Lie bialgebra, where the scalar product is given by the symmetric bilinear form. Using the identification End $\mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$ the corresponding r–matrix may be represented as

$$r^s = P_+ - P_- + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where P_+, P_- and $P_{\mathfrak{h}'}$ are the orthogonal projection operators onto the nilradical $\mathfrak{n}_+$ corresponding to Δ_+, the opposite nilradical $\mathfrak{n}_-$, and $\mathfrak{h}'$, respectively, in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h}' + \mathfrak{h}'^{\perp} + \mathfrak{n}_-,$$

and $\mathfrak{h}'^{\perp}$ is the orthogonal complement to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the symmetric bilinear form.

Let G be the connected simply connected semisimple Poisson–Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* the dual Poisson–Lie group.

Observe that G is an algebraic group (see e.g. §104, Theorem 12 in [114]). Note also that

$$r_+^s = P_+ + \frac{\kappa}{2}\frac{1+s}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{\mathfrak{h}}, \quad r_-^s = -P_- + \frac{\kappa}{2}\frac{1+s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}},$$

where $P_{\mathfrak{h}}$ is the orthogonal projection operator onto $\mathfrak{h} \subset \mathfrak{g}$ with respect to the symmetric bilinear form, and hence the subspaces $\mathfrak{i}_\pm$ and $\mathfrak{k}_\pm$ defined by (3.1.3) coincide with the Borel subalgebras $\mathfrak{b}_\pm$ in $\mathfrak{g}$ corresponding to $\Delta_\pm$ and their nilradicals $\mathfrak{n}_\pm$, respectively. Therefore every element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (n_+, n_-)(h_+, h_-), \tag{3.2.1}$$

where $n_\pm \in N_\pm$, $h_+ = exp((\frac{\kappa}{2}\frac{1+s}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}id)x)$, $h_- = exp((\frac{\kappa}{2}\frac{1+s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}id)x)$, $x \in \mathfrak{h}$. In particular, G^* is a solvable subgroup in $G \times G$. In general G^* does not need to be algebraic.

Our main object will be a certain algebra of functions on G^*, $\mathbb{C}[G^*]$. This algebra may be explicitly described as follows. Let π_V be a finite-dimensional representation of G. Then matrix elements of $\pi_V(L_\pm)$ are well–defined functions on G^*, and $\mathbb{C}[G^*]$ is the subspace in $C^\infty(G^*)$ generated by matrix elements of $\pi_V(L_\pm)$, where V runs through all finite–dimensional representations of G. The elements $L^{\pm,V} = \pi_V(L_\pm)$ may be viewed as elements of the space $\mathbb{C}[G^*] \otimes \mathrm{End}V$. For every two finite–dimensional $\mathfrak{g}$–modules V and W we denote $r_+^{s\,VW} = (\pi_V \otimes \pi_W) r_+^s$, where r_+^s is regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

Proposition 3.2.1. *$\mathbb{C}[G^*]$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$, the Poisson brackets of the elements $L^{\pm,V}$ are given by*

$$\begin{aligned} \{L_1^{\pm,V}, L_2^{\pm,W}\} &= -2[r_\pm^{s\,VW}, L_1^{\pm,V} L_2^{\pm,W}], \\ \{L_1^{-,V}, L_2^{+,W}\} &= -2[r_\pm^{s\,VW}, L_1^{-,V} L_2^{+,W}], \end{aligned} \tag{3.2.2}$$

where

$$L_1^{\pm,V} = L^{\pm,V} \otimes I_W, \quad L_2^{\pm,W} = I_V \otimes L^{\pm,W},$$

and I_X is the unit matrix in X.

Moreover, the map $\Delta : \mathbb{C}[G^*] \to \mathbb{C}[G^*] \otimes \mathbb{C}[G^*]$ dual to the multiplication in G^*,

$$\Delta(L_{ij}^{\pm,V}) = \sum_k L_{ik}^{\pm,V} \otimes L_{kj}^{\pm,V}, \tag{3.2.3}$$

is a homomorphism of Poisson algebras, and the map $S : \mathbb{C}[G^*] \to \mathbb{C}[G^*]$,

$$S(L_{ij}^{\pm,V}) = (L^{\pm,V})_{ij}^{-1}$$

is an antihomomorphism of Poisson algebras.

Remark 3.2.2. *Recall that a Poisson–Hopf algebra is a Poisson algebra which is also a Hopf algebra such that the comultiplication is a homomorphism of Poisson algebras and the antipode is an antihomomorphism of Poisson algebras. According to Proposition 3.2.1* $\mathbb{C}[G^*]$ *is a Poisson–Hopf algebra.*

Now we construct a quantization of the Poisson–Hopf algebra $\mathbb{C}[G^*]$. For any finite rank representation $\pi_V : U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \to V^{res}$, where V is a finite rank representation of $U_h(\mathfrak{g})$, one can define an action of elements H_i, $i = 1, \ldots, l$ on V^{res} by requiring that H_i acts on weight vectors of weight λ by multiplication by $\lambda(H_i)$. Then from the definition of the R–matrix $\mathcal{R}^s$ it follows that ${}^qL^{\pm,V}$ given by

$$ {}^qL^{-,V} = (id \otimes \pi_V){\mathcal{R}_{21}^s}^{-1} = (id \otimes \pi_V S^s)\mathcal{R}_{21}^s, \quad {}^qL^{+,V} = (id \otimes \pi_V)\mathcal{R}^s. $$

are well–defined invertible elements of $U_h^s(\mathfrak{g}) \otimes \mathrm{End}_{\mathcal{B}}(V^{res})$.

If we fix a basis in V^{res}, ${}^qL^{\pm,V}$ may be regarded as matrices with matrix elements $({}^qL^{\pm,V})_{ij}$ being elements of $U_h^s(\mathfrak{g})$.

We denote by $\mathbb{C}_{\mathcal{B}}[G^*]$ the $\mathcal{B}$–Hopf subalgebra in $U_h^s(\mathfrak{g})$ generated by matrix elements of $({}^qL^{\pm,V})^{\pm 1}$, where V runs through all finite rank representation of $U_h(\mathfrak{g})$.

From the definition of the elements ${}^qL^{\pm,V}$ and from formula (2.7.17) it follows that $\mathbb{C}_{\mathcal{B}}[G^*]$ is the $\mathcal{B}$–subalgebra in $U_h^s(\mathfrak{g})$ generated by the elements $q^{\pm(Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}, q^{\pm(-Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i = 1, \ldots, l$, $\tilde{f}_\beta = (1 - q_\beta^{-2})f_\beta$, $\tilde{e}_\beta = (1 - q_\beta^2)e_\beta e^{h\beta^\vee}$, $\beta \in \Delta_+$ (see Section 1.4 in [25] and Theorem 4.6 in [24] for a similar result in case of the quantum group associated to the standard bialgebra structure).

From the Yang–Baxter equation for $\mathcal{R}^s$ we get relations between ${}^qL^{\pm,V}$,

$$ R^{VW}\,{}^qL_2^{\pm,W}\,{}^qL_1^{\pm,V} = {}^qL_1^{\pm,V}\,{}^qL_2^{\pm,W} R^{VW}, \tag{3.2.4} $$

$$ R^{VW}\,{}^qL_2^{+,W}\,{}^qL_1^{-,V} = {}^qL_1^{-,V}\,{}^qL_2^{+,W} R^{VW}. \tag{3.2.5} $$

By ${}^qL_1^{\pm,W}$, ${}^qL_2^{\pm,V}$ we understand the following matrices in $V^{res} \otimes W^{res}$ with entries being elements of $U_h^s(\mathfrak{g})$

$$ {}^qL_1^{\pm,V} = {}^qL^{\pm,V} \otimes I_W, \quad {}^qL_2^{\pm,W} = I_V \otimes {}^qL^{\pm,W}, $$

where I_X is the unit matrix in X.

From (2.5.2) we can obtain the action of the comultiplication on the matrices ${}^qL^{\pm,V}$:

$$ \Delta_s({}^qL_{ij}^{\pm,V}) = \sum_k {}^qL_{ik}^{\pm,V} \otimes {}^qL_{kj}^{\pm,V} \tag{3.2.6} $$

and the antipode,

$$ S_s({}^qL_{ij}^{\pm,V}) = ({}^qL^{\pm,V})_{ij}^{-1}. \tag{3.2.7} $$

Since $\mathcal{R}^s = 1 \otimes 1 \pmod h$ relations (3.2.4) and (3.2.5) imply that the quotient algebra $\mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*]$ is commutative, and one can equip it with a Poisson structure given by

$$ \{x_1, x_2\} = \frac{1}{dr^2}\frac{[a_1, a_2]}{q^{\frac{1}{dr^2}} - 1} \pmod{(q^{\frac{1}{dr^2}} - 1)}, \tag{3.2.8} $$

where $a_1, a_2 \in \mathbb{C}_{\mathcal{B}}[G^*]$ reduce to $x_1, x_2 \in \mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*] \bmod (q^{\frac{1}{dr^2}} - 1)$.

Obviously, the comultiplication and the antipode in $\mathbb{C}_{\mathcal{B}}[G^*]$ induce a comultiplication and an antipode in $\mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*]$ compatible with the introduced Poisson structure, and the quotient $\mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*]$ becomes a Poisson–Hopf algebra.

Proposition 3.2.3. *The Poisson–Hopf algebra* $\mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*]$ *is isomorphic to* $\mathbb{C}[G^*]$ *as a Poisson–Hopf algebra.*

Proof. Denote by $p : \mathbb{C}_{\mathcal{B}}[G^*] \to \mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*] = \mathbb{C}[G^*]'$ the canonical projection, and let $\tilde{L}^{\pm,V} = (p \otimes p_V)({}^qL^{\pm,V}) \in \mathbb{C}[G^*]' \otimes \mathrm{End}V$, where $p_V : V^{res} \to \overline{V} = V^{res}/(q^{\frac{1}{dr^2}} - 1)V^{res}$ is the projection of V^{res} onto the corresponding module $\overline{V}$ over $U^{res}(\mathfrak{g}) = U_{\mathcal{B}}^{s,res}(\mathfrak{g})/(q^{\frac{1}{dr^2}} - 1)U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ equipped also with the natural action of $\mathfrak{h}$ as described above. We denote by $\pi_{\overline{V}}$ the corresponding representation of $U(\mathfrak{g})$.

Observe that the map
$$\imath : \mathbb{C}[G^*]' \to \mathbb{C}[G^*], \quad (\imath \otimes id)\tilde{L}^{\pm,V} = L^{\pm,\overline{V}}$$
is a well–defined linear isomorphism. Indeed, consider, for instance, element $\tilde{L}^{+,V}$. From (2.6.11) it follows that

$$\begin{array}{l} \tilde{L}^{+,V} = \prod_\beta exp[p((1-q_\beta^{-2})f_\beta) \otimes \pi_{\overline{V}}(X_\beta)] \times \\ \times (p\otimes id) \exp\left[\sum_{i=1}^l hH_i \otimes \pi_{\overline{V}}((\kappa \frac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i)\right]. \end{array} \tag{3.2.9}$$

On the other hand (3.2.1) implies that every element L_+ may be represented in the form

$$L_+ = \prod_\beta \exp[b_\beta X_\beta] \exp\left[\sum_{i=1}^l b_i(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i\right], \; b_i, b_\beta \in \mathbb{C},$$

and hence
$$L^{+,\overline{V}} = \prod_\beta exp[b_\beta \pi_{\overline{V}}(X_\beta)] exp\left[\sum_{i=1}^l b_i \pi_{\overline{V}}((\kappa\tfrac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i)\right].$$
Comparing this with (3.2.9) and recalling the definition of $\imath$ we deduce that $\imath$ is a linear isomorphism. We have to prove that $\imath$ is an isomorphism of Poisson–Hopf algebras.

Observe that $\mathcal{R}^s = 1 \otimes 1 - 2hr^s_-$ (mod h^2). Therefore from commutation relations (3.2.4), (3.2.5) it follows that $\mathbb{C}[G^*]'$ is a commutative algebra, and the Poisson brackets of matrix elements $\tilde{L}^{\pm,\overline{V}}_{ij}$ (see (3.2.8)) are given by (3.2.2), where $L^{\pm,\overline{V}}$ are replaced by $\tilde{L}^{\pm,\overline{V}}$. The factor $\frac{1}{dr^2}$ in formula (3.2.8) normalizes the Poisson bracket in such a way that bracket (3.2.8) is in agreement with (3.2.2).

From (3.2.6) we also obtain that the action of the comultiplication on the matrices $\tilde{L}^{\pm,\overline{V}}$ is given by (3.2.3), where $L^{\pm,\overline{V}}$ are replaced by $\tilde{L}^{\pm,\overline{V}}$. This completes the proof.

□

We shall call the map $p : \mathbb{C}_\mathcal{B}[G^*] \to \mathbb{C}_\mathcal{B}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_\mathcal{B}[G^*] = \mathbb{C}[G^*]$ the quasiclassical limit.

Now using the Hopf algebra $\mathbb{C}_\mathcal{B}[G^*]$ we shall define q-W–algebras. From the definition of the elements ${}^qL^{\pm,V}$ it follows that the matrix elements of ${}^qL^{\pm,V^{\pm 1}}$ form Hopf subalgebras $\mathbb{C}_\mathcal{B}[B_\pm] \subset \mathbb{C}_\mathcal{B}[G^*]$, and that $\mathbb{C}_\mathcal{B}[G^*]$ contains the subalgebra $\mathbb{C}_\mathcal{B}[N_+]$ generated by elements the $\tilde{f}_\beta = (1-q_\beta^{-2})f_\beta$, $\beta \in \Delta_+$.

Suppose that the positive root system Δ_+ and its ordering are associated to s. Denote by $\mathbb{C}_\mathcal{B}[M_+]$ the subalgebra in $\mathbb{C}_\mathcal{B}[N_+]$ generated by the elements $\tilde{f}_\beta$, $\beta \in \Delta_{\mathfrak{m}_+}$.

The linear subspace of $\mathfrak{g}$ generated by the root vectors X_α $(X_{-\alpha})$, $\alpha \in \Delta_{\mathfrak{m}_+}$ is in fact a Lie subalgebra $\mathfrak{m}_+ \subset \mathfrak{g}$ $(\mathfrak{m}_- \subset \mathfrak{g})$. By definition $\Delta_{\mathfrak{m}_+} \subset \Delta_+$, and hence $\mathfrak{m}_\pm \subset \mathfrak{n}_\pm$.

Note that one can consider $\mathfrak{n}_+$ and $\mathfrak{m}_\pm$ as Lie subalgebras in $\mathfrak{g}^*$ via embeddings

$$\mathfrak{n}_+ \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (x, 0),$$

$$\mathfrak{m}_+ \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (x, 0),$$

$$\mathfrak{m}_- \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (0, x),$$

Using these embeddings the algebraic subgroups $N_+, M_\pm \subset G$ corresponding to the algebraic Lie subalgebras $\mathfrak{n}_+, \mathfrak{m}_\pm \subset \mathfrak{g}$ can be regarded as Lie subgroups in G^* corresponding to the Lie subalgebras $\mathfrak{n}_+, \mathfrak{m}_\pm \subset \mathfrak{g}^*$. This way the algebra $\mathbb{C}_\mathcal{B}[N_+]$ becomes naturally a quantization of the algebra of regular functions on the subgroup $N_+ \subset G^*$, and $\mathbb{C}_\mathcal{B}[M_+]$ becomes a quantization of the algebra of regular functions on the subgroup $M_+ \subset G^*$ in the sense that $p(\mathbb{C}_\mathcal{B}[N_+]) = \mathbb{C}[N_+]$ and $p(\mathbb{C}_\mathcal{B}[M_+]) = \mathbb{C}[M_+]$. Here and below for every algebraic variety V we denote by $\mathbb{C}[V]$ the algebra of regular functions on V.

Note that M_- can also be regarded as a subgroup in G^* corresponding to the Lie subalgebra $\mathfrak{m}_- \subset \mathfrak{g}^*$.

The following proposition gives the most important property of the subalgebra $\mathbb{C}_\mathcal{B}[M_+]$ which plays the key role in the definition of q-W–algebras.

Proposition 3.2.4. *The defining relations in the subalgebra $\mathbb{C}_\mathcal{B}[M_+]$ for the generators $\tilde{f}_\beta = (1-q_\beta^{-2})f_\beta$, $\beta \in \Delta_{\mathfrak{m}_+} = \{\beta_1, \ldots, \beta_c\}$ are of the form*

$$\tilde{f}_\alpha \tilde{f}_\beta - q^{(\alpha,\beta)+\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}\tilde{f}_\beta\tilde{f}_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(k_1,\ldots,k_n)\tilde{f}_{\delta_1}^{k_1}\tilde{f}_{\delta_2}^{k_2}\ldots\tilde{f}_{\delta_n}^{k_n}, \; \alpha<\beta, \tag{3.2.10}$$

where $C(k_1,\ldots,k_n)\in\mathcal{B}$*, the products* $\tilde f_{\beta_1}^{k_1}\tilde f_{\beta_2}^{k_2}\ldots\tilde f_{\beta_c}^{k_c}$ *form a* $\mathcal{B}$*–basis of* $\mathbb{C}_{\mathcal{B}}[M_+]$ *and the products* $\tilde f_{\beta_1}^{k_1}\tilde f_{\beta_2}^{k_2}\ldots\tilde f_{\beta_D}^{k_D}$ *form a* $\mathcal{B}$*–basis of* $\mathbb{C}_{\mathcal{B}}[N_+]$*.*

If $\kappa=1$ *then for any* $k_i\in\mathcal{B}$*,* $i=1,\ldots,l'$ *the map* $\chi_q^s:\mathbb{C}_{\mathcal{B}}[M_+]\rightarrow\mathcal{B}$*,*

$$\chi_q^s(\tilde f_\beta)=\begin{cases}0 & \beta\not\in\{\gamma_1,\ldots,\gamma_{l'}\}\\ k_i & \beta=\gamma_i\end{cases}\quad, \tag{3.2.11}$$

is a character of $\mathbb{C}_{\mathcal{B}}[M_+]$ *vanishing on the r.h.s. and on the l.h.s. of relations (3.2.10).*

Assume that $\varepsilon^{2d_i}\neq 1$*, and* $\varepsilon^4\neq 1$ *if* $\mathfrak{g}$ *is of type* G_2*. Suppose also that there exists* $n\in\mathbb{Z}$ *such that* $\varepsilon^{nd-1}=1$*. Let* $\kappa=nd$*. Then the algebra* $\mathbb{C}_\varepsilon[M_+]=\mathbb{C}_{\mathcal{B}}[M_+]/(q^{\frac{1}{dr^2}}-\varepsilon^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}[M_+]$*, where* $\varepsilon^{\frac{1}{dr^2}}$ *is a root of* ε *of degree* $\frac{1}{dr^2}$*, is isomorphic to* $U_\varepsilon^s(\mathfrak{m}_-)$*, the elements* $f^{\mathbf{r}}=f_{\beta_1}^{r_1}\ldots f_{\beta_c}^{r_c}$*,* $r_i\in\mathbb{N}$*,* $i=1,\ldots d$ *form a linear basis of* $U_\varepsilon^s(\mathfrak{m}_-)$*, and for any* $c_i\in\mathbb{C}$*,* $i=1,\ldots,l'$ *the map* $\chi_\varepsilon^s:U_\varepsilon^s(\mathfrak{m}_-)\rightarrow\mathbb{C}$*,*

$$\chi_\varepsilon^s(f_\beta)=\begin{cases}0 & \beta\not\in\{\gamma_1,\ldots,\gamma_{l'}\}\\ c_i & \beta=\gamma_i\end{cases}\quad, \tag{3.2.12}$$

is a character of $U_\varepsilon^s(\mathfrak{m}_-)$*.*

Proof. By Step 1, Theorem 21.1 in [26] the products $\tilde f_{\beta_1}^{k_1}\tilde f_{\beta_2}^{k_2}\ldots\tilde f_{\beta_D}^{k_D}$ form a $\mathcal{B}$–basis of $\mathbb{C}_{\mathcal{B}}[N_+]$.

By Lemma 2.7.2 and Remark 2.7.3 any element of $\mathbb{C}_{\mathcal{B}}[M_+]$ can be uniquely represented as a $\mathbb{C}(q^{\frac{1}{dr^2}})$–linear combination of the elements $\tilde f_{\beta_1}^{k_1}\tilde f_{\beta_2}^{k_2}\ldots\tilde f_{\beta_c}^{k_c}$. By the uniqueness of the Poincaré–Birkhoff–Witt decomposition for $\mathbb{C}_{\mathcal{B}}[N_+]$ established above the coefficients of this decomposition must belong to $\mathcal{B}$.

From (2.7.11) we also obtain commutation relations (3.2.10) with $C(k_1,\ldots,k_n)\in\mathbb{C}(q^{\frac{1}{dr^2}})$. As we already proved the products $\tilde f_{\beta_1}^{k_1}\tilde f_{\beta_2}^{k_2}\ldots\tilde f_{\beta_c}^{k_c}$ form a $\mathcal{B}$–basis of $\mathbb{C}_{\mathcal{B}}[M_+]$. Therefore the coefficients $C(k_1,\ldots,k_n)$ in (3.2.10) belong to $\mathcal{B}$.

Assume that $\kappa=1$. In order to prove that the map $\chi_q^s:\mathbb{C}_{\mathcal{B}}[M_+]\rightarrow\mathcal{B}$ defined by (3.2.11) is a character we show that all relations (3.2.10) for $\tilde f_\alpha,\tilde f_\beta$ with $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$, which are obviously defining relations in the subalgebra $\mathbb{C}_{\mathcal{B}}[M_+]$, belong to the kernel of χ_q^s. By definition the only generators of $\mathbb{C}_{\mathcal{B}}[M_+]$ on which χ_q^s may not vanish are $\tilde f_{\gamma_i}$, $i=1,\ldots,l'$. By part (v) of Proposition 1.6.1 for any two roots $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$ such that $\alpha<\beta$ the sum $\alpha+\beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k\gamma_{i_k}$, where $c_k\in\mathbb{N}$ and $\alpha<\gamma_{i_1}<\ldots<\gamma_{i_t}<\beta$. Hence for any two roots $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$ such that $\alpha<\beta$ the value of the map χ_q^s on the right hand side of the corresponding commutation relation (3.2.10) is equal to zero.

Therefore it suffices to prove that

$$\chi_q^s(\tilde f_{\gamma_i}\tilde f_{\gamma_j}-q^{(\gamma_i,\gamma_j)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)}\tilde f_{\gamma_j}\tilde f_{\gamma_j})=k_ik_j(1-q^{(\gamma_i,\gamma_j)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)})=0,\ i<j.$$

The last identity holds provided $(\gamma_i,\gamma_j)+(\frac{1+s}{1-s}P_{\mathfrak{h}'}^*\gamma_i,\gamma_j)=0$ for $i<j$ which is indeed the case by Lemma 2.7.1.

Assume now that $\varepsilon^{2d_i}\neq 1$, and $\varepsilon^4\neq 1$ if $\mathfrak{g}$ is of type G_2. Suppose also that there exists $n\in\mathbb{Z}$ such that $\varepsilon^{nd-1}=1$. Let $\kappa=nd$.

Under these conditions imposed on ε the map $\mathbb{C}_\varepsilon[M_+]\rightarrow U_\varepsilon^s(\mathfrak{m}_-)$, $\tilde f_\alpha\mapsto(1-q_\alpha^{-2})f_\alpha$, $\alpha\in\Delta_{\mathfrak{m}_+}$ is obviously an algebra isomorphism.

By Lemma 2.7.2 and Remark 2.7.3 the elements $f^{\mathbf{r}}=f_{\beta_1}^{r_1}\ldots f_{\beta_c}^{r_c}$, $r_i\in\mathbb{N}$, $i=1,\ldots d$ form a linear basis of $U_\varepsilon^s(\mathfrak{m}_-)$.

From (2.7.11) we obtain the following commutation relations

$$f_\alpha f_\beta-\varepsilon^{(\alpha,\beta)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}f_\beta f_\alpha=\sum_{\alpha<\delta_1<\ldots<\delta_n<\beta}D(k_1,\ldots,k_n)f_{\delta_1}^{k_1}f_{\delta_2}^{k_2}\ldots f_{\delta_n}^{k_n},\ \alpha<\beta, \tag{3.2.13}$$

where $D(k_1,\ldots,k_n)\in\mathbb{C}$.

In order to show that the map $\chi_\varepsilon^s:U_\varepsilon^s(\mathfrak{m}_-)\rightarrow\mathbb{C}$ defined by (3.2.12) is a character we verify that all relations (3.2.13) for f_α,f_β with $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$, which are obviously defining relations in the subalgebra $U_\varepsilon^s(\mathfrak{m}_-)$, belong to the kernel of χ_ε^s. By definition the only generators of $U_\varepsilon^s(\mathfrak{m}_-)$ on which χ_ε^s may not vanish are f_{γ_i}, $i=1,\ldots,l'$. By part (v) of Proposition 1.6.1 for any two roots $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$ such that $\alpha<\beta$ the sum $\alpha+\beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k\gamma_{i_k}$, where $c_k\in\mathbb{N}$ and $\alpha<\gamma_{i_1}<\ldots<\gamma_{i_t}<\beta$. Hence for any two roots $\alpha,\beta\in\Delta_{\mathfrak{m}_+}$ such that $\alpha<\beta$ the value of the map χ_ε^s on the right hand side of the corresponding commutation relation (3.2.10) is equal to zero.

Therefore it suffices to prove that

$$\chi_\varepsilon^s(f_{\gamma_i}f_{\gamma_j}-\varepsilon^{(\gamma_i,\gamma_j)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)}f_{\gamma_j}f_{\gamma_j})=c_ic_j(1-\varepsilon^{(\gamma_i,\gamma_j)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)})=0,\ i<j.$$

By Lemma 2.7.1 $(\frac{1+s}{1-s}P_{\mathfrak{h}'}^*\gamma_i,\gamma_j)=-(\gamma_i,\gamma_j)$ for $i<j$, and hence

$$\chi_\varepsilon^s(f_{\gamma_i}f_{\gamma_j}-\varepsilon^{(\gamma_i,\gamma_j)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)}f_{\gamma_j}f_{\gamma_j})=c_ic_j(1-\varepsilon^{(\gamma_i,\gamma_j)(1-nd)})=0$$

for $i<j$ as by the assumption $\varepsilon^{nd-1}=1$. This completes the proof.

□

Next we define the algebra $\mathbb{C}_\mathcal{B}[G_*]$ and discuss its properties. For any finite rank representation V of $U_h(\mathfrak{g})$, let ${}^qL^V={}^qL^{-,V^{-1}}{}^qL^{+,V}=(id\otimes\pi_V)\mathcal{R}_{21}^s\mathcal{R}^s$. Let $\mathbb{C}_\mathcal{B}[G_*]$ be the $\mathcal{B}$–subalgebra in $\mathbb{C}_\mathcal{B}[G^*]$ generated by the matrix entries of ${}^qL^V$, where V runs over all finite rank representations of $U_h^s(\mathfrak{g})$. From the definition of $\mathcal{R}^s$ we have

$$\begin{array}{l}\mathcal{R}_{21}^s\mathcal{R}^s=\prod_\beta\sum_{k=0}^\infty q_\beta^{\frac{k(k+1)}{2}}[(1-q_\beta^{-2})e_\beta^k e^{-hk\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee}\otimes f_\beta^{(k)}]\times\\ \times\exp\left[2h\sum_{i=1}^l Y_i\otimes H_i\right]\prod_\beta\sum_{k=0}^\infty q_\beta^{\frac{k(k+1)}{2}}[(1-q_\beta^{-2})e^{hk\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee-hk\beta^\vee}f_\beta^k\otimes e_\beta^{(k)}q^{k\beta^\vee}].\end{array}\quad(3.2.14)$$

Using this formula one immediately checks that actually $\mathbb{C}_\mathcal{B}[G_*]\subset U_\mathcal{A}^s(\mathfrak{g})\cap\mathbb{C}_\mathcal{B}[G^*]$.

Define the right adjoint action of $U_q^s(\mathfrak{g})$ on $U_q^s(\mathfrak{g})$ by the formula

$$\mathrm{Ad}x(w)=S_s(x^1)wx^2,\quad(3.2.15)$$

and the left adjoint action of $U_q^s(\mathfrak{g})$ on $U_q^s(\mathfrak{g})$ by

$$\mathrm{Ad}'x(w)=x^1wS_s(x^2),\quad(3.2.16)$$

where we use the abbreviated Sweedler notation for the coproduct $\Delta_s(x)=x^1\otimes x^2$, $x,w\in U_q^s(\mathfrak{g})$.

Let $\mathbb{C}_\mathcal{B}[G]$ be the restricted Hopf algebra dual to $U_\mathcal{B}^{s,res}(\mathfrak{g})$ which is generated by the matrix elements of finite rank representations of $U_\mathcal{B}^{s,res}(\mathfrak{g})$ of the form V^{res}, where V is a finite rank representation of $U_h^s(\mathfrak{g})$. Action (3.2.16) induces a left adjoint action Ad of $U_\mathcal{B}^{s,res}(\mathfrak{g})$ on $\mathbb{C}_\mathcal{B}[G]$ defined by

$$(\mathrm{Ad}xf)(w)=f(\mathrm{Ad}'x(w)),f\in\mathbb{C}_\mathcal{B}[G],x,w\in U_\mathcal{B}^{s,res}(\mathfrak{g}).\quad(3.2.17)$$

One can also equip finite rank representations of $U_\mathcal{B}^{s,res}(\mathfrak{g})$ of the form V^{res}, where V is a finite rank representation of $U_h^s(\mathfrak{g})$, with a natural action of $\mathbb{C}_\mathcal{B}[G^*]$, where the elements

$$q^{\pm(Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)},q^{\pm(-Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)},\ i=1,\ldots l$$

act on a weight vector v_λ of weight λ by multiplication by the elements of

$$q^{\pm((Y_i,\lambda^\vee)-\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i,\lambda^\vee))},q^{\pm(-(Y_i,\lambda^\vee)-\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i,\lambda^\vee))}\in\mathcal{B},\ i=1,\ldots l,$$

respectively, and all the other generators of $\mathbb{C}_\mathcal{B}[G^*]$ which belong to $U_\mathcal{B}^{s,res}(\mathfrak{g})$ act in a natural way. Therefore adjoint action (3.2.17) can be extended to an action of $\mathbb{C}_\mathcal{B}[G^*]$, where elements $x\in\mathbb{C}_\mathcal{B}[G^*]$ act by the same formula (3.2.17).

Note that by Lemma 2.2 in [53]

$$\mathrm{Ad}x(wz)=\mathrm{Ad}x^1(w)\mathrm{Ad}x^2(z).\quad(3.2.18)$$

For $\varepsilon\in\mathbb{C}^*$ we define $\mathbb{C}_\varepsilon[G]=\mathbb{C}_\mathcal{B}[G]/(q^{\frac{1}{dr^2}}-\varepsilon^{\frac{1}{dr^2}})\mathbb{C}_\mathcal{B}[G]$, $\mathbb{C}_\varepsilon[G_*]=\mathbb{C}_\mathcal{B}[G_*]/(q^{\frac{1}{dr^2}}-\varepsilon^{\frac{1}{dr^2}})\mathbb{C}_\mathcal{B}[G_*]$, where $\varepsilon^{\frac{1}{dr^2}}$ is a root of ε of degree $\frac{1}{dr^2}$.

Proposition 3.2.5. *If ε is not a root of unity the algebra $\mathbb{C}_\varepsilon[G_*]$ can be identified with the* Ad *locally finite part $U_\varepsilon^s(\mathfrak{g})^{fin}$ of $U_\varepsilon^s(\mathfrak{g})$,*

$$U_\varepsilon^s(\mathfrak{g})^{fin}=\{x\in U_\varepsilon^s(\mathfrak{g}):\dim(\mathrm{Ad}U_\varepsilon^s(\mathfrak{g})(x))<+\infty\},$$

where the adjoint action of the algebra $U_\varepsilon^s(\mathfrak{g})$ on itself is defined by formula (3.2.15).

Moreover, the map

$$\mathbb{C}_\mathcal{B}[G]\to\mathbb{C}_\mathcal{B}[G_*],f\mapsto(id\otimes f)(\mathcal{R}_{21}^s\mathcal{R}^s)\quad(3.2.19)$$

is an isomorphism of $U_\mathcal{B}^{s,res}(\mathfrak{g})$ and $\mathbb{C}_\mathcal{B}[G^*]$–modules with respect to the adjoint actions Ad *defined by (3.2.15) and (3.2.17), respectively. In particular, $\mathbb{C}_\mathcal{B}[G_*]$ is stable under the adjoint action of $U_\mathcal{B}^{s,res}(\mathfrak{g})$ and $\mathbb{C}_\mathcal{B}[G^*]$.*

Proof. All statements except for the isomorphism $\mathbb{C}_\varepsilon[G_*]=U_\varepsilon^s(\mathfrak{g})^{fin}$ follow from the definitions and the discussion above.

It remains to establish the isomorphism $\mathbb{C}_\varepsilon[G_*]=U_\varepsilon^s(\mathfrak{g})^{fin}$. Let V_{ω_i}, $i=1,\ldots,l$ be the finite rank representation of $U_h(\mathfrak{g})$ with highest weight ω_i, $i=1,\ldots,l$. From formula (3.2.14) and from the definition of ${}^qL^V=(id\otimes\pi_V)\mathcal{R}_{21}^s\mathcal{R}^s$ it follows that the matrix element $(id\otimes v_i^*)\mathcal{R}_{21}^s\mathcal{R}^s(id\otimes v_i)$ of ${}^qL^{V_{\omega_i}}$ corresponding to the highest weight vector v_i of $V_{\omega_i}^{res}$ and to the lowest weight vector $v_i^*\in V_{\omega_i}^{res*}$ of the dual representation $V_{\omega_i}^{res*}$, normalized in such a way that $v_i^*(v_i)=1$, coincides with L_i^2. This implies that L_i^2 are elements of the algebra $\mathbb{C}_\varepsilon[G_*]\subset U_\varepsilon^s(\mathfrak{g})$ as well.

Denote by $\mathfrak{H}\subset\mathbb{C}_\varepsilon[G_*]\subset U_\varepsilon^s(\mathfrak{g})$ the subalgebra generated by the elements $L_i^2\in\mathbb{C}_\varepsilon[G_*]$, $i=1,\ldots,l$. Similarly to Theorem 7.1.6 and Lemma 7.1.16 in [52] one can obtain that $U_\varepsilon^s(\mathfrak{g})^{fin}=\mathrm{Ad}U_\varepsilon^s(\mathfrak{g})\mathfrak{H}$. Since $\mathbb{C}_\varepsilon[G_*]$ is stable under the adjoint action we have an inclusion, $U_\varepsilon^s(\mathfrak{g})^{fin}\subset\mathbb{C}_\varepsilon[G_*]$.

On the other hand the adjoint action of $U_\varepsilon^s(\mathfrak{g})$ on $\mathbb{C}_\varepsilon[G]$ is locally finite by the definition of $\mathbb{C}_\varepsilon[G]$ and of the adjoint action. Using isomorphism (3.2.19) we deduce that the adjoint action of $U_\varepsilon^s(\mathfrak{g})$ on $\mathbb{C}_\varepsilon[G_*]$ is locally finite as well. Hence $\mathbb{C}_\varepsilon[G_*]\subset U_\varepsilon^s(\mathfrak{g})^{fin}$, and $\mathbb{C}_\varepsilon[G_*]=U_\varepsilon^s(\mathfrak{g})^{fin}$.

□

Now we are ready to define q-W–algebras. In the rest of this section we assume that $\kappa=1$. Denote by $I_\mathcal{B}$ the left ideal in $\mathbb{C}_\mathcal{B}[G^*]$ generated by the kernel of χ_q^s, and by $\rho_{\chi_q^s}$ the canonical projection $\mathbb{C}_\mathcal{B}[G^*]\rightarrow\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}=Q'_\mathcal{B}$. Let $Q_\mathcal{B}$ be the image of $\mathbb{C}_\mathcal{B}[G_*]$ under the projection $\rho_{\chi_q^s}$.

We shall need the following property of the algebras $\mathbb{C}_\mathcal{B}[G^*]$, $\mathbb{C}_\mathcal{B}[G_*]$ and $\mathbb{C}_\mathcal{B}[G]$ and of $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$ and $Q_\mathcal{B}$.

Proposition 3.2.6. *$\mathbb{C}_\mathcal{B}[G^*]$, $\mathbb{C}_\mathcal{B}[G_*]$, $\mathbb{C}_\mathcal{B}[G]$, $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$ and $Q_\mathcal{B}$ are free $\mathcal{B}$–modules.*

Proof. Recall that $\mathbb{C}_\mathcal{B}[G^*]$ is the $\mathcal{B}$–subalgebra in $U_h^s(\mathfrak{g})$ generated by the elements $q^{\pm(Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $q^{\pm(-Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i=1,\ldots,l$, $\tilde{f}_\beta=(1-q_\beta^{-2})f_\beta$, $\tilde{e}_\beta=(1-q_\beta^2)e_\beta e^{h\beta^\vee}$, $\beta\in\Delta_+$. The subalgebra of $\mathbb{C}_\mathcal{B}[G^*]$ generated by the elements $q^{\pm(Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}, q^{\pm(-Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i=1,\ldots,l$ is in turn a subalgebra of the $\mathcal{B}$–subalgebra $U'_\mathcal{B}(\mathfrak{h})\subset U_h^s(\mathfrak{g})$ generated by the elements $U_i=q^{\frac{1}{dr^2}Y_i},U_i^{-1}$, $i=1,\ldots,l$. The last algebra is obviously $\mathcal{B}$–free with a basis consisting of the products $U_1^{n_1}\ldots U_l^{n_l}$, $n_1,\ldots,n_l\in\mathbb{Z}$. Since $\mathcal{B}$ is a principal ideal domain the subalgebra in $U'_\mathcal{B}(\mathfrak{h})$ generated by the elements $q^{\pm(Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}, q^{\pm(-Y_i-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i=1,\ldots,l$ is also $\mathcal{B}$–free by Theorem 6.5 in [87]. Denote by V_i, $i\in\mathbb{N}$ elements of some $\mathcal{B}$–basis of this subalgebra.

Now from Step 1 of the proof of the Theorem in Section 12.1 in [26] it follows that the elements

$$\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_1}^{k_1}$$

with $n_j,k_j,i\in\mathbb{N}$, $j=1,\ldots,D$ form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}[G^*]$. Using this basis and the definition of $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$ one immediately sees that the classes of the elements $\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_{c+1}}^{k_{c+1}}$ with $n_j,k_m,i\in\mathbb{N}$, $j=1,\ldots,D$, $m=c+1,\ldots,D$ form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$.

Since $\mathcal{B}$ is a principal ideal domain the subalgebra $\mathbb{C}_\mathcal{B}[G_*]\subset\mathbb{C}_\mathcal{B}[G^*]$ is also $\mathcal{B}$–free by Theorem 6.5 in [87], and isomorphism (3.2.19) implies that $\mathbb{C}_\mathcal{B}[G]$ is $\mathcal{B}$–free.

Finally, since $\mathcal{B}$ is a principal ideal domain the $\mathcal{B}$–submodule $Q_\mathcal{B}\subset\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$ is $\mathcal{B}$–free.

□

Remark 3.2.7. *The fact that the algebra $\mathbb{C}_q[G]=\mathbb{C}_\mathcal{B}[G]\otimes_\mathcal{B}\mathbb{C}(q^{\frac{1}{dr^2}})$ is $\mathbb{C}(q^{\frac{1}{dr^2}})$–free can be proved in a more straightforward way. Namely, according to the results of Section 7 in [69] $\mathbb{C}_q[G]$ is generated by the matrix elements of finite-dimensional $U_q^s(\mathfrak{g})\simeq U_q(\mathfrak{g})$–modules. By Theorem 10.1.7 in [18] all such modules are completely reducible, the irreducible components being highest weight modules V_λ^q generated by highest weight vectors of integral dominant weights $\lambda\in P_+$. Therefore*

$$\mathbb{C}_q[G]\simeq\bigoplus_{\lambda\in P_+}V_\lambda^{q*}\otimes V_\lambda^q.$$

This decomposition and the fact that all V_λ^q are $\mathbb{C}(q^{\frac{1}{dr^2}})$–free imply that $\mathbb{C}_q[G]$ is $\mathbb{C}(q^{\frac{1}{dr^2}})$–free.

Now observe that we have an inclusion $[\mathbb{C}_\mathcal{B}[M_+],\mathrm{Ker}\chi_q^s]\subset\mathrm{Ker}\chi_q^s$. Using this inclusion, formula (3.2.15), the fact that $\Delta_s(\mathbb{C}_\mathcal{B}[M_+])\subset\mathbb{C}_\mathcal{B}[B_+]\otimes\mathbb{C}_\mathcal{B}[M_+]$ (see formula (2.7.12)) we deduce that the adjoint action of $\mathbb{C}_\mathcal{B}[M_+]$ on $\mathbb{C}_\mathcal{B}[G_*]$ induces an action on $Q'_\mathcal{B}$ and on $Q_\mathcal{B}$ which we also call the adjoint action and denote it by Ad.

Let $\mathcal{B}_{\varepsilon_s}$ be the trivial representation of $\mathbb{C}_\mathcal{B}[M_+]$ given by the counit. Consider the space $W_\mathcal{B}^s(G)$ of Ad–invariants in $Q_\mathcal{B}$,

$$W_\mathcal{B}^s(G)=\mathrm{Hom}_{\mathbb{C}_\mathcal{B}[M_+]}(\mathcal{B}_{\varepsilon_s},Q_\mathcal{B}). \tag{3.2.20}$$

Proposition 3.2.8. *$W_{\mathcal{B}}^s(G)$ is isomorphic to the subspace of all $v+I_{\mathcal{B}}\in Q_{\mathcal{B}}$ such that $mv\in I_{\mathcal{B}}$ (or $[m,v]\in I_{\mathcal{B}}$) in $\mathbb{C}_{\mathcal{B}}[G^*]$ for any $m\in I_{\mathcal{B}}$, where $v\in \mathbb{C}_{\mathcal{B}}[G^*]$ is any representative of $v+I_{\mathcal{B}}\in Q_{\mathcal{B}}$.*

Multiplication in $\mathbb{C}_{\mathcal{B}}[G^*]$ induces a multiplication on the space $W_{\mathcal{B}}^s(G)$.

Proof. For the proof we shall firstly derive the formula for the adjoint action of the generators $\tilde{f}_\beta$. From (2.7.12) using linear independence of weight components and the fact that $\mathbb{C}_{\mathcal{B}}[B_+]$ is a Hopf algebra we obtain

$$\Delta_s(\tilde{f}_{\beta_k})=G_{\beta_k}^{-1}\otimes\tilde{f}_{\beta_k}+\tilde{f}_{\beta_k}\otimes 1+\sum_i\tilde{y}_i\otimes\tilde{x}_i, \tag{3.2.21}$$

where

$$G_\beta=e^{h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee-h\beta^\vee}\in\mathbb{C}_{\mathcal{B}}[B_+], \tilde{y}_i=e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\gamma_{x_i}^\vee+h\gamma_{x_i}^\vee}\overline{\overline{y}}_i,$$
$$\overline{\overline{y}}_i\in\mathbb{C}_{\mathcal{B}}([-\beta_{k+1},-\beta_D]),$$
$$\tilde{x}_i\in\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_{k-1}]),$$

$\overline{\overline{y}}_i,\tilde{x}_i$ belong to weight components and have non-zero weights, γ_{x_i} is the weight of $\tilde{x}_i$,

$$\mathbb{C}_{\mathcal{B}}([-\beta_{k+1},-\beta_D])(\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_{k-1}]))$$

is the subalgebra in $\mathbb{C}_{\mathcal{B}}[N_+]$ generated by $\tilde{f}_{\beta_{k+1}},\ldots,\tilde{f}_{\beta_D}$ $(\tilde{f}_{\beta_1},\ldots,\tilde{f}_{\beta_{k-1}})$.

By (3.2.21) we also have

$$S_s(\tilde{f}_\beta)=-G_\beta\tilde{f}_\beta-\sum_i S_s(\tilde{y}_i)\tilde{x}_i.$$

Combining this formula with (3.2.21) and using (3.2.15) we deduce

$$\mathrm{Ad}\tilde{f}_{\beta_k}w=-G_{\beta_k}[\tilde{f}_{\beta_k},w]-\sum_i S_s(\tilde{y}_i)[\tilde{x}_i,w]$$

The induced action of the elements $\tilde{f}_{\beta_k}\in\mathbb{C}_{\mathcal{B}}[M_+]$, $\beta_k\in\Delta_{\mathfrak{m}_+}$, on $Q'_{\mathcal{B}}$ takes the form

$$\mathrm{Ad}\tilde{f}_{\beta_k}v=-G_{\beta_k}(\tilde{f}_{\beta_k}-\chi_q^s(\tilde{f}_{\beta_k}))v-\sum_i S_s(\tilde{y}_i)(\tilde{x}_i-\chi_q^s(\tilde{x}_i))v,\beta_k\in\Delta_{\mathfrak{m}_+}. \tag{3.2.22}$$

We have to show that $W_{\mathcal{B}}^s(G)$ is isomorphic to the subspace of all $v\in Q_{\mathcal{B}}\subset Q'_{\mathcal{B}}$ such that $mv=0$ in $Q'_{\mathcal{B}}$ for any $m\in I_{\mathcal{B}}$.

The left ideal $I_{\mathcal{B}}$ is generated by elements $x-\chi_q^s(x)$, $x\in\mathbb{C}_{\mathcal{B}}[M_+]$. Therefore by (3.2.22) if for some $v\in Q_{\mathcal{B}}$ $mv=0$ in $Q'_{\mathcal{B}}$ for any $m\in I_{\mathcal{B}}$ then v is invariant with respect to the adjoint action of all generators $\tilde{f}_{\beta_k}$ of $\mathbb{C}_{\mathcal{B}}[M_+]$, and hence $v\in W_{\mathcal{B}}^s(G)$.

Now assume that $v\in W_{\mathcal{B}}^s(G)$. We shall prove that $mv=0$ in $Q'_{\mathcal{B}}$ for any $m\in I_{\mathcal{B}}$.

Since the left ideal $I_{\mathcal{B}}$ is generated by elements $x-\chi_q^s(x)$, $x\in\mathbb{C}_{\mathcal{B}}[M_+]$ it suffices to show that $(x-\chi_q^s(x))v=0$ for any $x\in\mathbb{C}_{\mathcal{B}}[M_+]$. We shall prove this statement by induction using the subalgebras $\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_k])$, $k=1,\ldots,c$, so that $\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_c])=\mathbb{C}_{\mathcal{B}}[M_+]$.

Observe that β_1 is a simple root, and hence from (3.2.22) we obtain

$$0=\mathrm{Ad}\tilde{f}_{\beta_1}v=-G_{\beta_1}(\tilde{f}_{\beta_1}-\chi_q^s(\tilde{f}_{\beta_1}))v.$$

Since the element $G_{\beta_1}\in\mathbb{C}_{\mathcal{B}}[N_+]$ is invertible this implies

$$(\tilde{f}_{\beta_1}-\chi_q^s(\tilde{f}_{\beta_1}))v=0,$$

i.e. $(x-\chi_q^s(x))v=0$ in $Q'_{\mathcal{B}}$ for any $x\in\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_1])$ as the subalgebra $\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_1])$ is generated by $\tilde{f}_{\beta_1}$.

Now suppose that for some $1<k\leq c$ $(x-\chi_q^s(x))v=0$ in $Q'_{\mathcal{B}}$ for any $x\in\mathbb{C}_{\mathcal{B}}([-\beta_1,-\beta_{k-1}])$. Then from (3.2.22) we obtain

$$0=\mathrm{Ad}\tilde{f}_{\beta_k}v=-G_{\beta_k}(\tilde{f}_{\beta_k}-\chi_q^s(\tilde{f}_{\beta_k}))v-\sum_i S_s(\tilde{y}_i)(\tilde{x}_i-\chi_q^s(\tilde{x}_i))v=-G_{\beta_k}(\tilde{f}_{\beta_k}-\chi_q^s(\tilde{f}_{\beta_k}))v$$

since $\tilde{x}_i \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}])$. The previous identity and the fact that the element $G_{\beta_k} \in \mathbb{C}_{\mathcal{B}}[N_+]$ is invertible yield

$$(\tilde{f}_{\beta_k} - \chi_q^s(\tilde{f}_{\beta_k}))v = 0.$$

Now observe that by Proposition 3.2.4 any element x of $\mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_k])$ can be uniquely represented in the form $x = \tilde{f}_{\beta_k} z + z'$, where $z, z' \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}])$. Therefore

$$xv = (\tilde{f}_{\beta_k} z + z')v = (\tilde{f}_{\beta_k} \chi_q^s(z) + \chi_q^s(z'))v = (\chi_q^s(\tilde{f}_{\beta_k})\chi_q^s(z) + \chi_q^s(z'))v = \chi_q^s(\tilde{f}_{\beta_k} z + z')v = \chi_q^s(x)v.$$

This establishes the induction step and proves the first claim of this proposition.

From the second description of the space $W_{\mathcal{B}}^s(G)$ it follows that if $v_1, v_2 \in \mathbb{C}_{\mathcal{B}}[G^*]$ are any representatives of elements $v_1 + I_q, v_2 + I_q \in W_{\mathcal{B}}^s(G)$ then the formula

$$(v_1 + I_q)(v_2 + I_q) = v_1 v_2 + I_q$$

defines a multiplication in $W_{\mathcal{B}}^s(G)$. This completes the proof.

□

The space $W_{\mathcal{B}}^s(G)$ equipped with the multiplication opposite to the one defined in the previous proposition is called the q-W–algebra associated to (the conjugacy class of) the Weyl group element $s \in W$.

In conclusion we obtain some results on the structure of $Q_{\mathcal{B}}$. Consider the Lie algebra $\mathfrak{L}_{\mathcal{B}}$ associated to the associative algebra $\mathbb{C}_{\mathcal{B}}[M_+]$, i.e. $\mathfrak{L}_{\mathcal{B}}$ is the Lie algebra which is isomorphic to $\mathbb{C}_{\mathcal{B}}[M_+]$ as a linear space, and the Lie bracket in $\mathfrak{L}_{\mathcal{B}}$ is given by the usual commutator of elements in $\mathbb{C}_{\mathcal{B}}[M_+]$.

Define an action of the Lie algebra $\mathfrak{L}_{\mathcal{B}}$ on the space $\mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$:

$$m \cdot (x + I_{\mathcal{B}}) = \rho_{\chi_q^s}([m, x]). \tag{3.2.23}$$

where $x \in \mathbb{C}_{\mathcal{B}}[G^*]$ is any representative of $x + I_{\mathcal{B}} \in \mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$ and $m \in \mathbb{C}_{\mathcal{B}}[M_+]$. The algebra $W_{\mathcal{B}}^s(G)$ can be regarded as the intersection of the space of invariants with respect to action (3.2.23) with the subspace $Q_{\mathcal{B}} \subset \mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$.

Note also that since χ_q^s is a character of $\mathbb{C}_{\mathcal{B}}[M_+]$ the ideal $I_{\mathcal{B}}$ is stable under the action of $\mathbb{C}_{\mathcal{B}}[M_+]$ on $\mathbb{C}_{\mathcal{B}}[G^*]$ by commutators.

Denote by $\mathcal{B}_{\chi_q^s}$ the rank one representation of the algebra $\mathbb{C}_{\mathcal{B}}[M_+]$ defined by the character χ_q^s. Using the description of the algebra $W_{\mathcal{B}}^s(G)$ in terms of action (3.2.23) and the isomorphism $\mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}} = \mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}$ one can also define the algebra $W_{\mathcal{B}}^s(G)$ as the intersection

$$W_{\mathcal{B}}^s(G) = \mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}[M_+]}(\mathcal{B}_{\chi_q^s}, \mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}) \cap Q_{\mathcal{B}}.$$

Using Frobenius reciprocity we also have

$$\mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}[M_+]}(\mathcal{B}_{\chi_q^s}, \mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}) = \mathrm{End}_{\mathbb{C}_{\mathcal{B}}[G^*]}(\mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}).$$

Hence the algebra $W_{\mathcal{B}}^s(G)$ acts on the space $\mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}$ from the right by operators commuting with the natural left $\mathbb{C}_{\mathcal{B}}[G^*]$–action on $\mathbb{C}_{\mathcal{B}}[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}[M_+]} \mathcal{B}_{\chi_q^s}$. By the definition of $W_{\mathcal{B}}^s(G)$ this action preserves $Q_{\mathcal{B}}$ and by the above presented arguments it commutes with the natural left $\mathbb{C}_{\mathcal{B}}[G_*]$–action on $Q_{\mathcal{B}}$.

Thus $Q_{\mathcal{B}}$ is a $\mathbb{C}_{\mathcal{B}}[G_*]$–$W_{\mathcal{B}}^s(G)$ bimodule equipped also with the adjoint action of $\mathbb{C}_{\mathcal{B}}[M_+]$. By (3.2.18) the adjoint action satisfies

$$\mathrm{Ad}x(yv) = \mathrm{Ad}x^1(y)\mathrm{Ad}x^2(v), \; x \in \mathbb{C}_{\mathcal{B}}[M_+], \; y \in \mathbb{C}_{\mathcal{B}}[G_*], v \in Q_{\mathcal{B}}, \tag{3.2.24}$$

and $\Delta_s(x) = x^1 \otimes x^2$.

Denote by $1 \in Q_{\mathcal{B}}$ the image of the element $1 \in \mathbb{C}_{\mathcal{B}}[G_*]$ in the quotient $Q_{\mathcal{B}}$ under the canonical projection $\mathbb{C}_{\mathcal{B}}[G_*] \to Q_{\mathcal{B}}$. Obviously 1 is the generating vector for $Q_{\mathcal{B}}$ as a module over $\mathbb{C}_{\mathcal{B}}[G_*]$. Using formula (3.2.24) and recalling that $Q_{\mathcal{B}}$ is a $\mathbb{C}_{\mathcal{B}}[G_*]$–$W_{\mathcal{B}}^s(G)$ bimodule, for $x \in \mathbb{C}_{\mathcal{B}}[M_+], y \in \mathbb{C}_{\mathcal{B}}[G_*]$, and for a representative $w \in \mathbb{C}_{\varepsilon}[G_*]$ of an element $w + I_{\mathcal{B}} \in W_{\mathcal{B}}^s(G)$ we have

$$\mathrm{Ad}x(wy1) = \mathrm{Ad}x(yw1) = \mathrm{Ad}x_1(y)\mathrm{Ad}x_2(w1) =$$

$$= \mathrm{Ad}x_1(y)\varepsilon_s(x_2)w1 = \mathrm{Ad}x(y)w1 = w\mathrm{Ad}x(y1).$$

Since $Q_{\mathcal{B}}$ is generated by the vector 1 over $\mathbb{C}_{\mathcal{B}}[G_*]$ the last relation implies that the action of $W_{\mathcal{B}}^s(G)$ on $Q_{\mathcal{B}}$ commutes with the adjoint action.

We can summarize the results of the discussion above in the following proposition.

Proposition 3.2.9. *The space $Q_{\mathcal{B}}$ is naturally equipped with the structure of a left $\mathbb{C}_{\mathcal{B}}[G_*]$–module, a right $\mathbb{C}_{\mathcal{B}}[M_+]$–module via the adjoint action and a right $W^s_{\mathcal{B}}(G)$–module in such a way that the left $\mathbb{C}_{\mathcal{B}}[G_*]$–action and the right $\mathbb{C}_{\mathcal{B}}[M_+]$–action commute with the right $W^s_{\mathcal{B}}(G)$–action and compatibility condition (3.2.24) is satisfied.*

Finally we remark that by specializing q to a particular value $\varepsilon \in \mathbb{C}$, $\varepsilon \neq 0$, one can define a complex associative algebra $\mathbb{C}_\varepsilon[G_*] = \mathbb{C}_{\mathcal{B}}[G_*]/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}[G_*]$, its subalgebra $\mathbb{C}_\varepsilon[M_+]$ with a nontrivial character χ^s_ε and the corresponding W–algebra

$$W^s_\varepsilon(G) = \mathrm{Hom}_{\mathbb{C}_\varepsilon[M_+]}(\mathbb{C}_{\varepsilon_s}, Q_\varepsilon), \tag{3.2.25}$$

where $\mathbb{C}_{\varepsilon_s}$ is the trivial representation of the algebra $\mathbb{C}_\varepsilon[M_+]$ induced by the counit, $Q_\varepsilon = Q_{\mathcal{B}}/Q_{\mathcal{B}}(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})$.

Obviously, for generic ε we have $W^s_\varepsilon(G) = W^s_{\mathcal{B}}(G)/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})W^s_{\mathcal{B}}(G)$.

3.3 Poisson reduction

In this section we recall basic facts on Poisson reduction. They will be used in the next section to describe Poisson q-W–algebras as reduced Poisson algebras.

Let $M,\ B,\ B'$ be Poisson manifolds. Two Poisson surjections

$$\begin{array}{ccccc} & & M & & \\ & \overset{\pi'}{\swarrow} & & \overset{\pi}{\searrow} & \\ B' & & & & B \end{array}$$

form a dual pair if the pullback $\pi'^* C^\infty(B')$ is the centralizer of $\pi^* C^\infty(B)$ in the Poisson algebra $C^\infty(M)$. In this case the sets $B'_b = \pi'\left(\pi^{-1}(b)\right),\ b \in B$ are Poisson submanifolds in B' called reduced Poisson manifolds.

Fix an element $b \in B$. Then the algebra of functions $C^\infty(B'_b)$ may be described as follows. Let I_b be the ideal in $C^\infty(M)$ generated by elements $\pi^*(f),\ f \in C^\infty(B),\ f(b) = 0$. Denote $M_b = \pi^{-1}(b)$. Then the algebra $C^\infty(M_b)$ is simply the quotient of $C^\infty(M)$ by I_b. Denote by $P_b : C^\infty(M) \to C^\infty(M)/I_b = C^\infty(M_b)$ the canonical projection onto the quotient.

Lemma 3.3.1. *Suppose that the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$. Then one can define an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$ by*

$$f \cdot \varphi = P_b(\{\pi^*(f), \tilde{\varphi}\}), \tag{3.3.1}$$

where $f \in C^\infty(B)$, $\varphi \in C^\infty(M_b)$ and $\tilde{\varphi} \in C^\infty(M)$ is a representative of φ in $C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Moreover, $C^\infty(B'_b)$ is the subspace of invariants in $C^\infty(M_b)$ with respect to this action.

Proof. Let $\varphi \in C^\infty(M_b)$. Choose a representative $\tilde{\varphi} \in C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, Hamiltonian vector fields of functions $\pi^*(f),\ f \in C^\infty(B)$ are tangent to the submanifold M_b. Therefore the right hand side of (3.3.1) only depends on φ but not on the representative $\tilde{\varphi}$, and hence formula (3.3.1) defines an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$.

Using the definition of the dual pair we obtain that $\varphi = {\pi'}^*(\psi)$ for some $\psi \in C^\infty(B'_b)$ if and only if $P_b(\{\pi^*(f), \tilde{\varphi}\}) = 0$ for every $f \in C^\infty(B)$.

Finally we obtain that $C^\infty(B'_b)$ is exactly the subspace of invariants in $C^\infty(M_b)$ with respect to action (3.3.1). □

Definition 3.3.2. *The algebra $C^\infty(B'_b)$ is called a reduced Poisson algebra. We also denote it by $C^\infty(M_b)^{C^\infty(B)}$.*

Remark 3.3.3. *Note that the description of the algebra $C^\infty(M_b)^{C^\infty(B)}$ obtained in Lemma 3.3.1 is independent of both the manifold B' and the projection π'. Observe also that the Hamiltonian vector fields of functions $\pi^*(f), f \in C^\infty(B)$ are tangent to M_b, and hence the reduced space B'_b may be identified with across–section of the action of the Poisson algebra $C^\infty(B)$ on M_b by Hamiltonian vector fields in the case when this action is free. In particular, in this case B'_b may be regarded as a submanifold in M_b.*

In the case when the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$ the Poisson structure on the algebra $C^\infty(B'_b)$ can be explicitly described as follows. Let $\varphi_1, \varphi_2 \in C^\infty(M_b)^{C^\infty(B)}$. Choose representatives $\tilde{\varphi}_{1,2} \in C^\infty(M)$ such that $P_b(\tilde{\varphi}_{1,2}) = \varphi_{1,2}$. Then

$$\{\varphi_1, \varphi_2\} = \{\tilde{\varphi}_1, \tilde{\varphi}_2\} \mod I_b. \tag{3.3.2}$$

By Lemma 3.3.1 the class in $C^\infty(M)/I_b = C^\infty(M_b)$ of the function in right hand side of this formula is $C^\infty(B)$–invariant and independent of the choice of the representatives $\tilde{\varphi}_{1,2} \in C^\infty(M)$.

An important example of dual pairs is provided by Poisson group actions. Recall that a (local) left Poisson group action of a Poisson–Lie group A on a Poisson manifold M is a (local) left group action $A \times M \to M$ which is also a Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure).

If the space M/A is a smooth manifold, there exists a unique Poisson structure on M/A such that the canonical projection $M \to M/A$ is a Poisson map.

Let $\mathfrak{a}$ be the Lie algebra of A. Denote by $\langle\cdot,\cdot\rangle$ the canonical paring between $\mathfrak{a}^*$ and $\mathfrak{a}$. A map $\mu : M \to A^*$ is called a moment map for a (local) left Poisson group action $M \times A \to M$ if

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X\rangle(\xi_\varphi), \tag{3.3.3}$$

where θ_{A^*} is the universal left–invariant Maurer–Cartan form on A^*, $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$.

Proposition 3.3.4. ([67], Theorem 4.9) *Let $A \times M \to M$ be a left (local) Poisson group action of a Poisson–Lie group A on a Poisson manifold M with moment map $\mu : M \to A^*$. Denote by Π_{A^*} the Poisson tensor of A^*. Then there exists a right invariant bivector field Λ on A^* such that $\Pi_\mu = \Pi_{A^*} + \Lambda$ is a Poisson tensor on A^* and the map $\mu : M \to A^*_\mu$ is Poisson, where A^*_μ the manifold A^* equipped with the Poisson structure associated to Π_μ.*

From the definition of the moment map it follows that if M/A is a smooth manifold then the canonical projection $M \to M/A$ and the moment map $\mu : M \to A^*$ form a dual pair.

The main example of Poisson group actions is the so–called dressing action. The dressing action may be described as follows.

Proposition 3.3.5. *Let G be a connected simply connected Poisson–Lie group with factorizable tangent Lie bialgebra, G^* the dual group. Then there exists a unique left local Poisson group action*

$$G \times G^* \to G^*, \quad ((L_+, L_-), g) \mapsto g \circ (L_+, L_-),$$

such that the identity mapping $\mu : G^* \to G^*$ is the moment map for this action.

Moreover, let $q : G^* \to G$ be the map defined by

$$q(L_+, L_-) = L_-^{-1}L_+.$$

Then

$$q(g \circ (L_+, L_-)) = gL_-L_+^{-1}g^{-1}. \tag{3.3.4}$$

The notion of Poisson group actions may be generalized as follows. Let $A \times M \to M$ be a Poisson group action of a Poisson–Lie group A on a Poisson manifold M. A subgroup $K \subset A$ is called admissible if the set $C^\infty(M)^K$ of K-invariants is a Poisson subalgebra in $C^\infty(M)$. If space M/K is a smooth manifold, we may identify the algebras $C^\infty(M/K)$ and $C^\infty(M)^K$. Hence there exists a Poisson structure on M/K such that the canonical projection $M \to M/K$ is a Poisson map.

Proposition 3.3.6. ([90], Theorem 6; [67], §2) *Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of a Poisson–Lie group A. A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is admissible if the annihilator $\mathfrak{k}^\perp$ of $\mathfrak{k}$ in $\mathfrak{a}^*$ is a Lie subalgebra $\mathfrak{k}^\perp \subset \mathfrak{a}^*$.*

We shall need the following particular example of dual pairs arising from Poisson group actions.

Let $A \times M \to M$ be a left (local) Poisson group action of a Poisson–Lie group A on a manifold M. Suppose that this action possesses a moment map $\mu : M \to A^*$. Let K be an admissible subgroup in A. Denote by $\mathfrak{k}$ the Lie algebra of K. Assume that $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra in $\mathfrak{a}^*$. Suppose also that there is a splitting $\mathfrak{a}^* = \mathfrak{t} + \mathfrak{k}^\perp$ (direct sum of vector spaces), and that $\mathfrak{t}$ is a Lie subalgebra in $\mathfrak{a}^*$. Then the linear space $\mathfrak{k}^*$ is naturally identified with $\mathfrak{t}$.

Assume that $A^* = TK^\perp$ as a manifold, where $K^\perp, T$ are the Lie subgroups of A^* corresponding to the Lie subalgebras $\mathfrak{k}^\perp, \mathfrak{t} \subset \mathfrak{a}^*$, respectively. For any $a^* = tk^\perp \in A^*$ with $k^\perp \in K^\perp$, $t \in T$ denote $\pi_{K^\perp}(a^*) = k^\perp$, $\pi_T(a^*) = t$. This defines maps $\pi_{K^\perp} : A^* \to K^\perp$, $\pi_T : A^* \to T$.

Proposition 3.3.7. *Suppose that for any $k^\perp \in K^\perp$ the transformation*

$$\begin{aligned} \mathfrak{t} &\to \mathfrak{t}, \\ t &\mapsto (\mathrm{Ad}(k^\perp)t)_{\mathfrak{t}}, \end{aligned} \tag{3.3.5}$$

where the subscript $\mathfrak{t}$ stands for the $\mathfrak{t}$–component with respect to the decomposition $\mathfrak{a}^* = \mathfrak{t} + \mathfrak{k}^\perp$, is invertible.

Define a map $\overline{\mu} : M \to T$ by

$$\overline{\mu} = \pi_T \mu.$$

Then

(i) $\overline{\mu}^*(C^\infty(T))$ is a Poisson subalgebra in $C^\infty(M)$, and hence one can equip T with a Poisson structure such that $\overline{\mu} : M \to T$ is a Poisson map.

(ii) Moreover, the algebra $C^\infty(M)^K$ is the centralizer of $\overline{\mu}^*(C^\infty(T))$ in the Poisson algebra $C^\infty(M)$. In particular, if M/K is a smooth manifold the maps

$$\begin{array}{ccccc} & & M & & \\ & \overset{\pi}{\swarrow} & & \overset{\overline{\mu}}{\searrow} & , \\ M/K & & & & T \end{array} \tag{3.3.6}$$

form a dual pair.

Proof. (i) We claim that multiplication in A^* gives rise to a right Poisson group action $A^*_\mu \times A^* \to A^*_\mu$. Indeed, for $g \in A^*$ denote by l_g, r_g the left (right) translation by g on A^*. By the definition of Π_μ

$$\Pi_\mu(gh) = \Pi_{A^*}(gh) + \Lambda(gh) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_{A^*}(g) + r_{h*}\Lambda(g) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_\mu(g), \tag{3.3.7}$$

where we used the fact that $\Pi_{A^*}(gh) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_{A^*}(g)$ as A^* is a Poisson–Lie group and that Λ is right invariant. Identity (3.3.7) is equivalent to the fact that multiplication in A^* gives rise to a right Poisson group action $A^*_\mu \times A^* \to A^*_\mu$.

Since $\mathfrak{k} \subset \mathfrak{a}$ is a Lie subalgebra and $\mathfrak{k}^{\perp\perp} = \mathfrak{k}$ the subgroup $K^\perp \subset A^*$ is admissible. Therefore restricting the action $A^*_\mu \times A^* \to A^*_\mu$ to $K^\perp$ we deduce that $C^\infty(A^*_\mu)^{K^\perp}$ is a Poisson subalgebra in $C^\infty(A^*_\mu)$, where the action of $K^\perp$ is induced by the action of $K^\perp \subset A^*$ on A^* by right translations.

Now recall that $A^* = TK^\perp$ as a manifold, and hence $C^\infty(A^*_\mu)^{K^\perp} = C^\infty(T)$. Thus T naturally becomes a Poisson manifold and the map $\pi_T : A^*_\mu \to T$ becomes Poisson. We deduce that the map $\overline{\mu} = \pi_T\mu$ is Poisson as the composition of the Poisson maps $\mu : M \to A^*_\mu$ and $\pi_T : A^*_\mu \to T$.

(ii) By the definition of the moment map we have

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X\rangle(\xi_\varphi), \tag{3.3.8}$$

where $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$. Since $A^* = TK^\perp$ the pullback of the left–invariant Maurer–Cartan form $\mu^*(\theta_{A^*})$ may be represented as follows

$$\mu^*(\theta_{A^*}) = Ad(\pi_{K^\perp}\mu)^{-1}(\overline{\mu}^*\theta_T) + (\pi_{K^\perp}\mu)^*\theta_{K^\perp},$$

where $(\pi_{K^\perp}\mu)^*\theta_{K^\perp} \in \mathfrak{k}^\perp$.

Now let $X \in \mathfrak{k}$. Then $\langle(\pi_{K^\perp}\mu)^*\theta_{K^\perp}), X\rangle = 0$ and formula (3.3.8) takes the form

$$\begin{aligned} L_{\widehat{X}}\varphi &= \langle Ad(\pi_{K^\perp}\mu)^{-1}(\overline{\mu}^*\theta_T), X\rangle(\xi_\varphi) = \\ &= \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)). \end{aligned} \tag{3.3.9}$$

Since by the assumption transformation (3.3.5) is invertible, $L_{\widehat{X}}\varphi = \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)) = 0$ for any $X \in \mathfrak{k}$ if and only if $\overline{\mu}_*(\xi_\varphi) = 0$. Thus the function $\varphi \in C^\infty(M)$ is K–invariant if and only if $\{\varphi, \overline{\mu}^*(\psi)\} = 0$ for any $\psi \in C^\infty(T)$. This completes the proof.

□

From the previous proposition, from Lemma 3.3.1 and Remark 3.3.3 we immediately obtain the following corollary.

Corollary 3.3.8. *Suppose that the conditions of Proposition 3.3.7 are satisfied. Let $t \in T$ be such that the map $f \mapsto f(t)$ is a character of the Poisson algebra $C^\infty(T)$. Then the action of K on M induces a (local) action on $\overline{\mu}^{-1}(t)$ and a (local) action on $C^\infty(\overline{\mu}^{-1}(t))$ given by*

$$X \circ \varphi = \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_{\tilde{\varphi}})), X \in \mathfrak{k}, \varphi \in C^\infty(\overline{\mu}^{-1}(t)),$$

where $\tilde{\varphi}$ is any representative of $\varphi \in C^\infty(\overline{\mu}^{-1}(t)) = C^\infty(M)/I_t$ in $C^\infty(M)$. The algebra $C^\infty(\overline{\mu}^{-1}(t))^K$ of invariants with respect to this action is isomorphic to the reduced Poisson algebra $C^\infty(\overline{\mu}^{-1}(t))^{C^\infty(T)}$.

3.4 Poisson reduction and q-W–algebras

In this section we realize the quasiclassical limit of the algebra $W^s_{\mathcal{B}}(G)$ as the algebra of functions on a reduced Poisson manifold. In this section we always assume that $\kappa = 1$.

Denote by χ^s the character of the Poisson subalgebra $\mathbb{C}[M_+]$ such that $\chi^s(p(x)) = \chi^s_q(x) \mod (q^{\frac{1}{dr^2}} - 1)$ for every $x \in \mathbb{C}_{\mathcal{B}}[M_+]$.

Note that the image of the algebra $\mathbb{C}_{\mathcal{B}}[G_*]$ under the projection $p : \mathbb{C}_{\mathcal{B}}[G^*] \to \mathbb{C}_{\mathcal{B}}[G^*]/(1 - q^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}[G^*]$ is a certain subalgebra of $\mathbb{C}[G^*]$ that we denote by $\mathbb{C}[G_*]$. By Proposition 3.2.5 we have $\mathbb{C}[G_*] \simeq \mathbb{C}[G]$ as algebras. Let $I = p(I_{\mathcal{B}})$ be the ideal in $\mathbb{C}[G^*]$ generated by the kernel of χ^s. Then by the discussion after formula (3.2.23) the Poisson algebra $W^s(G) = W^s_q(G)/(q^{\frac{1}{dr^2}} - 1)W^s_q(G)$ is the subspace of all $x + I \in Q_1$, $Q_1 = Q_{\mathcal{B}}/(1 - q^{\frac{1}{dr^2}})Q_{\mathcal{B}} \subset \mathbb{C}[G^*]/I$, such that $\{m, x\} \in I$ for any $m \in \mathbb{C}[M_-]$, and the Poisson bracket in $W^s(G)$ takes the form $\{(x+I), (y+I)\} = \{x, y\} + I$, $x + I, y + I \in W^s(G)$. We shall also write $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]} \cap Q_1$, where the action of the Poisson algebra $\mathbb{C}[M_+]$ on the space $\mathbb{C}[G^*]/I$ is defined as follows

$$x \cdot (v + I) = \rho_{\chi^s}(\{x, v\}), \tag{3.4.1}$$

$v \in \mathbb{C}[G^*]$ is any representative of $v + I \in \mathbb{C}[G^*]/I$ and $x \in \mathbb{C}[M_+]$.

One can describe the space of invariants $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]}$ with respect to this action by analyzing the related underlying manifolds and varieties. First observe that the algebra $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]}$ is a particular example of the reduced Poisson algebra introduced in Lemma 3.3.1 .

Indeed, recall that according to (3.2.1) any element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (n_+, n_-)(h_+, h_-), \tag{3.4.2}$$

where $n_\pm \in N_\pm$, $h_+ = exp((\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $h_- = exp((\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $x \in \mathfrak{h}$.

Formula (3.2.1) and a decomposition of elements of N_+ into products of elements which belong to one–dimensional subgroups corresponding to roots also imply that any element L_+ can be represented in the form

$$\begin{array}{l} L_+ = \prod_\beta exp[b_\beta X_\beta] \times \\ exp\left[\sum_{i=1}^l b_i(\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})H_i\right],\ b_i, b_\beta \in \mathbb{C}, \end{array} \tag{3.4.3}$$

where the product over roots is taken according to the normal ordering associated to s.

Now define a moment map $\mu_{M_-} : G^* \to M_+$ by

$$\mu_{M_-}(L_+, L_-) = m_+, \tag{3.4.4}$$

where for L_+ given by (3.4.3) m_+ is defined as follows

$$m_+ = \prod_{\beta \in \Delta_{\mathfrak{m}_+}} exp[b_\beta X_\beta],$$

and the product over roots is taken according to the normal order in the segment $\Delta_{\mathfrak{m}_+}$.

Note that by definition $\mathbb{C}[M_+] = \{\varphi \in \mathbb{C}[G^*] : \varphi = \varphi(m_+)\}$. Therefore $\mathbb{C}[M_+]$ is generated by the pullbacks of regular functions on M_+ with respect to the map μ_{M_-}. Since $\mathbb{C}[M_+]$ is a Poisson subalgebra in $\mathbb{C}[G^*]$, and regular functions on M_+ are dense in $C^\infty(M_+)$ on every compact subset, we can equip the manifold M_+ with the Poisson structure in such a way that μ_{M_-} becomes a Poisson mapping.

Let $u \in M_+$ be the element defined by

$$u = \prod_{i=1}^{l'} exp[t_i X_{\gamma_i}], t_i = k_i \ (\mathrm{mod}\ (q^{\frac{1}{dr^2}} - 1)), \tag{3.4.5}$$

where the product over roots is taken according to the normal order in the segment $\Delta_{\mathfrak{m}_+}$.

By Proposition 3.2.3 the elements $L^{\pm,V} = (p \otimes p_V)({}^q L^{\pm,V})$ belong to the space $\mathbb{C}[G^*] \otimes \mathrm{End}\overline{V}$, where $p_V : V^{res} \to \overline{V} = V^{res}/(q^{\frac{1}{dr^2}} - 1)V^{res}$ is the projection of the finite rank $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–module V^{res} onto the corresponding $\mathfrak{g}$–module $\overline{V}$, and the map

$$\mathbb{C}_{\mathcal{B}}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G^*] \to \mathbb{C}[G^*],\ L^{\pm,V} \mapsto L^{\pm,\overline{V}}$$

is an isomorphism. In particular, from (2.6.11) it follows that

$$\begin{array}{l} L^{+,\overline{V}} = \prod_\beta exp[p((1 - q_\beta^{-2}) f_\beta) \otimes \pi_{\overline{V}}(X_\beta)] \times \\ (p \otimes id) exp\left[\sum_{i=1}^{l} hH_i \otimes \pi_{\overline{V}}((\frac{2}{1-s} P_{\mathfrak{h}'} + P_{\mathfrak{h}'^\perp})Y_i)\right]. \end{array} \tag{3.4.6}$$

From (3.4.6) and the definition of χ^s we obtain that $\chi^s(\varphi) = \varphi(u)$ for every $\varphi \in \mathbb{C}[M_+]$. χ^s naturally extends to a character of the Poisson algebra $C^\infty(M_+)$.

Now applying Lemma 3.3.1 we can define a reduced Poisson algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ as follows. Denote by I_u the ideal in $C^\infty(G^*)$ generated by elements $\mu_{M_-}^* \psi,\ \psi \in C^\infty(M_+),\ \psi(u) = 0$. Let $P_u : C^\infty(G^*) \to C^\infty(G^*)/I_u = C^\infty(\mu_{M_-}^{-1}(u))$ be the canonical projection. Define the action of $C^\infty(M_+)$ on $C^\infty(\mu_{M_-}^{-1}(u))$ by

$$\psi \cdot \varphi = P_u(\{\mu_{M_-}^* \psi, \tilde{\varphi}\}), \tag{3.4.7}$$

where $\psi \in C^\infty(M_-),\ \varphi \in C^\infty(\mu_{M_-}^{-1}(u))$ and $\tilde{\varphi} \in C^\infty(G^*)$ is a representative of φ such that $P_u \tilde{\varphi} = \varphi$. The reduced Poisson algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ is the algebra of $C^\infty(M_+)$–invariants in $C^\infty(\mu_{M_-}^{-1}(u))$ with respect to action (3.4.7). The reduced Poisson algebra is naturally equipped with a Poisson structure induced from $C^\infty(G^*)$ as described in (3.3.2).

Lemma 3.4.1. *Let $\overline{q(\mu_{M_-}^{-1}(u))}$ be the closure of $q(\mu_{M_-}^{-1}(u))$ in G with respect to Zariski topology. Then $Q_1 \simeq \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$, and the algebra $W^s(G)$ is isomorphic to the algebra of regular functions on $\overline{q(\mu_{M_-}^{-1}(u))}$ pullbacks of which under the map q are invariant with respect to the action (3.4.7) of $C^\infty(M_+)$ on $C^\infty(\mu_{M_-}^{-1}(u))$, i.e.*

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)},$$

where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map $q^* : C^\infty(q(\mu_{M_-}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$ and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

Proof. First observe that by the definition $\mu_{M_-}^{-1}(u)$ is a submanifold in G^* and that $I = \mathbb{C}[G^*] \cap I_u$. Therefore by the definition of the algebra $\mathbb{C}[G^*]$ and of the map μ_{M_-} the quotient $\mathbb{C}[G^*]/I$ is identified with the algebra of functions on $\mu_{M_-}^{-1}(u)$ generated by the restrictions of elements of $\mathbb{C}[G^*]$ to $\mu_{M_-}^{-1}(u)$.

Also by the definition $Q_1 \subset \mathbb{C}[G^*]/I$ is the algebra generated by the restrictions to $\mu_{M_-}^{-1}(u)$ of the pullbacks of elements from the algebra of regular functions $\mathbb{C}[G]$ under the map $q : G^* \to G$. Therefore $Q_1 = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$.

From these observations we deduce that $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap Q_1 = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$.

Since $\mathbb{C}[M_-]$ is dense in $C^\infty(M_-)$ on every compact subset in M_- we have

$$C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)} \cong C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]}.$$

Now observe that action (3.4.7) of elements from $\mathbb{C}[M_-]$ coincides with action (3.4.1) when restricted to elements from $\mathbb{C}[G^*]/I$, and hence $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] = C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] = C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_-)} \cap \mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}]$. This completes the proof.

□

We shall realize the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ as the algebra of functions on a reduced Poisson manifold. In this construction we use the dressing action of the Poisson–Lie group G on G^*.

Consider the restriction of the dressing action $G \times G^* \to G^*$ to the subgroup $M_- \subset G$. Let G^*/M_- be the quotient of G^* with respect to the dressing action of M_-, $\pi : G^* \to G^*/M_-$ the canonical projection. Note that the space G^*/M_- is not a smooth manifold. However, it turns out that the subspace $\pi(\mu_{M_-}^{-1}(u)) \subset G^*/M_-$ is a smooth manifold. More precisely, we have the following lemma.

Lemma 3.4.2. *The preimage $\mu_{M_-}^{-1}(u) \subset G^*$ is locally stable under the (locally defined) dressing action of M_-, and the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$.*

Proof. The proof will be based on Corollary 3.3.8.

First observe that according to part (iv) of Proposition 3.1.2 $(\mathfrak{i}_-, \mathfrak{i}_+) = (\mathfrak{b}_-, \mathfrak{b}_+)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Therefore B_- is a Poisson–Lie subgroup in G.

By Proposition 3.3.5 and by the definition of the moment map we have for any $X \in \mathfrak{b}_-$, $\varphi \in C^\infty(G^*)$

$$L_{\widehat{X}}\varphi(L_+, L_-) = (\theta_{G^*}(L_+, L_-), X)(\xi_\varphi) = (r_+^{-1}\mu_{B_-}^*(\theta_{B_+}), X)(\xi_\varphi), \tag{3.4.8}$$

where $\widehat{X}$ is the corresponding vector field on G^*, ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(G^*)$, and the map $\mu_{B_-} : G^* \to B_+$ is defined by $\mu_{B_-}(L_+, L_-) = L_+$. Now from Proposition 3.1.2 (iv) and the definition of the moment map it follows that μ_{B_-} is a moment map for the restriction of the dressing action to the subgroup B_-.

Next observe that the complementary subset to $\Delta_{\mathfrak{m}_+}$ in Δ_+ is a minimal segment $\Delta_{\mathfrak{m}_+}^0$. Now using Proposition 3.1.2 (iv) the subspace $\mathfrak{m}_-^\perp$ in $\mathfrak{b}_+$ can be identified with the linear subspace in $\mathfrak{b}_+$ spanned by the Cartan subalgebra $\mathfrak{h}$ and by the root subspaces corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$. Using the fact that the adjoint action of $\mathfrak{h}$ normalizes root subspaces and Lemma 1.6.6 we deduce that $\mathfrak{m}_-^\perp \subset \mathfrak{b}_+$ is a Lie subalgebra, and hence $M_- \subset B_-$ is an admissible subgroup.

Moreover, the dual group B_+ can be uniquely factorized as $B_+ = M_+M_-^\perp$, where $M_-^\perp \subset B_+$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_-^\perp \subset \mathfrak{b}_+$, and $M_+ \subset B_+$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+$.

Now observe that $\mathfrak{m}_-^\perp = \mathfrak{h} + \mathfrak{m}_{-0}^\perp \subset \mathfrak{b}_+$ (direct sum of vector spaces), where $\mathfrak{m}_{-0}^\perp$ is the Lie subalgebra generated by the root vectors corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$. The Lie subalgebra $\mathfrak{m}_-$ is generated by the root subspaces corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}$. Since all root subspaces are invariant under the adjoint action of $\mathfrak{h}$ and the restriction of the adjoint action of the root vectors corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$ is nilpotent we deduce that for any $m_+ \in \mathfrak{m}_+$, $k^\perp \in M_-^\perp$, $k^\perp = hk_0^\perp$, $h \in H$, $k_0^\perp = \exp(x)$, $x \in \mathfrak{m}_{-0}^\perp$ one has

$$(\mathrm{Ad}(hk_0^\perp)(m_+))_{\mathfrak{m}_+} = \mathrm{Ad}h((\mathrm{Ad}k_0^\perp(m_+))_{\mathfrak{m}_+}) = \mathrm{Ad}h((\exp(\mathrm{ad}x)(m_+))_{\mathfrak{m}_+}) = \mathrm{Ad}h((\mathrm{Id} + V)(m_+)),$$

where the subscript $\mathfrak{m}_+$ stands for the $\mathfrak{m}_+$–component in the direct vector space decomposition $\mathfrak{b}_+ = \mathfrak{m}_+ + \mathfrak{m}_-^\perp$, and V is a linear nilpotent transformation of $\mathfrak{m}_+$.

The maps $\mathrm{Ad}h$ and $\mathrm{Id} + V$ are obviously invertible. Hence for any $k^\perp \in M_-^\perp$ the map

$$\mathfrak{m}_+ \to \mathfrak{m}_+, m_+ \mapsto (\mathrm{Ad}(hk_0^\perp)(m_+))_{\mathfrak{m}_+}$$

is invertible as well.

We conclude that all the conditions of Corollary 3.3.8 are satisfied with $A = B_-, K = M_-, A^* = B_+, T = M_+, K^\perp = M_-^\perp, \mu = \mu_{B_-}$. It follows that the preimage $\mu_{M_-}^{-1}(u) \subset G^*$ is locally stable under the (locally defined) dressing action of M_-, and the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$. This completes the proof.

□

Observe that by (3.3.4) under the map $q : G^* \to G$, $q(L_+, L_-) = L_-^{-1}L_+$ the dressing action becomes the action of G on itself by conjugations. Consider the restriction of this action to the subgroup M_+. Denote by $\pi_q : G \to G/M_-$ the canonical projection onto the quotient with respect to this action. We shall see that $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is an algebraic variety and $\mathbb{C}[\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})] \simeq W^s(G)$. We shall also obtain an explicit description of the variety $\overline{q(\mu_{M_-}^{-1}(u))}$ and of the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$.

First we describe the image of the level surface $\mu_{M_-}^{-1}(u)$ of the moment map μ_{M_-} under the map q. Let $X_\alpha(t) = \exp(tX_\alpha) \in G$, $t \in \mathbb{C}$ be the one–parameter subgroup in the algebraic group G corresponding to root $\alpha \in \Delta$. Recall that for any $\alpha \in \Delta_+$ and any $t \neq 0$ the element

$$s_\alpha(t) = X_\alpha(t)X_{-\alpha}(-t^{-1})X_\alpha(t) \in G \tag{3.4.9}$$

is a representative for the reflection s_α corresponding to the root α. Denote by $s \in G$ the following representative of the Weyl group element $s \in W$,

$$s = s_{\gamma_1}(t_1)\ldots s_{\gamma_{l'}}(t_{l'}), \tag{3.4.10}$$

where the numbers t_i are defined in (3.4.5), and we assume that $t_i \neq 0$ for any i.

We shall also use the following representatives for s^1 and s^2

$$s^1 = s_{\gamma_1}(t_1)\ldots s_{\gamma_n}(t_n),\ s^2 = s_{\gamma_{n+1}}(t_{n+1})\ldots s_{\gamma_{l'}}(t_{l'}).$$

The following Proposition is an improved version of Proposition 7.2 in [102] suitable for the purposes of quantization.

Proposition 3.4.3. *Let $q : G^* \to G$ be the map defined by*

$$q(L_+, L_-) = L_-^{-1}L_+.$$

Suppose that the numbers t_i defined in (3.4.5) are not equal to zero for all i. Then

$$q(\mu_{M_-}^{-1}(u)) \subset N_-sH_0Z_+M_- = N_-sH_0M_-Z_+ = (N_- \cap N)Z_-sH_0M_-Z_+ = \tag{3.4.11}$$

$$= (N_- \cap N)Z_-sH_0Z_+M_- \subset NsZN,$$

where H_0 is the subgroup corresponding to the orthogonal complement $\mathfrak{h}'^\perp$ of $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the symmetric bilinear form on $\mathfrak{g}$, $Z_\pm = Z \cap N_\pm$. The closure $\overline{q(\mu_{M_-}^{-1}(u))}$ of $q(\mu_{M_-}^{-1}(u))$ with respect to the Zariski topology is also contained in $NsZN$.

Proof. Using definition (3.4.4) of the map μ_{M_-} we can describe the preimage $\mu_{M_-}^{-1}(u)$ as follows

$$\mu_{M_-}^{-1}(u) = \{(uyh_+, n_-h_-) | n_- \in N_-, h_\pm = e^{r_\pm^s x}, x \in \mathfrak{h}, y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}\}, \tag{3.4.12}$$

where for any additively closed subset of roots $\Xi \subset \Delta$ we denote by N_Ξ the subgroup in G generated by the one–parameter subgroups corresponding to the roots from Ξ. Therefore

$$q(\mu_{M_-}^{-1}(u)) = \{h_-^{-1}n_-^{-1}uyh_+ | n_- \in N_-, h_\pm = e^{r_\pm^s x}, x \in \mathfrak{h}, y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}\}. \tag{3.4.13}$$

First we show that for any $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ and $n_- \in N_-$ the element $n_-^{-1}uy$ belongs to $N_-sMZ_+M_-$. Fix the circular normal ordering on Δ corresponding to the normal ordering of Δ_+ associated to s.

In the proof we shall frequently use the following lemma.

Lemma 3.4.4. *Let $[\alpha, \beta] \subset \Delta$ be a minimal segment and assume that $[\alpha, \beta] = [\alpha, \gamma] \cup [\delta, \beta]$, where the segments $[\alpha, \gamma]$ and $[\delta, \beta]$ are disjoint and minimal as well. Then any element $m \in N_{[\alpha,\beta]}$ can be uniquely factorized as $m = g_1g_2 = g_2'g_1'$, $g_1, g_1' \in N_{[\alpha,\gamma]}$, $g_2, g_2' \in N_{[\delta,\beta]}$. Moreover, if $\delta = \beta$ then for any $m' \in N_{[\alpha,\gamma]}$ and any $t \in \mathbb{C}$ one has $m'X_\beta(t) = X_\beta(t)m''$, where $m'' \in N_{[\alpha,\gamma]}$.*

Proof. The proof is obtained by straightforward application of Chevalley's commutation relations between one–parameter subgroups and Lemma 1.6.6.

□

Since the roots $\gamma_1, \ldots, \gamma_n$ are mutually orthogonal the adjoint action of $s_{\gamma_i}(t_i)$, $i = 1, \ldots, n$ on each of the root subspaces $\mathfrak{g}_{\gamma_j}$, $j = 1, \ldots, n, j \neq i$ is given by multiplication by a non–zero constant. Therefore there are non–zero constants $c_1, \ldots, c_n$ such that $X_{\gamma_k}(c_k)s_{\gamma_1}\ldots s_{\gamma_{k-1}} = s_{\gamma_1}\ldots s_{\gamma_{k-1}}X_{\gamma_k}(-t_k^{-1})$, $k = 2, \ldots, n$, and we define $c_1 = -t_1^{-1}$.

Obviously we have

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = X_{-\gamma_1}(-c_1)\ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n)\ldots X_{-\gamma_1}(c_1)X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) =$$

$$= n_1 X_{-\gamma_n}(c_n)\ldots X_{-\gamma_1}(c_1)X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n), \quad n_1 = X_{-\gamma_1}(-c_1)\ldots X_{-\gamma_n}(-c_n) \in N_{\Delta_-^n},$$

where $\Delta_-^n = \{\alpha \in \Delta_- : -\gamma_1 \leq \alpha \leq -\gamma_n\}$.

Using the relation $X_{-\gamma_1}(-t_1^{-1})X_{\gamma_1}(t_1) = X_{\gamma_1}(-t_1)s_{\gamma_1}$ one can rewrite the last identity as follows

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n_1 X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)s_{\gamma_1}X_{\gamma_2}(t_2)\ldots X_{\gamma_n}(t_n). \tag{3.4.14}$$

Now we can write

$$X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1) =$$
$$= X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2)\ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2).$$

The product $X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2)\ldots X_{-\gamma_n}(-c_n)$ belongs to the subgroup of G generated by the one–parameter subgroups corresponding to roots from the set $\Delta^1 = \{\alpha \in \Delta : -\gamma_2 \leq \alpha \leq \gamma_1, s^1\alpha = -\alpha\}$. By Lemma 1.6.6 the minimal segment $\{\alpha \in \Delta : -\gamma_2 \leq \alpha \leq \gamma_1\}$ is closed under addition of roots and the set of roots on which s^1 acts by multiplication by -1 is obviously closed under addition of roots. Hence Δ^1 is closed under addition of roots. Assume that the order of roots in Δ^1 is induced by the circular normal ordering of Δ. Using Lemma 1.6.6 and the fact that Δ^1 is closed under addition of roots the element $X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2)\ldots X_{-\gamma_n}(-c_n)$ can be represented as a product of elements from one–parameter subgroups corresponding to roots from Δ^1 ordered in the way described above. Denote the intersection of Δ^1 with Δ_+ by $\Delta_+^1 = \{\alpha \in \Delta_+ : \alpha \leq \gamma_1, s^1\alpha = -\alpha\}$ and let $\Delta_-^1 = \Delta^1 \cap \Delta_- = \{\alpha \in \Delta_- : -\gamma_2 \leq \alpha, s^1\alpha = -\alpha\}$. Then the above mentioned factorization yields

$$X_{-\gamma_n}(c_n)\ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2)\ldots X_{-\gamma_n}(-c_n) = n_2' x_1',$$

where $n_2' \in N_{\Delta_-^1}, x_1' \in N_{\Delta_+^1}$.

Substituting the last relation into (3.4.14) and using the definition of c_2 and the orthogonality of roots γ_1 and γ_2 we obtain

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n_2 x_1' X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)s_{\gamma_1}X_{-\gamma_2}(-t_2^{-1})X_{\gamma_2}(t_2)\ldots X_{\gamma_n}(t_n),$$

where $n_2 = n_1 n_2' \in N_{\Delta_-^{s^1}}$, $\Delta_-^{s^1} = \{\alpha \in \Delta_- : s^1\alpha = -\alpha\}$.

Now we can use the relation $X_{-\gamma_2}(-t_2^{-1})X_{\gamma_2}(t_2) = X_{\gamma_2}(-t_2)s_{\gamma_2}$, the orthogonality of roots γ_1 and γ_2, and apply similar arguments to get

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n_2 x_1' X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)s_{\gamma_1}s_{\gamma_2}X_{\gamma_3}(t_3)\ldots X_{\gamma_n}(t_n), a_2 \neq 0. \tag{3.4.15}$$

We can also write

$$X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2) =$$
$$= X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3)\ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3).$$

The product $X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3)\ldots X_{-\gamma_n}(-c_n)$ belongs to the subgroup of G generated by the one–parameter subgroups corresponding to roots from the set $\Delta^2 = \{\alpha \in \Delta : -\gamma_3 \leq \alpha \leq \gamma_2, s^1\alpha = -\alpha\}$. By Lemma 1.6.6 the minimal segment $\{\alpha \in \Delta : -\gamma_3 \leq \alpha \leq \gamma_2\}$ is closed under addition of roots and the set of roots on which s^1 acts by multiplication by -1 is obviously closed under addition of roots. Hence Δ^2 is closed under addition of roots. Assume that the order of roots in Δ^2 is induced by the circular normal ordering of Δ. Using Lemma 1.6.6 and the fact that Δ^2 is closed under addition of roots the element $X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3)\ldots X_{-\gamma_n}(-c_n)$ can be represented as a product of elements from one–parameter subgroups corresponding to roots from Δ^2 ordered in the way described above. Denote the intersection of Δ^2 with Δ_+ by $\Delta_+^2 = \{\alpha \in \Delta_+ : \alpha \leq \gamma_2, s^1\alpha = -\alpha\}$ and let $\Delta_-^2 = \Delta^2 \cap \Delta_- = \{\alpha \in \Delta_- : s^1\alpha = -\alpha, -\gamma_3 \leq \alpha\}$. Then the above mentioned factorization yields

$$X_{-\gamma_n}(c_n)\ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3)\ldots X_{-\gamma_n}(-c_n) = n_3' x_2'', \tag{3.4.16}$$

where $n_3' \in N_{\Delta_-^2}, x_2'' \in N_{\Delta_+^2}$.

Substituting the last relation into (3.4.15) and using the definition of c_3 and the orthogonality of roots γ_1, γ_2 and γ_3 we obtain

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n_2 x_1' n_3' x_2'' X_{-\gamma_n}(c_n)\ldots X_{-\gamma_4}(c_4)s_{\gamma_1}s_{\gamma_2}X_{\gamma_3}(-t_3^{-1})X_{\gamma_3}(t_3)\ldots X_{\gamma_n}(t_n) \tag{3.4.17}$$

Since $N_{\Delta_+^1} \subset N_{\Delta_+^2}$ we deduce $x_1' n_3' x_2'' \in N_{\Delta^2}$. Therefore using arguments applied above to obtain (3.4.16) we get $x_1' n_3' x_2'' = n_3'' x_2'$, $x_2' \in N_{\Delta_+^2}$, $n_3'' \in N_{\Delta_-^2}$, and (3.4.17) takes the form

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n_3 x_2' X_{-\gamma_n}(c_n)\ldots X_{-\gamma_4}(c_4) s_{\gamma_1} s_{\gamma_2} X_{-\gamma_3}(-t_3^{-1}) X_{\gamma_3}(t_3)\ldots X_{\gamma_n}(t_n),$$

where $n_3 = n_2 n_3'' \in N_{\Delta_-^{s^1}}$.

We can proceed in a similar way to obtain the following representation

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n\widetilde{x} s_{\gamma_1}\ldots s_{\gamma_n} = n\widetilde{x} s^1, \quad n \in N_{\Delta_-^{s^1}}, \widetilde{x} \in N_{\Delta_+^n}, \tag{3.4.18}$$

where $\Delta_+^n = \{\alpha \in \Delta_+ : \alpha \le \gamma_n, s^1\alpha = -\alpha\} = \{\alpha \in \Delta_+ : \gamma_1 \le \alpha \le \gamma_n\}$.

Note that s^1 acts by multiplication by -1 on these roots, so $\Delta_-^n = -\Delta_+^n = s^1(\Delta_+^n)$. Then $N_{\Delta_-^n} = (s^1)^{-1} N_{\Delta_+^n} s^1 \subset N_{\Delta_-^{s^1}}$ and (3.4.18) can be rewritten in the following form

$$X_{\gamma_1}(t_1)\ldots X_{\gamma_n}(t_n) = n s^1 (s^1)^{-1}\widetilde{x} s^1 = n s^1 n', \quad n \in N_{\Delta_-^{s^1}}, n' = (s^1)^{-1}\widetilde{x} s^1 \in N_{\Delta_-^n}. \tag{3.4.19}$$

Similarly one has

$$X_{-\gamma_{n+1}}(t_{n+1})\ldots X_{-\gamma_{l'}}(t_{l'}) = n'' s_{\gamma_{n+1}}\ldots s_{\gamma_{l'}} n''' = n'' s^2 n''', \quad n'' \in N_{\Delta_-^{s^2}}, n''' \in N_{\Delta_-^{l'}}, \tag{3.4.20}$$

where $\Delta_-^{s^2} = \{\alpha \in \Delta_- : s^2\alpha = -\alpha\}$, and $\Delta_-^{l'} = \{\alpha \in \Delta_- : -\gamma_{n+1} \le \alpha \le -\gamma_{l'}\}$.

Combining (3.4.19) and (3.4.20), using the definition of the circular normal ordering of the root system Δ associated to s, Lemmas 1.6.6, 3.4.4 and commutation relations between one–parameter subgroups corresponding to roots one can obtain

$$n_-^{-1} u y = n_-^{-1} n s^1 n' n'' s^2 n''' y = k s^1 g s^2 n''' y, g \in N_{\Delta_-^s \setminus \Delta_0}, k \in N_-. \tag{3.4.21}$$

By Proposition 1.6.1

$$s^2 \Delta_{s^1}^s \subset \Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0) \subset \Delta_+ \setminus \Delta_0,$$

and hence

$$s^2 \Delta_{s^1}^s \subset \Delta_{\mathfrak{m}_+} \subset \left(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0)\right) \cap (\Delta_+ \setminus \Delta_0), \tag{3.4.22}$$

Now the minimal segment $\Delta_+^s \setminus \Delta_0$ can be represented as the following disjoint union

$$\Delta_+^s \setminus \Delta_0 = \left(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_0)\right) \cup \Delta_{s^1}^s. \tag{3.4.23}$$

Note that since the decomposition $s = s^1 s^2$ is reduced we have

$$s^1(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_0)) \subset \Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_0) \subset \Delta_+ \setminus \Delta_0. \tag{3.4.24}$$

Observe that the segments $\Delta_{s^1}^s$, $\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_0)$ are minimal with respect to the circular normal ordering on Δ associated to s. Thus from (3.4.23) and Lemma 3.4.4 we deduce that the element $g \in N_{\Delta_-^s \setminus \Delta_0}$ from formula (3.4.21) can be uniquely decomposed as the product $g = g'g''$, where $g' \in N_{-\left(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_0)\right)}$, and $g'' \in N_{-\Delta_{s^1}^s}$. By (3.4.22), (3.4.24) we have $s^1 g' (s^1)^{-1} \in N_-$ and $(s^2)^{-1} g'' s^2 \in M_-$, and (3.4.21) takes the form

$$n_-^{-1} u y = k s^1 g' g'' s^2 n''' y = k s^1 g' (s^1)^{-1} s (s^2)^{-1} g'' s^2 n''' y = k' s \widehat{n} y, \widehat{n} \in M_-, k' \in N_-. \tag{3.4.25}$$

The element $\widehat{n} y$ belongs to the subgroup $N_{\Delta'}$, $\Delta' = \{\alpha \in \Delta : \gamma_{l'} < \alpha \le -\gamma_{l'}\}$. The images of all roots from the set $(\Delta' \setminus \Delta_0) \cap \Delta_+$under the action of s belong to Δ_- . The set complimentary to $(\Delta' \setminus \Delta_0) \cap \Delta_+$ in Δ' is $-\Delta_{\mathfrak{m}_+} \cup (\Delta_0 \cap \Delta_+)$. Therefore using the decomposition of elements $N_{\Delta'}$ as products of elements from one–parameter subgroups corresponding to roots and recalling Lemma 3.4.4 one can get $\widehat{n} y = y' z_+ m$, where $z_+ \in Z_+ = Z \cap N_+$, $s y' s^{-1} \in N_-$, $m \in M_-$, and

$$n_-^{-1} u y = k'' s z_+ m = k'' z_+' s m, m \in M_-, k'' \in N_-, z_+, z_+' = s z_+ s^{-1} \in Z_+. \tag{3.4.26}$$

Hence $n_-^{-1} u y \in N_- s Z_+ M_-$.

Now we show that

$$N_-sZ_+M_- = N_-sM_-Z_+ = (N_- \cap N)Z_-sM_-Z_+ = (N_- \cap N)Z_-sZ_+M_-.$$

Note that the elements of the subgroup $N_- \cap \overline{N}$ are transformed to M_- by the conjugation by s^{-1}. Therefore using a decomposition of elements N_- into products of elements from one–parameter subgroups corresponding to roots and Lemma 3.4.4 one can obtain that any $k'' \in N_-$ can be represented in the form $k'' = \overline{n}z_-k'''$, $\overline{n} \in N \cap N_-$, $z_- \in Z_- = Z \cap N_-$, $k''' \in N_- \cap \overline{N}$, $s^{-1}k'''s \in M_-$. Therefore for any $m \in M_-, z_+ \in Z_+$, we have

$$k''smz_+ = \overline{n}z_-sm'z_+, \overline{n} \in N \cap N_-, m' = s^{-1}k'''sm \in M_-, z_- \in Z_-,$$

and hence $N_-sM_-Z_+ \subset (N \cap N_-)Z_-sM_-Z_+$. The opposite inclusion is obvious. Thus $N_-sM_-Z_+ = (N \cap N_-)Z_-sM_-Z_+$. Similarly one obtains $N_-sZ_+M_- = N_-sM_-Z_+ = (N \cap N_-)Z_-sZ_+M_-$ and obviously $(N \cap N_-)Z_-sZ_+M_- \subset (N_- \cap N)sZM_- \subset NsZN$.

Next we prove that for any $n_- \in N_-, x \in \mathfrak{h}$ and $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ we have $h_-^{-1}n_-^{-1}uyh_+ \in (N \cap N_-)Z_-sH_0M_-Z_+ \subset NsZN$, where $h_\pm = e^{r_\pm^s x}$, i.e. $q(\mu_{M_-}^{-1}(u)) \subset (N \cap N_-)Z_-sH_0M_-Z_+ \subset NsZN$.

Let $H' \subset H$ be the subgroup corresponding to the Lie subalgebra $\mathfrak{h}' \subset \mathfrak{h}$. Recall that we denote by $H_0 \subset H$ the subgroup corresponding to the orthogonal complement $\mathfrak{h}'^\perp$ of $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the symmetric bilinear form on $\mathfrak{g}$. Note that $\mathfrak{h}'^\perp$ is the space of fixed points for the action of s on $\mathfrak{h}$. We obviously have $H = H'H_0$. From the definition of $r_\pm^s$ it follows that for any $h_0 \in H_0$ and $h' \in H'$ elements $h_+ = h_0h'$ and $h_- = h_0^{-1}s(h')$ are of the form $h_\pm = e^{r_\pm^s x}$ for some $x \in \mathfrak{h}$ and all elements $h_\pm$ are obtained in this way.

Next observe that the space $(N \cap N_-)Z_-sH_0M_-Z_+$ is invariant with respect to the following action of the subgroup of $H \times H$ formed by elements of the form $(h_+, h_-) = (h_0h', h_0^{-1}s(h'))$:

$$(h_+, h_-) \circ L = h_-^{-1}Lh_+, h = h_+ = h_0h', h_- = h_0^{-1}s(h'). \tag{3.4.27}$$

Indeed, let $L = vz_-skwz_+,\ v \in (N \cap N_-), w \in M_-, z_\pm \in Z_\pm, k \in H_0$ be an element of $(N \cap N_-)Z_-sH_0M_-Z_+$. Then

$$(h_+, h_-) \circ L = h_-^{-1}vz_-h_-h_-^{-1}skh_+h_+^{-1}wz_+h_+ = h_-^{-1}vz_-h_-skh_0^2h_+^{-1}wz_+h_+ \tag{3.4.28}$$

since $s^{-1}h_-^{-1}s = h_0h'^{-1}$. The r.h.s. of the last equality belongs to $(N \cap N_-)Z_-sH_0M_-Z_+$ because H normalizes $(N \cap N_-)$, M_- and $Z_\pm$.

Comparing action (3.4.27) with (3.4.13) and recalling that for any $n_- \in N_-$ and $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ one has $n_-^{-1}uy \in (N \cap N_-)Z_-sM_-Z_+ \subset (N \cap N_-)Z_-sH_0M_-Z_+$ we deduce $q(\mu_{M_-}^{-1}(u)) \subset (N \cap N_-)Z_-sH_0M_-Z_+$.

Similarly, using the fact that H normalizes the subgroups $N_-, N \cap N_-, M_-$ and $Z_\pm$ of G and recalling that for any $n_- \in N_-$ and $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ we also have $n_-^{-1}uy \in N_-sZ_+M_- = N_-sM_-Z_+ = (N \cap N_-)Z_-sZ_+M_- = (N \cap N_-)Z_-sM_-Z_+$, one can show that $q(\mu_{M_-}^{-1}(u)) \subset N_-sH_0Z_+M_- = N_-sH_0M_-Z_+ = (N \cap N_-)Z_-sH_0M_-Z_+ = (N \cap N_-)Z_-sH_0Z_+M_- = (N \cap N_-)sZ_-H_0Z_+M_- \subset NsZN$, where the last inclusion follows from the inclusions $Z_-H_0Z_+ \subset Z$, $M_- \subset N$.

The set $q(\mu_{M_-}^{-1}(u))$ is not Zariski closed in G. But by Proposition 1.3.1 $NsZN$ is Zariski closed in G. Therefore the Zariski closure $\overline{q(\mu_{M_-}^{-1}(u))}$ is contained in $NsZN$.

□

Now we are in a position to describe the closure $\overline{q(\mu_{M_-}^{-1}(u))}$, the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ and the algebra $W^s(G)$.

Theorem 3.4.5. *Suppose that the numbers t_i defined in (3.4.5) are not equal to zero for all i. Let $N'_s \subset N_- \cap N_s$ be the subgroup generated by one–parameter subgroups corresponding to the roots $\alpha \in \Delta_-^s \setminus \Delta_0$ satisfying the condition $-\gamma_{l'} < \alpha$, $M_-^s = M_- \cap N_s$. Then $\overline{q(\mu_{M_-}^{-1}(u))} = N_-sZM_- = N_-sZM_-^s$ is invariant under conjugations by elements of M_-, the conjugation action of M_- on $\overline{q(\mu_{M_-}^{-1}(u))}$ is free, the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is a smooth variety isomorphic to $N'_ssZM_-^s \simeq \Sigma_s = sZN_s$, $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) \simeq N'_ssZM_-^s$. Moreover, the conjugation action*

$$M_- \times N'_ssZM_-^s \to N_-sZM_- \tag{3.4.29}$$

is an isomorphism of varieties, and hence the algebra $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is isomorphic to $\mathbb{C}[M_-] \otimes \mathbb{C}[N'_ssZM_-^s]$.

The Poisson algebra $W^s(G)$ is isomorphic to the Poisson algebra of regular functions on $N'_ssZM_-^s$, $W^s(G) = \mathbb{C}[\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})] = \mathbb{C}[N'_ssZM_-^s] \simeq \mathbb{C}[\Sigma_s]$. Thus the algebra $W_{\mathcal{B}}^s(G)$ is a non–commutative deformation of the algebra of regular functions on the transversal slice $\Sigma_s \simeq N'_ssZM_-^s$.

Proof. Firstly, as we observed in Lemma 3.4.2 the preimage $\mu_{M_-}^{-1}(u)$ is locally stable under the (locally defined) dressing action of M_-, and hence $q(\mu_{M_-}^{-1}(u)) \subset NsZN$ is (locally) stable under the action of $M_- \subset N$ on $NsZN$ by conjugations. Since the conjugation action of N on $NsZN$ is free the (locally defined) conjugation action of M_- on $q(\mu_{M_-}^{-1}(u))$ is (locally) free as well.

Now observe that by Proposition 3.4.3 $\overline{q(\mu_{M_-}^{-1}(u))} \subset NsZN$. Since by Theorem 1.3.1 the conjugation action of N on $NsZN$ is free and regular, sZN_s being a cross–section for this action, and $\overline{q(\mu_{M_-}^{-1}(u))}$ is closed, the induced action of $M_- \subset N$ on $\overline{q(\mu_{M_-}^{-1}(u))}$ is globally defined and is free as well. Therefore the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is a smooth variety.

Next observe that the definitions of M_-^s and of N_s' and Lemma 3.4.4 imply the following factorization $N_s = M_-^s N_s'$. Thus if $szn_s \in sZN_s$, $z \in Z$, $n_s \in N_s$ then n_s can be uniquely factorized as $n_s = m_s n_s'$, $m_s \in M_-^s$, $n_s' \in N_s'$ and we have

$$szn_s = szm_s n_s'.$$

Conjugating this element by n_s' we deduce that szn_s is uniquely conjugated to the element

$$n_s' szm_s \in N_s' sZM_-^s$$

and hence $N_s' sZM_-^s \simeq sZN_s = \Sigma_s$ is a cross–section for the action of N on $NsZN$ as well.

We show now that the closure of $q(\mu_{M_-}^{-1}(u))$ contains the varieties $N_- sZM_-$ and $N_s' sZM_-^s$. Recall that by (3.4.11) $q(\mu_{M_-}^{-1}(u)) \subset N_- sH_0Z_+M_- = N_-H_0Z_+sM_-$. Observe that the proof of presentation (3.4.26), formula (3.4.27) and the definition of $q(\mu_{M_-}^{-1}(u))$ imply that it contains elements of the form ksn for some $n \in M_- \subset N$ and arbitrary $k \in N_-H_0Z_+$. This also follows from the fact that $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the right multiplication by arbitrary elements from Z_+ and with respect to the left multiplication by arbitrary elements from N_-, as $Z_+ \subset N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$, and $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the right multiplication by arbitrary elements from $N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$, and $q(\mu_{M_-}^{-1}(u))$ is also closed with respect to the restriction of action (3.4.27) to H_0.

Now recall that $\bar{h}_0(\alpha) > 0$ for $\alpha \in \Delta_+^s \setminus \Delta_0$ and $\bar{h}_0(\alpha) = 0$ for $\alpha \in \Delta_0$, and hence the $\mathbb{C}^*$–action on G induced by conjugations by the elements $h(t)$ from the one–parameter subgroup generated by $-\bar{h}_0 \in \mathfrak{h}$ is contracting on N and fixes all elements of Z. Applying action (3.4.27) with $h = h(t)$ to the elements ksn with arbitrary $k \in N_-H_0Z_+$ we immediately deduce, with the help of (3.4.27), that the M_-–component n can be contracted to the identity element using the above defined contracting action, and the closure of $q(\mu_{M_-}^{-1}(u))$ contains the variety N_-Zs as the closure of $Z_-H_0Z_+$ is Z.

By definition $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the left multiplication by arbitrary elements from N_-. Recall also that $M_- \subset N$ freely acts on $\overline{q(\mu_{M_-}^{-1}(u))}$ by conjugations. Therefore $\overline{q(\mu_{M_-}^{-1}(u))}$ also contains the variety $N_-ZsM_- = N_-sZM_-$. In particular, $N_s' sZM_-^s \subset \overline{q(\mu_{M_-}^{-1}(u))}$.

Note that

$$N_-sH_0Z_+M_- \subset N_-ZsM_- \subset \overline{q(\mu_{M_-}^{-1}(u))} \subset \overline{N_-sH_0Z_+M_-}.$$

This implies after taking closures that

$$\overline{N_-ZsM_-} = \overline{q(\mu_{M_-}^{-1}(u))} = \overline{N_-sH_0Z_+M_-}.$$

But the variety N_-ZsM_- is closed. Indeed, let $N_{s^{-1}}' = N_{s^{-1}} \cap N_-$, $N' = \{n \in N : s^{-1}ns \in N\}$. These definitions and Lemma 3.4.4 imply that $N \cap N_- = N_{s^{-1}}' N'$. Note that s normalizes Z and Z normalizes N'. Observe also that $s^{-1}N'sM_- = N_{\alpha \leq -\gamma_{l'}}$, where $N_{\alpha \leq -\gamma_{l'}}$ is the subgroup of N generated by one–parameter subgroups corresponding to the roots $\alpha \in \Delta_-^s$ from the segment defined by the condition $\alpha \leq -\gamma_{l'}$. Therefore every element $nzsm \in N_-ZsM_- = (N \cap N_-)ZsM_-$, $n \in N \cap N_-$, $z \in Z$, $m \in M_-$ with $n = n_1n_2$, $n_1 \in N_{s^{-1}}'$, $n_2 \in N'$ can be represented as follows

$$nzsm = n_1n_2zsm = n_1sz's^{-1}n_2'sm \in N_{s^{-1}}' sZN_{\alpha \leq -\gamma_{l'}}, z' \in Z, n_2' \in N',$$

and any element of $N_{s^{-1}}' sZN_{\alpha \leq -\gamma_{l'}}$ can be obtained this way. Thus $N_-ZsM_- = (N \cap N_-)ZsM_- = N_{s^{-1}}' sZN_{\alpha \leq -\gamma_{l'}}$.

But the variety $\overline{N}ZN$ is closed by Corollary 1.3.2 and

$$s^{-1}N_{s^{-1}}' sZN_{\alpha \leq -\gamma_{l'}} \subset \overline{N}ZN$$

is a closed subvariety of it as $s^{-1}N'_{s^{-1}}s \subset \overline{N}$, $N_{\alpha\leq-\gamma_{l'}} \subset N$ are closed subgroups. Therefore $N_-ZsM_- = (N\cap N_-)ZsM_- = N'_{s^{-1}}sZN_{\alpha\leq-\gamma_{l'}}$ is closed.

Hence

$$N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))} = \overline{N_-sH_0Z_+M_-}.$$

Using Lemma 1.3.1 one can also easily obtain that $N_-ZsM_- = N_-ZsM_-^s$.

Finally we show that $N'_sZsM_-^s$ is a cross–section for the free conjugation action of M_- on $N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$. Observe that any two points of $N'_sZsM_-^s$ are not M_-–conjugate. Indeed, we have an inclusion $N'_sZsM_-^s \subset \overline{q(\mu_{M_-}^{-1}(u))}$, and two points of $\overline{q(\mu_{M_-}^{-1}(u))}$ can not be M_--conjugate if they are not N–conjugate in $NsZN$. But $N'_sZsM_-^s$ is a cross–section for the action of N on $ZsZN$. Thus any two points of $N'_sZsM_-^s$ are not N–conjugate, and hence they are not M_-–conjugate. Therefore the closed variety $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ must contain the closed variety $N'_sZsM_-^s \simeq \Sigma_s$.

From formula (1.6.5) for the cardinality $\sharp\Delta_{\mathfrak{m}_+}$ of the set $\Delta_{\mathfrak{m}_+}$ and from the definitions of $\overline{q(\mu_{M_-}^{-1}(u))}$ and of $N'_sZsM_-^s$ we deduce that the dimension of the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is equal to the dimension of the variety $N'_sZsM_-^s$,

$$\dim\, \pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) = \dim\, G - 2\dim\, M_- = 2D + l - 2\sharp\Delta_{\mathfrak{m}_+} = 2D + l - 2\left(D - \frac{l(s)-l'}{2} - D_0\right) =$$
$$= l(s) + 2D_0 + l - l' = \dim\, N_s + \dim\, Z = \dim\, sZN_s = \dim\, \Sigma_s = \dim\, N'_sZsM_-^s.$$

Since π_q is a morphism of varieties $\pi_q^{-1}(N'_sZsM_-^s) = \pi_q^{-1}(\Sigma_s)$ is a closed smooth subvariety of the smooth variety $N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$. The number of connected components of $\pi_q^{-1}(N'_sZsM_-^s)$ is equal to the cardinality of the finite set Z/Z^0, where Z^0 is the identity component of Z. Each such component is irreducible.

The number of connected components of the smooth closed variety $N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$ is also equal to the cardinality of the finite set Z/Z^0. Each such component is irreducible. The last two observations together with the identity for the dimensions imply $\pi_q^{-1}(N'_sZsM_-^s) = N_-ZsM_-$.

Therefore $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) \simeq N'_sZsM_-^s$, $N'_sZsM_-^s$ is a cross–section for the action of M_- on $\overline{q(\mu_{M_-}^{-1}(u))}$, and the conjugation action

$$M_- \times N'_sZsM_-^s \to N_-ZsM_-$$

is an isomorphism of varieties. We conclude that the algebra $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is isomorphic to $\mathbb{C}[M_-]\otimes\mathbb{C}[N'_sZsM_-^s]$, $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cong \mathbb{C}[M_-]\otimes\mathbb{C}[N'_sZsM_-^s]$.

Now recall that by Lemma 3.4.1

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)},$$

where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map $q^* : C^\infty(q(\mu_{M_-}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$ and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

By Lemma 3.4.2 the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$, and hence

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)} = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{M_-}. \tag{3.4.30}$$

As we already proved the variety $\overline{q(\mu_{M_-}^{-1}(u))}$ is stable under the conjugation action of M_-, and the map $\pi_q : \overline{q(\mu_{M_+}^{-1}(u))} \to \pi_q\overline{q(\mu_{M_+}^{-1}(u))}$ is a morphism of varieties. Moreover, under the map $q: G^* \to G$ the local dressing action of M_- on G^* becomes the conjugation action on G. Therefore the map

$$\mathbb{C}[\pi_q\overline{q(\mu_{M_-}^{-1}(u))}] \to \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{M_-}, \quad \psi \mapsto \pi_q^*\psi \tag{3.4.31}$$

is an isomorphism, where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map

$$q^* : C^\infty(q(\mu_{M_+}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$$

and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

Combining (3.4.30) and (3.4.31) we obtain that $W^s(G) \cong \mathbb{C}[\pi_q\overline{q(\mu_{M_-}^{-1}(u))}]$. This completes the proof.

∎

3.5 Zhelobenko type operators for Poisson q-W–algebras

In this section we present the main result of this chapter, a formula for a projection operator $\Pi : \mathbb{C}[N_-ZsM_-] \rightarrow \mathbb{C}[N_-ZsM_-]^{M_-}$ onto the subspace of invariants $\mathbb{C}[N_-ZsM_-]^{M_-}$ which is isomorphic to $W^s(G)$ as an algebra according to Theorem 3.4.5. This formula has a direct quantum analogue which will be introduced in the next chapter.

The operator Π can be defined following the philosophy of [104] where a similar projection operator onto the subspace $\mathbb{C}[NZsN]^N \subset \mathbb{C}[NZsN]$ was defined and studied. More precisely, according to Theorem 3.4.5 any $g \in N_-ZsM_-$ can be uniquely represented in the form

$$g = nn_szsm_sn^{-1}, n \in M_-, n_s \in N_s', m_s \in M_-^s, z \in Z. \tag{3.5.1}$$

If for $f \in \mathbb{C}[N_-ZsM_-]$ we define $\Pi f \in \mathbb{C}[N_-ZsM_-]$ by

$$(\Pi f)(g) = f(n^{-1}gn) = f(n_szsm_s) \tag{3.5.2}$$

then Πf is an M_-–invariant function, and any M_-–invariant regular function on N_-ZsM_- can be obtained this way. Moreover, by the definition $\Pi^2 = \Pi$, i.e. Π is a projection onto $\mathbb{C}[N_-ZsM_-]^{M_-}$.

To obtain an explicit formula for the operator Π we firstly find an explicit formula for n in terms of g in (3.5.1). Denote by ω the Chevalley anti–involution on $\mathfrak{g}$ which is induced by the antiautomorphism ω of $U_h^s(\mathfrak{g})$ on $U(\mathfrak{g}) \simeq U_h^s(\mathfrak{g})/hU_h^s(\mathfrak{g})$. We also denote the corresponding anti–involution of G by the same letter. The following proposition is an analogue of Proposition 2.11 in [104], where a similar statement was proved for the action of N on $NZsN$.

Proposition 3.5.1. *Let $g = nn_szsm_sn^{-1} \in N_-ZsM_-$. Let α_i be the simple roots of a system of positive roots Δ_+ associated to s, s_i the corresponding simple reflections, $\beta_1, \ldots, \beta_D$, $\beta_j = s_{i_1} \ldots s_{i_{j-1}}\alpha_{i_j}$ a normal ordering of Δ_+ corresponding to s. Let ω_i, $i = 1, \ldots l$ be the fundamental weights corresponding to Δ_+, v_{ω_i} a non–zero highest weight vector in the irreducible highest weight G–module V_{ω_i} of highest weight ω_i, $(\cdot,\cdot)$ the contravariant non–degenerate bilinear form on V_{ω_i} such that $(v, xw) = (\omega(x)v, w)$ for any $v, w \in V_{\omega_i}$, $x \in \mathfrak{g}$ and $(v_{\omega_i}, v_{\omega_i}) = 1$. Then n can be uniquely factorized as $n = X_{-\beta_1}(t_1) \ldots X_{-\beta_c}(t_c)$ and the numbers t_i can be found inductively by the following formula*

$$t_p = c_p \frac{(w_pv_{\omega_{i_p}}, g_ps^{-1}w_{p-1}v_{\omega_{i_p}})}{(w_{p-1}v_{\omega_{i_p}}, g_ps^{-1}w_{p-1}v_{\omega_{i_p}})}, \tag{3.5.3}$$

where $w_p = s_{\beta_p} \ldots s_{\beta_1}$, $w_{p-1} = s_{\beta_{p-1}} \ldots s_{\beta_1}$, c_p is a non–zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector $X_{-\beta_p} \in \mathfrak{g}$, $g_p = n_p^{-1}gn_p$, $n_p = X_{-\beta_1}(t_1) \ldots X_{-\beta_{p-1}}(t_{p-1})$ and it is assumed that $n_1 = 1$, $w_0 = 1$.

Proof. The numbers t_p can be found by induction starting with $p = 1$. We shall establish the induction step. The case $p = 1$ corresponding to the base of the induction can be considered in a similar way.

Assume that $t_1, \ldots, t_{p-1}$ have already been found. Then

$$g_ps^{-1} = n_p^{-1}gn_ps^{-1} = X_{-\beta_p}(t_p) \ldots X_{-\beta_c}(t_c)n_szsm_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p)s^{-1},$$

where $n_p = X_{-\beta_1}(t_1) \ldots X_{-\beta_{p-1}}(t_{p-1})$. By Lemma 3.4.4 we can write

$$X_{-\beta_p}(t_p) \ldots X_{-\beta_c}(t_c)n_s = m_1X_{-\beta_p}(t_p), m_1 \in N_{[-\beta_{p+1},-\beta_D]}.$$

Since $\Delta_{w_p^{-1}} = \{\beta_1, \ldots, \beta_p\}$ we have $w_p^{-1}N_{[-\beta_{p+1},-\beta_D]}w_p \subset N_-$, and hence

$$\begin{aligned}(w_pv_{\omega_{i_p}}, g_ps^{-1}w_{p-1}v_{\omega_{i_p}}) &= (w_pv_{\omega_{i_p}}, m_1X_{-\beta_p}(t_p)zsm_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p)s^{-1}w_{p-1}v_{\omega_{i_p}}) = \\ &= (w_pv_{\omega_{i_p}}, X_{-\beta_p}(t_p)zsm_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p)s^{-1}w_{p-1}v_{\omega_{i_p}})\end{aligned}$$

as $v_{\omega_{i_p}}$ is a highest weight vector.

Now observe that $m_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p) \in M_-^sN_{[-\beta_p,-\beta_c]}$. Note that the properties of the normal ordering in Δ_+ associated to s imply that $M_-^s = N_{[-\beta_k,-\beta_c]}$ for some k and that the union $[-\beta_p, -\beta_c] \cup [-\beta_k, -\beta_c]$ is a minimal segment, so the subgroup $M_-^sN_{[-\beta_p,-\beta_c]}$ is generated by one–parameter subgroups corresponding to the roots from that segment. Thus using Lemma 3.4.4 one can uniquely factorize the element $m_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p)$ as

$$m_sX_{-\beta_c}(-t_c) \ldots X_{-\beta_p}(-t_p) = m_2m_3, m_3 \in N_{[-\beta_k,-\beta_c]}, m_2 \in N_{[-\beta_p,-\beta_{k-1}]},$$

where it is assumed that $N_{[-\beta_p,-\beta_{k-1}]}=1$ if $p>k-1$. If $\alpha\in[\beta_p,\beta_{k-1}]$ then $s\alpha\in\Delta_+^s$ and by the properties of the normal ordering (1.6.3) in Proposition 1.6.1 we have $s\alpha>\alpha$, and if $s\alpha+\alpha_0\in\Delta_+^s$ for $\alpha_0\in\Delta_0$ then $s\alpha+\alpha_0>\alpha$. Observing also that Z is generated by one–parameter subgroups corresponding to roots from Δ_0 and by the centralizer of s in H which normalizes all one–parameter subgroups corresponding to roots, we deduce $zsm_sX_{-\beta_c}(-t_c)\ldots X_{-\beta_p}(-t_p)s^{-1}=m_4zsm_3s^{-1}$, $m_4=zsm_2s^{-1}z^{-1}\in N_{[-\beta_{p+1},-\beta_D]}$.

Now using Lemma 3.4.4 we can uniquely factorize the element $X_{\beta_p}(t_p)m_4\in N_{[-\beta_p,-\beta_D]}$ as $X_{\beta_p}(t_p)m_4=m_5X_{\beta_p}(t_p)$, $m_5\in N_{[-\beta_{p+1},-\beta_D]}$. Remark that $w_p^{-1}N_{[-\beta_{p+1},-\beta_D]}w_p\subset N_-$ as $\Delta_{w_p^{-1}}=\{\beta_1,\ldots,\beta_p\}$ and hence

$$\begin{aligned}(w_pv_{\omega_{i_p}},g_ps^{-1}w_{p-1}v_{\omega_{i_p}})&=(w_pv_{\omega_{i_p}},X_{-\beta_p}(t_p)m_4zsm_3s^{-1}w_{p-1}v_{\omega_{i_p}})=\\&=(w_pv_{\omega_{i_p}},m_5X_{-\beta_p}(t_p)zsm_3s^{-1}w_{p-1}v_{\omega_{i_p}})=\\&=(w_pv_{\omega_{i_p}},X_{-\beta_p}(t_p)zsm_3s^{-1}w_{p-1}v_{\omega_{i_p}})\end{aligned}$$

as $v_{\omega_{i_p}}$ is a highest weight vector.

By the definition of m_3 we have $sm_3s^{-1}\in\overline{N}$. Denote by $\delta_1,\ldots,\delta_D$ the normal ordering (1.6.3) in Δ_+^s corresponding to s, so that $\beta_p=\delta_q$ for some q. Applying arguments similar to those used above we can factorize $sm_3s^{-1}=m_6m_7$, $m_6\in N_{[\delta_1,\delta_{q-1}]}$, $m_7\in N_{[\delta_q,\delta_D]}$. Note that $[\delta_q,\delta_D]\subset[\beta_p,\beta_D]$, and each root from this segment remains positive under the action of w_{p-1}^{-1}. Therefore we obtain that

$$\begin{aligned}(w_pv_{\omega_{i_p}},g_ps^{-1}w_{p-1}v_{\omega_{i_p}})&=(w_pv_{\omega_{i_p}},X_{-\beta_p}(t_p)zsm_3s^{-1}w_{p-1}v_{\omega_{i_p}})=\\=(w_pv_{\omega_{i_p}},X_{-\beta_p}(t_p)zm_6m_7w_{p-1}v_{\omega_{i_p}})&=(w_pv_{\omega_{i_p}},X_{-\beta_p}(t_p)zm_6w_{p-1}v_{\omega_{i_p}})=\\&=(X_{\beta_p}(t_p)w_pv_{\omega_{i_p}},zm_6w_{p-1}v_{\omega_{i_p}}),\end{aligned}$$

where we assume that the root vectors $X_{\pm\beta_p}$ are chosen in such a way that $\omega(X_{-\beta_p})=X_{\beta_p}$.

Since $\beta_p\not\in\Delta_{w_{p-1}^{-1}}$ we infer $X_{\beta_p}w_{p-1}v_{\omega_{i_p}}=0$, and hence $w_{p-1}v_{\omega_{i_p}}$ is a highest weight vector for the $\mathfrak{sl}_2$–triple generated by the elements $X_{\pm\beta_p}$. Moreover, since $w_{p-1}^{-1}(-\beta_p)=-\alpha_{i_p}$ the vector $X_{-\alpha_{i_p}}=w_{p-1}^{-1}X_{-\beta_p}w_{p-1}$ is a root vector corresponding to $-\alpha_{i_p}$, and hence $X_{-\beta_p}^2w_{p-1}v_{\omega_{i_p}}=w_{p-1}X_{-\alpha_{i_p}}^2v_{\omega_{i_p}}=0$ by the definition of $v_{\omega_{i_p}}$. Therefore $w_{p-1}v_{\omega_{i_p}}$ is a highest weight vector for the two–dimensional irreducible representation of the $\mathfrak{sl}_2$–triple generated by the elements $X_{\pm\beta_p}$, and $s_{\beta_p}w_{p-1}v_{\omega_{i_p}}$ is a non–zero lowest weight vector for that representation. Recalling the standard $\mathfrak{sl}_2$–representation theory, the fact that each weight space is an eigenspace for the action of H and that $X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}=X_{\beta_p}(t_p)s_{\beta_p}w_{p-1}v_{\omega_{i_p}}$ we deduce

$$\begin{aligned}X_{\beta_p}(t_p)w_pv_{\omega_{i_p}}=X_{\beta_p}(t_p)s_{\beta_p}w_{p-1}v_{\omega_{i_p}}&=s_{\beta_p}w_{p-1}v_{\omega_{i_p}}+t_pX_{\beta_p}s_{\beta_p}w_{p-1}v_{\omega_{i_p}}=\\&=s_{\beta_p}w_{p-1}v_{\omega_{i_p}}+\frac{t_p}{c_p}w_{p-1}v_{\omega_{i_p}},\end{aligned}$$

where c_p is a non–zero constant only depending on the choice of the representative $s_{\beta_p}\in G$ and on the choice of the root vectors $X_{\pm\beta_p}$. The first term in the last sum has weight $-\delta_q+w_{p-1}\omega_{i_p}$, and the second one $w_{p-1}\omega_{i_p}$.

The vector $zm_6w_{p-1}v_{\omega_{i_p}}$ is a linear combination of vectors of weights of the form $w_{p-1}\omega_{i_p}+\sum_{r=1}^{q-1}c_r\delta_r+\omega_0$, where $\omega_0\in\mathfrak{h}_0^*$, $c_r\in\{0,1,2,\ldots\}$. Since weight spaces corresponding to different weights are orthogonal with respect to the contravariant bilinear non–degenerate form on $V_{\omega_{i_p}}$ the only nontrivial contributions to the product $(X_{\beta_p}(t_p)w_pv_{\omega_{i_p}},zm_6w_{p-1}v_{\omega_{i_p}})$ come from the products of vectors of weights either $-\delta_q+w_{p-1}\omega_{i_p}$ or $w_{p-1}\omega_{i_p}$.

In the first case we must have $w_{p-1}\omega_{i_p}+\sum_{r=1}^{q-1}c_r\delta_r+\omega_0=-\delta_q+w_{p-1}\omega_{i_p}$, and hence $\sum_{r=1}^{q-1}c_r\delta_r+\omega_0=-\delta_q$. In particular, $\overline{h}_0(\sum_{r=1}^{q-1}c_r\delta_r+\omega_0)=\sum_{r=1}^{q-1}c_r\overline{h}_0(\delta_r)=-\overline{h}_0(\delta_q)$ which is impossible as $-\overline{h}_0(\delta_q)<0$ and $c_r\overline{h}_0(\delta_r)\geq 0$.

In the second case we must have $w_{p-1}\omega_{i_p}+\sum_{r=1}^{q-1}c_r\delta_r+\omega_0=w_{p-1}\omega_{i_p}$ or $\sum_{r=1}^{q-1}c_r\delta_r+\omega_0=0$. In particular, $\overline{h}_0(\sum_{r=1}^{q-1}c_r\delta_r+\omega_0)=\sum_{r=1}^{q-1}c_r\overline{h}_0(\delta_r)=0$ which forces $c_r=0$ for all r as $\overline{h}_0(\delta_r)>0$ and $c_r\in\{0,1,2,\ldots\}$, and hence $\omega_0=0$ as well.

We conclude that the only nontrivial contributions to the product

$$(X_{\beta_p}(t_p)w_pv_{\omega_{i_p}},zm_6w_{p-1}v_{\omega_{i_p}})$$

come from the products of vectors of weights $w_{p-1}\omega_{i_p}$. By the above considerations only terms of the form $zw_{p-1}v_{\omega_{i_p}}$ may give contributions of weight $w_{p-1}\omega_{i_p}$ in the weight decomposition of the element $zm_6w_{p-1}v_{\omega_{i_p}}$, and this yields

$$\begin{aligned}&(w_pv_{\omega_{i_p}},g_ps^{-1}w_{p-1}v_{\omega_{i_p}})=(X_{\beta_p}(t_p)w_pv_{\omega_{i_p}},zm_6w_{p-1}v_{\omega_{i_p}})=\\&=\tfrac{t_p}{c_p}(w_{p-1}v_{\omega_{i_p}},zw_{p-1}v_{\omega_{i_p}}).\end{aligned}$$

Therefore

$$t_p = c_p \frac{(w_p v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}})}{(w_{p-1} v_{\omega_{i_p}}, z w_{p-1} v_{\omega_{i_p}})}.$$

Similar arguments show that

$$(w_{p-1} v_{\omega_{i_p}}, z w_{p-1} v_{\omega_{i_p}}) = (w_{p-1} v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}}).$$

Combining the last two identities we obtain formula (3.5.3).

□

Formulas (3.5.3) have no direct quantum analogues, and for the purposes of quantization we shall need other formulas for the coefficients t_p introduced in the previous proposition. These formulas express the coefficients t_p in terms of other matrix elements of finite-dimensional irreducible representations of G. These matrix elements can be defined by specializing the results of Lemma 2.3 in [25] at $q = 1$. By this lemma there are integral dominant weights $\mu_p(\mu'_p)$, $p = 1, \ldots, D$ and elements $v_p \in V_{\mu_p}$ $(v'_p \in V_{\mu'_p})$ such that

$$(v_p, X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_1}^{(n_1)} v_{\mu_p}) = \begin{cases} 1 & \text{if } X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_1}^{(n_1)} = X_{-\beta_p} \\ 0 & \text{otherwise} \end{cases}, \quad (3.5.4)$$

$$(v'_p, X_{-\beta_1}^{(n_1)} \ldots X_{-\beta_D}^{(n_D)} v_{\mu'_p}) = \begin{cases} 1 & \text{if } X_{-\beta_1}^{(n_1)} \ldots X_{-\beta_D}^{(n_D)} = X_{-\beta_p} \\ 0 & \text{otherwise} \end{cases},$$

where for $\alpha \in \Delta$, $k \in \mathbb{N}$ $X_\alpha^{(k)} = \frac{X_\alpha^k}{k!}$.

Proposition 3.5.2. *Let $g = n n_s z s m_s n^{-1} \in N_- Z s M_-$ be an element of $N_- Z s M_-$ represented as in (3.5.1), where $n = X_{-\beta_1}(t_1) \ldots X_{-\beta_c}(t_c)$, $g_p = n_p^{-1} g n_p$, $n_p = X_{-\beta_1}(t_1) \ldots X_{-\beta_{p-1}}(t_{p-1})$. Then $g_p \in N_{[-\beta_p, -\beta_D]} Z s M_-^s$,*

$$g_p = X_{-\beta_p}(r_p) \ldots X_{-\beta_D}(r_D) z s m'_s, m'_s \in M_-^s, \quad (3.5.5)$$

$$r_p = t_p,$$

and the numbers t_p can be found inductively by the following formula

$$t_p = \frac{(v_p, g_p s^{-1} v_{\mu_p})}{(v_{\mu_p}, g_p s^{-1} v_{\mu_p})} = \frac{(v'_p, g_p s^{-1} v_{\mu'_p})}{(v_{\mu'_p}, g_p s^{-1} v_{\mu'_p})}. \quad (3.5.6)$$

Proof. We shall prove this proposition by induction. In the proof we shall use the notation and arguments from the proof of the previous proposition. Firstly we show that $g \in N_- Z s M_-^s$. Indeed, using Lemma 3.4.4 we can represent any element $m \in M_-$ as $m = m_1 m_2$, $m_1 \in N_{[-\beta_1, -\beta_{k-1}]}, m_2 \in M_-^s$. Next, if $\alpha \in [\beta_1, \beta_{k-1}]$ then $s\alpha \in \Delta_+^s$ and by the properties of the normal ordering (1.6.3) in Proposition 1.6.1 we have $s\alpha > \alpha$, and if $s\alpha + \alpha_0 \in \Delta_+^s$ for $\alpha_0 \in \Delta_0$ then $s\alpha + \alpha_0 > \alpha$, and in both cases $s\alpha, s\alpha + \alpha_0 \in \Delta_+$. Observing also that Z is generated by the one–parameter subgroups corresponding to roots from Δ_0 and by the centralizer of s in H which normalizes all one–parameter subgroups corresponding to roots, we deduce $z s m_1 s^{-1} \in N_- Z$ for any $z \in Z$. Therefore $z s m = z s m_1 s^{-1} s m_2 \in N_- Z s M_-^s$, so $Z s M_- \subset N_- Z s M_-^s$, and hence $g \in N_- Z s M_- \subset N_- Z s M_-^s$.

Now we shall prove the first two statements by induction. Assume that

$$g_p = X_{-\beta_p}(r_p) \ldots X_{-\beta_D}(r_D) z s m'_s, z \in Z, m'_s \in M_-^s. \quad (3.5.7)$$

Note that in the case when $p = 1$ this expression is already established as it directly follows from the fact that $g \in N_- Z s M_-^s$ proved above.

We show first that $r_p = t_p$. By Lemma 3.4.4 we can write

$$X_{-\beta_p}(r_p) \ldots X_{-\beta_D}(r_D) = m_1 X_{-\beta_p}(r_p), m_1 \in N_{[-\beta_{p+1}, -\beta_D]}.$$

Since $\Delta_{w_p^{-1}} = \{\beta_1, \ldots, \beta_p\}$ we have $w_p^{-1} N_{[-\beta_{p+1}, -\beta_D]} w_p \subset N_-$, and hence

$$(w_p v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}}) = (w_p v_{\omega_{i_p}}, m_1 X_{-\beta_p}(r_p) z s m'_s s^{-1} w_{p-1} v_{\omega_{i_p}}) =$$
$$(w_p v_{\omega_{i_p}}, X_{-\beta_p}(r_p) z s m'_s s^{-1} w_{p-1} v_{\omega_{i_p}})$$

as $v_{\omega_{i_p}}$ is a highest weight vector.

Now recall that $m'_s \in M_-^s$. Note that the properties of the normal ordering in Δ_+ associated to s imply that $M_-^s = N_{[-\beta_k, -\beta_c]}$ for some k.

By the definition of m'_s we have $sm'_s s^{-1} \in \overline{N}$. Denote by $\delta_1, \ldots, \delta_D$ the normal ordering in Δ_+^s corresponding to s, so that $\beta_p = \delta_q$ for some q. Using arguments similar to those used above we can factorize $sm'_s s^{-1} = mm'$, $m \in N_{[\delta_1, \delta_{q-1}]}$, $m' \in N_{[\delta_q, \delta_D]}$. Note that $[\delta_q, \delta_D] \subset [\beta_p, \beta_D]$, and each root from this segment remains positive under the action of w_{p-1}^{-1}. Therefore we obtain that

$$\begin{aligned}(w_p v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}}) = (w_p v_{\omega_{i_p}}, X_{-\beta_p}(t_p) z s m'_s s^{-1} w_{p-1} v_{\omega_{i_p}}) = \\ = (w_p v_{\omega_{i_p}}, X_{-\beta_p}(t_p) z m m' w_{p-1} v_{\omega_{i_p}}) = (w_p v_{\omega_{i_p}}, X_{-\beta_p}(t_p) z m w_{p-1} v_{\omega_{i_p}}) = \\ = (X_{\beta_p}(t_p) w_p v_{\omega_{i_p}}, z m w_{p-1} v_{\omega_{i_p}}),\end{aligned}$$

where we assume that the root vectors $X_{\pm\beta_p}$ are chosen in such a way that $\omega(X_{-\beta_p}) = X_{\beta_p}$.

Since $\beta_p \notin \Delta_{w_{p-1}^{-1}}$ we infer $X_{\beta_p} w_{p-1} v_{\omega_{i_p}} = 0$, and hence $w_{p-1} v_{\omega_{i_p}}$ is a highest weight vector for the $\mathfrak{sl}_2$–triple generated by the elements $X_{\pm\beta_p}$. Moreover, since $w_{p-1}^{-1}(-\beta_p) = -\alpha_{i_p}$ the vector $X_{-\alpha_{i_p}} = w_{p-1}^{-1} X_{-\beta_p} w_{p-1}$ is a root vector corresponding to $-\alpha_{i_p}$, and hence $X_{-\beta_p}^2 w_{p-1} v_{\omega_{i_p}} = w_{p-1} X_{-\alpha_{i_p}}^2 v_{\omega_{i_p}} = 0$ by the definition of $v_{\omega_{i_p}}$. Therefore $w_{p-1} v_{\omega_{i_p}}$ is a highest weight vector for the two–dimensional irreducible representation of the $\mathfrak{sl}_2$–triple generated by the elements $X_{\pm\beta_p}$, and $s_{\beta_p} w_{p-1} v_{\omega_{i_p}}$ is a non–zero lowest weight vector for that representation. Recalling the standard $\mathfrak{sl}_2$–representation theory, the fact that each weight space is an eigenspace for the action of H and that $X_{\beta_p}(t_p) w_p v_{\omega_{i_p}} = X_{\beta_p}(t_p) s_{\beta_p} w_{p-1} v_{\omega_{i_p}}$ we deduce

$$\begin{aligned}X_{\beta_p}(r_p) w_p v_{\omega_{i_p}} = X_{\beta_p}(r_p) s_{\beta_p} w_{p-1} v_{\omega_{i_p}} = s_{\beta_p} w_{p-1} v_{\omega_{i_p}} + r_p X_{\beta_p} s_{\beta_p} w_{p-1} v_{\omega_{i_p}} = \\ = s_{\beta_p} w_{p-1} v_{\omega_{i_p}} + \frac{r_p}{c_p} w_{p-1} v_{\omega_{i_p}},\end{aligned}$$

where c_p is a non–zero constant only depending on the choice of the representative $s_{\beta_p} \in G$ and on the choice of the root vector $X_{\pm\beta_p}$. The first term in the last sum has weight $-\delta_q + w_{p-1}\omega_{i_p}$, and the second one $w_{p-1}\omega_{i_p}$.

The vector $z m w_{p-1} v_{\omega_{i_p}}$ is a linear combination of vectors of weights of the form $w_{p-1}\omega_{i_p} + \sum_{u=1}^{q-1} c_u \delta_u + \omega_0$, where $\omega_0 \in \mathfrak{h}_0^*$, $c_u \in \{0, 1, 2, \ldots\}$. Since weight spaces corresponding to different weights are orthogonal with respect to the contravariant bilinear non–degenerate form on $V_{\omega_{i_p}}$ the only nontrivial contributions to the product $(X_{\beta_p}(r_p) w_p v_{\omega_{i_p}}, z m w_{p-1} v_{\omega_{i_p}})$ come from the products of vectors of weights either $-\delta_q + w_{p-1}\omega_{i_p}$ or $w_{p-1}\omega_{i_p}$.

In the first case we must have $w_{p-1}\omega_{i_p} + \sum_{u=1}^{q-1} c_u \delta_u + \omega_0 = -\delta_q + w_{p-1}\omega_{i_p}$, and hence $\sum_{u=1}^{q-1} c_u \delta_u + \omega_0 = -\delta_q$. In particular, $\overline{h}_0(\sum_{u=1}^{q-1} c_u \delta_u + \omega_0) = \sum_{u=1}^{q-1} c_u \overline{h}_0(\delta_u) = -\overline{h}_0(\delta_q)$ which is impossible as $-\overline{h}_0(\delta_q) < 0$ and $c_u \overline{h}_0(\delta_u) \geq 0$.

In the second case we must have $w_{p-1}\omega_{i_p} + \sum_{u=1}^{q-1} c_u \delta_u + \omega_0 = w_{p-1}\omega_{i_p}$ or $\sum_{u=1}^{q-1} c_u \delta_u + \omega_0 = 0$. In particular, $\overline{h}_0(\sum_{u=1}^{q-1} c_u \delta_u + \omega_0) = \sum_{u=1}^{q-1} c_u \overline{h}_0(\delta_u) = 0$ which forces $c_u = 0$ for all u as $\overline{h}_0(\delta_u) > 0$ and $c_u \in \{0, 1, 2, \ldots\}$, and hence $\omega_0 = 0$ as well.

We conclude that the only nontrivial contributions to the product

$$(X_{\beta_p}(r_p) w_p v_{\omega_{i_p}}, z m w_{p-1} v_{\omega_{i_p}})$$

come from the products of vectors of weights $w_{p-1}\omega_{i_p}$. By the above considerations only terms of the form $z w_{p-1} v_{\omega_{i_p}}$ may give contributions of weight $w_{p-1}\omega_{i_p}$ in the weight decomposition of the element $z m w_{p-1} v_{\omega_{i_p}}$, and this yields

$$\begin{aligned}&(w_p v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}}) = (X_{\beta_p}(r_p) w_p v_{\omega_{i_p}}, z m w_{p-1} v_{\omega_{i_p}}) = \\ &= \tfrac{r_p}{c_p} (w_{p-1} v_{\omega_{i_p}}, z w_{p-1} v_{\omega_{i_p}}).\end{aligned}$$

Therefore

$$r_p = c_p \frac{(w_p v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}})}{(w_{p-1} v_{\omega_{i_p}}, z w_{p-1} v_{\omega_{i_p}})}.$$

Similar arguments show that

$$(w_{p-1} v_{\omega_{i_p}}, z w_{p-1} v_{\omega_{i_p}}) = (w_{p-1} v_{\omega_{i_p}}, g_p s^{-1} w_{p-1} v_{\omega_{i_p}}).$$

Combining the last two identities we obtain formula (3.5.3) for r_p, i. e. $t_p = r_p$. At the same time when $p = 1$ this yields formula (3.5.3) for r_1 and establishes the base of induction.

g_{p+1} is defined by putting $t_p = 0$ in the formula

$$g_p = X_{-\beta_p}(t_p)\dots X_{-\beta_c}(t_c)n_s z s m_s X_{-\beta_c}(-t_c)\dots X_{-\beta_p}(-t_p), \quad (3.5.8)$$

and along with $t_p = r_p$ (3.5.7) yields

$$g_{p+1} = X_{-\beta_{p+1}}(r'_{p+1})\dots X_{-\beta_D}(r'_D)zsm'_s, z \in Z, m'_s \in M^s_-.$$

This establishes the induction step, and hence the first two statements of this proposition are proved.

Now we establish the second expression for t_p in (3.5.6), the first one can be justified in a similar way. By (3.5.5) we have

$$(v'_p, g_p s^{-1} v_{\mu'_p}) = (v'_p, X_{-\beta_p}(r_p)\dots X_{-\beta_D}(r_D)zsm'_s s^{-1} v_{\mu'_p}) \quad (3.5.9)$$

Recall that $sM^s_- s^{-1} \subset \overline{N}$ by the definition of M^s_-. Note that $Z\overline{N} \subset G$ is a Lie subgroup with the Lie algebra $\overline{\mathfrak{n}} + \mathfrak{z}$. Since $Z\overline{N}$ is generated by the one–parameter subgroups corresponding to the roots from the set $\Delta^s_+ \cup \Delta_0$ and by H_0 and the action of G on V_{μ_p} is locally finite we can write $zsm'_s s^{-1} v_{\mu_p} = x v_{\mu_p}$, where $x \in U(\overline{\mathfrak{n}} + \mathfrak{z})$.

Now observe that $\overline{\mathfrak{n}} + \mathfrak{z} = ((\overline{\mathfrak{n}} \cap \mathfrak{n}_-) + \mathfrak{z}_-) + \mathfrak{h}'^{\perp} + (\mathfrak{z}_+ + (\overline{\mathfrak{n}} \cap \mathfrak{n}_+))$, where $\mathfrak{z}_\pm = z \cap \mathfrak{n}_\pm$ and the terms $(\overline{\mathfrak{n}} \cap \mathfrak{n}_\pm) + \mathfrak{z}_\pm$, $\mathfrak{h}'^{\perp}$ in the direct linear sum in the right hand side are Lie algebras. Therefore by the Poincaré-Birkhoff-Witt theorem x is a linear combination of terms of the form $x_- x_0 x_+$, $x_\pm \in U((\overline{\mathfrak{n}} \cap \mathfrak{n}_\pm) + \mathfrak{z}_\pm)$ $x_0 \in U(\mathfrak{h}'^{\perp})$. Since v_{μ_p} is a highest weight vector only terms with $x_+ \in \mathbb{C}$ can contribute to the right hand side of (3.5.9). Rearranging scalar factors we can assume without loss of generality that for these terms $x_+ = 1$. Observe also that the roots $\alpha \in \Delta_-$ corresponding to the root vectors generating the Lie algebra $(\overline{\mathfrak{n}} \cap \mathfrak{n}_-) + \mathfrak{z}_-$ satisfy the condition $\alpha > -\beta_c$ and that $-\beta_p \leq -\beta_c$. Therefore (3.5.4) implies that only terms with $x_- \in \mathbb{C}$ can contribute to the right hand side of (3.5.9). Rearranging scalar factors we can assume without loss of generality that for these terms $x_- = 1$. Thus by (3.5.4) we deduce

$$(v'_p, g_p s^{-1} v_{\mu'_p}) = (v'_p, X_{-\beta_p}(r_p)\dots X_{-\beta_D}(r_D)x_0 v_{\mu'_p}) = \mu'_p(x_0) r_p.$$

Similarly we obtain

$$(v_{\mu'_p}, g_p s^{-1} v_{\mu'_p}) = (v_{\mu'_p}, x_0 v_{\mu'_p}) = \mu'_p(x_0).$$

The last two identities imply the second expression in (3.5.6).

□

Remark 3.5.3. *Note that $N_- Z s M_-$ is a closed subvariety in G. Therefore for each $p = 1, \dots, c$ the right hand side of (3.5.3) is a regular function on $N_- Z s M_-$ as the composition of the regular function*

$$n = X_{-\beta_1}(t_1)\dots X_{-\beta_c}(t_c) \mapsto t_p$$

defined on M_- and of isomorphism (3.4.29) of varieties. Hence the denominators in (3.5.3) and (3.5.6) must be canceled.

From Proposition 3.5.2 we immediately obtain the following corollary.

Corollary 3.5.4. *Let ϕ_k and ψ_k, $k = 1, \dots, D$ be the regular functions on G defined as follows*

$$\phi_k(g) = (v_k, g s^{-1} v_{\mu_k}), k = 1, \dots, D, g \in G,$$

$$\psi_k(g) = (v'_k, g s^{-1} v_{\mu'_k}), k = 1, \dots, D, g \in G.$$

Then the restrictions of the functions $\phi_k, k = 1, \dots, p-1$ to $N_- sZM_- = N_- sZM^s_-$ or the restrictions of the functions $\psi_k(g), k = 1, \dots, p-1$ to $N_- sZM_- = N_- sZM^s_-$ generate the vanishing ideal of $N_{[-\beta_p, -\beta_D]} sZM^s_- \subset N_- sZM^s_-$.

By repeating verbatim the arguments in the proof of Proposition 3.5.2 one can also obtain another corollary.

Corollary 3.5.5. *The restrictions of the functions $\phi_k, k = 1, \dots, p-1$ to $\overline{N_- sZHM_-} = \overline{N_- sZHM^s_-}$ or the restrictions of the functions $\psi_k(g), k = 1, \dots, p-1$ to $\overline{N_- sZHM_-} = \overline{N_- sZHM^s_-}$ generate the vanishing ideal of $\overline{N_{[-\beta_p, -\beta_D]} sZHM^s_-} \subset \overline{N_- sZHM^s_-}$.*

Let $\Delta'_s = (\Delta_{\mathfrak{m}_+} \cap \Delta_s) \cup \{\alpha \in \Delta_+ : \alpha > \gamma_{l'}\}$, $N'_- = N_{-\Delta'_s}$. Then the restrictions of the functions $\phi_k(g), k = 1, \dots, p-1$ to $\overline{N_- sZHN_-} = \overline{N_- sZHN'_-}$ or the restrictions of the functions $\psi_k(g), k = 1, \dots, p-1$ to $\overline{N_- sZHN_-} = \overline{N_- sZHN'_-}$ generate the vanishing ideal of $\overline{N_{[-\beta_p, -\beta_D]} sZHN'_-} \subset \overline{N_- sZHN'_-}$.

Observing that in the notation of Proposition 3.5.2 for $g = nn_szsm_sn^{-1} \in N_-ZsM_-$ we have $g_{c+1} = n_szsm_s = n_{c+1}^{-1}gn_{c+1}$, $n = n_{c+1} = X_{-\beta_1}(t_1)\ldots X_{-\beta_c}(t_c)$ and recalling the definition of the operator Π in (3.5.2) we infer the following theorem from Proposition 3.5.2.

Theorem 3.5.6. *Let G_p, $p = 1, \ldots, c$ be the rational function on G defined by*

$$G_p(g) = \frac{(v_p, gs^{-1}v_{\mu_p})}{(v_{\mu_p}, gs^{-1}v_{\mu_p})}, \tag{3.5.10}$$

and Π_p the operator on the space of rational functions on G induced by conjugation by the element $\exp(-G_pX_{-\beta_p})$,

$$\Pi_pf(g) = f(\exp(-G_p(g)X_{-\beta_p})g\exp(G_p(g)X_{-\beta_p})). \tag{3.5.11}$$

Then the restriction of the composition $\Pi_1\circ\ldots\circ\Pi_c$ to $\mathbb{C}[N_-ZsM_-]$ is equal to the projection operator Π onto the subspace $\mathbb{C}[N_-ZsM_-]^{M_-}$ of M_-–invariant regular functions on N_-ZsM_-, $\Pi : \mathbb{C}[N_-ZsM_-] \to \mathbb{C}[N_-ZsM_-]^{M_-}$,

$$\Pi = \Pi_1 \circ \ldots \circ \Pi_c. \tag{3.5.12}$$

This theorem has a quantum counterpart which will be formulated and proved in the next chapter. Corollary 3.5.5 and Theorem 3.5.6 are the only results of this chapter which will be used in Chapter 4.

3.6 Bibliographic comments

The results on Poisson–Lie groups used in this book can be found in [18], [28], [85], [91].

Proposition 3.1.1 is stated in [18] as Theorem 1.3.2 and Proposition 3.1.2 and the relevant properties of classical r-matrices can be found in [7], [89].

The result stated in Proposition 3.2.1 can be found in [91], Section 2.

Q-W–algebras for realizations of quantum groups associated to Weyl group elements were introduced in [95, 96] in the case of Coxeter elements and in [99] in the general situation. However, in the definitions given in those papers other forms of the quantum group are used. The definition of q-W–algebras in this book is more close to the one given in [101]; it uses the Ad locally finite part of the quantum group (see [52], Chapter 7) which reduces to the algebra of regular functions on G when $q = 1$. However, in this book we define all algebras over slightly different rings.

The exposition in Sections 3.2 and 3.4 follows [99, 101] with some appropriate modifications.

The presentation of the results on Poisson reduction in Section 3.3 is close to [97], Section 2.3. More details on the notion and the properties of dual pairs and Poisson reduction can be found in [90], and for statements related to the moment map for Poisson–Lie group actions the reader is referred to [67].

The original definition of the Poisson algebras $W^s(G)$ using the classical Poisson reduction only was given in [98].

The definition of the classical Zhelobenko type operator Π in Section 3.5 is a modified version of the definition given in [104].

Chapter 4

Zhelobenko type operators for q-W–algebras

In this chapter we define a quantum analogue Π^q of the operator Π and apply it to describe q-W–algebras. Observe that the operator Π is defined using the conjugation action and operators of multiplication by the functions G_p in the space $\mathbb{C}[N_-ZsM_-]$. The conjugation action has a natural quantum group analogue, the adjoint action. But multiplication by functions in $\mathbb{C}[G]$ is quite far from the multiplication in the algebra $\mathbb{C}_{\mathcal{B}}[G_*]$ which is used in the definition of q-W–algebras. However, using isomorphism (3.2.19) of Ad–modules $\mathbb{C}_{\mathcal{B}}[G_*]$ and $\mathbb{C}_{\mathcal{B}}[G]$ we can try to describe q-W–algebras in terms of the space $\mathbb{C}_{\mathcal{B}}[G]$ multiplication in which is more closely related to that of $\mathbb{C}[G]$. Therefore it is natural to expect that a quantum analogue of the operator Π, if it exists at all, should be defined in terms of the adjoint action and of operators of multiplication in $\mathbb{C}_{\mathcal{B}}[G]$ using appropriate quantum analogues of formulas (3.5.10), (3.5.11) and (3.5.12). We shall see that this conjecture is almost correct. In fact $\mathbb{C}_{\mathcal{B}}[G]$ should be replaced with a certain localization. More precisely, recall that the operator Π is defined using the functions G_p given by (3.5.10). Natural analogues of matrix elements which appear in formula (3.5.10) can be defined in the algebra $\mathbb{C}_{\mathcal{B}}[G]$. But formula (3.5.10) contains some artificial denominators which are canceled in the formula for Π (see Remark 3.5.3). It turns out that in the quantum case formulas similar to (3.5.10) make sense but the denominators in them are not canceled in the formula for Π^q, and we are forced to replace $\mathbb{C}_{\mathcal{B}}[G]$ with a localization containing all such denominators. This will also force us to replace the algebra $W^s_{\mathcal{B}}(G)$ with a certain localization $W^{s,loc}_{\mathcal{B}}(G)$ of it.

The main difficulty in defining a quantum analogue Π^q of the operator Π is that the proof of the fact that the operator defined by (3.5.12) is a projection operator onto $W^s(G)$ is based on isomorphism (3.4.29) a quantum counterpart of which does not make sense. Recall that $W^s_{\mathcal{B}}(G)$ is the space of invariants with respect to the adjoint action of $\mathbb{C}_{\mathcal{B}}[M_+]$ on $Q_{\mathcal{B}}$. Although quantum analogues of operators (3.5.11) can be defined the proof of the fact that their composition similar to (3.5.12) is a projection operator on the localization $W^{s,loc}_{\mathcal{B}}(G)$ of $W^s_{\mathcal{B}}(G)$ should only use the algebra structure of $\mathbb{C}_{\mathcal{B}}[G]$, the properties of the adjoint action of $\mathbb{C}_{\mathcal{B}}[M_+]$ on $Q_{\mathcal{B}}$, and the structure of $Q_{\mathcal{B}}$. These are the only technical tools in our disposal.

Thus our first task is to describe in terms of $\mathbb{C}_{\mathcal{B}}[G]$ the $\mathbb{C}_{\mathcal{B}}[M_+]$–module $Q_{\mathcal{B}}$ originally defined using $\mathbb{C}_{\mathcal{B}}[G_*]$. In the classical case this would correspond to describing the vanishing ideal of the closed subvariety $N_-ZsM_- \simeq N_-ZsM_-^s \subset G$ as by Lemma 3.4.1 and by Theorem 3.4.5 $Q_{\mathcal{B}}/(q^{\frac{1}{dr^2}}-1)Q_{\mathcal{B}} \simeq \mathbb{C}[N_-ZsM_-^s]$. It turns out that not all elements of $\mathbb{C}[G]$ generating the vanishing ideal of $N_-ZsM_-^s$ have nice quantum counterparts in $\mathbb{C}_{\mathcal{B}}[G]$. Recall that $\mathbb{C}[G]$ is $P_+\times P_+$–graded via the left and the right regular action of H on G. The subvariety $N_-ZsM_-^s \subset G$ is closed and some generators of its vanishing ideal belong to the graded components and some do not. It turns out that at least some of the generators of the latter type have no nice quantum counterparts. But for our purposes it suffices to replace $N_-ZsM_-^s$ with a larger variety $\overline{N_-ZHsM_-^s}$ the vanishing ideal of which has a nice quantum counterpart $\overline{J}^1_{\mathcal{B}}$ in $\mathbb{C}_{\mathcal{B}}[G]$. This counterpart is described in Proposition 4.1.2 and its image under the natural map $\mathbb{C}_{\mathcal{B}}[G] \simeq \mathbb{C}_{\mathcal{B}}[G_*] \to Q_{\mathcal{B}}$ is zero.

After recollecting some facts on the algebra $\mathbb{C}_{\mathcal{B}}[G]$ and on the adjoint action in Section 4.2 we study properties of $\overline{J}^1_{\mathcal{B}}$ in Section 4.3. For technical reasons related to the non–commutativity of the algebra $\mathbb{C}_{\mathcal{B}}[G]$ we also introduce and study a $\mathcal{B}$–submodule $J^1_{\mathcal{B}} \subset \overline{J}^1_{\mathcal{B}}$ which is a quantum counterpart of the vanishing ideal of $\overline{N_-sZHN_-'} \supset \overline{N_-ZHsM_-^s}$.

In order to show that Π^q is a projection operator on $W^{s,loc}_{\mathcal{B}}(G)$ we shall need more relations which resemble relations in the algebras $\mathbb{C}[\overline{N_{[-\beta_p,-\beta_D]}sZHM_-^s}], \mathbb{C}[\overline{N_{[-\beta_p,-\beta_D]}sZHN_-'}]$ of regular functions on the varieties $\overline{N_{[-\beta_p,-\beta_D]}sZHM_-^s} \subset \overline{N_-sZHM_-^s}$ and $\overline{N_{[-\beta_p,-\beta_D]}sZHN_-'} \subset \overline{N_-sZHN_-'}$ introduced in Corollary 3.5.5. These algebras are closely related to the algebras of regular functions on the subvarieties $N_{[-\beta_p,-\beta_D]}sZM_-^s \subset N_-sZM_-^s$. The use of these subvarieties in our setting comes from that fact that elements of $N_{[-\beta_p,-\beta_D]}sZM_-^s$ are obtained from elements $g \in N_-sZM_-^s$ by conjugation by n_p^{-1} as in Proposition 3.5.2. This conjugation corresponds to applying the operator $\Pi_1 \circ \ldots \circ \Pi_{p-1}$ in the algebra $\mathbb{C}[N_-ZsM_-^s]$, so that the vanishing ideal of $N_{[-\beta_p,-\beta_D]}sZM_-^s$ belongs to the kernel of this operator. For the purposes of quantization we have to replace the varieties $N_{[-\beta_p,-\beta_D]}sZM_-^s$ with the larger varieties $\overline{N_{[-\beta_p,-\beta_D]}sZHM_-^s} \subset \overline{N_-sZHM_-^s}, \overline{N_{[-\beta_p,-\beta_D]}sZHN_-'} \subset \overline{N_-sZHN_-'}$, $N_{[-\beta_p,-\beta_D]}sZM_-^s \subset \overline{N_{[-\beta_p,-\beta_D]}sZHM_-^s}, \overline{N_{[-\beta_p,-\beta_D]}sZHN_-'}$ due to the reasons mentioned above, and the vanishing ideals of these extended varieties have well–defined quantum analogues $\overline{J}_{\mathcal{B}}^p$ and $J_{\mathcal{B}}^p$ defined and studied in Section 4.4. Note that since we define all our algebras over the ring $\mathcal{B}$, for technical reasons, we shall have to consider a priori slightly larger ideals $\overline{I}_{\mathcal{B}}^p = (\overline{J}_{\mathcal{B}}^p \otimes \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G]$ and $I_{\mathcal{B}}^p = (J_{\mathcal{B}}^p \otimes \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G]$.

In Section 4.5 we introduce the localizations mentioned above and study their properties and the relevant properties of the adjoint action. The results obtained in Sections 4.1, 4.2, 4.3, 4.4 and 4.5 are prerequisites for the study of the properties of the quantum analogues P_p of the operators Π_p and of their compositions in Section 4.6, the main properties being summarized in Proposition 4.6.1. In Proposition 4.6.7 we also define quantum analogues of monomials in variables G_p, $p = 1, \ldots, c$ which play a crucial role in the study of equivariant modules over a quantum group and in the proof of the De Concini–Kac–Procesi conjecture.

Finally in Section 4.7 we prove that the image of the operator Π^q almost coincides with the localization $W_{\mathcal{B}}^{s,loc}(G)$ of the algebra $W_{\mathcal{B}}^s(G)$.

4.1 A quantum analogue of the level surface of the moment map for q-W–algebras

In this section we describe a quantum counterpart $\overline{J}_{\mathcal{B}}^1 \subset \mathbb{C}_{\mathcal{B}}[G]$ of the vanishing ideal of the variety $\overline{N_-ZHsM_-^s} \supset N_-ZsM_-^s$ containing the closure $\overline{q(\mu_{M_-}^{-1}(u))} = \mathbb{C}[N_-ZsM_-^s]$ of the level surface of the moment map μ_{M_-} corresponding to the element $u \in M_+$. As we mentioned in the introduction to this chapter isomorphism (3.2.19) of Ad–modules $\mathbb{C}_{\mathcal{B}}[G_*]$ and $\mathbb{C}_{\mathcal{B}}[G]$ plays a central role in the description of $\overline{J}_{\mathcal{B}}^1$. However, for technical reasons we replace the adjoint action of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ on $\mathbb{C}_{\mathcal{B}}[G]$ with the twisted adjoint action defined by

$$(\mathrm{Ad}_0 x f)(w) = f(\omega_0 S_s^{-1}(\mathrm{Ad}' x(S_s\omega_0 w))) = f((\omega_0 S_s^{-1})(x^1)w\omega_0(x^2)), \tag{4.1.1}$$

where $f \in \mathbb{C}_{\mathcal{B}}[G], x, w \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$, and consider isomorphism (3.2.19) twisted by $\omega_0 S_s^{-1}$,

$$\varphi : \mathbb{C}_{\mathcal{B}}[G] \to \mathbb{C}_{\mathcal{B}}[G_*], f \mapsto (id \otimes f)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s). \tag{4.1.2}$$

If $\kappa = 1$ it induces a homomorphism of $\mathbb{C}_{\mathcal{B}}[M_+]$–modules

$$\phi : \mathbb{C}_{\mathcal{B}}[G] \to Q_{\mathcal{B}}, \phi(f) = \varphi(f)1, \tag{4.1.3}$$

where $\mathbb{C}_{\mathcal{B}}[G]$ is equipped with the restriction of action (4.1.1) to $\mathbb{C}_{\mathcal{B}}[M_+]$ and $Q_{\mathcal{B}}$ with the action induced by the adjoint action Ad of $\mathbb{C}_{\mathcal{B}}[M_+]$ and 1 is the image of $1 \in \mathbb{C}_{\mathcal{B}}[G_*]$ in $Q_{\mathcal{B}}$ under the natural map $\mathbb{C}_{\mathcal{B}}[G_*] \to Q_{\mathcal{B}}$.

The following proposition is a quantum counterpart of Proposition 3.4.11. The proof of Proposition 3.4.11 was based on the Chevalley commutation relations between one–parameter subgroups in G as described in Lemma 1.3.1 and on formula (3.4.9) for representatives of Weyl group elements in G. In the quantum case instead of the Chevalley commutation relations we have commutation relations between quantum root vectors, and the Weyl group is replaced with the corresponding braid group generators of which are also expressed in terms on generators of the quantum group by formula (2.2.3). However, in the quantum case the generators of the braid group do not square to identity automorphisms of the quantum group and we are only allowed to use the braid group relations. The action of the braid group on quantum root vectors is also very difficult to control. It is much more complicated than the conjugation action of representatives in G of Weyl group elements on root vectors in $\mathfrak{g}$. All this brings additional complications to the proof of Proposition 4.1.2.

As before we assume that a Weyl group element $s \in W$ and an ordered system of positive roots Δ_+ associated to s are fixed, and denote by $\beta_1, \ldots, \beta_D$ the ordered roots in Δ_+ as in Section 1.6.

Let $U_q^{res}(w'(\mathfrak{b}_+))$ be the subalgebra in $U_q^{res}(\mathfrak{g})$ generated by the elements

$$(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})},\ldots,(X^-_{\beta_1})^{(n_1)},(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})},\ldots,(X^+_{\beta_D})^{(n_D)},n_i\in\mathbb{N},i=1,\ldots,D,$$

where $\beta_{k_{l'}}=\gamma_{l'}$, and by $U_q^{res}(H)$.

Below we denote the multiplication in the algebra $\mathbb{C}_\mathcal{B}[G]$ by $\otimes$. We shall also use the following notation for elements of $\mathbb{C}_\mathcal{B}[G]$. Let V^{res} be a $U_\mathcal{B}^{s,res}(\mathfrak{g})$–lattice in a finite rank $U_h(\mathfrak{g})$–module V. Recall that there is a contravariant non–degenerate bilinear form $(\cdot,\cdot)$ on V such that $(u,xv)=(\omega(x)u,v)$ for any $u,v\in V$, $x\in U_h(\mathfrak{g})$. Assume that u is such that $(u,w)\in\mathcal{B}$ for any $w\in V^{res}$. Then $(u,\cdot)$ is an element of the dual module V^{res*}. Since V and V^{res} are of finite ranks and $(\cdot,\cdot)$ is non–degenerate all elements of V^{res*} can be obtained this way. Clearly, for any $v\in V^{res}$ $(u,\cdot v)\in\mathbb{C}_\mathcal{B}[G]$, and by the definition $\mathbb{C}_\mathcal{B}[G]$ is generated by such elements.

Let $c_\beta\in\mathcal{B}$, $\beta\in\Delta_{\mathfrak{m}_+}$ be elements such that

$$c_\beta=\begin{cases}k_i\in\mathcal{B} & \text{if } \beta=\gamma_i, i=1,\ldots,l'\\ 0 & \text{otherwise}\end{cases}.$$

As we observed in the proof of Proposition 3.2.6 the elements $\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_1}^{k_1}$ with $n_j,k_j,i\in\mathbb{N}$, $j=1,\ldots,D$ form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}[G^*]$.

Clearly, the elements $\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_{c+1}}^{k_{c+1}}(\tilde{f}_{\beta_c}-c_{\beta_c})^{k_c}\ldots(\tilde{f}_{\beta_1}-c_{\beta_1})^{k_1}$ with $n_j,k_j,i\in\mathbb{N}$, $j=1,\ldots,D$ also form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}[G^*]$. Let $I_\mathcal{B}^\mathbf{k}$ be the $\mathcal{B}$–submodule in $\mathbb{C}_\mathcal{B}[G^*]$ generated by the elements

$$\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_{c+1}}^{k_{c+1}}(\tilde{f}_{\beta_c}-c_{\beta_c})^{k_c}\ldots(\tilde{f}_{\beta_1}-c_{\beta_1})^{k_1}$$

with $n_j,k_j,i\in\mathbb{N}$, $j=1,\ldots,D$, and where at least one $k_j>0$ for $j<c+1$. Since these elements are linearly independent they form a $\mathcal{B}$–basis in $I_\mathcal{B}^\mathbf{k}$.

Proposition 4.1.1. *Let $\overline{J}_\mathcal{B}^1$ be the left ideal in $\mathbb{C}_\mathcal{B}[G]$ generated by the elements $(u,\cdot v)\in\mathbb{C}_\mathcal{B}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v\in V^{res}$ is such that $(u,T_sxv)=0$ for any $x\in U_q^{res}(w'(\mathfrak{b}_+))$. Denote $\overline{I}_\mathcal{B}^1=(\overline{J}_\mathcal{B}^1\otimes_\mathcal{B}\mathbb{C}(q^{\frac{1}{dr^2}}))\cap\mathbb{C}_\mathcal{B}[G]$. Then $\varphi(\overline{J}_\mathcal{B}^1)\subset I_\mathcal{B}^\mathbf{k}\cap\mathbb{C}_\mathcal{B}[G_*]$ and $\varphi(\overline{I}_\mathcal{B}^1)\subset I_\mathcal{B}^\mathbf{k}\cap\mathbb{C}_\mathcal{B}[G_*]$.*

Let $Q_\mathcal{B}^\mathbf{k}$ is the image of $\mathbb{C}_\mathcal{B}[G_*]\subset\mathbb{C}_\mathcal{B}[G^*]$ under the canonical projection $\mathbb{C}_\mathcal{B}[G^*]\to\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}^\mathbf{k}$. Denote by $1\in Q_\mathcal{B}^\mathbf{k}$ the image of $1\in\mathbb{C}_\mathcal{B}[G_*]$ in $Q_\mathcal{B}^\mathbf{k}$.

If u is a highest weight vector in a finite rank indecomposable representation V_λ of $U_h(\mathfrak{g})$of highest weight λ such that $(u,u)=1$ then for any $f\in\mathbb{C}_\mathcal{B}[G]$ we have in $Q_\mathcal{B}^\mathbf{k}$

$$\varphi(f\otimes(u,\cdot T_s^{-1}u))=\varphi(\mathrm{Ad}_0(q^{-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1= \tag{4.1.4}$$
$$=q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}\varphi(\mathrm{Ad}_0(q^{(-\kappa\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f))1,$$

where the classes in the quotient $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}^\mathbf{k}$ of the elements in the right hand side of (4.1.10) belong to $Q_\mathcal{B}^\mathbf{k}$. In particular,

$$\phi((u,\cdot T_s^{-1}u))=q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1,$$

and $q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1$ should be understood as the class of the element $q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}\in\mathbb{C}_\mathcal{B}[G^*]$ in the quotient $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}^\mathbf{k}$. This class belongs to $Q_\mathcal{B}^\mathbf{k}$.

Proof. The proof of this proposition is based on Lemma 4.1.7 which will be proved in the end of this section.

First we obtain a useful expression for $(id\otimes\omega_0S_s^{-1})(\mathcal{R}_{21}^s\mathcal{R}^s)$. In order to do that we recall some properties of universal R-matrices,

$$(S_s\otimes id)\mathcal{R}^s=(id\otimes S_s^{-1})\mathcal{R}^s=\mathcal{R}^{s^{-1}},(S_s\otimes S_s)\mathcal{R}^s=\mathcal{R}^s,$$

Using the first identity above we can write

$$\mathcal{R}_{21}^s\mathcal{R}^s=\mathcal{R}_{21}^s(id\otimes S_s)(\mathcal{R}^{s^{-1}})=(id\otimes S_s^{-1})(\mathcal{R}_{21}^{s\,-1})(id\otimes S_s)(\mathcal{R}^{s^{-1}})=(id\otimes S_s)((id\otimes S_s^{-2})(\mathcal{R}_{21}^{s\,-1})\circ\mathcal{R}^{s^{-1}}),$$

where

$$(a\otimes b)\circ(c\otimes d)=ac\otimes db.$$

Now since ω_0 an algebra antiautomorphism we have

$$(id\otimes\omega_0S_s^{-1})(\mathcal{R}_{21}^s\mathcal{R}^s)=(id\otimes\omega_0)((id\otimes S_s^{-2})(\mathcal{R}_{21}^{s\,-1})\circ\mathcal{R}^{s^{-1}})=(id\otimes\omega_0S_s^{-2})(\mathcal{R}_{21}^{s\,-1})(id\otimes\omega_0)(\mathcal{R}^{s^{-1}}).$$

Recalling the definition of $\mathcal{R}^s$ we obtain

$$\mathcal{R}^{s-1} = exp\left[-h(\sum_{i=1}^{l} Y_i \otimes H_i - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right] \times \tag{4.1.5}$$
$$\times \prod_{\beta \in \Delta_+} exp_{q_\beta^{-1}}[(1-q_\beta^2) f_\beta \otimes e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee}] =$$
$$= \prod_{\beta \in \Delta_+} exp_{q_\beta^{-1}}[(1-q_\beta^2) e^{h(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} - id)\beta^\vee} f_\beta \otimes e_\beta q^{\beta^\vee}] \times$$
$$\times exp\left[-h(\sum_{i=1}^{l} Y_i \otimes H_i - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right]$$

and using the fact that

$$S_s^{-2} = \mathrm{Ad}\ q^{2\rho^\vee},$$

where ρ is a half of the sum of the positive roots, we also deduce

$$(id \otimes S_s^{-2})(\mathcal{R}_{21}^{s\ -1}) = \prod_{\beta \in \Delta_+} exp_{q_\beta^{-1}}[(1-q_\beta^2) q^{-2\beta(\rho^\vee)} e_\beta q^{\beta^\vee} \otimes e^{h\left(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} - id\right)\beta^\vee} f_\beta] \times \tag{4.1.6}$$
$$exp\left[-h(\sum_{i=1}^{l} (Y_i \otimes H_i) + \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right].$$

The order of the terms in the products in the formulas above is such that the α–term appears to the left of the β–term if $\alpha > \beta$ with respect to the normal ordering of Δ_+.

Combining (4.1.5) and (4.1.6) we arrive at the following expression for $(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s)$

$$(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s) = \prod_{\rightarrow}^{\leftarrow} exp_{q_\beta^{-1}}[(1-q_\beta^2) q^{-2\beta(\rho^\vee)} e_\beta q^{\beta^\vee} \otimes \omega_0(e^{h\left(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} - id\right)\beta^\vee} f_\beta)] \times$$
$$\times exp\left[h(\sum_{i=1}^{l} (Y_i \otimes H_i^f) + \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i^f)\right] \times \tag{4.1.7}$$
$$\times exp\left[h(\sum_{i=1}^{l} (Y_i \otimes H_i^r) - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i^r)\right] \times$$
$$\times \prod_{\rightarrow}^{\leftarrow} exp_{q_\beta^{-1}}[(1-q_\beta^2) f_\beta \otimes \omega_0(e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee})],$$

where in the product

$$\prod_{\rightarrow}^{\leftarrow}$$

the upper (the lower) arrow indicates the order of the terms in the first (the second) factor of the tensor product relative to the normal ordering of Δ_+, and superscripts $f(r)$ indicate that the corresponding term appears in the front (in the rear) of all the other terms in the product.

Assume now that $u \in V$ has highest weight λ and $v \in V^{res}$ is any vector of weight μ such that $(u, \cdot v) \in \mathbb{C}_\mathcal{B}[G]$. Observe that the elements of the subalgebra $\mathbb{C}_\mathcal{B}[M_+]$ appear in the first factor of the tensor product in formula (4.1.7) on the right. Then, since v is a highest weight vector, we have in $Q_\mathcal{B}^{\mathbf{k}}$ using (4.1.7), the definitions of $I_\mathcal{B}^{\mathbf{k}}$ and of $Q_\mathcal{B}^{\mathbf{k}} = \mathrm{Im}(\mathbb{C}_\mathcal{B}[G_*] \to \mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}^{\mathbf{k}})$, and the definition of the homomorphism φ

$$\varphi((u, \cdot v)) = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times$$
$$\times (id \otimes f)(\prod_{\overrightarrow{i\ =\ 1}}^{l'} exp_{q_{\gamma_i}^{-1}}[-q_{\gamma_i}^{-2} k_i \otimes \omega_0(e_{\gamma_i} e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\gamma_i^\vee})] \times \tag{4.1.8}$$
$$\times \prod_{\overrightarrow{\beta\ >\ \gamma_{l'}}}^{\leftarrow} exp_{q_\beta^{-1}}[(1-q_\beta^2) f_\beta \otimes \omega_0(e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee})]),$$

where $f = (u, \cdot v)$.

Now formula (4.1.8), the definitions of the subalgebra $U_q^{res}(w'(\mathfrak{b}_+))$ and of the elements $e_\beta = X_\beta^+ q^{K\beta^\vee}$, and Lemma 4.1.7 imply that in $Q_{\mathcal{B}}^{\mathbf{k}}$

$$\phi((u, \cdot v)) = \sum_i x_i(u, T_s y_i v)1, \tag{4.1.9}$$

where $x_i \in \mathbb{C}_{\mathcal{B}}[G_*]$, $y_i \in U_q^{res}(w'(\mathfrak{b}_+))$. Since every element of V^{res} is the sum of its weight components a formula of type (4.1.9) holds for arbitrary $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$, where u is a highest weight vector.

If v is chosen in such a way that $(u, T_s x v) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$ we deduce from (4.1.9) that $(u, T_s y_i v) = 0$. Thus $\phi((u, \cdot v)) = 0$ in $Q_{\mathcal{B}}^{\mathbf{k}}$.

Now using the properties

$$(\Delta_s \otimes id)\mathcal{R}^s = \mathcal{R}^s_{13}\mathcal{R}^s_{23}, (id \otimes \Delta_s)\mathcal{R}^s = \mathcal{R}^s_{13}\mathcal{R}^s_{12},$$

and the fact that ω_0 is a coautomorphism and S_s is an anti-coautomorphism we get

$$(id \otimes \Delta_s)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}^s_{21}\mathcal{R}^s) = (id \otimes \omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})(id \otimes \Delta_s^{opp})(\mathcal{R}^s_{21}\mathcal{R}^s) =$$

$$= (id \otimes \omega_0 S_s^{-2} \otimes \omega_0 S_s^{-2})(\mathcal{R}^s_{31}{}^{-1}\mathcal{R}^s_{21}{}^{-1})(id \otimes \omega_0 \otimes \omega_0)(\mathcal{R}^s_{12}{}^{-1}\mathcal{R}^s_{13}{}^{-1}).$$

From this identity we obtain, similarly to (4.1.9), that for any $f \in \mathbb{C}_{\mathcal{B}}[G]$ in $Q_{\mathcal{B}}^{\mathbf{k}}$

$$\varphi(f \otimes (u, \cdot v)) = \sum_i x_i' \varphi(f) x_i''(u, T_s y_i' v)1,$$

where $x_i', x_i'' \in \mathbb{C}_{\mathcal{B}}[G_*]$, $y_i' \in U_q^{res}(w'(\mathfrak{b}_+))$. Hence $\varphi(f \otimes (u, \cdot v)) = 0$ in $Q_{\mathcal{B}}^{\mathbf{k}}$ by the choice of v, i.e. $\varphi(\overline{J}^1_{\mathcal{B}}) \subset I_{\mathcal{B}}^{\mathbf{k}} \cap \mathbb{C}_{\mathcal{B}}[G_*]$.

In order to show that $\varphi(\overline{I}^1_{\mathcal{B}}) \subset I_{\mathcal{B}}^{\mathbf{k}} \cap \mathbb{C}_{\mathcal{B}}[G_*]$ we naturally extend φ to and Ad–module isomorphism $\varphi : \mathbb{C}_q[G] \to \mathbb{C}_q[G_*]$, where $\mathbb{C}_q[G] = \mathbb{C}_{\mathcal{B}}[G] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\mathbb{C}_q[G_*] = \mathbb{C}_{\mathcal{B}}[G_*] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. By the definition of $\overline{I}^1_{\mathcal{B}}$ we have $\varphi(\overline{I}^1_{\mathcal{B}}) \subset (I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G_*]$ as obviously $\varphi(\overline{J}^1_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \subset I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$ by the first part of the proof and $\varphi(\mathbb{C}_{\mathcal{B}}[G]) \subset \mathbb{C}_{\mathcal{B}}[G_*]$.

We also have $(I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G_*] \subset (I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G^*]$ as $\mathbb{C}_{\mathcal{B}}[G_*] \subset \mathbb{C}_{\mathcal{B}}[G^*]$.

Recall that by the definition the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{k_D} \ldots \tilde{f}_{\beta_{c+1}}^{k_{c+1}} (\tilde{f}_{\beta_c} - c_{\beta_c})^{k_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{k_1}$ with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$, and where at least one $k_j > 0$ for $j < c+1$ form a $\mathcal{B}$–basis in $I_{\mathcal{B}}^{\mathbf{k}}$, and this basis can be completed to a $\mathcal{B}$–basis of $\mathbb{C}_{\mathcal{B}}[G^*]$ which consists of the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{k_D} \ldots \tilde{f}_{\beta_{c+1}}^{k_{c+1}} (\tilde{f}_{\beta_c} - c_{\beta_c})^{k_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{k_1}$ with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$.

This implies $(I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G^*] = I_{\mathcal{B}}^{\mathbf{k}}$, and hence $\varphi(\overline{I}^1_{\mathcal{B}}) \subset (I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G_*] = ((I_{\mathcal{B}}^{\mathbf{k}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}[G^*]) \cap \mathbb{C}_{\mathcal{B}}[G_*] = I_{\mathcal{B}}^{\mathbf{k}} \cap \mathbb{C}_{\mathcal{B}}[G_*]$.

Arguments similar to those in the first part of the proof show that if u is a highest weight vector in a finite rank indecomposable representation V_λ of $U_h(\mathfrak{g})$ of highest weight λ then in $Q_{\mathcal{B}}^{\mathbf{k}}$

$$\varphi(f \otimes (u, \cdot T_s^{-1} u)) = q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \varphi(f) q^{s^{-1}(-\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} 1 =$$

$$= q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} \mathrm{Ad}(q^{(-\kappa \frac{1+s}{1-s} s^{-1} P_{\mathfrak{h}'} + s^{-1})\lambda^\vee})(\varphi(f))1 =$$

$$= q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} \varphi(\mathrm{Ad}_0(q^{(-\kappa \frac{1+s}{1-s} s^{-1} P_{\mathfrak{h}'} + s^{-1})\lambda^\vee})(f))1 =$$

$$= q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} \varphi(\mathrm{Ad}_0(q^{(-\kappa \frac{1+s}{1-s} s^{-1} P_{\mathfrak{h}'} + s^{-1})\lambda^\vee})(f)).$$

The formula above can also be rewritten as

$$\varphi(f \otimes (u, \cdot T_s^{-1} u)) = \mathrm{Ad}(q^{-(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee})(\varphi(f)) q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} 1 =$$

$$= \varphi(\mathrm{Ad}_0(q^{-(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee})(f)) q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} 1.$$

In particular, in $Q_{\mathcal{B}}^{\mathbf{k}}$

$$\varphi((u, \cdot T_s^{-1} u)) = q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{h}'})\lambda^\vee} 1.$$

Note that the classes in the quotient $\mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$ of the elements

$$q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}\varphi(\mathrm{Ad}_0(q^{(-\kappa\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f)), \varphi(\mathrm{Ad}_0(q^{-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}$$

and $q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}$ in the right hand sides of the formulas above a priori belong to $Q_{\mathcal{B}}^{\mathbf{k}}$. This completes the proof.

□

Now consider the case when $\kappa = 1$ and $k_i \in \mathcal{B}$, $i = 1, \ldots, l'$ are defined in (3.2.11). Recall that the elements $\tilde{e}_{\beta_1}^{n_1}\ldots\tilde{e}_{\beta_D}^{n_D}V_i\tilde{f}_{\beta_D}^{k_D}\ldots\tilde{f}_{\beta_{c+1}}^{k_{c+1}}(\tilde{f}_{\beta_c}-c_{\beta_c})^{k_c}\ldots(\tilde{f}_{\beta_1}-c_{\beta_1})^{k_1}$ with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}_{\mathcal{B}}[G^*]$ and observe that in the considered case the elements $(\tilde{f}_{\beta_c}-c_{\beta_c})^{k_c}\ldots(\tilde{f}_{\beta_1}-c_{\beta_1})^{k_1}$ with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$, and where at least one $k_j > 0$ for $j < c+1$, form a $\mathcal{B}$–basis of $\mathrm{Ker}\chi_q^s$. Therefore in the considered case $I_{\mathcal{B}}^{\mathbf{k}} = I_{\mathcal{B}}$ and $Q_{\mathcal{B}}^{\mathbf{k}} = Q_{\mathcal{B}}$, and we can apply the previous proposition to get the following statement.

Proposition 4.1.2. *Assume that $\kappa = 1$ and $k_i \in \mathcal{B}$ are defined in (3.2.11). Then $\overline{J}_{\mathcal{B}}^1$ and $\overline{I}_{\mathcal{B}}^1$ lie in the kernel of ϕ. Moreover, if u is a highest weight vector in a finite rank indecomposable representation V_λ of $U_h(\mathfrak{g})$ of highest weight λ such that $(u,u) = 1$ then for any $f \in \mathbb{C}_{\mathcal{B}}[G]$*

$$\phi(f \otimes (u, \cdot T_s^{-1}u)) = \varphi(\mathrm{Ad}_0(q^{-(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1 = \qquad (4.1.10)$$

$$= q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}\phi(\mathrm{Ad}_0(q^{(-\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f)),$$

where the classes in the quotient $\mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$ of the elements in the right hand side of (4.1.10) belong to $Q_{\mathcal{B}}$. In particular,

$$\phi((u, \cdot T_s^{-1}u)) = q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1,$$

and $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1$ should be understood as the class of the element $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee} \in \mathbb{C}_{\mathcal{B}}[G^*]$ in the quotient $\mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$. This class belongs to $Q_{\mathcal{B}}$.

The rest of this section will be devoted to the proof of Lemma 4.1.7. This proof is in turn split into several other lemmas.

Lemma 4.1.3. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$. Let $\overline{w} = s_{i_1}\ldots s_{i_D}$ be a reduced decomposition of the longest element of the Weyl group W. Then for any $\beta = s_{i_1}\ldots s_{i_{k-1}}\alpha_{i_k} \in \Delta_+$ and $k \in \mathbb{N}$*

$$(u, (X_\beta^+)^{(k)}v) = \sum_{p,p'}(u, K_{p,p'}(X_\beta^-)^{(p)}(X_\beta^+)^{(p')}T_\beta v), \qquad (4.1.11)$$

where the sum in the right hand side is finite, $X_\beta^\pm = T_{i_1}\ldots T_{i_{k-1}}X_{i_k}^\pm$, $K_{p,p'} \in \mathbb{C}[q,q^{-1}]^*$ are integer powers of q, and

$$T_\beta = T_{i_1}\ldots T_{i_{k-1}}T_{i_k}^{-1}T_{i_{k-1}}^{-1}\ldots T_{i_1}^{-1}.$$

Proof. First we assume that v is a weight vector of weight λ. Then the only nontrivial contribution to the left hand side of (4.1.11) comes from the component of u of weight $\lambda + k\beta$, so we can assume that u has weight $\lambda + k\beta$.

Conjugating (2.2.5) by $T_{i_1}\ldots T_{i_{k-1}}$ we get

$$exp'_{q_i}(-X_\beta^+) = exp'_{q_i^{-1}}(-q_iX_\beta^-K_\beta^{-1})q_i^{\frac{H_\beta(H_\beta+1)}{2}}exp'_{q_i^{-1}}(q_i^{-1}X_\beta^+)T_\beta, \qquad (4.1.12)$$

where $K_\beta = q^{\beta^\vee}$, $H_\beta = T_{i_1}\ldots T_{i_{k-1}}H_{i_k}$.

Evaluating this identity on the matrix element $(u, \cdot v)$ and using the commutation relations of the quantum group we obtain (4.1.11) for v of weight λ, where the sum in the right hand side is such that all terms $K_{p,p'}(X_\beta^-)^{(p)}(X_\beta^+)^{(p')}T_\beta v$ have weight $\lambda + k\beta$, i.e. $p' - p - \beta^\vee(\lambda) = k$. Adding identities of type (4.1.11) for weight vectors v we obtain identities of the same type for arbitrary v.

□

Next we obtain some useful relations in the Weyl group which lead to important formulas for the action of braid group elements on quantum root vectors. Recall that according to the arguments in the proof of Proposition 5.1 in [99] s^1 is the longest element in the Weyl group W_1 of the semisimple part $\mathfrak{m}^1$ of the standard Levi subalgebra the positive roots of which in Δ_+ form the set

$$\Delta_+^1=\{\gamma_1,\beta^1_{t+\frac{p-n}{2}+2},\ldots,\beta^1_{t+\frac{p-n}{2}+n_1},\gamma_2,\beta^1_{t+\frac{p-n}{2}+n_1+2}\ldots,\beta^1_{t+\frac{p-n}{2}+n_2},\gamma_3,\ldots,\gamma_n, \\ -\beta^1_{t+1},\ldots,-\beta^1_{t+\frac{p-n}{2}}\} \tag{4.1.13}$$

and s^1 acts on them by multiplication by -1. The roots in (4.1.13) are ordered as in the normal ordering of Δ_+ associated to s.

Let

$$s^1=s_{i_1}\ldots s_{i_p}=s_{i_{k_1}}\ldots s_{i_{k_2}}\ldots s_{i_{k_n}}s_{i_{k_n+1}}\ldots s_{i_p}, i_{k_1}=i_1$$

be the corresponding reduced decomposition of s^1, where $\gamma_m=s_{i_{k_1}}\ldots s_{i_{k_2}}\ldots s_{i_{k_m-1}}\alpha_{i_{k_m}}$, $m=1,\ldots,n$. Since s^1 is an involution we also have the following reduced decomposition

$$s^1=s_{i_p}\ldots s_{i_1}=s_{i_p}\ldots s_{i_{k_n+1}}s_{i_{k_n}}\ldots s_{i_{k_2}}\ldots s_{i_{k_1}}. \tag{4.1.14}$$

Let $\gamma_1\leq\beta_q\leq\gamma_n$, $\beta_q=s_{i_1}\ldots s_{i_{q-1}}\alpha_{i_q}$. Since $s^1=-1$ in W_1

$$s^1=s_{i_q}\ldots s_{i_1}s^1s_{i_1}\ldots s_{i_q}=s_{i_q}\ldots s_{i_1}s_{i_1}\ldots s_{i_p}s_{i_1}\ldots s_{i_q}=s_{i_{q+1}}\ldots s_{i_p}s_{i_1}\ldots s_{i_q},$$

and in the right hand side we obtain a reduced decomposition of s^1. Now

$$s^1\alpha_{i_q}=-\alpha_{i_q}=s_{i_{q+1}}\ldots s_{i_p}s_{i_1}\ldots s_{i_q}\alpha_{i_q}=-s_{i_{q+1}}\ldots s_{i_p}s_{i_1}\ldots s_{i_{q-1}}\alpha_{i_q},$$

and hence

$$s_{i_{q+1}}\ldots s_{i_p}s_{i_1}\ldots s_{i_{q-1}}\alpha_{i_q}=\alpha_{i_q}. \tag{4.1.15}$$

Form the expressions $\gamma_m=s_{i_{k_1}}\ldots s_{i_{k_2}}\ldots s_{i_{k_m-1}}\alpha_{i_{k_m}}$, $m=1,\ldots,n$ we deduce

$$s_{\gamma_1}\ldots s_{\gamma_{q-1}}=s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_{q-1}-1}}s_{i_{k_{q-1}}}s_{i_{k_{q-1}-1}}\ldots s_{i_1}.$$

In particular, we have reduced decompositions

$$s^1=s_{\gamma_1}\ldots s_{\gamma_n}=s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_n-1}}s_{i_{k_n}}s_{i_{k_n-1}}\ldots s_{i_1} \tag{4.1.16}$$

and

$$s^1=s_{i_{k_n}}\ldots s_{i_1}s^1s_{i_1}\ldots s_{i_{k_n}}=s_{i_{k_n}}s_{i_{k_n-1}}\ldots s_{i_1}s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_n-1}}. \tag{4.1.17}$$

Comparing the first expression above with (4.1.14) we obtain the following identity for reduced decompositions

$$s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_n-1}}=s_{i_p}\ldots s_{i_{k_n+1}}. \tag{4.1.18}$$

As the roots γ_m, $m=1,\ldots,n$ are mutually orthogonal we deduce

$$s_{\gamma_1}\ldots s_{\gamma_{q-1}}\gamma_q=s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_{q-1}-1}}s_{i_{k_{q-1}}}s_{i_{k_{q-1}-1}}\ldots s_{i_1}s_{i_1}\ldots s_{i_{k_q-1}}\alpha_{i_{k_q}}= \\ =s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_{q-1}-1}}s_{i_{k_{q-1}+1}}\ldots s_{i_{k_q-1}}\alpha_{i_{k_q}}=\gamma_q=s_{i_1}\ldots s_{i_{k_q-1}}\alpha_{i_{k_q}}.$$

Therefore

$$s_{i_{k_q-1}}\ldots s_{i_1}s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_{q-1}-1}}s_{i_{k_{q-1}+1}}\ldots s_{i_{k_q-1}}\alpha_{i_{k_q}}=\alpha_{i_{k_q}},$$

and

$$s_{i_{k_q-1}}\ldots s_{i_1}s_{i_2}\ldots s_{i_{k_2-1}}s_{i_{k_2+1}}\ldots s_{i_{k_{q-1}-1}}s_{i_{k_{q-1}+1}}\ldots s_{i_{k_q-1}}$$

is a reduced decomposition since it is a part of reduced decomposition (4.1.17).

The last two properties and (2.2.10) imply

$$T_{i_{k_q-1}}\ldots T_{i_1}T_{i_2}\ldots T_{i_{k_2-1}}T_{i_{k_2+1}}\ldots T_{i_{k_{q-1}-1}}T_{i_{k_{q-1}+1}}\ldots T_{i_{k_q-1}}X^{\pm}_{i_{k_q}}=X^{\pm}_{i_{k_q}},$$

and hence

$$T_{\gamma_1}\ldots T_{\gamma_{q-1}}X^{\pm}_{\gamma_q}=T_{i_2}\ldots T_{i_{k_2-1}}T_{i_{k_2+1}}\ldots T_{i_{k_{q-1}-1}}T^{-1}_{i_{k_{q-1}}}T^{-1}_{i_{k_{q-1}-1}}\ldots T^{-1}_{i_1}T_{i_1}\ldots T_{i_{k_q-1}}X^{\pm}_{i_{k_q}}= \\ =T_{i_2}\ldots T_{i_{k_2-1}}T_{i_{k_2+1}}\ldots T_{i_{k_{q-1}-1}}T_{i_{k_{q-1}+1}}\ldots T_{i_{k_q-1}}X^{\pm}_{i_{k_q}}=T^{-1}_{i_1}\ldots T^{-1}_{i_{k_q-1}}X^{\pm}_{i_{k_q}}=\overline{X}^{\pm}_{\gamma_q}. \tag{4.1.19}$$

Lemma 4.1.4. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$. Then for any $m_1, \ldots, m_n \in \mathbb{N}$*

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) = (u, K T_{\gamma_1} \ldots T_{\gamma_n} v), \tag{4.1.20}$$

where K is of the form

$$K = \sum_{\substack{p_1, \ldots, p_k \\ p_1', \ldots, p_m'}} K_{p_1', \ldots, p_m'}^{p_1, \ldots, p_k} (\overline{X}_{\delta_1}^-)^{(p_1)} \ldots (\overline{X}_{\delta_k}^-)^{(p_k)} (\overline{X}_{\beta_1}^+)^{(p_1')} \ldots (\overline{X}_{\beta_m}^+)^{(p_m')}, \tag{4.1.21}$$

the finite sum in the right hand side is over all $\delta_i, \beta_j \in \Delta_+^1$ such that $\delta_1 < \delta_2 < \ldots < \delta_k$, $\gamma_1 \leq \beta_1 < \beta_2 < \ldots < \beta_m \leq \gamma_n$, and $\overline{X}_{\delta_i}^-$, $\overline{X}_{\beta_j}^+$ are the quantum root vectors of $U_h(\mathfrak{m}^1) \subset U_h(\mathfrak{g})$ defined with the help of normal ordering (4.1.13) of Δ_+^1, $K_{p_1', \ldots, p_m'}^{p_1, \ldots, p_n} \in U_q^{res}(H)$.

Proof. By Lemma 4.1.3

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) =$$

$$= \sum_{\substack{c_1, \ldots, c_n \\ c_1', \ldots, c_n'}} K_{c_1', \ldots, c_n'}^{c_1, \ldots, c_n} (u, (X_{\gamma_1}^-)^{(c_1)} (X_{\gamma_1}^+)^{(c_1')} T_{\gamma_1} (X_{\gamma_2}^-)^{(c_2)} (X_{\gamma_2}^+)^{(c_2')} T_{\gamma_2} \ldots (X_{\gamma_n}^-)^{(c_n)} (X_{\gamma_n}^+)^{(c_n')} T_{\gamma_n} v),$$

where the sum in the right hand side is finite and $K_{c_1', \ldots, c_n'}^{c_1, \ldots, c_n}$ are integer powers of q.

Using (4.1.19) and observing that $X_{\gamma_1}^\pm = \overline{X}_{\gamma_1}^\pm$ as γ_1 the first simple root in the normal ordering of Δ_+ associated to s we obtain

$$(u, (X_{\gamma_1}^+)^{(k_1)} \ldots (X_{\gamma_n}^+)^{(k_n)} v) =$$

$$= \sum_{\substack{c_1, \ldots, c_n \\ c_1', \ldots, c_n'}} K_{c_1', \ldots, c_n'}^{c_1, \ldots, c_n} (u, (\overline{X}_{\gamma_1}^-)^{(c_1)} (\overline{X}_{\gamma_1}^+)^{(c_1')} (\overline{X}_{\gamma_2}^-)^{(c_2)} (\overline{X}_{\gamma_2}^+)^{(c_2')} \ldots (\overline{X}_{\gamma_n}^-)^{(c_n)} (\overline{X}_{\gamma_n}^+)^{(c_n')} T_{\gamma_1} \ldots T_{\gamma_n} v).$$

Now (4.1.20) follows from the previous identity by bringing the monomials

$$(\overline{X}_{\gamma_1}^-)^{(c_1)} (\overline{X}_{\gamma_1}^+)^{(c_1')} (\overline{X}_{\gamma_2}^-)^{(c_2)} (\overline{X}_{\gamma_2}^+)^{(c_2')} \ldots (\overline{X}_{\gamma_n}^-)^{(c_n)} (\overline{X}_{\gamma_n}^+)^{(c_n')}$$

in the right hand side to the form as in the right hand side of (4.1.21) with the help of Lemma 2.4.2.

□

Lemma 4.1.5. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$. Suppose that u is a highest weight vector. Then for any $m_1, \ldots, m_n \in \mathbb{N}$*

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) =$$
$$= (u, \sum_{q_1, \ldots, q_{k_n}} C_{q_1, \ldots, q_{k_n}} T_{\gamma_1} \ldots T_{\gamma_n} (X_{\beta_1}^-)^{(q_1)} \ldots (X_{\beta_{k_n}}^-)^{(q_{k_n})} v),$$

where $C_{q_1, \ldots, q_{k_n}} \in \mathbb{C}[q, q^{-1}]$.

Proof. Denote $T^1 = T_{\gamma_1} \ldots T_{\gamma_n}$. From the definition of $T_{\gamma_1}, \ldots, T_{\gamma_n}$ and (4.1.18) we obtain

$$T^1 = T_{\gamma_1} \ldots T_{\gamma_n} = T_{i_2} \ldots T_{i_{k_2 - 1}} T_{i_{k_2 + 1}} \ldots T_{i_{k_n - 1}} T_{i_{k_n}}^{-1} T_{i_{k_n - 1}}^{-1} \ldots T_{i_1}^{-1} = \tag{4.1.22}$$

$$= T_{i_p} \ldots T_{i_{k_n + 1}} T_{i_{k_n}}^{-1} T_{i_{k_n - 1}}^{-1} \ldots T_{i_1}^{-1}.$$

Therefore for $\gamma_1 \leq \beta_q \leq \gamma_n$, $\beta_q = s_{i_1} \ldots s_{i_{q-1}} \alpha_{i_q}$ we have

$$(T^1)^{-1} \overline{X}_{\beta_q}^+ = T_{i_1} \ldots T_{i_{k_n}} T_{i_{k_n + 1}}^{-1} \ldots T_{i_p}^{-1} T_{i_1}^{-1} \ldots T_{i_{q-1}}^{-1} X_{i_q}^+. \tag{4.1.23}$$

By (4.1.15), (2.2.10)

$$T_{i_{q+1}}^{-1} \ldots T_{i_p}^{-1} T_{i_1}^{-1} \ldots T_{i_{q-1}}^{-1} X_{i_q}^+ = X_{i_q}^+,$$

and hence

$$T_{i_{k_n + 1}}^{-1} \ldots T_{i_p}^{-1} T_{i_1}^{-1} \ldots T_{i_{q-1}}^{-1} X_{i_q}^+ = T_{i_{k_n}} \ldots T_{i_{q+1}} X_{i_q}^+.$$

Combining this with (4.1.23) we infer

$$(T^1)^{-1}\overline{X}^+_{\beta_q} = T_{i_1}\ldots T_{i_{k_n}}T_{i_{k_n}}\ldots T_{i_{q+1}}X^+_{i_q} \tag{4.1.24}$$

Since u is a highest weight vector, Lemma 4.1.4 and the definition of the bilinear form on V imply

$$(u,(X^+_{\gamma_1})^{(m_1)}\ldots(X^+_{\gamma_n})^{(m_n)}v) = (u, \sum_{p_1',\ldots,p_m'} K^{0,\ldots,0}_{p_1',\ldots,p_m'}(\overline{X}^+_{\beta_1})^{(p_1')}\ldots(\overline{X}^+_{\beta_m})^{(p_m')}T_{\gamma_1}\ldots T_{\gamma_n}v).$$

Combining this with (4.1.24) we get

$$(u,(X^+_{\gamma_1})^{(m_1)}\ldots(X^+_{\gamma_n})^{(m_n)}v) = \tag{4.1.25}$$
$$= (u, \sum_{p_1',\ldots,p_m'} K^{0,\ldots,0}_{p_1',\ldots,p_{k_n}'}T_{\gamma_1}\ldots T_{\gamma_n}(T_{i_1}\ldots T_{i_{k_n}})((X^+_{\delta_1})^{(p_1')}\ldots(X^+_{\delta_{k_n}})^{(p_m')})v),$$

where for $q=1,\ldots,k_n$ we denote $\delta_q = s_{i_{k_n}}\ldots s_{i_{q+1}}\alpha_{i_q}$, and $X^+_{\delta_q} = T_{i_{k_n}}\ldots T_{i_{q+1}}X^+_{i_q}$.

Let $U_q^{res}(\mathfrak{m}^1)$ be the restricted specialization of $U_h(\mathfrak{m}^1)$, $U_q^{res}(\mathfrak{m}^1_+)$ the subalgebra of $U_q^{res}(\mathfrak{m}^1)$ generated by $(X^+_\beta)^{(k)}$ for simple roots $\beta\in\Delta^1_+$ and $k\geq 0$. Then for $q=1,\ldots,k_n$ and $k\geq 0$ we have $(X^+_{\delta_q})^{(k)}\in U_q^{res}(\mathfrak{m}^1_+)$.

Since

$$s^1 = s_{i_{k_n}}\ldots s_{i_1}s^1 s_{i_1}\ldots s_{i_{k_n}} = s_{i_{k_n}}\ldots s_{i_1}s_{i_p}\ldots s_{i_1}s_{i_1}\ldots s_{i_{k_n}} = s_{i_{k_n}}\ldots s_{i_1}s_{i_p}\ldots s_{i_{k_n+1}},$$

and in the right hand side we have a reduced decomposition, the elements

$$\begin{aligned} Y_{\phi_{k_n+1}} &= T^{-1}_{i_{k_n}}\ldots T^{-1}_{i_1}T^{-1}_{i_p}\ldots T^{-1}_{i_{k_n+2}}X^+_{i_{k_n+1}},\\ Y_{\phi_{k_n+2}} &= T^{-1}_{i_{k_n}}\ldots T^{-1}_{i_1}T^{-1}_{i_p}\ldots T^{-1}_{i_{k_n+3}}X^+_{i_{k_n+2}},\ldots,\\ Y_{\phi_p} &= T^{-1}_{i_{k_n}}\ldots T^{-1}_{i_1}X^+_{i_p},\\ Y_{\phi_1} &= T^{-1}_{i_{k_n}}\ldots T^{-1}_{i_2}X^+_{i_1},\ldots,\\ Y_{\phi_{k_n}} &= X^+_{i_{k_n}}\end{aligned}$$

belong to $U_q^{res}(\mathfrak{m}^1_+)$ and the products

$$(Y_{\phi_{k_n+1}})^{(q_{k_n+1})}(Y_{\phi_{k_n+2}})^{(q_{k_n+2})}\ldots(Y_{\phi_p})^{(q_p)}(Y_{\phi_1})^{(q_1)}\ldots(Y_{\phi_{k_n}})^{(q_{k_n})}$$

form a $\mathbb{C}[q,q^{-1}]$–basis of $U_q^{res}(\mathfrak{m}^1_+)$. Expanding each monomial $(X^+_{\delta_1})^{(p_1')}\ldots(X^+_{\delta_m})^{(p_{k_n}')}$ in (4.1.25) with respect to this basis we get

$$(u,(X^+_{\gamma_1})^{(m_1)}\ldots(X^+_{\gamma_n})^{(m_n)}v) = (u,\sum_{q_1,\ldots,q_p}K_{q_1,\ldots,q_p}T_{\gamma_1}\ldots T_{\gamma_n}(T_{i_1}\ldots T_{i_{k_n}})((Y_{\phi_{k_n+1}})^{(q_{k_n+1})}(Y_{\phi_{k_n+2}})^{(q_{k_n+2})}\ldots$$
$$\ldots(Y_{\phi_p})^{(q_p)}(Y_{\phi_1})^{(q_1)}\ldots(Y_{\phi_{k_n}})^{(q_{k_n})})v), \tag{4.1.26}$$

where $K_{q_1,\ldots,q_p}\in\mathbb{C}[q,q^{-1}]$.

Now for $r=k_n+1,\ldots p$

$$(T_{i_1}\ldots T_{i_{k_n}})(Y_{\phi_r}) = T^{-1}_{i_p}\ldots T^{-1}_{i_{r+1}}X^+_{i_r}, \tag{4.1.27}$$

and for $r=1,\ldots k_n$

$$(T_{i_1}\ldots T_{i_{k_n}})(Y_{\phi_r}) = T_{i_1}\ldots T_{i_r}X^+_{i_r} = -T_{i_1}\ldots T_{i_{r-1}}X^-_{i_r}K_{i_r} = -X^-_{\beta_r}K_{\beta_r}, \tag{4.1.28}$$

where $K_{\beta_r} = T_{i_1}\ldots T_{i_{r-1}}K_{i_r}$.

Since u is a highest weight vector, by Lemma 2.7.4 we have $T_{\gamma_1}\ldots T_{\gamma_1}u = \omega(c)T_{i_p}\ldots T_{i_1}u$, where $c\in\mathbb{C}[q,q^{-1}]^*$. Therefore for any element $w\in V$, $x\in U_h(\mathfrak{g})$ we can write

$$(u,T_{\gamma_1}\ldots T_{\gamma_n}xw) = c(u,T_{i_1}\ldots T_{i_p}xw).$$

Combining this with (4.1.27) yields for $r=k_n+1,\ldots p$

$$\begin{aligned}(u,T_{\gamma_1}\ldots T_{\gamma_n}(T_{i_1}\ldots T_{i_{k_n}})(Y_{\phi_r})xw) &= c(u,T_{i_1}\ldots T_{i_p}(T^{-1}_{i_p}\ldots T^{-1}_{i_{r+1}})(X^+_{i_r})xw) =\\ = c(u,(T_{i_1}\ldots T_{i_r})(X^+_{i_r})T_{i_1}\ldots T_{i_p}xw) &= -c(u,(T_{i_1}\ldots T_{i_{r-1}})(X^-_{i_r}K_{i_r})T_{i_1}\ldots T_{i_p}xw) =\\ &= -c(u,X^-_{\beta_r}K_{\beta_r}T_{i_1}\ldots T_{i_p}xw) = 0.\end{aligned}$$

From this formula, (4.1.26), (4.1.28) and the commutation relations between the elements K_{β_r} and the quantum root vectors we deduce

$$(u,(X^+_{\gamma_1})^{(m_1)}\ldots(X^+_{\gamma_n})^{(m_n)}v)=$$
$$=(u,\sum_{q_1,\ldots,q_{k_n}}C_{q_1,\ldots,q_{k_n}}T_{\gamma_1}\ldots T_{\gamma_n}(X^-_{\beta_1})^{(q_1)}\ldots(X^-_{\beta_{k_n}})^{(q_{k_n})}v),$$

where $C_{q_1,\ldots,q_{k_n}}\in\mathbb{C}[q,q^{-1}]$.

□

Recall now that according to the results obtained in the proof of Proposition 5.1 in [99] s^2 is the longest element in the Weyl group W_2 of the semisimple part $\mathfrak{m}^2$ of the Levi subalgebra the positive roots of which in Δ_+ form the set

$$\begin{aligned}\Delta_+^2=\{\gamma_{n+1},\beta^2_{q+2},\ldots,\beta^2_{q+m_1},\gamma_{n+2},\beta^2_{q+m_1+2},\ldots,\beta^2_{q+m_2},\gamma_{n+3},\ldots,\\ \gamma_{l'},\beta^2_{q+m_{l(s^2)}+1},\ldots,\beta^2_{2q+2m_{l(s^2)}-(l'-n)}\},\end{aligned}\tag{4.1.29}$$

and s^2 acts on them by multiplication by -1. The roots in (4.1.29) are ordered as in the normal ordering of Δ_+ associated to s.

As above we denote $\gamma_m=s_{i_1}\ldots s_{i_{k_{n+1}}}\ldots s_{i_{k_m-1}}\alpha_{i_{k_m}}$, $m=n+1,\ldots,l'$.

Let $w=s_{i_1}\ldots s_{i_{k_{n+1}-1}}$, $T_w=T_{i_1}\ldots T_{i_{k_{n+1}-1}}$. Then $\widetilde{\Delta}_+^2=w^{-1}(\Delta_+^2)$ is the set of positive roots of the semisimple part $\widetilde{\mathfrak{m}}^2$ of a standard Levi subalgebra of $\mathfrak{g}$, $\widetilde{s}^2=w^{-1}s^2w$ is the longest element in the Weyl group $\widetilde{W}_2$ of $\widetilde{\mathfrak{m}}^2$.

For any root $\beta\in\Delta_+$ denote

$$\widetilde{X}^\pm_\beta=T_w^{-1}(X^\pm_\beta),\ \ \widetilde{\overline{X}}^\pm_\beta=\overline{T}_w^{-1}(\overline{X}^\pm_\beta),\ \ \widetilde{T}_\beta=T_w^{-1}T_\beta T_w.$$

Similarly to Lemma 4.1.4 we have

Lemma 4.1.6. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u,v\in V$. Then for any $m_{n+1},\ldots,m_{l'}\in\mathbb{N}$*

$$(u,(\widetilde{X}^+_{\gamma_{n+1}})^{(m_{n+1})}\ldots(\widetilde{X}^+_{\gamma_{l'}})^{(m_{l'})}v)=(u,\widetilde{K}\widetilde{T}_{\gamma_{n+1}}\ldots\widetilde{T}_{\gamma_{l'}}v),\tag{4.1.30}$$

where $\widetilde{K}$ is of the form

$$\widetilde{K}=\sum_{\substack{p_1,\ldots,p_k\\p'_1,\ldots,p'_m}}M^{p_1,\ldots,p_k}_{p'_1,\ldots,p'_m}(\widetilde{X}^-_{\delta_1})^{(p_1)}\ldots(\widetilde{X}^-_{\delta_k})^{(p_k)}(\widetilde{\overline{X}}^+_{\beta'_1})^{(p'_1)}\ldots(\widetilde{\overline{X}}^+_{\beta'_m})^{(p'_m)},\tag{4.1.31}$$

the finite sum in the right hand side is over all $\delta_i,\beta_j\in\Delta_+^2$ such that $\delta_1<\delta_2<\ldots<\delta_k$, $\gamma_{n+1}\le\beta'_1<\beta'_2<\ldots<\beta'_m\le\gamma_{l'}$, $M^{p_1,\ldots,p_n}_{p'_1,\ldots,p'_m}\in U_q^{res}(H)$.

Proof. Similarly to Lemma 4.1.4 we deduce that formula (4.1.30) holds with

$$\widetilde{K}=\sum_{\substack{p_1,\ldots,p_k\\p'_1,\ldots,p'_m}}L^{p_1,\ldots,p_k}_{p'_1,\ldots,p'_m}(\widetilde{\overline{X}}^-_{\delta_1})^{(p_1)}\ldots(\widetilde{\overline{X}}^-_{\delta_k})^{(p_k)}(\widetilde{\overline{X}}^+_{\beta'_1})^{(p'_1)}\ldots(\widetilde{\overline{X}}^+_{\beta'_m})^{(p'_m)},\tag{4.1.32}$$

where $L^{p_1,\ldots,p_n}_{p'_1,\ldots,p'_m}\in U_q^{res}(H)$.

Let $U_q^{res}(\widetilde{\mathfrak{m}}^2)$ be the restricted specialization of $U_h(\widetilde{\mathfrak{m}}^2)$, $U_q^{res}(\widetilde{\mathfrak{m}}^2_+)$ the subalgebra of $U_q^{res}(\widetilde{\mathfrak{m}}^2)$ generated by $(X^+_\beta)^{(k)}$ for simple roots $\beta\in\widetilde{\Delta}_+^2$ and $k\ge0$. Then the monomials $(\widetilde{\overline{X}}^-_{\delta_1})^{(p_1)}\ldots(\widetilde{\overline{X}}^-_{\delta_k})^{(p_k)}$, $\delta_i,\in\Delta_+^2$, $\delta_1<\delta_2<\ldots<\delta_k$ form a linear basis of $U_q^{res}(\widetilde{\mathfrak{m}}^2_+)$ and the monomials $(\widetilde{X}^-_{\delta_1})^{(p_1)}\ldots(\widetilde{X}^-_{\delta_k})^{(p_k)}$ $\delta_i,\in\Delta_+^2$, $\delta_1<\delta_2<\ldots<\delta_k$ form a linear basis of $U_q^{res}(\widetilde{\mathfrak{m}}^2_+)$ as well. Thus $\widetilde{K}$ can be rewritten in form (4.1.31) using the latter basis.

□

Lemma 4.1.7. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$. Suppose that u is a highest weight vector. Then for any $m_1, \ldots, m_{l'} \in \mathbb{N}$*

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(m_{l'})} v) =$$

$$= \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)} v) =$$

$$= c \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(u, T_s (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)} v) =$$

$$= c' \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(u, \overline{T}_s (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)} v),$$

where $F_{n_1,\ldots,n_D} \in \mathbb{C}[q, q^{-1}]$, $c, c' \in \mathbb{C}[q, q^{-1}]^*$ are integer powers of q up to constant factors.

Proof. By Lemmas 4.1.5 and 4.1.6 we have

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(m_{l'})} v) =$$

$$= (u, \sum_{q_1,\ldots,q_{k_n}} C_{q_1,\ldots,q_{k_n}} T_{\gamma_1} \ldots T_{\gamma_n} (X^-_{\beta_1})^{(q_1)} \ldots (X^-_{\beta_{k_n}})^{(q_{k_n})} \times$$

$$\times T_w \sum_{\substack{p_1,\ldots,p_k \\ p_1',\ldots,p_m'}} M^{p_1,\ldots,p_k}_{p_1',\ldots,p_m'} (\widetilde{X}^-_{\delta_1})^{(p_1)} \ldots (\widetilde{X}^-_{\delta_k})^{(p_k)} (\widetilde{\overline{X}}^+_{\beta_1'})^{(p_1')} \ldots (\widetilde{\overline{X}}^+_{\beta_m'})^{(p_m')} \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v) =$$

$$= (u, \sum_{q_1,\ldots,q_{k_n}} C_{q_1,\ldots,q_{k_n}} T_{\gamma_1} \ldots T_{\gamma_n} (X^-_{\beta_1})^{(q_1)} \ldots (X^-_{\beta_{k_n}})^{(q_{k_n})} \times$$

$$\times \sum_{\substack{p_1,\ldots,p_k \\ p_1',\ldots,p_m'}} M^{p_1,\ldots,p_k}_{p_1',\ldots,p_m'} (X^-_{\delta_1})^{(p_1)} \ldots (X^-_{\delta_k})^{(p_k)} T_w (\widetilde{\overline{X}}^+_{\beta_1'})^{(p_1')} \ldots (\widetilde{\overline{X}}^+_{\beta_m'})^{(p_m')} \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v).$$

Observe that $(X^-_\beta)^{(k)}$, $\beta_1 \leq \beta \leq \gamma_{l'}$ generate a $\mathbb{C}[q, q^{-1}]$–subalgebra for which the monomials

$$(X^-_{\beta_{k_{l'}}})^{(q_{k_{l'}})} \ldots (X^-_{\beta_1})^{(q_1)}$$

form a linear basis. Rewriting the monomials

$$(X^-_{\beta_1})^{(q_1)} \ldots (X^-_{\beta_{k_n}})^{(q_{k_n})} (X^-_{\delta_1})^{(p_1)} \ldots (X^-_{\delta_k})^{(p_k)}$$

in terms of this basis we arrive at

$$(u, (X^+_{\gamma_1})^{(k_1)} \ldots (X^+_{\gamma_n})^{(k_n)} (X^+_{\gamma_{n+1}})^{(k_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(k_{l'})} v) = \tag{4.1.33}$$

$$= (u, T_{\gamma_1} \ldots T_{\gamma_n} \sum_{\substack{q_1,\ldots,q_{k_{l'}} \\ p_1',\ldots,p_m'}} N^{q_1,\ldots,q_{k_{l'}}}_{p_1',\ldots,p_m'} (X^-_{\beta_{k_{l'}}})^{(q_{k_{l'}})} \ldots (X^-_{\beta_1})^{(q_1)} T_w (\widetilde{\overline{X}}^+_{\beta_1'})^{(p_1')} \ldots$$

$$\ldots (\widetilde{\overline{X}}^+_{\beta_m'})^{(p_m')} \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v),$$

where $N^{q_1,\ldots,q_{k_{l'}}}_{p_1',\ldots,p_m'} \in \mathbb{C}[q, q^{-1}]$.

Now observe that if $\gamma_n < \beta \leq \gamma_{l'}$ then $\beta \notin \Delta^1_+$. The root β can be written in the form $\beta = n\alpha + c$, where $\alpha \notin \Delta^1_+$ is a simple root, $n > 0$ and c is a linear combination of other simple roots. Since W_1 is parabolic in W, $s^1 \in W_1$ and $\mathfrak{m}_1 \subset \mathfrak{g}$ is a standard Levi subalgebra we deduce $s^1\beta = n\alpha + c'$, where c' is a linear combination of other simple roots. This implies $s^1\beta \in \Delta_+$. Thus $\omega(T_{\gamma_1} \ldots T_{\gamma_n}(X^-_\beta))u = 0$ for otherwise the vector $\omega(T_{\gamma_1} \ldots T_{\gamma_n}(X^-_\beta))u$ has weight greater than that of u which is impossible as u is a highest weight vector. Applying this observation in (4.1.33) we deduce

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(m_{l'})} v) = \tag{4.1.34}$$

$$= (u, T_{\gamma_1} \dots T_{\gamma_n} \sum_{\substack{q_1, \dots, q_{k_{l'}} \\ p'_1, \dots, p'_m}} K^{q_1,\dots,q_{k_n}}_{p'_1,\dots,p'_m} (X^-_{\beta_{k_n}})^{(q_{k_n})} \dots (X^-_{\beta_1})^{(q_1)} T_w (\widetilde{\overline{X}}^+_{\beta'_1})^{(p'_1)} \dots (\widetilde{\overline{X}}^+_{\beta'_m})^{(p'_m)} \widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v),$$

where $K^{q_1,\dots,q_{k_n}}_{p'_1,\dots,p'_m} \in \mathbb{C}[q, q^{-1}]$.

Denote $\widetilde{T}^2 = \widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}}$. For $\gamma_{n+1} \le \beta_q \le \gamma_{l'}$ we have $\beta_q = w s_{k_{n+1}} \dots s_{i_{q-1}} \alpha_{i_q}$, $q = k_{n+1}, \dots, k_{l'}$. Similarly to (4.1.24) we infer

$$(\widetilde{T}^2)^{-1} \widetilde{\overline{X}}^+_{\beta_q} = T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}} T_{i_{k_{l'}}} \dots T_{i_{q+1}} X^+_{i_q}. \tag{4.1.35}$$

Denote $\widetilde{X}^+_{\delta_q} = T_{i_{k_{l'}}} \dots T_{i_{q+1}} X^+_{i_q}$, $q = k_{n+1}, \dots k_{l'}$. Then from (4.1.34) and (4.1.35) we obtain

$$(u, (X^+_{\gamma_1})^{(m_1)} \dots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \dots (X^+_{\gamma_{l'}})^{(m_{l'})} v) = \tag{4.1.36}$$

$$= (u, T_{\gamma_1} \dots T_{\gamma_n} \sum_{\substack{q_1, \dots, q_{k_{l'}} \\ p'_{k_{n+1}}, \dots, p'_{k_{l'}}}} K^{q_1,\dots,q_{k_n}}_{p'_{k_{n+1}},\dots,p'_{k_{l'}}} (X^-_{\beta_{k_n}})^{(q_{k_n})} \dots (X^-_{\beta_1})^{(q_1)} T_w \widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} (T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}}) \times$$

$$\times ((\widetilde{X}^+_{\delta_{k_{n+1}}})^{(p'_{k_{n+1}})} \dots (\widetilde{X}^+_{\delta_{k_{l'}}})^{(p'_{k_{l'}})}) T_w^{-1} v) =$$

$$= (u, T_{\gamma_1} \dots T_{\gamma_n} \sum_{\substack{q_1, \dots, q_{k_{l'}} \\ p'_{k_{n+1}}, \dots, p'_{k_{l'}}}} K^{q_1,\dots,q_{k_n}}_{p'_{k_{n+1}},\dots,p'_{k_{l'}}} (X^-_{\beta_{k_n}})^{(q_{k_n})} \dots (X^-_{\beta_1})^{(q_1)} T_w \widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}} \times$$

$$\times (\widetilde{X}^+_{\delta_{k_{n+1}}})^{(p'_{k_{n+1}})} \dots (\widetilde{X}^+_{\delta_{k_{l'}}})^{(p'_{k_{l'}})} T_{w'}^{-1} v),$$

where $K^{q_1,\dots,q_{k_n}}_{p_{k'_{n+1}},\dots,p'_{k_{l'}}} \in \mathbb{C}[q, q^{-1}]$ and

$$T_{w'} = T_w T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}} = T_{i_1} \dots T_{i_{k_{l'}}}.$$

Now if

$$\overline{w} = s_{i_1} \dots s_{i_{k_{n+1}}} \dots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \dots s_{i_{p'}} \dots s_{i_D}$$

is the reduced decomposition of the longest element of the Weyl group corresponding to the normal ordering in Δ_+ associated to s, so that $\widetilde{s}^2 = s_{i_{k_{n+1}}} \dots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \dots s_{i_{p'}}$ is the corresponding reduced decomposition of $\widetilde{s}^2$, then, similarly to (4.1.22), we have

$$\widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} = T_{i_{p'}} \dots T_{i_{k_{l'}+1}} T^{-1}_{i_{k_{l'}}} T^{-1}_{i_{k_{l'}-1}} \dots T^{-1}_{i_{k_{n+1}}},$$

and hence

$$\widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}} = T_{i_{p'}} \dots T_{i_{k_{l'}+1}}. \tag{4.1.37}$$

Observe that

$$T = T_w \widetilde{T}_{\gamma_{n+1}} \dots \widetilde{T}_{\gamma_{l'}} T_{i_{k_{n+1}}} \dots T_{i_{k_{l'}}} = T_{i_1} \dots T_{i_{k_{n+1}-1}} T_{i_{p'}} \dots T_{i_{k_{l'}+1}}$$

is the braid group element corresponding to the reduced decomposition

$$s_{i_1} \dots s_{i_{k_{n+1}-1}} s_{i_{p'}} \dots s_{i_{k_{l'}+1}} \tag{4.1.38}$$

which is a part of the reduced decomposition obtained from the reduced decomposition

$$s_{i_1} \dots s_{i_{k_{n+1}-1}} s_{i_{k_{n+1}}} \dots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \dots s_{i_{p'}}$$

by inverting the part $\widetilde{s}^2 = s_{i_{k_{n+1}}} \dots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \dots s_{i_{p'}}$. This inversion gives a reduced decomposition again because $\widetilde{s}^2 = -1$ is the longest element in $\widetilde{W}_2$.

Now for $\beta_1 \le \beta_q \le \beta_{k_n}$ we have

$$T^{-1}(X^-_{\beta_q}) = T^{-1}_{i_{k_{l'}+1}} \dots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \dots T^{-1}_{i_{q+1}} T^{-1}_{i_q} \dots T^{-1}_{i_1} T_{i_1} \dots T_{i_{q-1}} X^-_{i_q} = \tag{4.1.39}$$

$$= T^{-1}_{i_{k_{l'}+1}} \ldots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \ldots T^{-1}_{i_{q+1}} T^{-1}_{i_q} X^-_{i_q} =$$

$$= -T^{-1}_{i_{k_{l'}+1}} \ldots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \ldots T^{-1}_{i_{q+1}} (X^+_{i_q} K_{i_q}) = T^{-1}_{i_{k_{l'}+1}} \ldots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \ldots T^{-1}_{i_{q+1}} (X^+_{i_q}) R_q,$$

where

$$R_q = T^{-1}_{i_{k_{l'}+1}} \ldots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \ldots T^{-1}_{i_{q+1}} (K_{i_q}).$$

Since $s_{i_1} \ldots s_{i_{k_{n+1}-1}} s_{i_{p'}} \ldots s_{i_{k_{l'}+1}}$ is a reduced decomposition

$$T^{-1}_{i_{k_{l'}+1}} \ldots T^{-1}_{i_{p'}} T^{-1}_{i_{k_{n+1}-1}} \ldots T^{-1}_{i_{q+1}} (X^+_{i_q})^{(k)} \in U_q^{res}(\mathfrak{n}_+),$$

where $U_q^{res}(\mathfrak{n}_+)$ is the subalgebra of $U_q^{res}(\mathfrak{g})$ generated by $(X_i^+)^{(k)}$ $i = 1, \ldots, l,$ $k = 0, 1, \ldots$. Applying this observation, (4.1.39), commutation relations between elements R_q and the quantum root vectors, and the fact that u is a weight vector we obtain from (4.1.36)

$$\begin{aligned}
&(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(m_{l'})} v) = \\
&= (u, T_{\gamma_1} \ldots T_{\gamma_n} T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}} X T^{-1}_{w'} v) = \\
&= (u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_w T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}} X T^{-1}_{w'} v) = \\
&= (u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_{w'} X T^{-1}_{w'} v), X \in U_q^{res}(\mathfrak{n}_+),
\end{aligned} \tag{4.1.40}$$

where we also used the fact that all monomials $(\widetilde{X}^+_{\delta_{k_{n+1}}})^{(p'_{k_{n+1}})} \ldots (\widetilde{X}^+_{\delta_{k_{l'}}})^{(p'_{k_{l'}})}$ in (4.1.34) belong to $U_q^{res}(\mathfrak{n}_+)$ since $s_{i_{k_{n+1}}} \ldots s_{i_{k_{l'}}}$ is a reduced decomposition.

Now consider the reduced decomposition $\overline{w} = s_{i_{k_{l'}}} \ldots s_{i_1} s_{p_{k_{l'}+1}} \ldots s_{p_D}$, the corresponding normal ordering $\beta'_{k_{l'}}, \ldots, \beta'_1 \beta'_{p_{k_{l'}+1}}, \ldots, \beta'_{p_D}$ of Δ_+, the quantum root vectors

$$\overline{X}^+_{\beta'_q} = \begin{cases} T^{-1}_{i_{k_{l'}}} \ldots T^{-1}_{i_{q-1}} X^+_{i_q} & 1 \le q \le k_{l'} \\ T^{-1}_{w'} T^{-1}_{p_{k_{l'}+1}} \ldots T^{-1}_{p_{q-1}} X^+_{p_q} & k_{l'} + 1 \le q \le D \end{cases},$$

and the linear basis of $U_q^{res}(\mathfrak{n}_+)$,

$$(\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})} \ldots (\overline{X}^+_{\beta'_1})^{(n_1)} (\overline{X}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (\overline{X}^+_{\beta'_D})^{(n_D)}. \tag{4.1.41}$$

Consider also another reduced decomposition $\overline{w} = s_{i_{k_{l'}+1}} \ldots s_{i_D} s_{p_1} \ldots s_{p_{k_{l'}}}$, the corresponding normal ordering $\beta'_{k_{l'}}, \ldots, \beta'_1 \beta'_{p_{k_{l'}+1}}, \ldots, \beta'_{p_D}$ of Δ_+, the quantum root vectors

$$X^+_{\beta''_q} = \begin{cases} T_{w''} T_{i_{p_1}} \ldots T_{p_{q-1}} X^+_{p_q} & 1 \le q \le k_{l'} \\ T_{i_{k_{l'}+1}} \ldots T_{i_{q-1}} X^+_{i_q} & k_{l'} + 1 \le q \le D \end{cases},$$

where $T_{w''} = T_{i_{k_{l'}+1}} \ldots T_{i_D}$, and the linear basis of $U_q^{res}(\mathfrak{n}_+)$,

$$(X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta''_D})^{(n_D)} (X^+_{\beta''_1})^{(n_1)} \ldots (X^+_{\beta''_{k_{l'}}})^{(n_{k_{l'}})}. \tag{4.1.42}$$

We can represent the monomials $(\overline{X}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (\overline{X}^+_{\beta'_D})^{(n_D)}$ using basis (4.1.42),

$$(\overline{X}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (\overline{X}^+_{\beta'_D})^{(n_D)} = \tag{4.1.43}$$

$$= \sum_{q_1, \ldots q_D} C_{q_1, \ldots q_D} (X^+_{\beta''_{k_{l'}+1}})^{(q_{k_{l'}+1})} \ldots (X^+_{\beta''_D})^{(q_D)} (X^+_{\beta''_1})^{(q_1)} \ldots (X^+_{\beta''_{k_{l'}}})^{(q_{k_{l'}})},$$

where $C_{q_1, \ldots q_D} \in \mathbb{C}[q, q^{-1}]$. Applying $T_{w'}$ to this identity we get

$$(\widetilde{\overline{X}}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (\widetilde{\overline{X}}^+_{\beta'_D})^{(n_D)} = \tag{4.1.44}$$

$$= \sum_{q_1,\ldots q_D} C_{q_1,\ldots q_D} (X^+_{\beta_{k_{l'}+1}})^{(q_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(q_D)} (\widetilde{X}^+_{\beta''_1})^{(q_1)} \ldots (\widetilde{X}^+_{\beta''_{k_{l'}}})^{(q_{k_{l'}})},$$

where for $k_{l'}+1 \leq q \leq D$

$$\widetilde{\overline{X}}^+_{\beta'_q} = T_{w'} \overline{X}^+_{\beta'_q} = T^{-1}_{p_{k_{l'}+1}} \ldots T^{-1}_{p_{q-1}} X^+_{p_q} \in U_q^{res}(\mathfrak{n}_+), \tag{4.1.45}$$

and for $1 \leq q \leq k_{l'}$

$$\widetilde{X}^+_{\beta''_q} = T_{w'} X^+_{\beta''_q} = T_{w'} T_{w''} T_{i_{p_1}} \ldots T_{p_{q-1}} X^+_{p_q} = T_{\overline{w}} T_{i_{p_1}} \ldots T_{p_{q-1}} X^+_{p_q} \in U_q^{res}(\mathfrak{n}_-) U_q(H), \tag{4.1.46}$$

$U_q^{res}(\mathfrak{n}_-)U_q(H)$ is the algebra generated by $U_q^{res}(\mathfrak{n}_-)$ and by $U_q(H)$, and we used the fact that $T_{i_{p_1}} \ldots T_{p_{q-1}} X^+_{p_q} \in U_q^{res}(\mathfrak{n}_+)$ and $T_{\overline{w}} U_q^{res}(\mathfrak{n}_+) \subset U_q^{res}(\mathfrak{n}_-)U_q(H)$. Observing that the elements $\widetilde{X}^+_{\beta''_q} \in U_q^{res}(\mathfrak{n}_-)U_q(H)$, $1 \leq q \leq k_{l'}$ have strictly negative weights we deduce from (4.1.44), (4.1.45), (4.1.46) and (2.4.1) in Lemma 2.4.2 that (4.1.43) takes the form

$$(\overline{X}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (\overline{X}^+_{\beta'_D})^{(n_D)} = \sum_{q_{k_{l'}+1},\ldots q_D} C_{q_{k_{l'}+1},\ldots q_D} (X^+_{\beta''_{k_{l'}+1}})^{(q_{k_{l'}+1})} \ldots (X^+_{\beta''_D})^{(q_D)}, \tag{4.1.47}$$

where $C_{q_{k_{l'}+1},\ldots q_D} \in \mathbb{C}[q,q^{-1}]$. Recalling basis (4.1.41) we infer that every element of $U_q^{res}(\mathfrak{n}_+)$ is a $\mathbb{C}[q,q^{-1}]$–linear combination of monomials of the form

$$(\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})} \ldots (\overline{X}^+_{\beta'_1})^{(n_1)} (X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta''_D})^{(n_D)}. \tag{4.1.48}$$

Kostant's formula shows that they form a linear basis of $U_q^{res}(\mathfrak{n}_+)$.

Now for $1 \leq q \leq k_{l'}$ we have

$$T_{w'} \overline{X}^+_{\beta'_q} = T_{w'} T^{-1}_{i_{k_{l'}}} \ldots T^{-1}_{i_{q-1}} X^+_{i_q} = T_{i_1} \ldots T_{i_q} X^+_{i_q} = -T_{i_1} \ldots T_{i_{q-1}} X^-_{i_q} K_{i_q} = -X^-_{\beta_q} K_{\beta_q},$$

where

$$K_{\beta_q} = T_{i_1} \ldots T_{i_{q-1}} K_{i_q},$$

and for $k_{l'}+1 \leq q \leq D$

$$T_{w'} X^+_{\beta''_q} = T_{w'} T_{i_{k_{l'}+1}} \ldots T_{i_{q-1}} X^+_{i_q} = X^+_{\beta_q}.$$

The last two identities and commutation relations between elements K_{β_q} and quantum root vectors imply

$$T_{w'} \left((\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})} \ldots (\overline{X}^+_{\beta'_1})^{(n_1)} (X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta''_D})^{(n_D)} \right) =$$

$$= Q_{n_1,\ldots,n_{k_{l'}}} (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)},$$

where $Q_{n_1,\ldots,n_{k_{l'}}}$ is a monomial in $K_1^{\pm 1}, \ldots K_l^{\pm 1}$.

Recalling basis (4.1.48), and using the previous formula we deduce that in (4.1.40)

$$T_{w'} X T^{-1}_{w'} = \sum_{n_1,\ldots,n_D} F'_{n_1,\ldots,n_D} (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)},$$

where $F'_{n_1,\ldots,n_D} \in U_q(H)$, and (4.1.40) takes the form

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} (X^+_{\gamma_{n+1}})^{(m_{n+1})} \ldots (X^+_{\gamma_{l'}})^{(m_{l'})} v) = \tag{4.1.49}$$

$$= \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D} (u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} (X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})} \ldots (X^-_{\beta_1})^{(n_1)} (X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})} \ldots (X^+_{\beta_D})^{(n_D)} v),$$

where $F_{n_1,\ldots,n_D} \in \mathbb{C}[q,q^{-1}]$. This proves the first formula in the statement of this lemma.

To justify the last two formulas in the statement of the lemma we observe that $T_{\gamma_{l'}} \ldots T_{\gamma_{n+1}} T_{\gamma_n} \ldots T_{\gamma_1}$, T_s^{-1} and $\overline{T}_s^{-1}$ act as the same transformations of $\mathfrak{h} \subset U_h(\mathfrak{h})$ and apply Lemma 2.7.4. This completes the proof.

□

In the course of the proof of the previous lemma we obtained the following result.

Corollary 4.1.8. *The products*

$$(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)}$$

or

$$(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}$$

form a $U_q^{res}(H)$–basis in the subalgebra $U_q^{res}(w'(\mathfrak{b}_+))$ in $U_q^{res}(\mathfrak{g})$ generated over $U_q^{res}(H)$ by the elements

$$(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})},\ldots,(X^-_{\beta_1})^{(n_1)},(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})},\ldots,(X^+_{\beta_D})^{(n_D)}, n_i\in\mathbb{N}, i=1,\ldots,D.$$

4.2 Some auxiliary results on the quantized algebra of regular functions on an algebraic Poisson–Lie group

In this short section we give several formulas related to the adjoint action and commutation relations in the algebra $\mathbb{C}_{\mathcal{B}}[G]$.

Firstly, following [10], Theorem I.8.16 we recall the commutation relations in the algebra $\mathbb{C}_{\mathcal{B}}[G]$ which follow from the fact that $U_h^s(\mathfrak{g})$ is quasitriangular. Namely, if V, V' are finite rank representations of $U_h(\mathfrak{g})$, $(V)_\eta,(V')_\rho$, $(V)_\beta,(V')_\gamma$ their weight subspaces of weights η,ρ,β and γ, respectively, and $v\in(V)_\eta, v_1\in(V')_\rho$, $u\in(V)_\beta, u_1\in(V')_\gamma$ then evaluating the identity $\Delta_s^{opp}(x)\mathcal{R}^s=\mathcal{R}^s\Delta_s(x)$ on the matrix element $(u,\cdot v)\otimes(u_1,\cdot v_1)$ and recalling formula (2.6.11) we obtain

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\eta^\vee,\rho^\vee)}\left((u_1,\cdot v_1)\otimes(u,\cdot v)+\sum_{\nu,i}(u_1,\cdot u_{\nu,i}v_1)\otimes(u,\cdot u_{-\nu,i}v)\right)= \tag{4.2.1}$$

$$=q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\beta^\vee,\gamma^\vee)}(u,\cdot v)\otimes(u_1,\cdot v_1)+\sum_{\nu,i}q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)(\beta^\vee+\nu^\vee),\gamma^\vee-\nu^\vee)}(\omega(u_{-\nu,i})u,\cdot v))\otimes(\omega(u_{\nu,i})u_1,\cdot v_1),$$

where

$$u_{-\nu,i}=c_{-\nu,i}f_{\beta_1}^{(n_1)}\ldots f_{\beta_D}^{(n_D)}, n_1\beta_1+\ldots+n_D\beta_D=\nu,$$
$$u_{\nu,i}=c_{\nu,i}e_{\beta_1}^{n_1}\ldots e_{\beta_D}^{n_D}, n_1\beta_1+\ldots+n_D\beta_D=\nu,$$

$c_{\pm\nu,i}\in\mathcal{B}$, and similarly evaluating the identity $\Delta_s^{opp}(x){\mathcal{R}_{21}^s}^{-1}={\mathcal{R}_{21}^s}^{-1}\Delta_s(x)$ on the matrix element $(u,\cdot v)\otimes(u_1,\cdot v_1)$ we get

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\eta^\vee,\rho^\vee)}\left((u_1,\cdot v_1)\otimes(u,\cdot v)+\sum_{\nu,i}(u_1,\cdot u'_{-\nu,i}v_1)\otimes(u,\cdot u'_{\nu,i}v)\right)=$$

$$=q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta^\vee,\gamma^\vee)}(u,\cdot v)\otimes(u_1,\cdot v_1)+\sum_{\nu,i}q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)(\beta^\vee+\nu^\vee),\gamma^\vee-\nu^\vee)}(\omega(u'_{\nu,i})u,\cdot v))\otimes(\omega(u'_{-\nu,i})u_1,\cdot v_1),$$

where

$$u'_{-\nu,i}=c'_{-\nu,i}f_{\beta_D}^{(n_D)}\ldots f_{\beta_1}^{(n_1)}, n_1\beta_1+\ldots+n_D\beta_D=\nu,$$
$$u'_{\nu,i}=c'_{\nu,i}e_{\beta_D}^{n_D}\ldots e_{\beta_1}^{n_1}, n_1\beta_1+\ldots+n_D\beta_D=\nu,$$

$c'_{\pm\nu,i}\in\mathcal{B}$.

If $v=T_{s^{-1}}v_\lambda\in(V)_{s^{-1}\lambda}, v_1=T_{s^{-1}}v_\mu\in(V')_{s^{-1}\mu}$, $u\in(V)_\beta, u_1=v_\mu\in(V')_\mu$, where $v_\lambda\in V$ and $v_\mu\in V'$ are highest weight vectors, then the previous identity yields

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\lambda^\vee,\mu^\vee)}(v_\mu,\cdot T_{s^{-1}}v_\mu)\otimes(u,\cdot T_{s^{-1}}v_\lambda)= \tag{4.2.2}$$

$$=q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta^\vee,\mu^\vee)}(u,\cdot T_{s^{-1}}v_\lambda)\otimes(v_\mu,\cdot T_{s^{-1}}v_\mu).$$

The next lemma shows how the adjoint action behaves with respect to the multiplication in $\mathbb{C}_{\mathcal{B}}[G]$.

Lemma 4.2.1. *For any* $f, g \in \mathbb{C}_{\mathcal{B}}[G], x \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ *we have*

$$\mathrm{Ad}_0 x(f \otimes g) = \mathrm{Ad}_0 x^2 f \otimes g((\omega_0 S_s^{-1})(x^1) \cdot \omega_0 x^3),$$

where $\Delta_s^2 x = (\Delta_s \otimes id)\Delta_s x = x^1 \otimes x^2 \otimes x^3$ *in the Sweedler notation.*

In particular,

$$\mathrm{Ad}_0 f_\beta(f \otimes g) = f \otimes g((\omega_0 S_s^{-1})(f_\beta)\cdot) + \mathrm{Ad}_0 f_\beta f \otimes g(G_\beta \cdot) + \sum_i \mathrm{Ad}_0 x_i f \otimes g((\omega_0 S_s^{-1})(y_i)\cdot) + \tag{4.2.3}$$

$$+f(G_\beta \cdot G_\beta^{-1}) \otimes g(G_\beta \cdot \omega_0(f_\beta)) + \sum_i \mathrm{Ad}_0 y_i^2 f \otimes g((\omega_0 S_s^{-1})(y_i^1) \cdot \omega_0 x_i) =$$

$$= f \otimes g((\omega_0 S_s^{-1})(f_\beta)\cdot) + \mathrm{Ad}_0 f_\beta f \otimes g(G_\beta \cdot) + f(G_\beta \cdot G_\beta^{-1}) \otimes g(G_\beta \cdot \omega_0(f_\beta)) +$$

$$+ \sum_i \mathrm{Ad}_0 y_i f \otimes g(G_\beta \cdot \omega_0 x_i) + \sum_i \mathrm{Ad}_0 x_i^1 f \otimes g((\omega_0 S_s^{-1})(y_i) \cdot \omega_0 x_i^2)$$

and

$$\mathrm{Ad}_0 f_\beta^{(n)}(f \otimes g) = \sum_{k=0}^{n} \sum_{p=0}^{n-k} q_\beta^{-k(n-k)-p(n-k-p)} \mathrm{Ad}_0(G_\beta^{-k} f_\beta^{(p)})(f) \otimes g(\omega_0 S_s^{-1}(G_\beta^{-k-p} f_\beta^{(n-k-p)}) \cdot \omega_0(f_\beta^{(k)})) +$$

$$+ \sum_{k=0}^{n-1} \sum_i q_\beta^{-k(n-k)} \mathrm{Ad}_0(G_\beta^{-k} x_i^{(n-k)})(f) \otimes g((\omega_0 S_s^{-1})(G_\beta^{-k} y_i^{(n-k)}) \cdot \omega_0(f_\beta^{(k)})) + \tag{4.2.4}$$

$$+ \sum_i \mathrm{Ad}_0({y_i^{(n)}}^2)(f) \otimes g((\omega_0 S_s^{-1})({y_i^{(n)}}^1) \cdot \omega_0(x_i^{(n)})),$$

where G_β, x_i, y_i, $x_i^{(p)}$, $y_i^{(p)}$ *are defined in (2.7.12) and (2.7.13), and* $\Delta_s(x_i) = x_i^1 \otimes x_i^2$, $\Delta_s(y_i) = y_i^1 \otimes y_i^2$, $\Delta_s(y_i^{(p)}) = {y_i^{(p)}}^1 \otimes {y_i^{(p)}}^2$ *in the Sweedler notation.*

Proof. Denote using the Sweedler notation

$$\Delta_s^3 x = (\Delta_s \otimes id \otimes id)(\Delta_s \otimes id)\Delta_s x = x^1 \otimes x^2 \otimes x^3 \otimes x^4$$

and observe that the definition of Ad' implies that for any $x, z \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$

$$\Delta_s^{opp}(\mathrm{Ad}' xz) = (x^2 \otimes x^1)(z^2 \otimes z^1)(S_s x^3 \otimes S_s x^4) = \mathrm{Ad}' x^2 z^2 \otimes x^1 z^1 S_s x^3.$$

Let $z = S_s \omega_0 y$, $y \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Then, since $\omega_0 S_s^{-1}$ is an algebra homomorphism and a coalgebra anti-homomorphism, we deduce

$$\Delta_s \omega_0 S_s^{-1}(\mathrm{Ad}' x S_s \omega_0 y) = (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\Delta_s^{opp}(\mathrm{Ad}' xz) = (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\mathrm{Ad}' x^2 z^2 \otimes x^1 z^1 S_s x^3 =$$

$$= (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\mathrm{Ad}' x^2 (S_s \omega_0)(y^1) \otimes x^1 (S_s \omega_0)(y^2) S_s x^3 = \omega_0 S_s^{-1} \mathrm{Ad}' x^2((S_s \omega_0)(y^1)) \otimes (\omega_0 S_s^{-1})(x^1) y^2 \omega_0 x^3.$$

Evaluating the last identity on $f \otimes g$ we get the first formula in the statement of the lemma. (4.2.3) and (4.2.4) are obtained from it using (2.7.12) and (2.7.13).

□

4.3 Properties of the quantized vanishing ideal of the level surface of the moment map for q-W–algebras

In this section we study the properties of the quantum counterpart $\overline{J}_{\mathcal{B}}^1 \subset \mathbb{C}_{\mathcal{B}}[G]$ of the vanishing ideal of the variety $\overline{N_- Z H s M_-^s}$, and the properties of a $\mathcal{B}$–submodule $J_{\mathcal{B}}^1 \subset \overline{J}_{\mathcal{B}}^1$ which is a quantum counterpart of the vanishing ideal of $\overline{N_- s Z H N_-'} \supset \overline{N_- Z H s M_-^s}$.

We start with a technical lemma which will allow us to use properties of the vanishing ideals of the varieties $\overline{N_- Z H s M_-^s}$ and $\overline{N_- s Z H N_-'}$ to prove some properties of their quantum counterparts $\overline{J}_{\mathcal{B}}^1$ and $J_{\mathcal{B}}^1$.

Lemma 4.3.1. *Let X be a free $\mathcal{B}$–module, $V \subset V' \subset X$ two its submodules such that $V = V'$ mod $(q^{\frac{1}{dr^2}} - 1)V'$. Let $V_q = V \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $V'_q = V' \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. Then $V_q = V'_q$.*

Proof. First observe that by Theorem 6.5 in [87] $V' \subset X$ is $\mathcal{B}$–free, and $V \subset V'$ is a free submodule of V'. Let e_a, $a \in A$ be a basis of V'. Then as shown in the proof of Theorem 6.5 in [87] V has a basis elements of which has the form $f_a = \sum_{b \leq a} c_a^b e_b$, $a \in B \subset A$, where $c_a^a \neq 0$, $c_a^b \in \mathcal{B}$. Since $V = V'$ mod $(q^{\frac{1}{dr^2}} - 1)V'$ we must have $B = A$, and $c_a^a \neq 0$ mod $(q^{\frac{1}{dr^2}} - 1)$, i.e. $c_a^a = 1 + (q^{\frac{1}{dr^2}} - 1)h_a$, $h_a \in \mathcal{B}$.

Now using a simple transfinite induction we can express the elements of the basis e_a regarded as elements of V'_q in terms of f_a. Indeed, if $a_0 \in A$ is the minimal element then $e_{a_0} = c_{a_0}^{a_0 \, -1} f_{a_0}$, and assuming that $e_d = \sum_{b \leq d} g_d^b f_b$, $g_d^b \in \mathbb{C}(q^{\frac{1}{dr^2}})$, $g_d^d \neq 0$ holds for all $d < a$ we get $e_a = c_a^{a\, -1}(f_a - \sum_{b<a} c_a^b e_b) = c_a^{a\, -1}(f_a - \sum_{b<a} c_a^b \sum_{h \leq b} g_b^h f_h)$, where all sums are finite. This establishes the base of the induction and completes the proof.

□

The following lemma describes $\overline{I}^1_{\mathcal{B}}$ as a vanishing ideal, in complete analogy with the classical situation.

Lemma 4.3.2. *Let $\overline{J}^1_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, T_s x v) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$.*

Let $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ be the subalgebra in $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ generated by the elements $f_\beta^{(n)}$, $\beta \in \Delta_{\mathfrak{m}_+}$, $n \in \mathbb{N}$. Then $\overline{J}^1_{\mathcal{B}}$ and $\overline{I}^1_{\mathcal{B}} \supset \overline{J}^1_{\mathcal{B}}$ are stable under the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$. Thus $\mathbb{C}_{\mathcal{B}}[G]/\overline{J}^1_{\mathcal{B}}$ and $\mathbb{C}_{\mathcal{B}}[G]/\overline{I}^1_{\mathcal{B}}$ are naturally equipped with the $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$–action induced by the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ on $\mathbb{C}_{\mathcal{B}}[G]$.

Let $\Delta^s_{\mathfrak{m}_+} = \Delta_{\mathfrak{m}_+} \cap \Delta^s_s$ and $\Delta_0^+ = \Delta_0 \cap \Delta_+$. Both $\Delta^s_{\mathfrak{m}_+}, \Delta_0^+ \subset \Delta_+$ are minimal segments. Denote by $U_q^{res}(-\Delta^s_{\mathfrak{m}_+})$ and $U_q^{res}(\Delta_0^+)$ the subalgebras of $U_q^{res}(\mathfrak{g})$ corresponding to $-\Delta^s_{\mathfrak{m}_+}$ and Δ_0^+, respectively. Let $\overline{J}^{1\,\prime}_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements vanishing on $xT_s h z_+ x'$ with arbitrary $x \in U_q^{res}(\mathfrak{n}_-), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$. Denote $\overline{J}^1_q = \overline{J}^1_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\overline{J}^{1\,\prime}_q = \overline{J}^{1\,\prime}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. Then $\overline{J}^1_{\mathcal{B}} \subset \overline{J}^{1\,\prime}_{\mathcal{B}}$ and $\overline{J}^1_q = \overline{J}^{1\,\prime}_q$. Thus $\overline{I}^1_{\mathcal{B}} = \overline{J}^1_q \cap \mathbb{C}_{\mathcal{B}}[G] = \overline{J}^{1\,\prime}_q \cap \mathbb{C}_{\mathcal{B}}[G]$.

Moreover, $\overline{J}^1_{\mathcal{B}} = \overline{J}^{1\,\prime}_{\mathcal{B}} = \overline{J}$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^1_{\mathcal{B}}$, where $\overline{J}$ is the vanishing ideal of $\overline{N_- sZHM_-^s}$.

Proof. From formula (4.2.4) with $g(\cdot) = (u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, T_s x v) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$, and arbitrary f we have, since u is a highest weight vector,

$$\begin{aligned}\mathrm{Ad}_0 f_\beta^{(n)}(f \otimes g) = &\sum_{k=0}^{n}\sum_{p=0}^{n-k} q_\beta^{-k(n-k)-p(n-k-p)} \mathrm{Ad}_0(G_\beta^{-k} f_\beta^{(p)})(f) \otimes g(\omega_0 S_s^{-1}(G_\beta^{-k-p} f_\beta^{(n-k-p)}) \cdot \omega_0(f_\beta^{(k)})) + \\ &+ \sum_{k=0}^{n-1}\sum_i q_\beta^{-k(n-k)} \mathrm{Ad}_0(G_\beta^{-k} x_i^{(n-k)})(f) \otimes g((\omega_0 S_s^{-1})(G_\beta^{-k} y_i^{(n-k)}) \cdot \omega_0(f_\beta^{(k)})) + \\ &+ \sum_i \mathrm{Ad}_0({y_i^{(n)}}^2)(f) \otimes g((\omega_0 S_s^{-1})({y_i^{(n)}}^1) \cdot \omega_0(x_i^{(n)})) = \\ &= \sum_{k=0}^{n} q_\beta^{-k(n-k)} \mathrm{Ad}_0(G_\beta^{-k} f_\beta^{(n-k)})(f) \otimes g(G_\beta^n \cdot \omega_0(f_\beta^{(k)})).\end{aligned} \tag{4.3.1}$$

Since u is a highest weight vector and $U_q^{res}(w'(\mathfrak{b}_+))$ is invariant under multiplication by $\omega_0((X_\beta^-)^{(k)})$ from the right, $f_\beta = q^{-K\beta^\vee} X_\beta^-$, and $q^{-K\beta^\vee}$ normalizes $w'(\mathfrak{b}_+)$, elements $g(G_\beta^n \cdot \omega_0(f_\beta^{(k)}))$ have the same properties as g. Now from the definition of $\overline{J}^1_{\mathcal{B}}$ it follows that $\mathrm{Ad}_0 f_\beta^{(n)}(f \otimes g) \in \overline{J}^1_{\mathcal{B}}$. Since elements $f \otimes g$, with $f, g \in \mathbb{C}_{\mathcal{B}}[G]$ as above, span $\overline{J}^1_{\mathcal{B}}$ we deduce $\mathrm{Ad}_0 f_\beta^{(n)}(\overline{J}^1_{\mathcal{B}}) \subset \overline{J}^1_{\mathcal{B}}$.

If we naturally extend the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ to $\mathbb{C}_q[G]$ the last inclusion implies $\mathrm{Ad}_0 f_\beta^{(n)}(\overline{I}^1_q) \subset \overline{I}^1_q$, and hence $\mathrm{Ad}_0 f_\beta^{(n)}(\overline{I}^1_{\mathcal{B}}) \subset \overline{I}^1_{\mathcal{B}}$ as clearly $\mathrm{Ad}_0 f_\beta^{(n)}(\mathbb{C}_{\mathcal{B}}[G]) \subset \mathbb{C}_{\mathcal{B}}[G]$ and $\mathrm{Ad}_0 f_\beta^{(n)}(\overline{I}^1_{\mathcal{B}}) = \mathrm{Ad}_0 f_\beta^{(n)}(\overline{I}^1_q \cap \mathbb{C}_{\mathcal{B}}[G]) \subset \overline{I}^1_q \cap \mathbb{C}_{\mathcal{B}}[G] = \overline{I}^1_{\mathcal{B}}$.

Now we justify the second claim of the lemma. Let $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$ be such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$. First note that if $\alpha \in \Delta_{\mathfrak{m}_+} \setminus \Delta^s_{\mathfrak{m}_+}$ then $s\alpha \in \Delta_+$ and if $\alpha \in \Delta_0^+$

then $s\alpha = \alpha$. Therefore applying the Poincaré–Birkhoff–Witt theorem for $U_q^{res}(w'(\mathfrak{b}_+))$ as stated in Corollary 4.1.8 and Lemma 2.4.2 we obtain that for $x \in U_q^{res}(w'(\mathfrak{b}_+))$

$$(u, T_s xv) = (u, T_s \sum_i z_i^+ x_i' v), z_i^+ \in U_q^{res}(\Delta_0^+), x_i' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s).$$

Since for all i $z_i^+ x_i' \in U_q^{res}(w'(\mathfrak{b}_+))$ we deduce that $(u, T_s xv) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$ if and only if $(u, T_s z_+ x' v) = 0$ for any $z_+ \in U_q^{res}(\Delta_0^+)$, $x' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)$, and hence $\overline{J}_{\mathcal{B}}^1$ coincides with the left ideal generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$ such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ satisfies $(u, T_s z_+ x' v) = 0$ for any $z_+ \in U_q^{res}(\Delta_0^+)$, $x' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)$.

Obviously $\overline{J}_{\mathcal{B}}^1 \subset \overline{J}_{\mathcal{B}}^{1\,\prime}$.

On the other hand $\overline{J}_{\mathcal{B}}^1 = \overline{J} \bmod (q^{\frac{1}{dr^2}} - 1)\overline{J}_{\mathcal{B}}^1$, where $\overline{J}$ is the ideal in $\mathbb{C}[G]$ generated by matrix elements of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $U(\mathfrak{g})$, and $v \in V$ is such that $(u, s z_+ x' v) = 0$ for any $z_+ \in U(\Delta_0^+) = U_q^{res}(\Delta_0^+)/(q^{\frac{1}{dr^2}} - 1)U_q^{res}(\Delta_0^+)$, $x' \in U(-\Delta_{\mathfrak{m}_+}^s) = U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)/(q^{\frac{1}{dr^2}} - 1)U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)$. We claim that $\overline{J}$ coincides with the ideal of regular functions vanishing on the closure of $N_- s Z H M_-^s$.

The proof of this claim is based on the description of closed subvarieties in $B_- \setminus G$ in terms of the so–called generalized Plücker coordinates.

Recall that matrix elements of the form $(u, \cdot v)$, where u is a highest weight vector in a finite-dimensional representation V of $U(\mathfrak{g})$, and $v \in V$, can be regarded as sections of line bundles on $B_- \setminus G$ (see [32]). They are also called generalized Plücker coordinates on $B_- \setminus G$.

Lemma 4.3.3. *Any closed subvariety in $B_- \setminus G$ is the zero locus of a finite set of generalized Plücker coordinates.*

Proof. Indeed, the flag variety $B_- \setminus G$ can be realized as the G–orbit $\mathcal{O}$ of the line $[u]$ defined by a non–zero lowest weight vector u in the projectivisation $P(V^*)$ of a finite-dimensional irreducible representation V^* of G dual to a highest weight irreducible representation V with a regular dominant highest weight (see e.g. [8], Section 2 or [40], §4). If we identify V with V^* using the contravariant bilinear form then a highest weight vector u of V becomes a lowest weight vector in V^* and the class $[g] \in B_- \setminus G$ of an element $g \in G$ corresponds to $\omega(g)[u] = [\omega(g)u] \in \mathcal{O} \subset P(V^*)$.

Now let $v \in V$. Then

$$(u, gv) = (\omega(g)u, v) = (x, v), x = \omega(g)u \in V^*, \tag{4.3.2}$$

so that $[x] = [\omega(g)u] \in \mathcal{O} \subset P(V^*)$.

Any $y \in V^*$ can be written in the form $y = \sum_{n=1}^S c_n e_n \in V^*$, where e_n, $n = 1, \ldots, S$ is a weight basis of V^*. The functions $\phi_n(y) = c_n$ are linear coordinates on V^*. Moreover, since $V \simeq V^*$ one can find elements $v_n \in V$ such that $\phi_n(y) = (y, v_n)$. In particular, ϕ_n generate the algebra of polynomial functions on V^*, and any closed subvariety in $P(V^*)$, and hence in $\mathcal{O}$, is the zero locus of a finite collection of some polynomials homogeneous in ϕ_n (see [44], Ch. 1, §2).

If $f(\phi_1, \ldots \phi_S)$ is such a polynomial of degree d then the equation $f(\phi_1(y), \ldots \phi_S(y)) = 0$ is well–defined in $P(V^*)$, i.e. if $y \in V^*$ is its solution then any element of the line $[y] \in P(V^*)$ is also its solution. Now using (4.3.2), the definition of ϕ_n and the homogeneity of f we deduce that for $[x] \in \mathcal{O}$ $f(\phi_1(x), \ldots \phi_S(x)) = 0$ if and only if $f((u, gv_1), \ldots, (u, gv_S)) = 0$ for any $g \in G$, where $[g] \in B_- \setminus G$ corresponds to $[x]$ under the isomorphism $\mathcal{O} \simeq B_- \setminus G$. Note also that using algebraic rules for matrix elements we immediately obtain that $f((u, \cdot v_1), \ldots, (u, \cdot v_S))$ is a generalized Plücker coordinate defined using the representation $V^{\otimes d}$ and the highest weight vector

$$u^{\otimes d} = \underbrace{u \otimes \ldots \otimes u}_{d \text{ times}} \in V^{\otimes d},$$

i.e. $f((u, \cdot v_1), \ldots, (u, \cdot v_S)) = (u^{\otimes d}, \cdot v)$ for some $v \in V^{\otimes d}$. The last two facts imply that any closed subvariety in $B_- \setminus G$ is the zero locus of a finite set of generalized Plücker coordinates.

□

We proceed with the proof of Lemma 4.3.2. Let $Z_\pm = Z \cap N_\pm$. Then $Z_- H Z_+$ is dense in ZH, and hence the closure of $N_- s Z H M_-^s$ coincides with the closure of $N_- s Z_- H Z_+ M_-^s$. Since s fixes Δ_0 we can also write $N_- s Z_- H Z_+ M_-^s = B_- s Z_+ M_-^s$ and

$$\overline{N_- s Z H M_-^s} = \overline{B_- s Z_+ M_-^s}. \tag{4.3.3}$$

Since $B_-sZ_+M_-^s$ is the preimage of $B_- \setminus B_-sZ_+M_-^s$ with respect to the natural projection $G \to B_- \setminus G$, $(u, gv) = 0$ for any $g \in B_-sZ_+M_-^s$ if and only if $(u, \cdot v)$ regarded as a Plücker coordinate vanishes on $B_- \setminus B_-sZ_+M_-^s$. By Lemma 4.3.3 the closure of $B_- \setminus B_-sZ_+M_-^s$ is the zero locus of a finite set of generalized Plücker coordinates vanishing on $B_- \setminus B_-sZ_+M_-^s$ which is dense in its closure. Hence the vanishing ideal of $\overline{B_-sZ_+M_-^s}$ is generated by the matrix elements $(u, \cdot v)$ which correspond to these Plücker coordinates.

If $(u, \cdot v)$ is any generalized Plücker coordinate vanishing on $B_- \setminus B_-sZ_+M_-^s$ all its derivatives in the directions tangent to $B_- \setminus B_-sZ_+M_-^s$ must vanish as well. Therefore $(u, sz_+x'v) = 0$ for any $x' \in U(-\Delta^s_{\mathfrak{m}_+})$ and $z_+ \in U(\Delta_0^+)$.

Conversely, any generalized Plücker coordinate of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $U(\mathfrak{g})$ and $v \in V$, obeying the last property gives rise to a generalized Plücker coordinate vanishing on $B_- \setminus B_-sZ_+M_-^s$ and hence on the closure of $B_- \setminus B_-sZ_+M_-^s$.

Thus the closure of $B_- \setminus B_-sZ_+M_-^s$ is the zero locus of the set of the generalized Plücker coordinates of the above described type, and by the results proved above the corresponding matrix elements also generate the vanishing ideal of $\overline{B_-sZ_+M_-^s}$. By (4.3.3) we deduce that this ideal also coincides with the vanishing ideal of $\overline{N_-sZHM_-^s}$. Hence the vanishing ideal of $\overline{N_-sZHM_-^s}$ is generated by functions of the form $(u, \cdot v) \in \mathbb{C}[G]$, where where u is a highest weight vector in a finite-dimensional representation V of $U(\mathfrak{g})$, and $v \in V$ satisfies $(u, sz_+x'v) = 0$ for any $x' \in U(-\Delta^s_{\mathfrak{m}_+})$ and $z_+ \in U(\Delta_0^+)$, i.e. $\overline{J}$ coincides with the ideal of regular functions vanishing on the closure of $N_-sZHM_-^s$.

All derivatives of elements of $\overline{J}$ in the directions tangent to $N_-sZHM_-^s$ must vanish. Therefore every element of $\overline{J}$ must vanish on elements $xshz_+x'$ for any $x \in U(\mathfrak{n}_-), z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$.

Conversely, any element of $\mathbb{C}[G]$ vanishing on on elements $xshz_+x'$ for any $x \in U(\mathfrak{n}_-), z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$ must also vanish on the closure of $N_-sZHM_-^s$, and hence from the definition of $\overline{J}$ we deduce that $\overline{J}$ coincides with the ideal generated by the elements vanishing on $xshz_+x'$ for arbitrary $x \in U(\mathfrak{n}_-), z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$. Therefore $\overline{J}^{1\,\prime}_{\mathcal{B}} = \overline{J}^1_{\mathcal{B}} = \overline{J} \text{ mod } (q^{\frac{1}{dr^2}} - 1)\overline{J}^{1\,\prime}_{\mathcal{B}}$.

Note that $\overline{J}^1_{\mathcal{B}} \subset \overline{J}^{1\,\prime}_{\mathcal{B}}$ are submodules of the $\mathcal{B}$–module $\mathbb{C}_{\mathcal{B}}[G]$ which is free by Proposition 3.2.6. Therefore by Lemma 4.3.1 $\overline{J}^1_q = \overline{J}^{1\,\prime}_q$. This completes the proof.

□

The following lemma will be needed later to cancel some terms in formula (4.2.4) for the adjoint action.

Lemma 4.3.4. *For any u from the augmentation ideal of $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$, $u \notin U_{\mathcal{B}}^{s,res}(\Delta_0^-)$ we have $(v_\mu, \cdot uT_{s^{-1}}v_\mu) \in \overline{J}^{1\,\prime}_{\mathcal{B}}$.*

Proof. It suffices to verify the statement for $u = f_{\beta_1}^{(n_1)} \ldots f_{\beta_D}^{(n_D)} \neq 1$, $u \notin U_{\mathcal{B}}^{s,res}(\Delta_0^-)$. For any $x \in U_q^{res}(\mathfrak{n}_-), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$ we have

$$(v_\mu, xT_shz_+x'uT_{s^{-1}}v_\mu) = 0$$

if x is from the augmentation ideal of $U_q^{res}(\mathfrak{n}_-)$. Thus without loss of generality we can assume that $x = 1$. In this case

$$\begin{aligned}(v_\mu, xT_shz_+x'uT_{s^{-1}}v_\mu) &= (v_\mu, T_shz_+x'uT_{s^{-1}}v_\mu) = k(v_\mu, T_sz_+x'uT_{s^{-1}}v_\mu) = \\ &= k(v_\mu, T_sz_+x'f_{\beta_1}^{(n_1)} \ldots f_{\beta_m}^{(n_m)}T_{s^{-1}}v_\mu), k \in \mathcal{B}. \qquad (4.3.4)\end{aligned}$$

Let $\{\beta_1, \ldots, \beta_m\} = \Delta_+ \cap \Delta_+^s$. Since $s(\beta_p) \in \Delta_-$ for $p > m$, the right hand side of (4.3.4) vanishes if there exists $p > m$ such that $n_p \neq 0$.

If $n_p = 0$ for all $p > m$ then, assuming without loss of generality that z_+ and x' are monomials in quantum root vectors, we infer that the weight of the element $z_+x'f_{\beta_1}^{(n_1)} \ldots f_{\beta_m}^{(n_m)}$ is not zero as $(\overline{h}_0, wt(z_+x'f_{\beta_1}^{(n_1)} \ldots f_{\beta_m}^{(n_m)})) < 0$. Hence the right hand side of (4.3.4) vanishes in this case as well.

Thus $(v_\mu, T_shz_+x'uT_{s^{-1}}v_\mu) = 0$ in all cases and by the definition of $\overline{J}^{1\,\prime}_{\mathcal{B}}$ given in Lemma 4.3.2 $(v_\mu, \cdot uT_{s^{-1}}v_\mu) \in \overline{J}^{1\,\prime}_{\mathcal{B}}$.

□

Next, as it was promised in the beginning of this section, we define a $\mathcal{B}$–submodule $J^1_{\mathcal{B}} \subset \overline{J}^1_{\mathcal{B}}$ which is a quantum counterpart of the vanishing ideal of $\overline{N_-sZHN_-'} \supset \overline{N_-ZHsM_-^s}$. The next three lemmas are similar to Lemmas 4.3.2 and 4.3.4 for $\overline{J}^1_{\mathcal{B}}$.

Lemma 4.3.5. *Let $J^1_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements $(u,\cdot v)\in \mathbb{C}_{\mathcal{B}}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v\in V$ is such that $(u,T_sxv)=0$ for any $x\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$. Denote $J^1_q=J^1_{\mathcal{B}}\otimes_{\mathcal{B}}\mathbb{C}(q^{\frac{1}{dr^2}})$ and $I^1_{\mathcal{B}}=J^1_q\cap\mathbb{C}_{\mathcal{B}}[G]$. Then $J^1_{\mathcal{B}}\subset\overline{J}^1_{\mathcal{B}}$ and $I^1_{\mathcal{B}}\subset\overline{I}^1_{\mathcal{B}}$ are two sided ideals in $\mathbb{C}_{\mathcal{B}}[G]$ stable under the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$. Thus $\mathbb{C}_{\mathcal{B}}[G]/J^1_{\mathcal{B}}$ and $\mathbb{C}_{\mathcal{B}}[G]/I^1_{\mathcal{B}}$ are naturally equipped with the $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$–action induced by the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ on $\mathbb{C}_{\mathcal{B}}[G]$.*

Proof. From commutation relations (4.2.1) it follows that $J^1_{\mathcal{B}}$ is in fact a two sided ideal. First without loss of generality we can assume that v belongs to a weight subspace. Indeed, for any given V and $u\in V$ as in the statement of this lemma $U_q(H)$ naturally acts on the subspace of V spanned by elements $v\in V$ such that $(u,T_sxv)=0$ for any $x\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$. Therefore this subspace is P–graded.

From (4.2.1) with u,v as in the statement of the lemma and arbitrary $(u_1,\cdot v_1)\in\mathbb{C}_{\mathcal{B}}[G]$ such that $v\in(V)_\eta, v_1\in(V')_\rho$, $u\in(V)_\beta, u_1\in(V')_\gamma$ we have

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\eta^\vee,\rho^\vee)}\left((u_1,\cdot v_1)\otimes(u,\cdot v)+\sum_{\nu,i}(u_1,\cdot u_{\nu,i}v_1)\otimes(u,\cdot u_{-\nu,i}v)\right)= \tag{4.3.5}$$

$$=q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\beta^\vee,\gamma^\vee)}(u,\cdot v)\otimes(u_1,\cdot v_1)$$

since u is a highest weight vector. Observe that $U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$ is invariant with respect to the right multiplication by the elements $u_{-\nu,i}$, and hence $(u,\cdot u_{-\nu,i}v)$ vanishes on elements of the form T_sx for any $x\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$. We deduce that $(u_1,\cdot u_{\nu,i}v_1)\otimes(u,\cdot u_{-\nu,i}v)\in J^1_{\mathcal{B}}$, and hence (4.3.5) implies $(u,\cdot v)\otimes(u_1,\cdot v_1)\in J^1_{\mathcal{B}}$, i.e. $J^1_{\mathcal{B}}$ is a right ideal as well.

This immediately implies that J^1_q is a two sided ideal in $\mathbb{C}_q[G]$, and hence $I^1_{\mathcal{B}}=J^1_q\cap\mathbb{C}_{\mathcal{B}}[G]$ is a two sided ideal in $\mathbb{C}_{\mathcal{B}}[G]$.

Similarly to the proof of an analogous property for $\overline{J}^1_{\mathcal{B}}$ in Lemma 4.3.2 (see formula (4.3.1)) one can deduce that $J^1_{\mathcal{B}}$ and $I^1_{\mathcal{B}}$ are stable with respect to the Ad_0-action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$.

□

Similarly to Lemma 4.3.2 one can describe $J^1_{\mathcal{B}}$ as a vanishing ideal.

Lemma 4.3.6. *Let $\Delta'_s=(\Delta_{\mathfrak{m}_+}\cap\Delta_s^s)\cup\{\alpha\in\Delta_+:\alpha>\gamma_{l'}\}$ and $\Delta_0^+=\Delta_0\cap\Delta_+$. Both $\Delta'_s,\Delta_0^+\subset\Delta_+$ are minimal segments. Denote by $U_q^{res}(-\Delta'_s)$ and $U_q^{res}(\Delta_0^+)$ the subalgebras of $U_q^{res}(\mathfrak{g})$ corresponding to $-\Delta'_s$ and Δ_0^+, respectively. Let $J^{1'}_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements vanishing on xT_shz_+x' with arbitrary $x\in U_q^{res}(\mathfrak{n}_-), z_+\in U_q^{res}(\Delta_0^+), x'\in U_q^{res}(-\Delta'_s), h\in U_q^{res}(H)$. Denote $J^1_q=J^1_{\mathcal{B}}\otimes_{\mathcal{B}}\mathbb{C}(q^{\frac{1}{dr^2}})$, $J^{1'}_q=J^{1'}_{\mathcal{B}}\otimes_{\mathcal{B}}\mathbb{C}(q^{\frac{1}{dr^2}})$. Then $J^1_{\mathcal{B}}\subset J^{1'}_{\mathcal{B}}$ and $J^1_q=J^{1'}_q$. In particular, $I^1_{\mathcal{B}}=J^1_q\cap\mathbb{C}_{\mathcal{B}}[G]=J^{1'}_q\cap\mathbb{C}_{\mathcal{B}}[G]$.*

Moreover, $J^1_{\mathcal{B}}=J^{1'}_{\mathcal{B}}=J$ mod $(q^{\frac{1}{dr^2}}-1)J^1_{\mathcal{B}}$, where J is the vanishing ideal of the variety $\overline{N_-sZHN_-}=\overline{N_-sZHN'_-}$.

Proof. Let u be a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, $v\in V$.

First note that if $\alpha\in\Delta_{\mathfrak{m}_+}\setminus\Delta'_s$ then $s\alpha\in\Delta_+$ and if $\alpha\in\Delta_0^+$ then $s\alpha=\alpha$. Therefore applying the Poincaré–Birkhoff–Witt theorem for $U_q^{res}(w'(\mathfrak{b}_+))$ as stated in Corollary 4.1.8 and Lemma 2.4.2 we obtain that for any $x\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$

$$(u,T_sxv)=(u,T_s\sum_i z_i^+x'v),z_i^+\in U_q^{res}(\Delta_0^+),x'_i\in U_q^{res}(-\Delta'_s).$$

Since for all i $z_i^+x'_i\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$ we deduce that $(u,T_sxv)=0$ for any $x\in U_q^{res}(w'(\mathfrak{b}_+))U_q^{res}(\mathfrak{n}_-)$ if and only if $(u,T_sz_+x'v)=0$ for any $x'\in U_q^{res}(-\Delta'_s)$, $z_+\in U_q^{res}(\Delta_0^+)$, and hence $J^1_{\mathcal{B}}$ coincides with the left ideal generated by the elements $(u,\cdot v)$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, $v\in V$ is such that $(u,T_sz_+x'v)=0$ for any $z_+\in U_q^{res}(\Delta_0^+)$, $x'\in U_q^{res}(-\Delta'_s)$.

Obviously $J^1_{\mathcal{B}}\subset J^{1'}_{\mathcal{B}}$.

On the other hand $J^1_{\mathcal{B}}=J$ mod $(q^{\frac{1}{dr^2}}-1)J^1_{\mathcal{B}}$, where J is the ideal in $\mathbb{C}[G]$ generated by matrix elements of the form $(u,\cdot v)\in\mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $U(\mathfrak{g})$, and $v\in V$ is such that $(u,sz_+x'v)=0$ for any $z_+\in U(\Delta_0^+)=U_q^{res}(\Delta_0^+)/(q^{\frac{1}{dr^2}}-1)U_q^{res}(\Delta_0^+)$, $x'\in U(-\Delta'_s)=$

$U_q^{res}(-\Delta_s')/(q^{\frac{1}{dr^2}}-1)U_q^{res}(-\Delta_s')$. Similarly to the proof of the second claim of Lemma 4.3.2 one can show that J coincides with the ideal of regular functions vanishing on $\overline{N_-sZHN_-}=\overline{N_-sZHN_-'}=\overline{B_-sZ_+N_-'}$.

All derivatives of elements of J in the directions tangent to N_-sZHN_-' must vanish. Therefore every element of J must vanish on elements $xshz_+x'$ with arbitrary $x\in U(\mathfrak{n}_-), z_+\in U(\Delta_0^+), x'\in U(-\Delta_s'), h\in U(\mathfrak{h})$. Conversely, if $(u',\cdot v')$ is a regular function such that $(u',xshz_+x'v')=0$ for any $x\in U(\mathfrak{n}_-)$, $h\in U(\mathfrak{h})$, $x'\in U(-\Delta_s')$ and $z_-\in U(-\Delta_0^+)$ then $(u',\cdot v')$ vanishes on N_-sZHN_-', and hence on its closure. Thus J coincides with the ideal of regular functions vanishing on elements $xshz_+x'$ with arbitrary $x\in U(\mathfrak{n}_-), z_+\in U(\Delta_0^+), x'\in U(-\Delta_s'), h\in U(\mathfrak{h})$. Therefore $J_{\mathcal{B}}^{1}{}'=J \bmod (q^{\frac{1}{dr^2}}-1)J_{\mathcal{B}}^{1}{}'$.

Recall that $J_{\mathcal{B}}^1=J \bmod (q^{\frac{1}{dr^2}}-1)J_{\mathcal{B}}^1$ and $J_{\mathcal{B}}^1\subset J_{\mathcal{B}}^{1}{}'$. Note also that $J_{\mathcal{B}}^1$ and $J_{\mathcal{B}}^1$ are submodules of the $\mathcal{B}$–module $\mathbb{C}_{\mathcal{B}}[G]$ which is free by Proposition 3.2.6. Therefore by Lemma 4.3.1 $J_q^1=J_q^{1}{}'$. This completes the proof.

□

The following lemma will be needed later to cancel some terms in formula (4.2.4) for the adjoint action.

Lemma 4.3.7. *For any $\beta\in\Delta_{\mathfrak{m}_+}\setminus\Delta_{s^{-1}}^s$ and $n>0$ we have $(v_\mu,\cdot f_\beta^{(n)}T_{s^{-1}}v_\mu)\in J_{\mathcal{B}}^{1}{}'$.*

Proof. For any $x\in U_q^{res}(\mathfrak{n}_-), z_+\in U_q^{res}(\Delta_0^+), x'\in U_q^{res}(-\Delta_s'), h\in U_q^{res}(H)$ we have

$$(v_\mu, xT_shz_+x'f_\beta^{(n)}T_{s^{-1}}v_\mu)=0$$

if x is from the augmentation ideal of $U_q^{res}(\mathfrak{n}_-)$. Thus without loss of generality we can assume that $x=1$. We can also assume without loss of generality that z_+ and x' are monomials in quantum root vectors. In the proof of Lemma 4.4.3 we shall consider a more general situation and show that in this case the weight of the element $T_sz_+x'f_\beta^{(n)}T_{s^{-1}}$ is not zero (see the arguments after formula (4.4.7)). Thus

$$(v_\mu, xT_shz_+x'f_\beta^{(n)}T_{s^{-1}}v_\mu)=(v_\mu, T_shz_+x'f_\beta^{(n)}T_{s^{-1}}v_\mu)=k(v_\mu, T_sz_+x'f_\beta^{(n)}T_{s^{-1}}v_\mu)=0,$$

where $k\in\mathcal{B}$, and hence by Lemma 4.3.6 $(v_\mu,\cdot f_\beta^{(n)}T_{s^{-1}}v_\mu)\in J_{\mathcal{B}}^{1}{}'$.

□

4.4 Higher quantized vanishing ideals

The next step in our approach is, as it was mentioned in the introduction to this chapter, the definition and the study of quantum analogues $\overline{J}_{\mathcal{B}}^p$ and $J_{\mathcal{B}}^p$ of the vanishing ideals of the varieties

$$\overline{N_{[-\beta_p,-\beta_D]}sZHM_-^s}\subset\overline{N_-sZHM_-^s}, \overline{N_{[-\beta_p,-\beta_D]}sZHN_-'}\subset\overline{N_-sZHN_-'}.$$

Recall that according to Corollary 3.5.5 the vanishing ideals of these varieties are defined in terms of functions ϕ_k introduced in Corollary 3.5.4. Therefore firstly we have to define quantum analogues A_p of these functions. This is done with the help of quantum counterparts of matrix elements (3.5.4). More precisely, by Proposition 8.3 in [42] there exist integral dominant weights μ_p, $p=1,\ldots,D$, and elements $v_p\in V_{\mu_p}$ such that $(v_p,\cdot v_{\mu_p})\in\mathbb{C}_{\mathcal{B}}[G]$, and

$$(v_p,\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\ldots f_{\beta_1}^{(n_1)})v_{\mu_p})=\begin{cases}1 & \text{if } f_{\beta_D}^{(n_D)}\ldots f_{\beta_1}^{(n_1)}=f_{\beta_p}\\ 0 & \text{otherwise}\end{cases}, \tag{4.4.1}$$

where the highest weight vectors $v_{\mu_p}\in V_{\mu_p}$ are normalized by the condition $(v_{\mu_p},v_{\mu_p})=1$.

This definition implies $\omega(\omega_0S_s^{-1}(f_{\beta_p}))v_p=v_{\mu_p}$, i.e. v_p has weight $\mu_p-\beta_p$, $(v_{\mu_p},\cdot v_{\mu_p})\in\mathbb{C}_{\mathcal{B}}[G]$, and

$$(v_p,\omega_0S_s^{-1}(f_{\beta_p})\cdot T_{s^{-1}}v_{\mu_p})=(v_{\mu_p},\cdot T_{s^{-1}}v_{\mu_p}). \tag{4.4.2}$$

Let

$$A_p=(v_p,\cdot T_{s^{-1}}v_{\mu_p}), A_p^0=(v_{\mu_p},\cdot T_{s^{-1}}v_{\mu_p}). \tag{4.4.3}$$

From (4.2.2) we deduce

$$(v_{\mu_p},\cdot T_{s^{-1}}v_{\mu_p})\otimes(v_p,\cdot T_{s^{-1}}v_{\mu_p})=q^{((-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\beta_p^\vee,\mu_p^\vee)}(v_p,\cdot T_{s^{-1}}v_{\mu_p})\otimes(v_{\mu_p},\cdot T_{s^{-1}}v_{\mu_p}),$$

i.e.

$$A_p^0\otimes A_p=q^{((-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\beta_p^\vee,\mu_p^\vee)}A_p\otimes A_p^0. \tag{4.4.4}$$

Now we can define $\overline{I}_{\mathcal{B}}^p$ and characterize it as a vanishing ideal.

Lemma 4.4.1. *For $p = 1, \ldots, c+1$, let $\overline{J}^p_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by $\overline{J}^1_{\mathcal{B}}$ and by $A_1, \ldots, A_{p-1}$. Let $\overline{J}^{p\,\prime}_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements vanishing on xT_shz_+x' with arbitrary $x = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$, $n_i \in \mathbb{N}$. Let $\overline{J}^{p\,\prime\prime}_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements vanishing on $x\overline{T}_shz_+x'$ with arbitrary $x = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$, $n_i \in \mathbb{N}$.*

Denote $\overline{J}^p_q = \overline{J}^p_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\overline{J}^{p\prime}_q = \overline{J}^p_{\mathcal{B}} \otimes'_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\overline{J}^{p\prime\prime}_q = \overline{J}^{p\,\prime\prime}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\overline{I}^p_{\mathcal{B}} = \overline{J}^p_q \cap \mathbb{C}_{\mathcal{B}}[G]$. Then $\overline{I}^p_{\mathcal{B}}$ is a left ideal in $\mathbb{C}_{\mathcal{B}}[G]$, $\overline{J}^p_{\mathcal{B}} \subset \overline{J}^{p\,\prime}_{\mathcal{B}}$, $\overline{J}^p_{\mathcal{B}} \subset \overline{J}^{p\,\prime\prime}_{\mathcal{B}}$, $\overline{J}^p_q = \overline{J}^{p\prime}_q = \overline{J}^{p\prime\prime}_q$, and hence $\overline{I}^p_{\mathcal{B}} = \overline{J}^{p\prime}_q \cap \mathbb{C}_{\mathcal{B}}[G] = \overline{J}^{p\prime\prime}_q \cap \mathbb{C}_{\mathcal{B}}[G]$.

Moreover, $\overline{J}^p_{\mathcal{B}} = \overline{J}^{p\,\prime}_{\mathcal{B}} = \overline{J}^p_1$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^{p\,\prime}_{\mathcal{B}}$, $\overline{J}^p_{\mathcal{B}} = \overline{J}^{p\,\prime\prime}_{\mathcal{B}} = \overline{J}^p_1$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^{p\,\prime\prime}_{\mathcal{B}}$, where $\overline{J}^p_1$ is the vanishing ideal of $\overline{N_{[-\beta_p,-\beta_D]}sZHM^s_-}$.

Proof. Firstly we claim that $\overline{J}^p_{\mathcal{B}} \subset \overline{J}^{p\,\prime}_{\mathcal{B}}$. Indeed, any element from $\overline{J}^1_{\mathcal{B}}$ belongs to $\overline{J}^{p\,\prime}_{\mathcal{B}} \supset \overline{J}^{1\,\prime}_{\mathcal{B}}$ as by Lemma 4.3.2 $\overline{J}^1_{\mathcal{B}} \subset \overline{J}^{1\,\prime}_{\mathcal{B}}$.

Next, for $q < p$ and $x = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$, $n_i \in \mathbb{N}$ we have, since v_{μ_q} is a highest weight vector,

$$A_q(xT_shz_+x') = (v_q, xT_shz_+x'T_{s^{-1}}v_{\mu_q}) = (v_q, x\omega_0 S_s^{-1}(y)v_{\mu_q}) \tag{4.4.5}$$

for some $y \in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$. We claim that that the right hand side of (4.4.5) vanishes.

Indeed, if $x \neq 1$ then $x\omega_0 S_s^{-1}(y) = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}y)$ and $f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}y$ belongs to the right ideal in $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ generated by $f_{\beta_u}^{(n_u)}$, $u \geq p > q$ and $n_u > 0$. So by (4.4.1) and by Lemma 2.7.2 we have

$$A_q(xT_shz_+x') = (v_q, \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}y)v_{\mu_q}) = 0$$

in this case.

If $x = 1$ we can assume without loss of generality that z_+ and x' are monomials in quantum root vectors. From (4.4.1) it follows that $A_q(xT_shz_+x') \neq 0$ only if $A_q(xT_shz_+x') = (v_q, T_shz_+x'T_{s^{-1}}v_{\mu_q}) = (v_q, c\omega_0 S_s^{-1}(f_{\beta_q})v_{\mu_q})$ for some $c \in \mathcal{B}$, $c \neq 0$.

Now observe that from the definitions of Δ_0 and $-\Delta^s_{\mathfrak{m}_+}$ it follows that the weight of the element $T_shz_+x'T_{s^{-1}}$ is a linear combination of simple roots from Δ_+^s with non-negative integer coefficients, and this collection of weights does not contain $-\beta_q \in \Delta_-^s$, the weight of $\omega_0 S_s^{-1}(f_{\beta_q})$. Therefore $A_q(xT_shz_+x') = 0$ in all cases.

We deduce that the elements generating $\overline{J}^p_{\mathcal{B}}$ obey the defining property of $\overline{J}^{p\,\prime}_{\mathcal{B}}$, and hence $\overline{J}^p_{\mathcal{B}} \subset \overline{J}^{p\,\prime}_{\mathcal{B}}$.

Observe that by Lemma 4.3.2 $\overline{J}^1_{\mathcal{B}} = \overline{J}$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^1_{\mathcal{B}}$, where $\overline{J}$ is the vanishing ideal of $\overline{N_-sZHM^s_-}$, and by the definition $A_n = \phi_n$ mod $(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}[G]$ for any n. Hence, since $\overline{J}^p_{\mathcal{B}}$ is generated by $\overline{J}^1_{\mathcal{B}}$ and by $A_1, \ldots, A_{p-1}$, Corollary 3.5.5 implies that $\overline{J}^p_{\mathcal{B}} = \overline{J}^p_1$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^p_{\mathcal{B}}$, where $\overline{J}^p_1$ is the vanishing ideal of $\overline{N_{[-\beta_p,-\beta_D]}sZHM^s_-}$,

All derivatives of elements of $\overline{J}^p_1$ in the directions tangent to $N_{[-\beta_p,-\beta_D]}sZHM^s_-$ must vanish. Therefore every element of $\overline{J}^p_1$ must vanish on elements $xshz_+x'$ with arbitrary $x = X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_p}^{(n_p)}$, $z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$.

Conversely, any element of $\mathbb{C}[G]$ vanishing on $xshz_+x'$ with arbitrary $x = X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_p}^{(n_p)}, z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$ must vanish on $N_{[-\beta_p,-\beta_D]}sZHM^s_-$, and hence on $\overline{N_{[-\beta_p,-\beta_D]}sZHM^s_-}$. Thus $\overline{J}^p_1$ coincides with the ideal in $\mathbb{C}[G]$ generated by the regular functions vanishing on $xshz_+x'$ with arbitrary $x = X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_p}^{(n_p)}, z_+ \in U(\Delta_0^+), x' \in U(-\Delta^s_{\mathfrak{m}_+}), h \in U(\mathfrak{h})$. Therefore $\overline{J}^{p\,\prime}_{\mathcal{B}} = \overline{J}^p_{\mathcal{B}} = \overline{J}^p_1$ mod $(q^{\frac{1}{dr^2}} - 1)\overline{J}^{p\,\prime}_{\mathcal{B}}$. Recalling that $\overline{J}^p_{\mathcal{B}} \subset \overline{J}^{p\,\prime}_{\mathcal{B}}$ we deduce with the help of Lemma 4.3.1 that $\overline{J}^p_q = \overline{J}^{p\prime}_q$.

The case of the left ideal generated by the elements of $\mathbb{C}_{\mathcal{B}}[G]$ vanishing on elements $x\overline{T}_shz_+x'$ with arbitrary $x = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}), z_+ \in U_q^{res}(\Delta_0^+), x' \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+}), h \in U_q^{res}(H)$, $n_i \in \mathbb{N}$ is treated in a similar way since by Lemma 2.7.4 for a highest weight vector $v_\mu \in V_\mu$ there is a non–zero multiple k of a power of q such that $T_{s^{-1}}v_\mu = k\overline{T}_{s^{-1}}v_\mu$.

Finally observe that $\overline{I}^p_{\mathcal{B}}$ is a left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ as clearly $\overline{J}^p_q$ is a left ideal in $\mathbb{C}_q[G]$ and $\overline{I}^p_{\mathcal{B}} = \overline{J}^p_{\mathcal{B}} \cap \mathbb{C}_{\mathcal{B}}[G]$. This completes the proof.

□

Similarly one can define quantum analogues $J^p_{\mathcal{B}}$ of the vanishing ideals of the varieties $\overline{N_{[-\beta_p,-\beta_D]}sZHN'_-}$.

Lemma 4.4.2. *For $p=1,\ldots,c+1$, let $J^p_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by $J^1_{\mathcal{B}}$ and by $A_1,\ldots,A_{p-1}$. Let $J^{p\prime}_{\mathcal{B}}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}[G]$ generated by the elements vanishing on xT_shz_+x' with arbitrary $x=\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\ldots f_{\beta_p}^{(n_p)}),z_+\in U_q^{res}(\Delta_0^+),x'\in U_q^{res}(-\Delta_s'),h\in U_q^{res}(H)$, $n_i\in\mathbb{N}$. Denote $J^p_q=J^p_{\mathcal{B}}\otimes_{\mathcal{B}}\mathbb{C}(q^{\frac{1}{dr^2}})$, $J^{p\prime}_q=J^{p\prime}_{\mathcal{B}}\otimes_{\mathcal{B}}\mathbb{C}(q^{\frac{1}{dr^2}})$ and let $I^p_{\mathcal{B}}=J^p_q\cap\mathbb{C}_{\mathcal{B}}[G]$. Then $I^p_{\mathcal{B}}$ is a left ideal in $\mathbb{C}_{\mathcal{B}}[G]$, $J^p_q=J^{p\prime}_q$, and hence $I^p_{\mathcal{B}}=J^{p\prime}_q\cap\mathbb{C}_{\mathcal{B}}[G]$.*

Moreover, $J^p_{\mathcal{B}}=J^{p\prime}_{\mathcal{B}}=J^p_1$ mod $(q^{\frac{1}{dr^2}}-1)J^{p\prime}_{\mathcal{B}}$, where J^p_1 is the vanishing ideal of $\overline{N_{[-\beta_p,-\beta_D]}sZHN'_-}$.

Proof. Firstly as in the proof of Lemma 4.4.1 we deduce that $J^p_{\mathcal{B}}\subset J^{p\prime}_{\mathcal{B}}$.

Similarly to the proof of Lemma 4.4.1 we also obtain, using the second assertion in Corollary 3.5.5, that $J^p_{\mathcal{B}}=J^{p\prime}_{\mathcal{B}}=J^p_1$ mod $(q^{\frac{1}{dr^2}}-1)J^{p\prime}_{\mathcal{B}}$, where J^p_1 is the vanishing ideal of $\overline{N_{[-\beta_p,-\beta_D]}sZHN'_-}$, and hence by Lemma 4.3.1 $J^p_q=J^{p\prime}_q$.

Finally using the same arguments as in the proof of Lemma 4.4.1 we deduce that $I^p_{\mathcal{B}}$ is a left ideal in $\mathbb{C}_{\mathcal{B}}[G]$. ∎

We conclude this section with a technical lemma which will be used to cancel some terms in formula (4.2.4) for the adjoint action.

Lemma 4.4.3. *For any $n>0$, $A_p(\cdot f_{\beta_q}^{(n)})\in J^{p\prime}_{\mathcal{B}}$ for $q\geq p$, $\beta_p\not\in\Delta^s_{s^1}$, $q\in\{1,\ldots,c\}$ and $A_p(\cdot f_{\beta_q}^{(n)})\in \overline{J}^{p\prime}_{\mathcal{B}}$ for $q\geq p$, $q\in\{1,\ldots,c\}$.*

Proof. We prove the first statement. The second one can be justified in a similar way with the help of Lemma 4.4.1.

For $q\geq p$, $\beta_p\not\in\Delta^s_{s^1}$ and any $x=\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\ldots f_{\beta_p}^{(n_p)}),z_+\in U_q^{res}(\Delta_0^+),x'\in U_q^{res}(-\Delta_s'),h\in U_q^{res}(H)$ we have, since v_{μ_p} is a highest weight vector,

$$(v_p,xT_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=(v_p,x\omega_0S_s^{-1}(y)v_{\mu_p})$$

for some $y\in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$. We claim that

$$(v_p,xT_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=0$$

if $x\neq 1,\omega_0S_s^{-1}(f_{\beta_p})$.

Indeed, let $x\neq 1,\omega_0S_s^{-1}(f_{\beta_p})$, so $x\omega_0S_s^{-1}(y)=\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\ldots f_{\beta_p}^{(n_p)}y)$. Using commutation relations (2.7.11) the element $f_{\beta_D}^{(n_D)}\ldots f_{\beta_p}^{(n_p)}y$ can be uniquely represented as a $\mathbb{C}(q^{\frac{1}{dr^2}})$–linear combination of monomials $f_{\beta_D}^{(n_D)}\ldots f_{\beta_1}^{(n_1)}$ with $n_p>1$ or at least one $n_t>0$ for $t>p$. This presentation is unique by the Poincaré–Birkhoff–Witt decomposition for $U_q^s(\mathfrak{n}_-)$, and by the uniqueness of the Poincaré–Birkhoff–Witt decomposition for $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ the coefficients in this decomposition belong to $\mathcal{B}$ (see Lemma 2.7.2).

So by (4.4.1) we have

$$A_q(xT_shz_+x')=(v_q,\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\ldots f_{\beta_p}^{(n_p)}y)v_{\mu_q})=0$$

in this case.

If $x=1$ then

$$(v_p,xT_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=(v_p,T_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=c(v_p,T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p}),c\in\mathcal{B},\tag{4.4.6}$$

and if $x=S_s^{-1}(f_{\beta_p})$ then

$$(v_p,xT_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=(v_{\mu_p},T_shz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})=c'(v_{\mu_p},T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p}),c'\in\mathcal{B}.\tag{4.4.7}$$

We claim that in both cases the right hand side in the formulas above vanishes. This is obviously the case if $s\beta_q\in\Delta_-$. Suppose that $\beta_p\in\Delta_{i_j}$ and $s\beta_q\in\Delta_+$. Without loss of generality we can assume that z_+ and x' are monomials in quantum root vectors corresponding to roots from Δ_0^+ and $-\Delta_s'$, respectively, so $z_+x'=(X^+_{\delta_a})^{(r_a)}\ldots(X^+_{\delta_b})^{(r_b)}(X^-_{\epsilon_g})^{(t_g)}\ldots(X^-_{\epsilon_h})^{(t_h)}$, $\delta_a,\ldots,\delta_b\in\Delta_0^+\subset\Delta_0$, $\epsilon_g,\ldots,\epsilon_h\in\Delta_s'$. If $(v_p,T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})$ is not zero then the weight of the element $T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}$ must be equal to $-\beta_p$ as the weight of v_p is $\mu_p-\beta_p$ and if $(v_{\mu_p},T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}v_{\mu_p})$ is not zero then the weight of the element $T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}$ must be equal to zero.

Let k be minimal possible such that $\beta_q, \epsilon_g, \ldots \epsilon_h \in \overline{\Delta}_{i_k}$. Denote by $\lambda = r_a\delta_a + \ldots + r_b\delta_b - t_g\epsilon_g - \ldots - t_h\epsilon_h - n\beta_q$ the weight of $z_+x'f_{\beta_q}^{(n)}$. Note that $i_k > 0$ as $q \in \{1, \ldots, c\}$ and hence $\beta_q \notin \Delta_0$.

The corresponding plane $\mathfrak{h}_{i_k}$ is shown at Figure 6 where we use the same notation as at Figure 2, Section 1.2. We denote by $\Delta^s_{s^{-1}}$ the open sector which is transformed into the lower half plane under the action of s^{-1}, by $s\Delta^s_{s^{-1}}$ its image under the action of s, and by $-\Delta^s_{s^{-1}}$, $-s\Delta^s_{s^{-1}}$ their images under the multiplication by -1. We also denote by $\Delta^1_{i_k}$ the sector which appeared at Figure 2 under the same name, by $s\Delta^1_{i_k}$ its image under the action of s, and by $-s\Delta^1_{i_k}$ its image under the multiplication by -1. The orthogonal projections of the roots from the set Δ_0 onto $\mathfrak{h}_{i_k}$ are zeroes and by the definition of the set Δ'_s and by the properties of normal ordering (1.6.3) of the root system Δ^s_+ in Proposition 1.6.1 the orthogonal projections of the roots $-s\epsilon_g, \ldots, -s\epsilon_h \in s(-\Delta'_s)$ onto $\mathfrak{h}_{i_k}$ belong to the sectors labeled $\Delta^s_{s^{-1}}$ and $s\Delta^1_{i_k}$ at Figure 6.

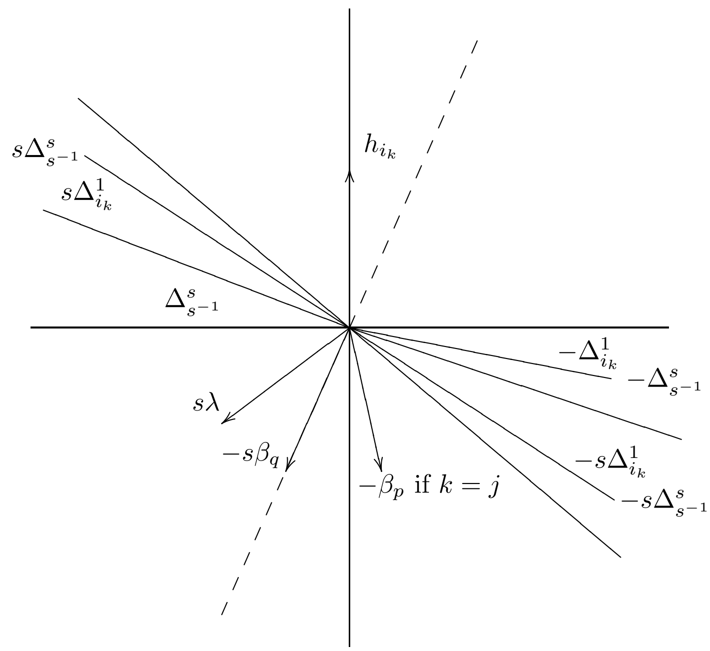


Fig. 6

Since $\beta_q \geq \beta_p \notin \Delta^s_{s^1}$ and $\Delta^s_{s^1}$ is an initial segment in Δ^s_+ we also have $\beta_q \notin \Delta^s_{s^1}$ and also by the assumption $s\beta_q \in \Delta_+$. Therefore the orthogonal projection of $-s\beta_q$ onto $\mathfrak{h}_{i_k}$ belongs to the lower half plane or possibly to the sector labeled $\Delta^s_{s^{-1}}$ at Figure 6 and does not belong to the sectors labeled $-\Delta_{s^{-1}}$ and $-s\Delta^1_{i_k}$ at Figure 6 This projection is labeled $-s\beta_q$ at Figure 6.

If the orthogonal projection of $-s\beta_q$ onto $\mathfrak{h}_{i_k}$ belongs to the sector labeled $\Delta^s_{s^{-1}}$ at Figure 6 then the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_k}$ belongs to the interior of the closure of the union of the sectors $\Delta^s_{s^{-1}}$ and $s\Delta^1_{i_k}$, and the intersection of this union with the lower half plane is at most $-\Delta^1_{i_k}$. Remark that in this case the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_k}$ is not zero, and hence $\lambda \neq 0$.

If $k = j$ note that the orthogonal projection of $-\beta_p$ onto $\mathfrak{h}_{i_j}$ belongs to the open lower half plane and does not belong to $-\Delta^1_{i_k}$. Thus in this case $s\lambda \neq -\beta_p$.

If the orthogonal projection of $-s\beta_q$ onto $\mathfrak{h}_{i_k}$ belongs to the lower half plane and does not belong to the sectors labeled $-\Delta^s_{s^{-1}}$ and $-s\Delta^1_{i_k}$ at Figure 6 then the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_j}$ belongs to the closed left half plane bounded by the dashed line at Figure 6. This projection labeled $s\lambda$ is shown at Figure 6 as well. Observe that in this case the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_k}$ is not zero, and hence $\lambda \neq 0$.

If $k = j$, recall that $\beta_p \leq \beta_q$ where $\beta_p, \beta_q \in \Delta_{i_j}$, and hence by the last property of the normal ordering in Δ^s_+ in Proposition 1.6.1 the orthogonal projection of $-\beta_p$ onto $\mathfrak{h}_{i_j}$ labeled by the same letter at Figure 6 belongs to the other open half plane bounded by the dashed line at Figure 6. Thus again $s\lambda \neq -\beta_p$.

In all cases above, if $k > j$ then the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_k}$ is not zero while the orthogonal projection of $-\beta_p$ onto $\mathfrak{h}_{i_k}$ is zero, and hence $s\lambda \neq -\beta_p$. If $k < j$ then the orthogonal projection of $s\lambda$ onto $\mathfrak{h}_{i_j}$ is zero while the orthogonal projection of $-\beta_p$ onto $\mathfrak{h}_{i_j}$ is zero, and hence again $s\lambda \neq -\beta_p$.

We deduce that in all cases the weight of the element $T_sz_+x'f_{\beta_q}^{(n)}T_{s^{-1}}$ is not equal to $-\beta_p$ or to zero. Therefore the right hand side in formulas (4.4.6), (4.4.7) vanishes, and by the definition of $J^{p'}_{\mathcal{B}}$ in Lemma 4.4.2 $A_p(\cdot f_{\beta_q}^{(n)}) \in J^{p'}_{\mathcal{B}}$

for $q \geq p$, $\beta_p \notin \Delta^s_{s^1}$, $q \in \{1, \ldots, c\}$.

□

4.5 Localizations

In order co define quantum counterparts of formulas (3.5.10) we have to introduce a certain localization $\mathbb{C}^{loc}_{\mathcal{B}}[G]$ of $\mathbb{C}_{\mathcal{B}}[G]$. However, $\mathbb{C}^{loc}_{\mathcal{B}}[G]$ is not a localization in the sense of localization for non–commutative algebras, and it is rather impossible to define a proper localization of the algebra $\mathbb{C}_{\mathcal{B}}[G]$ which contains the quantum counterparts of the denominators in formula (3.5.10) given by (4.4.3). But in fact we shall only need right denominators and $\mathbb{C}^{loc}_{\mathcal{B}}[G]$ will be defined as a "right" localization of $\mathbb{C}_{\mathcal{B}}[G]$. The exact meaning of this term will be explained below. Along with $\mathbb{C}^{loc}_{\mathcal{B}}[G]$ we shall also define "right" localizations $\overline{I}^{p\,loc}_{\mathcal{B}}$ and $I^{p\,loc}_{\mathcal{B}}$ of $\overline{I}^{p}_{\mathcal{B}}$ and $I^{p}_{\mathcal{B}}$ and study their properties.

We start by introducing a subalgebra $\mathbb{C}_{\mathcal{B}}[G]_0 \subset \mathbb{C}_{\mathcal{B}}[G]$ a proper localization of which contains all the required denominators.

Lemma 4.5.1. *The set of elements $(u, \cdot T_{s^{-1}}v) \in \mathbb{C}_{\mathcal{B}}[G]$, where v is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$ and $u \in V$ form a subalgebra $\mathbb{C}_{\mathcal{B}}[G]_0$ in $\mathbb{C}_{\mathcal{B}}[G]$.*

Proof. It suffices to show that the product of two elements from $\mathbb{C}_{\mathcal{B}}[G]_0$ belongs to $\mathbb{C}_{\mathcal{B}}[G]_0$. Let $s^{-1} = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition of s^{-1}. Then by (2.6.12)

$$\Delta_s(T_{s^{-1}}) = T_{s^{-1}} \otimes T_{s^{-1}} q^{\sum_{i=1}^{l}(-T_sY_i \otimes T_sKH_i + Y_i \otimes KH_i)} \prod_{p=1}^{k} \overline{\theta}^s_{\beta'_p}, \tag{4.5.1}$$

where for $p = 1, \ldots, k$

$$\beta'_p = s_{i_k} \ldots s_{i_{p+1}} \alpha_{i_p}, \overline{X}^{\pm}_{\beta'_p} = T^{-1}_{i_k} \ldots T^{-1}_{i_{p+1}} X^{\pm}_{i_p},$$

$$\overline{e}_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}(\overline{X}^{+}_{\beta'_p} e^{hK\beta'^{\vee}_p}), \overline{f}_{\beta'_p} = \psi^{-1}_{\{n_{ij}\}}(e^{-hK\beta'^{\vee}_p} \overline{X}^{-}_{\beta'_p}),\ K_{\beta'_p} = T^{-1}_{i_k} \ldots T^{-1}_{i_{p+1}} K_{i_p},$$

$$\overline{\theta}^s_{\beta'_p} = exp_{q_{\beta'_p}}[(1 - q^{-2}_{\beta'_p}) K^{-1}_{\beta'_p} e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta'^{\vee}_p} \overline{f}_{\beta'_p} \otimes \overline{e}_{\beta'_p} K_{\beta'_p}].$$

Since $Kh = \frac{\kappa}{2} \frac{1+s}{1-s} P_{\mathfrak{h}'} h$ for any $h \in \mathfrak{h}$, we have $Ks = sK$, and hence

$$q^{\sum_{i=1}^{l}(-T_sY_i \otimes T_sKH_i + Y_i \otimes KH_i)} = q^{\sum_{i=1}^{l}(-T_sY_i \otimes KT_sH_i + Y_i \otimes KH_i)} = q^{\sum_{i=1}^{l}(-Y_i \otimes KH_i + Y_i \otimes KH_i)} = 1 \tag{4.5.2}$$

where we also used the identity $\sum_{i=1}^{l} T_sY_i \otimes KT_sH_i = \sum_{i=1}^{l} Y_i \otimes KH_i$ which holds since T_sY_i, T_sH_i, $i = 1, \ldots, l$ is a pair of dual bases in $\mathfrak{h}$.

Now (4.5.1) takes the form

$$\Delta_s(T_{s^{-1}}) = T_{s^{-1}} \otimes T_{s^{-1}} \prod_{p=1}^{k} \overline{\theta}^s_{\beta'_p},$$

and for two highest weight vectors $v \in V, v' \in V'$ we have

$$\Delta_s(T_{s^{-1}}) v \otimes v' = T_{s^{-1}} v \otimes T_{s^{-1}} v'. \tag{4.5.3}$$

Therefore for any $u \in V, u' \in V'$ we have by (4.5.3)

$$(u, \cdot T_{s^{-1}}v) \otimes (u', \cdot T_{s^{-1}}v') = (u \otimes u', \cdot T_{s^{-1}}v \otimes T_{s^{-1}}v') = (u \otimes u', \cdot T_{s^{-1}}(v \otimes v')). \tag{4.5.4}$$

Since $v \otimes v'$ is a highest weight vector in $V \otimes V'$, the last identity implies $(u, \cdot T_{s^{-1}}v) \otimes (u', \cdot T_{s^{-1}}v') \in \mathbb{C}_{\mathcal{B}}[G]_0$. This completes the proof.

□

From formulas (4.2.2) and (4.5.4) it follows that the set $S = \{\Delta_\mu = (v_\mu, \cdot T_{s^{-1}}v_\mu)\} | \mu \in P_+\}$, where $v_\mu \in V_\mu$ are highest weight vectors normalized by $(v_\mu, v_\mu) = 1$, is a multiplicative set of normal elements in $\mathbb{C}_{\mathcal{B}}[G]_0$. Let $\mathbb{C}^{loc}_{\mathcal{B}}[G]_0$ be the localization of $\mathbb{C}_{\mathcal{B}}[G]_0$ by S.

We shall need more information on the structure of subalgebras N_k in $\mathbb{C}^{loc}_{\mathcal{B}}[G]_0$ generated by the elements

$$B_p = A^{0\,-1}_p \otimes A_p \in \mathbb{C}^{loc}_{\mathcal{B}}[G]_0, \tag{4.5.5}$$

$p = 1, \ldots, k$. The elements B_p are quantum analogues of functions (3.5.10). The following Lemma is similar to Proposition 8.3 in [42].

Lemma 4.5.2. *Let N_k be the subalgebra in $\mathbb{C}_{\mathcal{B}}^{loc}[G]_0$ generated by the elements $B_p = A_p^{0^{-1}} \otimes A_p \in \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$, $p = 1, \ldots, k$, and $U_{\mathcal{B}}^k$ the subalgebra in $\mathbb{C}_{\mathcal{B}}[G_*]$ generated by the elements $e_p = (q_{\beta_p}^{-1} - q_{\beta_p})q^{\beta_p^\vee} e_{\beta_p}$, $p = 1, \ldots, k$. Then the map*

$$B_p \mapsto e_p$$

gives rise to an algebra isomorphism $\vartheta : N_k \to U_{\mathcal{B}}^k$.

In particular, the elements B_p satisfy the following commutation relations

$$B_p B_r - q^{(\beta_p,\beta_r)+(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\beta_p,\beta_r)} B_r B_p = \sum_{k_{p+1},\ldots,k_{r-1}} C(k_{p+1}, \ldots, k_{r-1}) B_{p+1}^{k_{p+1}} \ldots B_{r-1}^{k_{r-1}},\ p < r, \tag{4.5.6}$$

where $C(k_{p+1}, \ldots, k_{r-1}) \in \mathcal{B}$.

Proof. First observe that the map

$$\psi : \mathbb{C}_{\mathcal{B}}[G] \to \mathbb{C}_{\mathcal{B}}[B_-], (u, \cdot v) \mapsto (u \otimes id, (\omega_0 \otimes id)(\mathcal{R})v \otimes id)$$

is an algebra homomorphism. Indeed, since ω_0 is a coautomorphism

$$(\Delta_s \otimes id)((\omega_0 \otimes id)(\mathcal{R}^s)) = (\omega_0 \otimes \omega_0 \otimes id)(\Delta_s \otimes id)\mathcal{R}^s = (\omega_0 \otimes \omega_0 \otimes id)(\mathcal{R}_{13}^s \mathcal{R}_{23}^s) = (\omega_0 \otimes id)(\mathcal{R}_{13}^s)(\omega_0 \otimes id)(\mathcal{R}_{23}^s),$$

and hence

$$\psi((u, \cdot v) \otimes (u', \cdot v')) = (u \otimes id, (\omega_0 \otimes id)(\mathcal{R}_{13}^s)v \otimes id)(u' \otimes id, (\omega_0 \otimes id)(\mathcal{R}_{23}^s)v' \otimes id) = \psi((u, \cdot v))\psi((u', \cdot v')),$$

i.e. ψ is an algebra homomorphism.

Recall that by Lemma 4.5.1 elements of the form $(u, \cdot T_{s^{-1}} v) \in \mathbb{C}_{\mathcal{B}}[G]$, where v is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$ and $u \in V$, form a subalgebra $\mathbb{C}_{\mathcal{B}}[G]_0$ in $\mathbb{C}_{\mathcal{B}}[G]$.

From (4.5.3) we also deduce that the map

$$\mathbb{C}_{\mathcal{B}}[G]_0 \to \mathbb{C}_{\mathcal{B}}[G], (u, \cdot T_{s^{-1}} v) \mapsto (u, \cdot v)$$

is an algebra homomorphism. Composing this map with ψ we obtain another algebra homomorphism ψ^0.

Next using (4.1.5) and the definition of A_p we obtain

$$\psi^0(A_p) = (v_p \otimes id, (\omega_0 S_s^{-1} \otimes id)(\mathcal{R})v_{\mu_p} \otimes id) = (v_p \otimes id, (\omega_0 \otimes id)(\mathcal{R}^{s^{-1}})v_{\mu_p} \otimes id) =$$

$$= (q_{\beta_p}^{-1} - q_{\beta_p})q^{(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\mu_p^\vee} q^{\beta_p^\vee} e_{\beta_p}.$$

Similarly

$$\psi^0(A_p^0) = q^{(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\mu_p^\vee}.$$

From the last two formulas we deduce that ψ^0 gives rise to an algebra homomorphism $\vartheta : N_k \to U_{\mathcal{B}}^k$ such that

$$\vartheta(B_p) = \vartheta(A_p^{0^{-1}} \otimes A_p)) = (q_{\beta_p}^{-1} - q_{\beta_p})q^{\beta_p^\vee} e_{\beta_p} = e_p.$$

This homomorphism is surjective by construction. ϑ is also injective as ψ^0 is injective (see Proposition 8.3 in [42] and [113], Theorem 2.6 for the proof).

Commutation relations (4.5.6) follow from (3.2.10) by applying ω and by multiplying by $q^{\beta_p^\vee} q^{\beta_r^\vee}$.

□

Denote $\mathbb{C}_{\mathcal{B}}^{loc}[G] = \mathbb{C}_{\mathcal{B}}[G] \otimes_{\mathbb{C}_{\mathcal{B}}[G]_0} \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$. $\mathbb{C}_{\mathcal{B}}^{loc}[G]$ is naturally a left $\mathbb{C}_{\mathcal{B}}[G]$–module and a right $\mathbb{C}_{\mathcal{B}}^{loc}[G]_0$–module. We denote by $\otimes$ both the left $\mathbb{C}_{\mathcal{B}}[G]$–action and the right $\mathbb{C}_{\mathcal{B}}^{loc}[G]_0$–action on $\mathbb{C}_{\mathcal{B}}^{loc}[G]$ and call these action multiplications.

Let $S^* = \{\Delta_\lambda \otimes \Delta_\mu^{-1} \in \mathbb{C}_{\mathcal{B}}^{loc}[G]_0 | \lambda, \mu \in P_+\}$, $S^{-1} = \{\Delta_\mu^{-1} \in \mathbb{C}_{\mathcal{B}}^{loc}[G]_0 | \mu \in P_+\}$, $\overline{I}_{\mathcal{B}}^{p\,loc}$ and $I_{\mathcal{B}}^{p\,loc}$ the images of $\overline{I}_{\mathcal{B}}^p \otimes S^* \subset \mathbb{C}_{\mathcal{B}}[G] \otimes \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$ and of $I_{\mathcal{B}}^p \otimes S^* \subset \mathbb{C}_{\mathcal{B}}[G] \otimes \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$, respectively, in $\mathbb{C}_{\mathcal{B}}^{loc}[G]$ under the projection $\mathbb{C}_{\mathcal{B}}[G] \otimes \mathbb{C}_{\mathcal{B}}^{loc}[G]_0 \to \mathbb{C}_{\mathcal{B}}[G] \otimes_{\mathbb{C}_{\mathcal{B}}[G]_0} \mathbb{C}_{\mathcal{B}}^{loc}[G]_0 = \mathbb{C}_{\mathcal{B}}^{loc}[G]$.

For $p = 1, \ldots, c+1$ let

$$\mathbb{C}_p^{loc}[G] = \begin{cases} \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc} & \text{if } \beta_p \notin \Delta_{s^1}^s \\ \mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc} & \text{if } \beta_p \in \Delta_{s^1}^s \end{cases}.$$

$\mathbb{C}_p^{loc}[G]$ is naturally a left $\mathbb{C}_{\mathcal{B}}[G]$–module.

Note that $\overline{I}_{\mathcal{B}}^{p\,loc} \subset \overline{I}_{\mathcal{B}}^{p+1^{loc}}$, $I_{\mathcal{B}}^{p\,loc} \subset I_{\mathcal{B}}^{p+1^{loc}}$ for $p = 1, \ldots, c$, and $I_{\mathcal{B}}^{p\,loc} \subset \overline{I}_{\mathcal{B}}^{p\,loc}$ for $p = 1, \ldots, c+1$.

Here and everywhere below for the technical reasons which already became manifest in Lemma 4.3.7 we have to distinguish two cases in our consideration, $\beta_p \not\in \Delta_{s^1}^s$ and $\beta_p \in \Delta_{s^{-1}}^s$. Conceptually this complication is related to the non–commutativity of the multiplication in $\mathbb{C}_{\mathcal{B}}[G]$.

By the definition $\overline{I}_{\mathcal{B}}^{p\,loc}$ and $I_{\mathcal{B}}^{p\,loc}$ are left $\mathbb{C}_{\mathcal{B}}[G]$–submodules in $\mathbb{C}_{\mathcal{B}}^{loc}[G]$. The following lemma shows that $\overline{I}_{\mathcal{B}}^{p}$, $I_{\mathcal{B}}^{p}$, $\overline{I}_{\mathcal{B}}^{p\,loc}$ and $I_{\mathcal{B}}^{p\,loc}$ have some nice properties with respect to multiplication by the elements $A_r, \Delta_\mu, B_r \in \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$ from the right as well.

Lemma 4.5.3. *For any $p \in \{1, \ldots, c+1\}$ $I_{\mathcal{B}}^{p\,loc}$ is invariant with respect to multiplication from the right by A_p and B_p, and $I_{\mathcal{B}}^{p\,loc}A_r = I_{\mathcal{B}}^{p\,loc}B_r \subset I_{\mathcal{B}}^{r\,loc}$ for any $r > p$, $r \in \{1, \ldots, c+1\}$.*

For any $p \in \{1, \ldots, c+1\}$ and any $\mu \in P_+$ $I_{\mathcal{B}}^{p}$ is invariant with respect to multiplication from the right by Δ_μ.

For any $p \in \{1, \ldots, c+1\}$ such that $\beta_p \in \Delta_{s^{-1}}^s$ $\overline{I}_{\mathcal{B}}^{p\,loc}$ is invariant with respect to multiplication from the right by A_p and B_p, and for $\beta_r \in \Delta_{s^{-1}}^s$ and any $r > p$, $r \in \{1, \ldots, c+1\}$ we have $\overline{I}_{\mathcal{B}}^{p\,loc}A_r = \overline{I}_{\mathcal{B}}^{p\,loc}B_r \subset \overline{I}_{\mathcal{B}}^{r\,loc}$.

For any $p \in \{1, \ldots, c+1\}$ such that $\beta_p \in \Delta_{s^{-1}}^s$ and any $\mu \in P_+$ $\overline{I}_{\mathcal{B}}^{p}$ is invariant with respect to multiplication from the right by Δ_μ.

Thus multiplication from the right induces a natural action of A_p and B_p on $\mathbb{C}_p^{loc}[G]$ and for $r \geq p$, $r \in \{1, \ldots, c+1\}$ right multiplication by A_r or B_r gives rise to well–defined homomorphisms of left $\mathbb{C}_{\mathcal{B}}[G]$–modules $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc} \to \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{r\,loc}$, and, if in addition $\beta_r \in \Delta_{s^{-1}}^s$, right multiplication by A_r or B_r gives rise to well–defined homomorphisms of left $\mathbb{C}_{\mathcal{B}}[G]$–modules $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc} \to \mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{r\,loc}$.

Proof. The proof of the first claim follows from the definitions of $I_{\mathcal{B}}^{p}$ and $I_{\mathcal{B}}^{p\,loc}$, commutation relations (4.2.2), (4.5.6) and from the fact that $I_{\mathcal{B}}^{1}$ is a two–sided ideal in $\mathbb{C}_{\mathcal{B}}[G]$. For the same reason $I_{\mathcal{B}}^{p}$ is invariant with respect to the multiplication from the right by Δ_μ for any $\mu \in P_+$.

From commutation relations (4.2.2), (4.5.6) it also follows that to establish the second claim it suffices to show that for $r \leq p$ with $\beta_p \in \Delta_{s^{-1}}^s$ we have $\overline{J}_q^{1}A_r \subset \overline{J}_q^{p}$, and to establish the last claim it suffices to verify that $\overline{J}_q^{1}\Delta_\mu \subset \overline{J}_q^{1}$ for any $\mu \in P_+$.

Indeed, recall that $\overline{J}_q^{1}$ is the left ideal in $\mathbb{C}_q[G]$ generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, \overline{T}_s xv) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$.

From formula (2.6.13) with k equal to the length of s we have

$$\Delta_s(\overline{T}_s) = \prod_{p=1}^{k} \overline{\theta}_{\beta_p}^{s}{}' q^{\sum_{i=1}^{l}(-Y_i \otimes KH_i + T_s Y_i \otimes T_s KH_i)} \overline{T}_s \otimes \overline{T}_s = \prod_{p=1}^{k} \overline{\theta}_{\beta_p}^{s}{}' \overline{T}_s \otimes \overline{T}_s \tag{4.5.7}$$

where we used (4.5.2).

Let $y = x\overline{T}_s hzx'$, $x = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)})$, $z \in U_q^{res}(\Delta_0^+)$, $x' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s), h \in U_q^{res}(H)$. Since u is a highest weight vector we obtain from (4.5.7), the definition of $\overline{\theta}_{\beta_p}^{s}{}'$ after (2.6.13), (2.7.12) and the fact that $\omega_0 S_s^{-1}$ is an anticoautomorphism that

$$((u, \cdot v) \otimes A_r)(y) = (u, \overline{T}_s h_1 z_1 x_1' v)(v_r, x\overline{T}_s h_2 z_2 x_2' T_s^{-1} v_{\mu_r}) =$$

$$= t(u, \overline{T}_s h_1 z_1 x_1' v)(v_r, x\overline{T}_s h_2 z_2 x_2' \overline{T}_s^{-1} v_{\mu_r}),$$

where t is a non–zero multiple of a power of q defined in Lemma 2.7.4, and we use the Sweedler notation for the coproducts, so for any $a \in U_q^{res}(\mathfrak{g}) \subset U_{\mathcal{B}}^{res}(\mathfrak{g}) \simeq U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ $\Delta_s(a) = a_1 \otimes a_2$.

As in the proof of Lemma 4.4.1 (see the discussion after formula (4.4.5)) we also deduce that $(v_r, x\overline{T}_s h_2 z_2 x_2' \overline{T}_s^{-1} v_{\mu_r}) = 0$ if $x \neq 1, \omega_0 S_s^{-1}(f_{\beta_r})$, i.e $((u, \cdot v) \otimes A_r)(y) = 0$ if $x \neq 1, \omega_0 S_s^{-1}(f_{\beta_r})$.

Let $x = 1$. Recall that $e_\beta = X_\beta^+ q^{K\beta^\vee}$ and $f_\beta = q^{-K\beta^\vee} X_\beta^-$, and hence

$$\Delta_s(X_\beta^+) = \Delta_s(e_\beta) q^{-K\beta^\vee} \otimes q^{-K\beta^\vee}, \Delta_s(X_\beta^-) = q^{K\beta^\vee} \otimes q^{K\beta^\vee} \Delta_s(f_\beta). \tag{4.5.8}$$

Combining these expressions with formulas (2.7.12) and (2.7.14) we deduce that

$$h_2z_2x_2' \in U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}([\Delta_0^+,-\gamma_{l'}]),$$

where $[\Delta_0^+,-\gamma_{l'}]$ is the minimal segment in Δ containing Δ_0^+ as its initial segment and $-\gamma_{l'}$ is its last element. But $\beta_r \in \Delta_{s^{-1}}^s$, so $s^{-1}(-\beta_r) \in \Delta_s^s$. Note also that $[\Delta_0^+,-\gamma_{l'}]$ is a subset of a system of positive roots $\Delta_0^+ \cup (\Delta_-^s \setminus \Delta_0^-)$ and $\Delta_s^s \cap (\Delta_0^+ \cup (\Delta_-^s \setminus \Delta_0^-)) = \emptyset$. Therefore the element $\overline{T}_s h_2z_2x_2'\overline{T}_{s^{-1}}$ has no components of weight $-\beta_r$ in its weight decomposition. Since the weight of v_r is $\mu_r - \beta_r$ we deduce that

$$((u,\cdot v)\otimes A_r)(y) = (u,\overline{T}_s h_1z_1x_1'v)(v_r,\overline{T}_s h_2z_2x_2'\overline{T}_s^{-1}v_{\mu_r}) = 0$$

in this case.

If $x = \omega_0 S_s^{-1}(f_{\beta_r})$ we have by the definition of v_r

$$((u,\cdot v)\otimes A_r)(y) = (u,\overline{T}_s h_1z_1x_1'v)(v_{\mu_r},\overline{T}_s h_2z_2x_2'\overline{T}_s^{-1}v_{\mu_r}), \tag{4.5.9}$$

and hence only zero weight components of the element $h_2z_2x_2'$ will contribute to the right hand side of the formula above. Recall that $h_2z_2x_2' \in U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}([\Delta_0^+,-\gamma_{l'}])$ and as we observed above $[\Delta_0^+,-\gamma_{l'}]$ is a subset of the system of positive roots $\Delta_0^+ \cup (\Delta_-^s \setminus \Delta_0^-)$. Therefore only zero degree monomials in quantum root vectors in the decomposition of $h_2z_2x_2'$ will contribute to the right hand side of (4.5.9). If we assume, without loss of generality, that z and x' are monomials in quantum root vectors then from (4.5.8), (2.7.12) and (2.7.14) one can see that the above mentioned contribution amounts to

$$((u,\cdot v)\otimes A_r)(y) = (u,\overline{T}_s h_1z_1x_1'v)(v_{\mu_r},\overline{T}_s h_2z_2x_2'\overline{T}_s^{-1}v_{\mu_r}) = c'(u,\overline{T}_s zx'v), c' \in \mathcal{B}.$$

Finally observe that $zx' \in U_q^{res}(w'(\mathfrak{b}_+))$, and hence $(u,\overline{T}_s zx'v) = t'(u,T_s zx'v) = 0$ by the choice of u and v. Here t' is a non–zero multiple of a power of q given by Lemma 2.7.4.

So in all cases $((u,\cdot v)\otimes A_r)(y) = 0$, and hence $\overline{J}_q^1 A_r \subset \overline{J}_q^p$ by Lemma 4.4.1.

Similar arguments show that for any $\mu \in P_+$ and $y = x\overline{T}_s hzx'$ with any $x \in U_q^{res}(\mathfrak{n}_-)$, $z \in U_q^{res}(\Delta_0^+)$, $x' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s), h \in U_q^{res}(H)$ one has

$$((u,\cdot v)\otimes \Delta_\mu)(y) = 0,$$

and hence $\overline{J}_q^1$ is invariant with respect to multiplication from the right by Δ_μ for any $\mu \in P_+$ by Lemma 4.3.2.

□

In the next two lemmas we study the properties of $\overline{I}_{\mathcal{B}}^p$, $I_{\mathcal{B}}^p$, and $\mathbb{C}_p^{loc}[G]$ with respect to the adjoint action. These properties will be needed to study properties of quantum analogues of the operators Π_p defined in (3.5.11).

Lemma 4.5.4. *For any $p = 1,\ldots,c$, $n \in \mathbb{N}$ and $f \in \mathbb{C}_{\mathcal{B}}[G]$ we have*

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f\otimes\Delta_\mu) = q^{n((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f)\otimes\Delta_\mu \tag{4.5.10}$$

in $\mathbb{C}_{\mathcal{B}}[G]/\overline{I}_{\mathcal{B}}^1$. Thus the adjoint action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ on $\mathbb{C}_{\mathcal{B}}[G]/\overline{I}_{\mathcal{B}}^1$ defined in Lemma 4.3.2 induces an action on $\mathbb{C}_1^{loc}[G]$ satisfying

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f\otimes\Delta_\mu^{-1}) = q^{-n((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f)\otimes\Delta_\mu^{-1}, f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{1\,loc} = \mathbb{C}_1^{loc}[G]. \tag{4.5.11}$$

This action is locally finite.

For any $p = 1,\ldots,c$ with $\beta_p \notin \Delta_{s^1}^s$, $n \in \mathbb{N}$ and $f \in \mathbb{C}_{\mathcal{B}}[G]$ we have

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f\otimes\Delta_\mu) = q^{n((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f)\otimes\Delta_\mu \tag{4.5.12}$$

in $\mathbb{C}_{\mathcal{B}}[G]/I_{\mathcal{B}}^1$. Thus the adjoint action of $f_{\beta_p}^{(n)}$ on $\mathbb{C}_{\mathcal{B}}[G]/I_{\mathcal{B}}^1$ defined in Lemma 4.3.5 induces a locally finite action on $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{1\,loc}$ satisfying

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f\otimes\Delta_\mu^{-1}) = q^{-n((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f)\otimes\Delta_\mu^{-1}, f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{1\,loc}. \tag{4.5.13}$$

Proof. The proof of the first part follows from formula (4.2.4) applied to $g=\Delta_\mu$. All terms in the r.h.s. of (4.2.4), except for the ones with $p=n-k$ in the first sum, vanish since v_μ is a highest weight vector. Lemma 4.3.4 implies vanishing all the remaining terms in the first sum in the r.h.s. of (4.2.4) modulo $\overline{I}_{\mathcal{B}}^1$, except for $k=0$. The only non-vanishing term in the first sum in the r.h.s. of (4.2.4) corresponds to $p=n$, $k=0$. It gives the r.h.s. of (4.5.10).

Local finiteness of the action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ on $\mathbb{C}_1^{loc}[G]$ follows from the local finiteness of the action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ on $\mathbb{C}_{\mathcal{B}}[G]/\overline{I}_{\mathcal{B}}^1$ and from formula (4.5.11).

The second part is proved similarly using Lemma 4.3.7.

□

Lemma 4.5.5. *For any $r\in\{1,\ldots,c\}$, $p\leq r$ and $n\in\mathbb{N}$ the left ideal $\overline{I}_{\mathcal{B}}^p\subset\mathbb{C}_{\mathcal{B}}[G]$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable, and, if in addition $\beta_r\notin\Delta_{s^1}^s$, then $I_{\mathcal{B}}^p$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable.*

For any $r\in\{1,\ldots,c\}$, $p\leq r$ and $n\in\mathbb{N}$ ${\overline{I}_{\mathcal{B}}^p}^{loc}/{\overline{I}_{\mathcal{B}}^1}^{loc}\subset\mathbb{C}_{\mathcal{B}}^{loc}[G]/{\overline{I}_{\mathcal{B}}^1}^{loc}\simeq\mathbb{C}_1^{loc}[G]$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable, and, if in addition $\beta_r\notin\Delta_{s^1}^s$, then ${I_{\mathcal{B}}^p}^{loc}/{I_{\mathcal{B}}^1}^{loc}\subset\mathbb{C}_{\mathcal{B}}^{loc}[G]/{I_{\mathcal{B}}^1}^{loc}$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable.

Thus for any $r\in\{1,\ldots,c\}$, $p\leq r$ and $n\in\mathbb{N}$ $\mathbb{C}_{\mathcal{B}}^{loc}[G]/{\overline{I}_{\mathcal{B}}^p}^{loc}$ is naturally equipped with an $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–action, and, if in addition $\beta_r\notin\Delta_{s^1}^s$ $\mathbb{C}_{\mathcal{B}}^{loc}[G]/{I_{\mathcal{B}}^p}^{loc}$ is naturally equipped with an $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–action.

In particular, $\mathbb{C}_p^{loc}[G]$ is naturally equipped with an $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–action.

Proof. By (4.5.11) and (4.5.13) to establish the first two claims it suffices to show that $\overline{J}_q^p$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable for any $r\in\{1,\ldots,c\}$, $p\leq r$ and $n\in\mathbb{N}$ and J_q^p is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable for any $r\in\{1,\ldots,c\}$ with $\beta_r\notin\Delta_{s^1}^s$, $p\leq r$ and $n\in\mathbb{N}$.

We show that $\overline{J}_q^p$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable for any $r\in\{1,\ldots,c\}$, $p\leq r$ and $n\in\mathbb{N}$. Since $\overline{J}_q^p$ is generated by $\overline{J}_q^1$ and by $A_1,\ldots,A_{p-1}$ and by Lemma 4.3.2 $\overline{J}_q^1$ is $\mathrm{Ad}_0 f_{\beta_r}^{(n)}$–stable it suffices to prove that $\mathrm{Ad}_0 f_{\beta_r}^{(n)}(f\otimes A_t)\in\overline{J}_q^p$, for any $f\in\mathbb{C}_{\mathcal{B}}[G]$, $t=1,\ldots,p-1$. By (4.2.3) we have

$$\begin{gathered}
\mathrm{Ad}_0 f_{\beta_r}^{(n)}(f\otimes A_t)=\sum_{k=0}^{n}\sum_{p=0}^{n-k}q_{\beta_r}^{-k(n-k)-p(n-k-p)}\mathrm{Ad}_0(G_{\beta_r}^{-k}f_{\beta_r}^{(p)})(f)\otimes A_t(\omega_0S_s^{-1}(G_{\beta_r}^{-k-p}f_{\beta_r}^{(n-k-p)})\cdot\omega_0(f_{\beta_r}^{(k)}))+\\
+\sum_{k=0}^{n-1}\sum_i q_{\beta_r}^{-k(n-k)}\mathrm{Ad}_0(G_{\beta_r}^{-k}x_i^{(n-k)})(f)\otimes A_t((\omega_0S_s^{-1})(G_{\beta_r}^{-k}y_i^{(n-k)})\cdot\omega_0(f_{\beta_r}^{(k)}))+ \qquad (4.5.14)\\
+\sum_i\mathrm{Ad}_0({y_i^{(n)}}^2)(f)\otimes A_t((\omega_0S_s^{-1})({y_i^{(n)}}^1)\cdot\omega_0(x_i^{(n)}))=\\
=\sum_{k=0}^{n}q_{\beta_r}^{-k(n-k)}q^{n(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_r-\beta_r,\mu_t-\beta_t)}\mathrm{Ad}_0(G_{\beta_r}^{-k}f_{\beta_r}^{(n-k)})(f)\otimes A_t(\cdot\omega_0(f_{\beta_r}^{(k)}))+\\
+\sum_{k=0}^{n}\sum_{p=0}^{n-k-1}q_{\beta_r}^{-k(n-k)-p(n-k-p)}\mathrm{Ad}_0(G_{\beta_r}^{-k}f_{\beta_r}^{(p)})(f)\otimes A_t(\omega_0S_s^{-1}(G_{\beta_r}^{-k-p}f_{\beta_r}^{(n-k-p)})\cdot\omega_0(f_{\beta_r}^{(k)}))+\\
+\sum_{k=0}^{n-1}\sum_i q_{\beta_r}^{-k(n-k)}\mathrm{Ad}_0(G_{\beta_r}^{-k}x_i^{(n-k)})(f)\otimes A_t((\omega_0S_s^{-1})(G_{\beta_r}^{-k}y_i^{(n-k)})\cdot\omega_0(f_{\beta_r}^{(k)}))+\\
+\sum_i\mathrm{Ad}_0({y_i^{(n)}}^2)(f)\otimes A_t((\omega_0S_s^{-1})({y_i^{(n)}}^1)\cdot\omega_0(x_i^{(n)})).
\end{gathered}$$

We claim that all terms in the right hand side of the formula above belong to $\overline{J}_q^p$. Indeed, by Lemma 4.4.3

$$q_{\beta_r}^{-k(n-k)}q^{n(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_r-\beta_r,\mu_t-\beta_t)}\mathrm{Ad}_0(G_{\beta_r}^{-k}f_{\beta_r}^{(n-k)})(f)\otimes A_t(\cdot\omega_0(f_{\beta_r}^{(k)}))\in\overline{J}_q^t\subset\overline{J}_q^p$$

for $k>0$. If $k=0$

$$q^{n(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_r-\beta_r,\mu_t-\beta_t)}\mathrm{Ad}_0(f_{\beta_r}^{(n)})(f)\otimes A_t\in\overline{J}_q^{t+1}\subset\overline{J}_q^p.$$

To establish the same claim for all the other terms in the right hand side of (4.5.14) we shall apply Lemma 4.4.1. The arguments are similar for all these terms. Consider, for instance, the terms in the last sum. For xT_shz_+x' with arbitrary $x=\omega_0S_s^{-1}(f_{\beta_D}^{(n_D)}\dots f_{\beta_p}^{(n_p)}), z_+\in U_q^{res}(\Delta_0^+), x'\in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s), h\in U_q^{res}(H)$, $n_i\in\mathbb{N}_0$ we have

$$A_t((\omega_0S_s^{-1})({y_i^{(n)}}^1)xT_shz_+x'\omega_0x_i^{(n)})=(v_t,(\omega_0S_s^{-1})({y_i^{(n)}}^1)xT_shz_+x'\omega_0x_i^{(n)}T_{s^{-1}}v_{\mu_t})=$$

$$=c(v_t,(\omega_0S_s^{-1})({\overline{y}_i^{(n)}}^1f_{\beta_D}^{(n_D)}\dots f_{\beta_p}^{(n_p)})T_shz_+x'\omega_0x_i^{(n)}T_{s^{-1}}v_{\mu_t})=$$

$$=c(v_t,(\omega_0S_s^{-1})({\overline{y}_i^{(n)}}^1f_{\beta_D}^{(n_D)}\dots f_{\beta_p}^{(n_p)})T_shz_+x'\omega_0x_i^{(n)}T_{s^{-1}}v_{\mu_t})=(v_t,(\omega_0S_s^{-1})({\overline{y}_i^{(n)}}^1f_{\beta_D}^{(n_D)}\dots f_{\beta_p}^{(n_p)}y)v_{\mu_t}),$$

where $c\in\mathcal{B}$, $y\in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ and for the last implication we used the fact that v_t is a weight vector and v_{μ_t} is a highest weight vector. Since ${\overline{y}_i^{(n)}}^1\in I_r^>$, $r\geq p$ and the weight of ${\overline{y}_i^{(n)}}^1$ is not zero, Lemma 2.7.2 applied to ${\overline{y}_i^{(n)}}^1f_{\beta_D}^{(n_D)}\dots f_{\beta_p}^{(n_p)}y$ implies that this element belongs to the right ideal in $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ generated by the elements $f_{\beta_D}^{(n_D)}\dots f_{\beta_{r+1}}^{(n_{r+1})}$, where at least one $n_i>0$. This ideal does not contain elements proportional to f_{β_t} as $1\leq t<p\leq r$. Therefore by the definition of A_t (see (4.4.1)) $A_t((\omega_0S_s^{-1})({y_i^{(n)}}^1)xT_shz_+x'\omega_0x_i^{(n)})=0$ and hence by Lemma 4.4.1 $f\otimes A_t((\omega_0S_s^{-1})({y_i^{(n)}}^1)\cdot\omega_0x_i^{(n)})\in\overline{J}_q^p$. Note that this statement holds for $t=p$ as well. This proves that $\overline{J}_q^p$ is $\mathrm{Ad}_0f_{\beta_r}$–stable for $r\geq p$.

To show that J_q^p is $\mathrm{Ad}_0f_{\beta_r}$–stable for any $r\in\{1,\dots,c\}$ with $\beta_r\notin\Delta_{s^1}^s$, $p\leq r$ and $n\in\mathbb{N}$ one has to apply similar arguments in conjunction with Lemmas 4.3.5, 4.4.2 and 4.4.3.

□

4.6 Zhelobenko type operators for q-W–algebras

This section is central in this part and in the whole book. We are going to introduce and study some quantum analogues of the operators Π_p and Π defined in (3.5.11) and (3.5.12). It turns out that the analogues P_p of Π_p can be obtained by proper extrapolation of the expansion of the conjugation operator in (3.5.11) in terms of the adjoint action operator and by replacing the coefficients G_p with their quantum counterparts B_p introduced in (4.5.5). However, the proof of Proposition 4.6.1 which asserts that the image of their composition P consists of invariant elements with respect to the adjoint action of $U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ is rather complicated. It entirely relies on the properties of the subspaces $\overline{I}_{\mathcal{B}}^{p\,loc},I_{\mathcal{B}}^{p\,loc}\subset\mathbb{C}_{\mathcal{B}}^{loc}[G]$, of the quotients $\mathbb{C}_p^{loc}[G]$ and of the adjoint action, which were obtained in the previous sections of this chapter.

The point is that in the quantum case we do not have in our disposal the isomorphism (3.4.29) which plays a crucial role in the proof of a similar property for the operator Π given by (3.5.11), (3.5.12) as one can see from the proofs of Propositions 3.5.1 and 3.5.2.

We start with the definition of quantum analogues P_p of the operators Π_p. For technical reasons we shall also need more general operators P_p^k. More precisely, by Lemmas 4.5.3 and 4.5.5 for $\beta_p\notin\Delta_{s^1}^s$, $f\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$, $p=1,\dots,c$, $n,k=0,1,\dots$, $n\geq k$, we have $\mathrm{Ad}_0f_{\beta_p}^{(n-k)}(f)\otimes B_p^n\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$. Note that by Lemma 4.5.4 for each $f\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ $\mathrm{Ad}_0f_{\beta_p}^{(n-k)}(f)=0$ for $n>N(f)$, where $N(f)\in\mathbb{N}$ depends on f. Therefore we can define define an element $P_p^k(f)\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ by

$$P_p^k(f)=\sum_{n=k}^{\infty}(-1)^nq_{\beta_p}^{\frac{(n-1)(n-2k)}{2}}\mathrm{Ad}_0f_{\beta_p}^{(n-k)}(f)\otimes B_p^n. \tag{4.6.1}$$

Similarly, for $\beta_p\in\Delta_{s^{-1}}^s$, $f\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$, $p=1,\dots,c$, $n,k=0,1,\dots$, $n\geq k$, we have $\mathrm{Ad}_0f_{\beta_p}^{(n-k)}(f)\otimes B_p^n\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$. Note that by Lemma 4.5.4 for each $f\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ $\mathrm{Ad}_0f_{\beta_p}^{(n-k)}(f)=0$ for $n>N(f)$, where $N(f)\in\mathbb{N}$ depends on f. Therefore we can define define an element $P_p^k(f)\in\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ by formula (4.6.1).

Let β_e be the first root of $\Delta_{s^{-1}}^s$ in normal ordering (1.6.3) which does not belong to $\Delta_{s^1}^s$. Then, using the imbedding $I_{\mathcal{B}}^{e\,loc}\subset\overline{I}_{\mathcal{B}}^{e\,loc}$ we can define the canonical projection $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{e\,loc}\to\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$, and hence in this case for $f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{e\,loc} = \mathbb{C}_e^{loc}[G]$ projecting $P_e^k(f) \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{e\,loc}$ defined in the former situation above onto $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$ we can introduce an operator

$$P_e^k : \mathbb{C}_e^{loc}[G] \to \mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}.$$

From the discussion above it follows that for all other indexes $p = 1, \ldots, c$, $p \neq e$ formula (4.6.1) can be used to define operators

$$P_p^k : \mathbb{C}_p^{loc}[G] \to \mathbb{C}_p^{loc}[G].$$

Denote $P_p^0 = P_p$.

The following proposition summarizes the main properties of the operators P_p.

Proposition 4.6.1. *For any $p = 1, \ldots, c$ the composition $P_{\leq p} = P_1 \ldots P_p : \mathbb{C}_p^{loc}[G] \to \mathbb{C}_1^{loc}[G]$ is well–defined. In fact for $p \neq e-1$ $P_{\leq p}$ is well–defined as an operator with the domain $\mathbb{C}_{p+1}^{loc}[G]$ and $P_{\leq e-1}$ is well–defined as an operator with the domain $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$.*

For $q \leq p$, $n > 0$ and for any $f \in \mathbb{C}_p^{loc}[G]$ we have

$$\mathrm{Ad}_0 f_{\beta_q}^{(n)}(P_{\leq p}(f)) = 0. \tag{4.6.2}$$

Moreover,

$$\mathrm{Ad}_0 f_{\beta_{p+1}}^{(n)}(P_{\leq p}(f)) = P_{\leq p}(\mathrm{Ad}_0 f_{\beta_{p+1}}^{(n)} f). \tag{4.6.3}$$

For $\beta_p \in \Delta_{s^1}^s$ and any $f \in \mathbb{C}_1^{loc}[G]$ such that $\mathrm{Ad}_0 x(f) = \varepsilon_s(x) f \quad \forall x \in U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p])$ we have in $\mathbb{C}_1^{loc}[G]$

$$P_{\leq p}(\bar{f}) = f, \tag{4.6.4}$$

where $\bar{f}$ is the class of f in $\mathbb{C}_p^{loc}[G]$.

For $\beta_p \notin \Delta_{s^1}^s$ and any $f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{1\,loc}$ such that $\mathrm{Ad}_0 x(f) = \varepsilon_s(x) f \quad \forall x \in U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p])$ we have

$$P_{\leq p}(\bar{f}) = f \ (mod\ \overline{I}_{\mathcal{B}}^{1\,loc}), \tag{4.6.5}$$

where $\bar{f}$ is the class of f in $\mathbb{C}_p^{loc}[G]$.

In particular, if $P = P_{\leq c}$ then for any $f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{1\,loc}$ such that $\mathrm{Ad}_0 x(f) = \varepsilon_s(x) f \quad \forall x \in U_{\mathcal{B}}^{s,res}(\mathfrak{m}_-)$ we have

$$P(\bar{f}) = f \ (mod\ \overline{I}_{\mathcal{B}}^{1\,loc}), \tag{4.6.6}$$

where $\bar{f}$ is the class of f in $\mathbb{C}_c^{loc}[G]$.

The proof of Proposition 4.6.1 is quite long. It will be split into several lemmas. The strategy of the proof can be explained by looking at a similar situation in the classical case.

Namely, from the definition of the operators Π_p in (3.5.11), Proposition 3.5.2 and Corollary 3.5.4 it follows that the kernel of the operator $\Pi_1 \ldots \Pi_p : \mathbb{C}[N_- sZM_-^s] \to \mathbb{C}[N_- sZM_-^s]$ coincides with the vanishing ideal of $N_{[-\beta_{p+1}, -\beta_D]} sZM_-^s \subset N_- sZM_-^s$. Similarly, in the quantum case we shall show that $P_{\leq p}$ is well–defined as an operator with the domain of definition $\mathbb{C}_{p+1}^{loc}[G]$, i.e. $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p+1\,loc}$ or $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p+1\,loc}$ depending on p.

Denote by L_p the the derivative of the action induced by the conjugation action by the elements of the one–parameter subgroup $\exp(-tX_{-\beta_p})$ at $t = 0$. From Propositions 3.5.1, 3.5.2 and Corollary 3.5.4 one can immediately deduce that $L_p \circ \Pi_p : \mathbb{C}[N_- sZM_-^s] \to \mathbb{C}[N_- sZM_-^s]$ is the zero operator modulo the vanishing ideal of $N_{[-\beta_p, -\beta_D]} sZM_-^s \subset N_- sZM_-^s$. In the quantum case we shall show that for $n > 0$ $\mathrm{Ad}_0 f_{\beta_p}^{(n)} P_p$ is the zero operator modulo $\overline{I}_{\mathcal{B}}^{p\,loc}$ or $I_{\mathcal{B}}^{p\,loc}$ depending on p. This is done using an explicit calculation and observing that $\mathrm{Ad}_0 f_{\beta_p}^{(n)}$ has some properties analogues to properties of derivatives of order n.

Finally to show that for $q \leq p$ and $n > 0$ $\mathrm{Ad}_0 f_{\beta_q}^{(n)}(P_{\leq p}(f)) = 0$ modulo $\overline{I}_{\mathcal{B}}^{1\,loc}$ we shall use the above mentioned property of the images of operators P_p and commutation relations between P_p and $\mathrm{Ad}_0 f_{\beta_q}^{(n)}$ for $q > p$ which, in particular, lead to commutation relations (4.6.3).

Firstly, we are going to show by an explicit calculation that for $n > 0$ $\mathrm{Ad}_0 f_{\beta_p}^{(n)} P_p$ is the zero operator modulo $\overline{I}_{\mathcal{B}}^{p\,loc}$ (or $I_{\mathcal{B}}^{p\,loc}$). Since the right hand side of formula (4.6.1) contains products of elements from $\mathbb{C}_q^{loc}[G]$ and of B_p^n, we have to study the adjoint action on such products.

Lemma 4.6.2. *Let $B_p = A_p^{0^{-1}} \otimes A_p \in \mathbb{C}_{\mathcal{B}}^{loc}[G]_0$, $f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]$, $p = 1, \ldots, c$. Let either $\beta_p \in \Delta_{s^{-1}}^s$ or $\beta_p \notin \Delta_{s^1}^s$. Then the identity*

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(\bar{f} \otimes B_p^k) = \sum_{r=0}^{\min(n,k)} q_{\beta_p}^{(2n-r)k+\frac{r(r-1)}{2}-rn} \left[\begin{array}{c} k \\ r \end{array} \right]_{q_{\beta_p}} \mathrm{Ad}_0 f_{\beta_p}^{(n-r)}(\bar{f}) \otimes B_p^{k-r} \quad (4.6.7)$$

holds in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ in the former case or in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ in the latter case, where $\bar{f}$ is the class of f in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ or in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$, respectively.

For any $f \in \mathbb{C}_{\mathcal{B}}^{loc}[G]$, $p, q = 1, \ldots, c$, $q > p$ let either $\beta_p \in \Delta_{s^{-1}}^s$ or $\beta_q \notin \Delta_{s^1}^s$. For any two roots $\alpha, \beta \in \Delta$ we denote

$$c_{\alpha\beta} = q^{((\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\alpha, \beta)}.$$

Then the identity

$$\mathrm{Ad}_0 f_{\beta_q}^{(m)}(\bar{f} \otimes B_p^n) = c_{\beta_p\beta_q}^{mn} \mathrm{Ad}_0 f_{\beta_q}^{(m)}(\bar{f}) \otimes B_p^n \quad (4.6.8)$$

holds in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ in the former case or in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ in the latter case, where $\bar{f}$ is the class of f in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ or in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$, respectively.

Proof. We prove (4.6.7) by induction over k. Firstly by (4.2.4), similarly to (4.5.14), we have for any $f \in \mathbb{C}_{\mathcal{B}}[G]$

$$\begin{aligned}
\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f \otimes A_p) &= q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_p - \beta_p, \mu_p - \beta_p)} \mathrm{Ad}_0(f_{\beta_p}^{(n)})(f) \otimes A_p + \\
&+ q^{(n-1)(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_p - \beta_p, \mu_p - \beta_p)} q_{\beta_p}^{-(n-1)} \mathrm{Ad}_0(f_{\beta_p}^{(n-1)})(f) \otimes A_p(\omega_0 S_s^{-1}(f_{\beta_p})\cdot) + \\
&+ \sum_{k=1}^{n} q_{\beta_p}^{-k(n-k)} q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_p - \beta_p, \mu_p - \beta_p)} \mathrm{Ad}_0(G_{\beta_p}^{-k} f_{\beta_p}^{(n-k)})(f) \otimes A_p(\cdot \omega_0(f_{\beta_p}^{(k)})) + \\
&+ \sum_{k=1}^{n} q_{\beta_p}^{-k(n-k)-(n-k-1)} \mathrm{Ad}_0(G_{\beta_p}^{-k} f_{\beta_p}^{(n-k-1)})(f) \otimes A_p(\omega_0 S_s^{-1}(G_{\beta_p}^{-n+1} f_{\beta_p}) \cdot \omega_0(f_{\beta_p}^{(k)})) + \\
&+ \sum_{k=0}^{n} \sum_{p=0}^{n-k-2} q_{\beta_p}^{-k(n-k)-p(n-k-p)} \mathrm{Ad}_0(G_{\beta_p}^{-k} f_{\beta_p}^{(p)})(f) \otimes A_p(\omega_0 S_s^{-1}(G_{\beta_p}^{-k-p} f_{\beta_p}^{(n-k-p)}) \cdot \omega_0(f_{\beta_p}^{(k)})) + \\
&+ \sum_{k=0}^{n-1} \sum_i q_{\beta_p}^{-k(n-k)} \mathrm{Ad}_0(G_{\beta_p}^{-k} x_i^{(n-k)})(f) \otimes A_p((\omega_0 S_s^{-1})(G_{\beta_p}^{-k} y_i^{(n-k)}) \cdot \omega_0(f_{\beta_p}^{(k)})) + \\
&+ \sum_i \mathrm{Ad}_0({y_i^{(n)}}^2)(f) \otimes A_p((\omega_0 S_s^{-1})({y_i^{(n)}}^1) \cdot \omega_0(x_i^{(n)})).
\end{aligned} \quad (4.6.9)$$

Using arguments similar to those in the proof of Lemma 4.5.5 one can show that all terms in the right hand side of (4.6.9), except for the first two, vanish in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ in the former case or in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ in the latter case mentioned in the statement. Also by (4.4.2) we have $A_p((\omega_0 S_s^{-1})(f_{\beta_p})\cdot) = A_p^0$, and hence in both cases we have in the corresponding quotient

$$\begin{aligned}
\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f \otimes A_p) &= q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_p - \beta_p, \mu_p - \beta_p)} \mathrm{Ad}_0(f_{\beta_p}^{(n)})(f) \otimes A_p + \\
&+ q^{(n-1)(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_p - \beta_p, \mu_p - \beta_p)} q_{\beta_p}^{-(n-1)} \mathrm{Ad}_0(f_{\beta_p}^{(n-1)})(f) \otimes A_p^0.
\end{aligned}$$

Since by definition elements of S^{-1} naturally act on $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$ in the former case or on $\mathbb{C}_{\mathcal{B}}^{loc}[G]/I_{\mathcal{B}}^{p\,loc}$ in the latter case by multiplication from the right one can multiply the last identity by $(A_p^0)^{-1}$ from the right. Using commutation relations (4.2.2), formulas (4.5.11), (4.5.13) and recalling the definition of B_p we finally obtain in in the corresponding quotient

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(f \otimes B_p) = q_{\beta_p}^{n-1} \mathrm{Ad}_0 f_{\beta_p}^{(n-1)}(f) + q_{\beta_p}^{2n} \mathrm{Ad}_0 f_{\beta_p}^{(n)}(f) \otimes B_p. \quad (4.6.10)$$

Since by definition elements of S^{-1} naturally act on both $\mathbb{C}^{loc}_{\mathcal{B}}[G]/\overline{I}^{p\,loc}_{\mathcal{B}}$ in the former case and on $\mathbb{C}^{loc}_{\mathcal{B}}[G]/I^{p\,loc}_{\mathcal{B}}$ in the latter case by multiplication from the right one can multiply the last identity by Δ_μ^{-1} from the right. Using commutation relations (4.2.2), formulas (4.5.11), (4.5.13) and recalling the definition of B_p we finally obtain in the corresponding quotient

$$\mathrm{Ad}_0 f^{(n)}_{\beta_p}(\overline{f\otimes\Delta_\mu^{-1}}\otimes B_p)=q^{n-1}_{\beta_p}\mathrm{Ad}_0 f^{(n-1)}_{\beta_p}(\overline{f\otimes\Delta_\mu^{-1}})+q^{2n}_{\beta_p}\mathrm{Ad}_0 f^{(n)}_{\beta_p}(\overline{f\otimes\Delta_\mu^{-1}})\otimes B_p,$$

where $\overline{f\otimes\Delta_\mu^{-1}}$ should be regarded now as the class of $f\otimes\Delta_\mu^{-1}\in\mathbb{C}^{loc}_{\mathcal{B}}[G]$ in the corresponding quotient. Since such classes generate these quotients formula (4.6.10) holds for any $f\in\mathbb{C}^{loc}_{\mathcal{B}}[G]$ This establishes the base of induction.

Now we assume that (4.6.7) holds for some natural k and for all natural n and prove that it holds for $k+1$ and all natural n. The arguments given below are the same for both cases, and we consider them simultaneously. Since by Lemma 4.5.3 right multiplication gives rise to a well–defined action of B_p on the corresponding quotient introduced in the statement, we have in this quotient for any $f\in\mathbb{C}^{loc}_{\mathcal{B}}[G]$ using the base of induction and the induction assumption

$$\mathrm{Ad}_0 f^{(n)}_{\beta_p}(\bar f\otimes B_p^{k+1})=q^{n-1}_{\beta_p}\mathrm{Ad}_0 f^{(n-1)}_{\beta_p}(\bar f\otimes B_p^{k})+q^{2n}_{\beta_p}\mathrm{Ad}_0 f^{(n)}_{\beta_p}(\bar f\otimes B_p^{k})\otimes B_p=$$

$$=q^{n-1}_{\beta_p}\sum_{r=0}^{\min(n-1,k)}q_{\beta_p}^{(2(n-1)-r)k+\frac{r(r-1)}{2}-r(n-1)}\begin{bmatrix}k\\ r\end{bmatrix}_{q_{\beta_p}}\mathrm{Ad}_0 f^{(n-r-1)}_{\beta_p}(\bar f)\otimes B_p^{k-r}+$$

$$+q^{2n}_{\beta_p}\sum_{r=0}^{\min(n,k)}q_{\beta_p}^{(2n-r)k+\frac{r(r-1)}{2}-rn}\begin{bmatrix}k\\ r\end{bmatrix}_{q_{\beta_p}}\mathrm{Ad}_0 f^{(n-r)}_{\beta_p}(\bar f)\otimes B_p^{k-r+1}=$$

$$=\sum_{r=0}^{\min(n,k+1)}q_{\beta_p}^{(2n-r)(k+1)+\frac{r(r-1)}{2}-rn}\left(q^{r}_{\beta_p}\begin{bmatrix}k\\ r\end{bmatrix}_{q_{\beta_p}}+q_{\beta_p}^{r-(k+1)}\begin{bmatrix}k\\ r-1\end{bmatrix}_{q_{\beta_p}}\right)\mathrm{Ad}_0 f^{(n-r)}_{\beta_p}(\bar f)\otimes B_p^{k-r+1}=$$

$$=\sum_{r=0}^{\min(n,k+1)}q_{\beta_p}^{(2n-r)(k+1)+\frac{r(r-1)}{2}-rn}\begin{bmatrix}k+1\\ r\end{bmatrix}_{q_{\beta_p}}\mathrm{Ad}_0 f^{(n-r)}_{\beta_p}(\bar f)\otimes B_p^{k+1-r},$$

where we used the identity

$$q^{r}_{\beta_p}\begin{bmatrix}k\\ r\end{bmatrix}_{q_{\beta_p}}+q_{\beta_p}^{r-(k+1)}\begin{bmatrix}k\\ r-1\end{bmatrix}_{q_{\beta_p}}=\begin{bmatrix}k+1\\ r\end{bmatrix}_{q_{\beta_p}}$$

which can be found e.g. in [55], Proposition 6.1. This establishes the induction step and completes the proof of (4.6.7).

Formula (4.6.8) is proved in a similar way by induction using (4.2.4), (4.4.2), the definition of B_p, (4.4.1) and Lemmas 4.4.3 and 4.5.3.

□

The previous lemma shows that in an appropriate quotient the operator $\mathrm{Ad}_0 f^{(n)}_{\beta_p}$ acts on $f\otimes B_p^k$ as a quantum analogue of a derivation of order n such that the derivative of B_p by $\mathrm{Ad}_0 f_{\beta_p}$ is equal to one. Recalling the definition of the classical counterpart G_p of B_p in (3.5.10) one can observe that similar formula can be obtained in the classical case as well. It will be given by specializing (4.6.7) at $q^{\frac{1}{dr^2}}=1$, and the $\mathrm{Ad}_0 f^{(n)}_{\beta_p}$–action will be replaced with $\frac{1}{n!}L_p^n$.

In the next lemma we show that for $n>0$ the composition $\mathrm{Ad}_0 f^{(n)}_{\beta_p}P_p$ is the zero operator modulo $\overline{I}^{p\,loc}_{\mathcal{B}}$ (or $I^{p\,loc}_{\mathcal{B}}$). At the same time this lemma lays a basis for the proof of the fact that the kernel of $P_{\leq p}$ contains $\overline{I}^{p+1\,loc}_{\mathcal{B}}$ (or $I^{p+1\,loc}_{\mathcal{B}}$). For $k=0$ formula (4.6.11) is a quantum analogue of the statement in the line before (3.5.8).

Lemma 4.6.3. *For any $f\in\mathbb{C}^{loc}_{p}[G]$, $p=1,\ldots,c$, $k=0,1,\ldots$ we have*

$$P_p^k(f\otimes B_p)=0, \tag{4.6.11}$$

$$\mathrm{Ad}_0 f^{(n)}_{\beta_p}(P_p(f))=0,\ n>0. \tag{4.6.12}$$

Proof. Since by Lemma 4.5.3 right multiplication by B_p gives rise to an action of B_p on $\mathbb{C}_p^{loc}[G]$ and on the image of P_p^k we have by (4.6.7) in the image of P_p^k

$$P_p^k(f\otimes B_p)=\sum_{n=k+1}^{\infty}(-1)^n q_{\beta_p}^{\frac{(n-1)(n-2k)}{2}}q_{\beta_p}^{n-k-1}\mathrm{Ad}_0 f_{\beta_p}^{(n-k-1)}(f)\otimes B_p^n+$$

$$+\sum_{n=k}^{\infty}(-1)^n q_{\beta_p}^{\frac{(n-1)(n-2k)}{2}}q_{\beta_p}^{2(n-k)}\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(f)\otimes B_p^{n+1}=$$

$$=-\sum_{n=k}^{\infty}(-1)^n q_{\beta_p}^{\frac{n(n+1-2k)}{2}+n-k}\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(f)\otimes B_p^{n+1}+$$

$$+\sum_{n=k}^{\infty}(-1)^n q_{\beta_p}^{\frac{(n-1)(n-2k)}{2}+2n-2k}\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(f)\otimes B_p^{n+1}=0$$

which proves (4.6.11).

Similarly by (4.6.7) we obtain in in the image of P_p^k

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(P_p(f))=\mathrm{Ad}_0 f_{\beta_p}^{(n)}\left(\sum_{k=0}^{\infty}(-1)^k q_{\beta_p}^{\frac{(k-1)k}{2}}\mathrm{Ad}_0 f_{\beta_p}^{(k)}(f)\otimes B_p^k\right)=$$

$$=\sum_{k=0}^{\infty}\sum_{t=0}^{\min(n,k)}(-1)^k q_{\beta_p}^{\frac{(k-1)k}{2}+(2n-t)k+\frac{t(t-1)}{2}-tn}\left[\begin{array}{c}k\\t\end{array}\right]_{q_{\beta_p}}\mathrm{Ad}_0(f_{\beta_p}^{(k)}f_{\beta_p}^{(n-t)})(f)\otimes B_p^{k-t}.$$

Introducing a new variable of summation $k-t=r$ and using the identity

$$f_{\beta_p}^{(k)}f_{\beta_p}^{(n-t)}=\left[\begin{array}{c}n+k-t\\k\end{array}\right]_{q_{\beta_p}}f_{\beta_p}^{(n+k-t)}$$

we get

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(P_p(f))=$$

$$=\sum_{r=0}^{\infty}\sum_{t=0}^{n}(-1)^{r+t}q_{\beta_p}^{\frac{(r+t-1)(r+t)}{2}+(2n-t)(r+t)+\frac{t(t-1)}{2}-tn}\left[\begin{array}{c}r+t\\t\end{array}\right]_{q_{\beta_p}}\left[\begin{array}{c}r+n\\r+t\end{array}\right]_{q_{\beta_p}}\mathrm{Ad}_0(f_{\beta_p}^{(n+r)})(f)\otimes B_p^r.$$

Now recalling that

$$\left[\begin{array}{c}r+t\\t\end{array}\right]_{q_{\beta_p}}\left[\begin{array}{c}r+n\\r+t\end{array}\right]_{q_{\beta_p}}=\left[\begin{array}{c}n\\t\end{array}\right]_{q_{\beta_p}}\left[\begin{array}{c}r+n\\r\end{array}\right]_{q_{\beta_p}}$$

we obtain

$$\mathrm{Ad}_0 f_{\beta_p}^{(n)}(P_p(f))=$$

$$=\sum_{r=0}^{\infty}(-1)^r q_{\beta_p}^{\frac{(r-1)r}{2}+2nr}\left[\begin{array}{c}r+n\\r\end{array}\right]_{q_{\beta_p}}\left(\sum_{t=0}^{n}(-1)^t q_{\beta_p}^{-t+tn}\left[\begin{array}{c}n\\t\end{array}\right]_{q_{\beta_p}}\right)\mathrm{Ad}_0(f_{\beta_p}^{(n+r)})(f)\otimes B_p^r=0,$$

where we used the identity

$$\sum_{t=0}^{n}(-1)^t q_{\beta_p}^{-t+tn}\left[\begin{array}{c}n\\t\end{array}\right]_{q_{\beta_p}}=0$$

which follows from the q–binomial theorem (see e.g. [37], Ch. 1).

This proves (4.6.12).

□

As we observed above, the definition of the operators Π_p in (3.5.11), Proposition 3.5.2 and Corollary 3.5.4 imply that the kernel of the operator $\Pi_1\ldots\Pi_p:\mathbb{C}[N_-sZM_-^s]\to\mathbb{C}[N_-sZM_-^s]$ coincides with the vanishing ideal of $N_{[-\beta_{p+1},-\beta_D]}sZM_-^s\subset N_-sZM_-^s$. The following lemma is a quantum counterpart of this property.

Lemma 4.6.4. *Let $k \in \mathbb{N}$. If $\beta_p \not\in \Delta_{s^1}^s$ then $P_p^k(I_{\mathcal{B}}^{p+1^{loc}}/I_{\mathcal{B}}^{p\,loc}) = 0$, if $\beta_p \in \Delta_{s^1}^s$ then $P_p^k(\overline{I}_{\mathcal{B}}^{p+1^{loc}}/\overline{I}_{\mathcal{B}}^{p\,loc}) = 0$. Thus for $p \neq e-1$ P_p^k is well–defined as an operator with the domain $\mathbb{C}_{p+1}^{loc}[G]$ and P_{e-1}^k is well–defined as an operator with the domain $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$.*

Moreover, for any $p, q = 1, \ldots, c$ $q \leq p$ and any $k_q, \ldots, k_p \in \mathbb{N}$ the composition $P_q^{k_q} \ldots P_p^{k_p}$ is well–defined and gives rise to an operator with domain $\mathbb{C}_{p+1}^{loc}[G]$ if $p \neq e-1$ and $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$ if $p = e-1$, and the target space being the target space of $P_q^{k_q}$

In particular, the composition $P_{\leq p} = P_1 \ldots P_p$ is well–defined and gives rise to an operator with domain $\mathbb{C}_{p+1}^{loc}[G]$ if $p \neq e-1$ and $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{e\,loc}$ if $p = e-1$, and the target space being $\mathbb{C}_1^{loc}[G]$.

Proof. Let us show, for instance, that if $\beta_p \in \Delta_{s^1}^s$ then $P_p^k(\overline{I}_{\mathcal{B}}^{p+1^{loc}}/\overline{I}_{\mathcal{B}}^{p\,loc}) = 0$. The other case is considered in a similar way.

Indeed, by the definition of $\overline{I}_{\mathcal{B}}^{p+1}$ and by commutation relations (4.2.2) for any element $f \in \overline{I}_{\mathcal{B}}^{p+1^{loc}}$ there exists an element $u \in \mathcal{B}$, $u \neq 0$ and $g, h \in \overline{I}_{\mathcal{B}}^{p\,loc}$ such that $uf = g + h \otimes B_p$. Let $\bar{f}, \bar{g}$, $\overline{h \otimes B_p}$ and $\bar{h}$ be the classes of the corresponding elements in $\mathbb{C}_{\mathcal{B}}^{loc}[G]/\overline{I}_{\mathcal{B}}^{p\,loc}$. Note that $\overline{h \otimes B_p} = \bar{h} \otimes B_p$ and $\bar{g} = 0$, and by the previous Lemma $P_p^{k_p}(\overline{h \otimes B_p}) = P_p^{k_p}(\bar{h} \otimes B_p) = 0$. Thus $P_p^{k_p}(\overline{uf}) = uP_p^{k_p}(\bar{f}) = 0$. Since $u \neq 0$ this implies $P_p^{k_p}(\bar{f}) = 0$. This completes the proof in the considered case.

The remaining statements of the lemma are simple corollaries of the first assertion.

□

Next, we are going to study how the adjoint action of quantum root vectors commutes with the operators P_p. For this purpose we shall need some commutation relations between quantum root vectors stated in the following lemma.

Lemma 4.6.5. *For any $\alpha < \beta$, $\alpha, \beta \in \Delta_+$ and any $m, n \in \mathbb{N}$ we have*

$$f_\alpha^{(m)} f_\beta^{(n)} = c_{\alpha\beta}^{mn} \sum_{p=0}^{m} q_\alpha^{p(m-1)} \sum_{\alpha < \delta_1 < \ldots < \delta_n \leq \beta} d_{p_1,\ldots,p_n}^p f_{\delta_1}^{(p_1)} \ldots f_{\delta_n}^{(p_n)} f_\alpha^{(m-p)}, \tag{4.6.13}$$

where the coefficients $d_{p_1,\ldots,p_n}^p \in \mathcal{B}$ do not depend on m.

Proof. To prove this lemma it suffices to show that

$$f_\alpha^m f_\beta^n = c_{\alpha\beta}^{mn} \sum_{p=0}^{m} \sum_{\alpha < \delta_1 < \ldots < \delta_n \leq \beta} c_{p_1,\ldots,p_n}^p S_m^p f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-p}, \tag{4.6.14}$$

where

$$S_m^p = q_\alpha^{p(m-1)} \begin{bmatrix} m \\ p \end{bmatrix}_{q_\alpha},$$

and the coefficients $c_{p_1,\ldots,p_n}^p \in \mathcal{A}$ do not depend on m.

Indeed, dividing (4.6.14) by $[n]_{q_\alpha}![m]_{q_\beta}!$ we arrive at an identity of the form (4.6.13) where the coefficients $d_{p_1,\ldots,p_n}^p$ a priori belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$. But by the uniqueness of the Poincaré-Birkhoff-Witt decomposition in $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ (see Lemma 2.7.2) we have $d_{p_1,\ldots,p_n}^p \in \mathcal{B}$.

Now we establish (4.6.14). Firstly we consider the case $n = 1$. By commutation relations (2.7.11) we have

$$f_\alpha^m f_\beta = c_{\alpha\beta}^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} c_{\alpha\beta}^{m-k_1-1} f_\alpha^{k_1} \sum_{\alpha < \delta_1 < \ldots < \delta_n < \beta} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-k_1-1}, \tag{4.6.15}$$

where $C(p_1, \ldots, p_n) \in \mathcal{A}$. The first term in the right hand side of this formula agrees with the term in the right hand side of (4.6.14) corresponding to $p = 0$. The other terms in the right hand side of (4.6.15) will contribute to the terms in the right hand side of (4.6.14) with $p > 0$.

Denote

$$D_\beta f_\alpha = \sum_{\alpha < \delta_1 < \ldots < \delta_n < \beta} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n}.$$

Then (4.6.15) takes the form

$$f_\alpha^m f_\beta = c_{\alpha\beta}^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} c_{\alpha\beta}^{m-k_1-1} f_\alpha^{k_1} D_\beta f_\alpha f_\alpha^{m-k_1-1}. \quad (4.6.16)$$

To get the term in the right hand side of (4.6.14) corresponding to $p=1$ we have to move $f_\alpha^{k_1}$ to the right in (4.6.15) using commutation relations (2.7.11),

$$f_\alpha f_\delta = c_{\alpha,\delta} f_\delta f_\alpha + \sum_{\alpha<\delta_1<\ldots<\delta_n<\delta} C(p_1,\ldots,p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2}\ldots f_{\delta_n}^{p_n} = c_{\alpha,\delta} f_\delta f_\alpha + D_\delta f_\alpha,\ \alpha<\delta, \quad (4.6.17)$$

and keep the leading term $c_{\alpha,\delta} f_\delta f_\alpha$ for each $\delta=\delta_1,\ldots,\delta_n$. Since for weight reasons $\delta_1 p_1+\ldots+\delta_n p_n=\alpha+\beta$, so the weight of $D_\beta f_\alpha$ is $\alpha+\beta$, this yields

$$f_\alpha^m f_\beta = c_{\alpha\beta}^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} c_{\alpha\beta}^{m-k_1-1} c_{\alpha\alpha+\beta}^{k_1} D_\beta f_\alpha f_\alpha^{m-1} + R, \quad (4.6.18)$$

where R stands for the terms contributing to the terms in the right hand side of (4.6.14) with $p>1$. Now

$$\sum_{k_1=0}^{m-1} c_{\alpha\beta}^{m-k_1-1} c_{\alpha\alpha+\beta}^{k_1} = c_{\alpha\beta}^{m-1} \sum_{k_1=0}^{m-1} c_{\alpha\beta}^{-k_1} c_{\alpha\beta}^{k_1} c_{\alpha\alpha}^{k_1} = c_{\alpha\beta}^{m-1} \sum_{k_1=0}^{m-1} q_\alpha^{2k_1} = c_{\alpha\beta}^{m-1} \frac{q_\alpha^{2m}-1}{q_\alpha^2-1} = c_{\alpha\beta}^{m-1} q_\alpha^{m-1}[m]_{q_\alpha} = c_{\alpha\beta}^{m-1} S_m^1,$$

and (4.6.18) takes the form

$$f_\alpha^m f_\beta = c_{\alpha\beta}^m f_\beta f_\alpha^m + c_{\alpha\beta}^{m-1} S_m^1 \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(p_1,\ldots,p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2}\ldots f_{\delta_n}^{p_n} f_\alpha^{m-1} + R. \quad (4.6.19)$$

The second term in the right hand side of this formula agrees with the term in the right hand side of (4.6.14) corresponding to $p=1$.

To get the term in the right hand side of (4.6.14) corresponding to $p=2$ we have to move $f_\alpha^{k_1}$ to the right in (4.6.15) using commutation relations (2.7.11),

$$f_\alpha^{k_1} f_{\delta_l} = c_{\alpha,\delta}^k f_{\delta_l} f_\alpha^{k_1} + \sum_{k_2=0}^{k_1-1} c_{\alpha\beta}^{k_1-k_2-1} f_\alpha^{k_2} D_{\delta_l} f_\alpha f_\alpha^{k_1-k_2-1}, \quad (4.6.20)$$

and keep the terms containing one "differentiation" $D_{\delta_l} f_\alpha$ for some $l=1,\ldots,n$. Since the weight of $D_{\delta_l} f_\alpha$ is $\alpha+\delta_l$, this yields

$$R = \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1}\sum_{l=1}^{n}\sum_{q=1}^{p_l} c_{\alpha\beta}^{m-k_1-1} c_{\alpha\delta_1}^{k_1p_1}\ldots c_{\alpha\delta_{l-1}}^{k_1p_{l-1}} c_{\alpha\delta_l}^{k_1(p_l-q)} c_{\alpha\delta_l}^{k_1-k_2-1} c_{\alpha\alpha+\delta_l}^{k_2} c_{\alpha\delta_l}^{(k_1-1)(q-1)} c_{\alpha\delta_{l+1}}^{(k_1-1)p_{l+1}}\ldots c_{\alpha\delta_n}^{(k_1-1)p_n} \times \quad (4.6.21)$$

$$\times C(p_1,\ldots,p_n) f_{\delta_1}^{p_1}\ldots f_{\delta_{l-1}}^{p_{l-1}} f_{\delta_l}^{p_l-q} D_{\delta_l} f_\alpha f_{\delta_l}^{q-1} f_{\delta_{l+1}}^{p_{l+1}}\ldots f_{\delta_n}^{p_n} f_\alpha^{m-2} + R_1,$$

where R_1 contains only terms with more than one "differentiation" $D_{\delta_l} f_\alpha$ for some $l=1,\ldots,n$.

Now by the definition of the coefficients $c_{\alpha\beta}$ and using the identity $\delta_1 p_1+\ldots+\delta_n p_n=\alpha+\beta$ we have

$$c_{\alpha\beta}^{m-k_1-1} c_{\alpha\delta_1}^{k_1p_1}\ldots c_{\alpha\delta_{l-1}}^{k_1p_{l-1}} c_{\alpha\delta_l}^{k_1(p_l-q)} c_{\alpha\delta_l}^{k_1-k_2-1} c_{\alpha\alpha+\delta_l}^{k_2} c_{\alpha\delta_l}^{(k_1-1)(q-1)} c_{\alpha\delta_{l+1}}^{(k_1-1)p_{l+1}}\ldots c_{\alpha\delta_n}^{(k_1-1)p_n} =$$

$$= c_{\alpha\beta}^{m-1} c_{\alpha\alpha}^{k_1+k_2} c_{\alpha\delta_l}^{-q} c_{\alpha\delta_{l+1}}^{-p_{l+1}}\ldots c_{\alpha\delta_n}^{-p_n},$$

so

$$\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1} c_{\alpha\beta}^{m-k_1-1} c_{\alpha\delta_1}^{k_1p_1}\ldots c_{\alpha\delta_{l-1}}^{k_1p_{l-1}} c_{\alpha\delta_l}^{k_1(p_l-q)} c_{\alpha\delta_l}^{k_1-k_2-1} c_{\alpha\alpha+\delta_l}^{k_2} c_{\alpha\delta_l}^{(k_1-1)(q-1)} c_{\alpha\delta_{l+1}}^{(k_1-1)p_{l+1}}\ldots c_{\alpha\delta_n}^{(k_1-1)p_n} =$$

$$= c_{\alpha\beta}^{m-1} c_{\alpha\delta_l}^{-q} c_{\alpha\delta_{l+1}}^{-p_{l+1}}\ldots c_{\alpha\delta_n}^{-p_n} \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1} c_{\alpha\alpha}^{k_1+k_2} =$$

$$= c_{\alpha\beta}^{m-1} c_{\alpha\delta_l}^{-q} c_{\alpha\delta_{l+1}}^{-p_{l+1}} \ldots c_{\alpha\delta_n}^{-p_n} \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} q_\alpha^{2(k_1+k_2)},$$

and, after combining terms containing the same monomials $f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n}$, equation (4.6.21) takes the form

$$R = c_{\alpha\beta}^m \sum_{\alpha<\delta_1<\ldots<\delta_n\leq\beta} c_{p_1,\ldots,p_n}^2 S_m^2 f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-2} + R_1, \tag{4.6.22}$$

where

$$S_m^2 = \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} q_\alpha^{2(k_1+k_2)},$$

and the coefficients $c_{p_1,\ldots,p_n}^2 \in \mathcal{A}$ do not depend on m. The first term in (4.6.22) will agree with with the term in the right hand side of (4.6.14) corresponding to $p = 2$ if we show that $S_m^2 = q_\alpha^{2(m-1)} \left[\begin{array}{c} m \\ 2 \end{array} \right]_{q_\alpha}$.

Now we can continue in the same way taking into account more "derivatives" $D_{\delta_l} f_\alpha$ to obtain formula (4.6.14) with $n = 1$, where

$$S_m^p = \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_p=0}^{k_{p-1}-1} q_\alpha^{2(k_1+k_2+\ldots+k_p)}.$$

From this formula it follows that the coefficients S_m^p satisfy the following relations

$$S_m^p = S_{m-1}^p + q_\alpha^{2(m-1)} S_{m-1}^{p-1}. \tag{4.6.23}$$

On the other hand for the q-binomial coefficients we have

$$\left[\begin{array}{c} m \\ p \end{array} \right]_{q_\alpha} = q_\alpha^{-p} \left[\begin{array}{c} m-1 \\ p \end{array} \right]_{q_\alpha} + q_\alpha^{m-p} \left[\begin{array}{c} m-1 \\ p-1 \end{array} \right]_{q_\alpha}.$$

If we denote

$$C_m^p = q_\alpha^{p(m-1)} \left[\begin{array}{c} m \\ p \end{array} \right]_{q_\alpha}$$

this implies

$$C_m^p = C_{m-1}^p + q_\alpha^{2(m-1)} C_{m-1}^{p-1},$$

i.e. the coefficients C_m^p satisfy the same relations (4.6.23) as the coefficients S_m^p. Since these relations completely determine them once S_2^p are known and $C_2^0 = S_2^0 = 1$, $C_2^2 = S_2^2 = q_\alpha^2$, $C_2^1 = S_2^1 = q_\alpha [2]_{q_\alpha}$, we deduce that $C_m^p = S_m^p$ for all m and p (compare with Theorem 6.1 in [55]). Thus formula (4.6.14) is established for $n = 1$.

For $n > 1$ we argue by induction over n. The base of the induction is already established. Now assuming that the statement is true for $n - 1$ we show that it holds for n. Indeed,

$$f_\alpha^m f_\beta^n = c_{\alpha\beta}^{m(n-1)} \sum_{q=0}^{m} \sum_{\alpha<\delta_1<\ldots<\delta_n\leq\beta} c_{p_1,\ldots,p_n}^q S_m^q f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-q} f_\beta = \tag{4.6.24}$$

$$= c_{\alpha\beta}^{m(n-1)} \sum_{q=0}^{m} \sum_{k=0}^{m-q} \sum_{\alpha<\delta_1<\ldots<\delta_n\leq\beta} c_{\alpha\beta}^{m-p} c'_{p'_1,\ldots,p'_n} S_m^q S_{m-q}^k f_{\delta_1}^{p'_1} \ldots f_{\delta_n}^{p'_n} f_\alpha^{m-q-k},$$

where $c'_{p'_1,\ldots,p'_n} \in \mathcal{A}$ do not depend on m.

Now

$$S_m^q S_{m-q}^k = \frac{q_\alpha^{-kq}}{[k]_{q_\alpha}!} S_m^{k+q}.$$

Introducing new summation variable $p = k + q$ in (4.6.24) we can rewrite (4.6.24) in the form

$$f_\alpha^m f_\beta^n = c_{\alpha\beta}^{mn} \sum_{p=0}^{m} \sum_{q=0}^{p} \sum_{\alpha<\delta_1<\ldots<\delta_n\leq\beta} c_{\alpha\beta}^{-p} \frac{q_\alpha^{-(p-q)q}}{[p-q]_{q_\alpha}!} S_m^p c'_{p'_1,\ldots,p'_n} f_{\delta_1}^{p'_1} \ldots f_{\delta_n}^{p'_n} f_\alpha^{m-p}.$$

Combining terms containing the same monomials $f_{\delta_1}^{p_1'}f_{\delta_2}^{p_2'}\ldots f_{\delta_n}^{p_n'}$ we arrive at an identity of the form (4.6.14) where the coefficients $c^p_{p_1,\ldots,p_n}$ a priori belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$. But by the uniqueness of the Poincaré-Birkhoff-Witt decomposition in $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ (see Lemma 2.7.2) we have $c^p_{p_1,\ldots,p_n}\in\mathcal{A}$. This completes the proof of (4.6.14).

□

The next lemma shows how the adjoint action of the quantum root vectors commutes with the operators P_p.

Lemma 4.6.6. *For any $p,q=1,\ldots,c$, $q>p$ and any f from the domain of P_p we have*

$$\mathrm{Ad}_0 f_{\beta_q}^{(m)}(P_p(f))=\sum_{k=0}^{\infty}P_p^k(\mathrm{Ad}_0(\sum_{p_{p+1},\ldots p_q}d^k_{p_{p+1},\ldots,p_q}f_{\beta_{p+1}}^{(p_{p+1})}\ldots f_{\beta_q}^{(p_q)})(f)),\qquad(4.6.25)$$

where only a finite number of terms in the sum are non-zero, $d^k_{p_{p+1},\ldots,p_q}\in\mathcal{B}$,

$$d^k_{p_{p+1},\ldots,p_{q-1},m}=\begin{cases}1 & \text{if } \ k=p_{p+1}=\ldots=p_{q-1}=0\\ 0 & \text{otherwise}\end{cases},$$

and for $n<m$ and any k

$$d^k_{0,\ldots,0,n}=0.$$

In particular,

$$\mathrm{Ad}_0 f_{\beta_{p+1}}^{(m)}(P_p(f))=P_p(\mathrm{Ad}_0(f_{\beta_{p+1}}^{(m)})(f)).\qquad(4.6.26)$$

Proof. By formula (4.6.8) and by the definition of P_p we have

$$\mathrm{Ad}_0 f_{\beta_q}^{(m)}(P_p(f))=\sum_{n=0}^{\infty}(-1)^n q_{\beta_p}^{\frac{(n-1)n}{2}}d^{mn}_{\beta_p\beta_q}\mathrm{Ad}_0(f_{\beta_p}^{(n)}f_{\beta_q}^{(m)})(f)\otimes B_p^n.$$

Using (4.6.13) we can rewrite this formula as follows

$$\mathrm{Ad}_0 f_{\beta_q}^{(m)}(P_p(f))=$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n q_{\beta_p}^{\frac{(n-1)n}{2}}q_{\beta_p}^{-k(n-1)}c^{mn}_{\beta_p\beta_q}\omega_0(c^{mn}_{\beta_p\beta_q})\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(\mathrm{Ad}_0(\sum_{\beta_p<\delta_1<\ldots<\delta_n\leq\beta_q}d^k_{p_1,\ldots,p_n}f_{\delta_1}^{(p_1)}\ldots f_{\delta_n}^{(p_n)})(f)).$$

Now recalling that $\omega_0(q)=q^{-1}$ and swapping the order of summation in the last formula we get

$$\mathrm{Ad}_0 f_{\beta_q}^{(m)}(P_p(f))=$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n q_{\beta_p}^{\frac{(n-1)n}{2}}q_{\beta_p}^{-k(n-1)}\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(\mathrm{Ad}_0(\sum_{\beta_p<\delta_1<\ldots<\delta_n\leq\beta_q}d^k_{p_1,\ldots,p_n}f_{\delta_1}^{(p_1)}\ldots f_{\delta_n}^{(p_n)})(f))=$$

$$=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^n q_{\beta_p}^{\frac{(n-1)(n-2k)}{2}}\mathrm{Ad}_0 f_{\beta_p}^{(n-k)}(\mathrm{Ad}_0(\sum_{\beta_p<\delta_1<\ldots<\delta_n\leq\beta_q}d^k_{p_1,\ldots,p_n}f_{\delta_1}^{(p_1)}\ldots f_{\delta_n}^{(p_n)})(f))=$$

$$=\sum_{k=0}^{\infty}P_p^k(\mathrm{Ad}_0(\sum_{\beta_p<\delta_1<\ldots<\delta_n\leq\beta_q}d^k_{p_1,\ldots,p_n}f_{\delta_1}^{(p_1)}\ldots f_{\delta_n}^{(p_n)})(f)).$$

(4.6.26) is obtained in a similar way using the relation $f_{\beta_p}^{(n)}f_{\beta_{p+1}}^{(m)}=c^{mn}_{\beta_p\beta_{p+1}}f_{\beta_{p+1}}^{(m)}f_{\beta_p}^{(n)}$.

□

Now we have all prerequisites to prove Proposition 4.6.1.

Proof of Proposition 4.6.1 By Lemmas 4.6.4, 4.6.6, the first claim in Lemma 4.5.5 and formula (4.6.12) we have for any f from the domain of P_p

$$\mathrm{Ad}_0 f_{\beta_q}^{(n)}(P_{\leq p}(f))=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\mathrm{Ad}_0(\sum_{p_2,\ldots,p_q}d^{k_1}_{p_2,\ldots,p_n}f_{\beta_2}^{(p_2)}\ldots f_{\beta_q}^{(p_q)})(P_2\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_2,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{p_2,\ldots,p_n}f_{\beta_3}^{(p_3)}\ldots f_{\beta_q}^{(p_q)})\mathrm{Ad}_0f_{\beta_2}^{(p_2)}(P_2\ldots P_p(f))\right)=$$

$$\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_3,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,p_3,\ldots,p_n}f_{\beta_3}^{(p_3)}\ldots f_{\beta_q}^{(p_q)})(P_2\ldots P_p(f))\right),$$

where, by weight counting in the left hand side and in the right hand side, for each k_1 $d^{k_1}_{0,0\ldots,0}=0$.

Similarly,

$$\mathrm{Ad}_0f_{\beta_q}^{(n)}(P_{\leq p}(f))=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_3,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,p_3,\ldots,p_n}f_{\beta_3}^{(p_3)}\ldots f_{\beta_q}^{(p_q)})(P_2\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_3,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,p_3,\ldots,p_n}f_{\beta_4}^{(p_4)}\ldots f_{\beta_q}^{(p_q)})(P_2\mathrm{Ad}_0(f_{\beta_3}^{(p_3)})P_3\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_4,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,0,p_4,\ldots,p_n}f_{\beta_4}^{(p_4)}\ldots f_{\beta_q}^{(p_q)})(P_2P_3\ldots P_p(f))\right).$$

Now we proceed along the same line,

$$\mathrm{Ad}_0f_{\beta_q}^{(n)}(P_{\leq p}(f))=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_1^{k_1}\left(\sum_{p_4,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,0,p_4,\ldots,p_n}f_{\beta_5}^{(p_5)}\ldots f_{\beta_q}^{(p_q)})(P_2^{k_2}\sum_{p_3',p_4'}\mathrm{Ad}_0(d^{k_2}_{p_3',p_4'}f_{\beta_3}^{(p_3')}f_{\beta_4}^{(p_4')})P_3\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_1^{k_1}\left(\sum_{p_4,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,0,p_4,\ldots,p_n}f_{\beta_5}^{(p_5)}\ldots f_{\beta_q}^{(p_q)})(P_2^{k_2}\sum_{p_4'}\mathrm{Ad}_0(d^{k_2}_{0,p_4'}f_{\beta_4}^{(p_4')})P_3\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_1^{k_1}\left(\sum_{p_4,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,0,p_4,\ldots,p_n}f_{\beta_5}^{(p_5)}\ldots f_{\beta_q}^{(p_q)})(P_2^{k_2}\sum_{p_4'}P_3\mathrm{Ad}_0(d^{k_2}_{0,p_4'}f_{\beta_4}^{(p_4')})P_4\ldots P_p(f))\right)=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_1^{k_1}\left(\sum_{p_4,\ldots,p_q}\omega_0(d^{k_2}_{0,0})\mathrm{Ad}_0(d^{k_1}_{0,0,p_4,\ldots,p_n}f_{\beta_5}^{(p_5)}\ldots f_{\beta_q}^{(p_q)})(P_2^{k_2}P_3P_4\ldots P_p(f))\right).$$

But $d^{k_2}_{0,0}$ is not zero only if $p_4=0$ and we infer

$$\mathrm{Ad}_0f_{\beta_q}^{(n)}(P_{\leq p}(f))=$$

$$=\sum_{k_1=0}^{\infty}P_1^{k_1}\left(\sum_{p_5,\ldots,p_q}\mathrm{Ad}_0(d^{k_1}_{0,0,0,p_5,\ldots,p_n}f_{\beta_5}^{(p_5)}\ldots f_{\beta_q}^{(p_q)})(P_2P_3\ldots P_p(f))\right).$$

Repeating the same arguments we arrive at

$$\mathrm{Ad}_0f_{\beta_q}^{(n)}(P_{\leq p}(f))=\sum_{k_1=0}^{\infty}\omega_0(d^{k_1}_{0,0,\ldots,0})P_1^{k_1}P_2\ldots P_p(f)=0$$

as $d^{k_1}_{0,0,\ldots,0}=0$ for all k_1. This proves (4.6.2).

(4.6.3) is established in a similar way using (4.6.25), Lemma 4.6.4, the first claim in Lemma 4.5.5 and formula (4.6.12).

Formulas (4.6.4) and (4.6.5) follow from the definition of $P_{\leq p}$, Lemma 4.5.3, the first part of Lemma 4.5.5 and Lemma 4.6.4 .

□

In conclusion, using the operators P_p and the elements B_p we define proper quantum analogues of monomials in functions G_p introduced in (3.5.10) (they were called $t_i(x)$ in the introduction). They will play a crucial rule in establishing a quantum group version of the Skryabin equivalence in Section 5.2 and in the proof of the De–Concini–Kac–Procesi conjecture.

Proposition 4.6.7. *Denote the class of $B_c^{n_c}$ in the domain of definition of P_{c-1} by the same letter. Then for $n_1, \ldots, n_c, k_1, \ldots, k_c \in \mathbb{N}$ the elements*

$$B_{n_1 \ldots n_c} = P_1(P_2(\ldots P_{c-2}(P_{c-1}(B_c^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_2^{n_2}) \otimes B_1^{n_1} \in \mathbb{C}_1^{loc}[G]$$

satisfy

$$\mathrm{Ad}_0(f_{\beta_1}^{(k_1)} \ldots f_{\beta_c}^{(k_c)})(B_{n_1 \ldots n_c}) = \begin{cases} \prod_{p=1}^c q_{\beta_p}^{\frac{n_p(n_p-1)}{2}} & \text{if } \ n_p = k_p \ \text{ for } \ p = 1, \ldots, c \\ 0 \ \text{ if } \ k_i = n_i, i = 1, \ldots, p-1 & \text{and } \ k_p > n_p \ \text{ for some } \ p \in \{1, \ldots, c\} \end{cases},$$

and hence

$$\mathrm{Ad}_0(f_{\beta_1}^{k_1} \ldots f_{\beta_c}^{k_c})(B_{n_1 \ldots n_c}) = \begin{cases} \prod_{p=1}^c q_{\beta_p}^{\frac{n_p(n_p-1)}{2}} [n_p]_{q_{\beta_p}}! \ \text{ if } \ n_p = k_p \ \text{ for } \ p = 1, \ldots, d \\ 0 \ \text{ if } \ k_i = n_i, i = 1, \ldots, p-1 \ \text{ and } \ k_p > n_p \ \text{ for some } \ p \in \{1, \ldots, c\} \end{cases}.$$

In particular, the elements $B_{n_1 \ldots n_c}$ are linearly independent.

Proof. The proof follows from Lemmas 4.5.3, 4.5.5, 4.6.4, 4.6.6, formula (4.6.12) and Proposition 4.6.1. We shall prove the statement by induction.

First observe that by Proposition 4.6.1 $\mathrm{Ad}_0 f_{\beta_1}^{(k)}(P_1(P_2(\ldots P_{c-2}(P_{c-1}(B_d^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_2^{k_2})) = 0$ for any $k > 0$ and recall that right multiplication by B_1 gives rise to an operator on $\mathbb{C}_1^{loc}[G]$ by Lemma 4.5.3. Therefore from (4.6.7) for $p = 1$ we have

$$\mathrm{Ad}_0 f_{\beta_1}^{(k_1)}(B_{n_1 \ldots n_c}) =$$
$$= q_{\beta_1}^{\frac{k_1(k_1-1)}{2}} P_{\leq 1}(P_2(P_3(\ldots P_{c-2}(P_{c-1}(B_c^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_3^{n_3}) \otimes B_2^{n_2}).$$

if $k_1 = n_1$ and

$$\mathrm{Ad}_0 f_{\beta_1}^{(k_1)}(B_{n_1 \ldots n_c}) = 0$$

if $k_1 > n_1$.

Now assume that for some p

$$\mathrm{Ad}_0(f_{\beta_1}^{(k_1)} \ldots f_{\beta_p}^{(k_p)})(B_{k_1 \ldots k_p n_{p+1} \ldots n_c}) = \tag{4.6.27}$$
$$= \prod_{m=1}^p q_{\beta_m}^{\frac{k_m(k_m-1)}{2}} P_{\leq p}(P_{p+1}(P_{p+2}(\ldots P_{c-2}(P_{c-1}(B_d^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_{p+2}^{n_{p+2}}) \otimes B_{p+1}^{n_{p+1}}).$$

Then by (4.6.3) and by the induction assumption

$$\mathrm{Ad}_0(f_{\beta_1}^{(k_1)} \ldots f_{\beta_{p+1}}^{(k_{p+1})})(B_{k_1 \ldots k_p n_{p+1} \ldots n_c}) =$$
$$= \prod_{m=1}^p q_{\beta_m}^{\frac{k_m(k_m-1)}{2}} P_{\leq p} \mathrm{Ad}_0 f_{\beta_{p+1}}^{(k_{p+1})}(P_{p+1}(P_{p+2}(\ldots P_{c-2}(P_{c-1}(B_c^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_{p+2}^{n_{p+2}}) \otimes B_{p+1}^{n_{p+1}}).$$

The last formula can be simplified using (4.6.7), the fact that

$$\mathrm{Ad}_0 f_{\beta_{p+1}}^{(k)} P_{p+1}(P_{p+2}(\ldots P_{c-2}(P_{c-1}(B_c^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_{p+2}^{n_{p+2}}) = 0$$

for any $k > 0$, Lemma 4.6.4 and recalling that right multiplication by B_{p+1} gives rise to an operator on the image of P_{p+1} by Lemma 4.5.3. This yields

$$\mathrm{Ad}_0(f_{\beta_1}^{(k_1)} \ldots f_{\beta_{p+1}}^{(k_{p+1})})(B_{k_1 \ldots k_p n_{p+1} \ldots n_c}) =$$
$$= \prod_{m=1}^{p+1} q_{\beta_m}^{\frac{k_m(k_m-1)}{2}} P_{\leq p+1}(P_{p+2}(\ldots P_{c-2}(P_{c-1}(B_d^{n_c}) \otimes B_{c-1}^{n_{c-1}}) \ldots) \otimes B_{p+2}^{n_{p+2}})$$

if $k_{p+1} = n_{p+1}$ and

$$\mathrm{Ad}_0(f_{\beta_1}^{(k_1)} \ldots f_{\beta_{p+1}}^{(k_{p+1})})(B_{k_1 \ldots k_p n_{p+1} \ldots n_c}) = 0$$

if $k_{p+1} > n_{p+1}$. This establishes the induction step and completes the proof of the proposition.

□

4.7 A description of q-W–algebras in terms of Zhelobenko type operators

Now we are in a position to describe q-W–algebras in terms of the operator P introduced in the previous section. Recall that q-W–algebras are only defined when the value of the parameter κ is equal to one. Therefore in this section we always assume that $\kappa = 1$. As a $\mathcal{B}$–module the q-W–algebra $W_{\mathcal{B}}^s(G)$ is the space of $\mathbb{C}_{\mathcal{B}}[M_+]$–invariants in $Q_{\mathcal{B}}$ with respect to the adjoint action. In order to use the operator P for the description of this space we shall transfer the results of Proposition 4.6.1 from $\mathbb{C}_1^{loc}[G]$ to a localization $Q_{\mathcal{B}}^{loc}$ of $Q_{\mathcal{B}}$ using a natural extension of the $\mathbb{C}_{\mathcal{B}}[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}[G] \to Q_{\mathcal{B}}$ to a homomorphism $\mathbb{C}_{\mathcal{B}}^{loc}[G] \to Q_{\mathcal{B}}^{loc}$. Recall that according to Proposition 4.1.2 $\overline{I}_{\mathcal{B}}^1$ belongs to the kernel of the homomorphism ϕ, and as we shall see $\overline{I}_{\mathcal{B}}^{1\ loc}$ belongs to the kernel of the extension of ϕ to $\mathbb{C}_{\mathcal{B}}^{loc}[G]$. Therefore one can compose this extension with the operator P, and by Proposition 4.6.1 the image of this composition is invariant with respect to the natural extension of the $\mathbb{C}_{\mathcal{B}}[M_+]$–adjoint action to $Q_{\mathcal{B}}^{loc}$. The operator Π is a classical counterpart of this composition and using the description of Π given in Theorem 3.5.6 we shall show that the image of the composition is a localization of the algebra $W_{\mathcal{B}}^s(G)$.

More precisely, formula (4.1.10) and the surjectivity of the map ϕ imply that one can define a natural action of the algebra generated by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee} \in \mathbb{C}_{\mathcal{B}}^{loc}[G^*]$, $\lambda \in P_+$ on $Q_{\mathcal{B}}$ as follows

$$q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}\phi(f) = \varphi(\mathrm{Ad}_0(q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee})(f))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1 = \tag{4.7.1}$$

$$= \mathrm{Ad}(q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee})(\varphi(f))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1,$$

where the last identity follows from Proposition 3.2.5. Let $Q_{\mathcal{B}}^{loc}$ be the localization of $Q_{\mathcal{B}}$ by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}$, $\lambda \in P_+$.

Now consider the subalgebra $\mathbb{C}_{\mathcal{B}}^{loc}[G_*] \subset \mathbb{C}_{\mathcal{B}}[G^*]$ generated by $\mathbb{C}_{\mathcal{B}}[G_*]$ and by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}$, $\lambda \in P$. Note that the adjoint action of these elements normalizes $\mathbb{C}_{\mathcal{B}}[G_*]$ in $\mathbb{C}_{\mathcal{B}}[G^*]$ as $\mathbb{C}_{\mathcal{B}}[G_*]$ is the direct sum of its weight components. Therefore $Q_{\mathcal{B}}^{loc}$ is the image of $\mathbb{C}_{\mathcal{B}}^{loc}[G_*]$ under the natural projection $\rho_{\chi_q^s} : \mathbb{C}_{\mathcal{B}}[G^*] \to \mathbb{C}_{\mathcal{B}}[G^*]/I_{\mathcal{B}}$, and hence the adjoint action of $\mathbb{C}_{\mathcal{B}}[M_+]$ on $Q_{\mathcal{B}}$ naturally extends to $Q_{\mathcal{B}}^{loc}$.

Lemma 4.7.1. *Let $\kappa = 1$. Then ϕ extends to a $\mathbb{C}_{\mathcal{B}}[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}^{loc}[G] \to Q_{\mathcal{B}}^{loc}$,*

$$\phi(f \otimes \Delta_\lambda^{-1}) = q^{2(P_{\mathfrak{h}'}\lambda^\vee,\lambda^\vee)-2P_{\mathfrak{h}'^\perp}\lambda^\vee}\phi(\mathrm{Ad}_0(q^{(\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}-s^{-1})\lambda^\vee})(f)), f \in \mathbb{C}_{\mathcal{B}}[G], \tag{4.7.2}$$

and $\overline{I}_{\mathcal{B}}^{1\ loc}$ belongs to the kernel of this homomorphism, so

$$\phi : \mathbb{C}_1^{loc}[G] \to Q_{\mathcal{B}}^{loc}.$$

Proof. From formula (4.1.10) it follows that ϕ extends to a $\mathbb{C}_{\mathcal{B}}[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}^{loc}[G] \to Q_{\mathcal{B}}^{loc}$ which is defined by (4.7.2), and by Proposition 4.1.2 $\overline{I}_{\mathcal{B}}^{1\ loc}$ belongs to the kernel of this homomorphism.

□

By (4.5.10) for $\lambda \in P_+$ Δ_λ is $\mathbb{C}_{\mathcal{B}}[M_+]$–invariant with respect to the Ad_0–action on $\mathbb{C}_1^{loc}[G]$, and hence $\phi(\Delta_\lambda) = q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1$ is $\mathbb{C}_{\mathcal{B}}[M_+]$–invariant with respect to the Ad–action on $Q_{\mathcal{B}}^{loc}$. Thus by (4.1.10) and (3.2.24) we have for $\beta \in \Delta_{\mathfrak{m}_+}$

$$\phi(\mathrm{Ad}_0(\tilde{f}_\beta)(f \otimes \Delta_\lambda)) = \mathrm{Ad}(\tilde{f}_\beta)(\varphi(\mathrm{Ad}_0(q^{-(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f)))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1,$$

and

$$\phi(\mathrm{Ad}_0(\tilde{f}_\beta)(f \otimes \Delta_\lambda^{-1})) = \mathrm{Ad}(\tilde{f}_\beta)(\varphi(\mathrm{Ad}_0(q^{(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f)))q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee}1.$$

The last formula completely determines the adjoint action of $\mathbb{C}_{\mathcal{B}}[M_+]$ on $Q_{\mathcal{B}}^{loc}$.

Now we can describe q-W–algebras in terms of the operator P.

Theorem 4.7.2. *Suppose that $\kappa = 1$ and for $i = 1, \ldots l'$ $k_i \in \mathcal{B}$ are defined in (3.2.11). Then the composition ϕP gives rise to a well–defined operator*

$$\Pi^q = \phi P : \mathbb{C}_{c+1}^{loc}[G] \to W_{\mathcal{B}}^{s,loc}(G), P = P_1 \ldots P_c \tag{4.7.3}$$

where $W_{\mathcal{B}}^{s,loc}(G) = \mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}[M_+]}(\mathbb{C}_{\mathcal{B}}, Q_{\mathcal{B}}^{loc})$.

Moreover, we have an imbedding of $\mathbb{C}_\mathcal{B}[M_+]$–modules, $Q_\mathcal{B}^{loc} \subset \mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$, multiplication in $\mathbb{C}_\mathcal{B}[G^*]$ induces a multiplication in $W_\mathcal{B}^{s,loc}(G)$ and we have an imbedding of algebras $W_\mathcal{B}^{s}(G) \subset W_\mathcal{B}^{s,loc}(G)$.

Assume furthermore that $k_i \neq 0$ mod $(q^{\frac{1}{dr^2}}-1)$ for $i=1,\ldots l'$. Let $W_q^{s,loc}(G) = W_\mathcal{B}^{s,loc}(G) \otimes_\mathcal{B} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\mathrm{Im}\Pi^q$ the image of Π^q and denote $\mathrm{Im}_q\Pi^q = \mathrm{Im}\Pi^q \otimes_\mathcal{B} \mathbb{C}(q^{\frac{1}{dr^2}})$.

Then $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi^q$ and $W_\mathcal{B}^{s,loc}(G) = W_\mathcal{B}^{s,loc}(G) \cap \mathrm{Im}_q\Pi^q$.

Proof. By Lemma 4.7.1 and by Proposition 4.6.1 the composition $\Pi^q = \phi P : \mathbb{C}_c^{loc}[G] \to W_\mathcal{B}^{s,loc}(G)$ is well–defined. All other claims of this theorem, except for the isomorphisms $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi^q$ and $W_\mathcal{B}^{s,loc}(G) = W_\mathcal{B}^{s,loc}(G) \cap \mathrm{Im}_q\Pi^q$ follow from the definitions and are established similarly to Proposition 3.2.8.

Using Lemma 4.3.1 and arguments similar to those in the last part of the proof of Lemma 4.3.2 one sees that in order to establish these isomorphisms it suffices to verify that the specialization of Π^q at $q^{\frac{1}{dr^2}}=1$ is surjective.

In order to do that we observe that by the definition and by Theorem 3.4.5 the specialization of $Q_\mathcal{B}^{loc}$ at $q^{\frac{1}{dr^2}}=1$ is isomorphic to the localization $\mathbb{C}^{loc}[N_-ZsM_-]$ of the algebra $\mathbb{C}[N_-ZsM_-]$ by the classical counterparts of the elements Δ_λ which we denote by the same symbols, $\Delta_\lambda = (v_\lambda, \cdot s^{-1}v_\lambda)$.

By (4.5.10) for $\lambda \in P_+$ Δ_λ regarded as elements of $\mathbb{C}_1^{loc}[G]$ are $\mathbb{C}_\mathcal{B}[M_+]$–invariant with respect to the Ad_0–action on $\mathbb{C}_1^{loc}[G]$, and hence $\phi(\Delta_\lambda) = q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1$ is $\mathbb{C}_\mathcal{B}[M_+]$–invariant with respect to the Ad–action on $Q_\mathcal{B}^{loc}$. Therefore their classical counterparts $\Delta_\lambda = (v_\lambda, \cdot s^{-1}v_\lambda) \in \mathbb{C}[N_-ZsM_-]$ are M_-–invariant, and hence $\mathbb{C}^{loc}[N_-ZsM_-]^{M_-}$ is the localization of $\mathbb{C}[N_-ZsM_-]^{M_-}$ by the elements Δ_λ, $\lambda \in P_+$.

This result and explicit formulas (3.5.11), (3.5.12), (3.5.6) for the operator Π, formulas (4.6.1) with $k=0$ and the definition of the operator P imply that the specialization of the operator Π^q at $q^{\frac{1}{dr^2}}=1$ gives rise to a natural extension of the projection operator $\Pi : \mathbb{C}[N_-ZsM_-] \to \mathbb{C}[N_-ZsM_-]^{M_-}$ to a projection operator $\Pi^{loc} : \mathbb{C}^{loc}[N_-ZsM_-] \to \mathbb{C}^{loc}[N_-ZsM_-]^{M_-}$ given by

$$\Pi^{loc}(f\Delta_\lambda^{-1}) = \Pi(f)\Delta_\lambda^{-1}, f \in \mathbb{C}[N_-ZsM_-].$$

Since the operator Π is surjective, Π^{loc} is also surjective.

From the surjectivity of the specialization of the operator Π^q at $q^{\frac{1}{dr^2}}=1$ proved above it follows that $W_\mathcal{B}^{s,loc}(G) = \mathrm{Im}\Pi^q$ mod $(q^{\frac{1}{dr^2}}-1)W_\mathcal{B}^{s,loc}(G)$. Note also that $\mathrm{Im}\Pi^q \subset W_\mathcal{B}^{s,loc}(G)$ are submodules of the $\mathcal{B}$–module $Q_\mathcal{B}^{loc}$, and $Q_\mathcal{B}^{loc}$ is a $\mathcal{B}$–submodule of the $\mathcal{B}$–module $\mathbb{C}_\mathcal{B}[G^*]/I_\mathcal{B}$ which is free over $\mathcal{B}$ by Proposition 3.2.6. Since $\mathcal{B}$ is a principal ideal domain $Q_\mathcal{B}^{loc}$ is $\mathcal{B}$–free by Theorem 6.5 in [87].

The properties mentioned in the previous paragraph and Lemma 4.3.1 imply that $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi^q$ and $W_\mathcal{B}^{s,loc}(G) = W_\mathcal{B}^{s,loc}(G) \cap \mathrm{Im}_q\Pi^q$. This completes the proof.

□

4.8 Bibliographic comments

The results presented in this chapter are entirely new.

Commutation relations in the algebra $\mathbb{C}_\mathcal{B}[G]$ which appear in Section 4.2 can be found, for instance, in [10], Theorem I.8.16.

The definition of the Zhelobenko type operators for q-W–algebras was inspired by the construction of extremal projection operators and of the Zhelobenko operators due to Zhelobenko. The definitions and the statements in this chapter are conceptually close to the definition and the properties of the Zhelobenko operators introduced and studied in [115]–[123] (see also [57]). Below, for the convenience of the reader who is familiar with these papers, we give references to similar statements from them. However, the results of [115]–[123] and [57] are not used in this book and not directly related to it.

For $k=0$ the operators P_p are counterparts of the Zhelobenko operators q_α introduced in [115], §2 and §5, in [123], Definition 5.2.1, and for $k>0$ the operators P_p^k are counterparts of the operators $q_{\alpha,\mathfrak{m}}^{(k)}$ defined in [57], formula (4.9).

Properties of the Zhelobenko operators similar to those of the operators P_p mentioned in Proposition 4.6.1 can be found in [115], §5, Proposition 1 (iii) and (iv), [123], Proposition 5.2.4 (b) and (c), [57], Lemma 4.5 (ii), (iv) and (v).

Zhelobenko operators q_w conceptually analogues to $P_{\leq p}$ were defined in [115], §5, Definition 1, [123], Definition 5.2.4.

Properties (4.6.11) and (4.6.12) are counterparts of Proposition 3, parts (i) and (iii) in [115], §2, and of properties (α) and (β) in Section 5.2.4 in [123].

Lemma 4.6.4 is analogous to a similar property for the Zhelobenko operators stated in [123], Proposition and Corollary 5.2.3, and in [57], Lemmas 4.3 and 4.5 (iii).

Formulas similar to (4.6.25) are used in the proof of Proposition 5.2.4 in [123] and in Proposition 4.4 in [57].

Theorem 4.7.2 is analogous to Theorem 2 in [115], §6 and to Theorem 5.5.1 in [123] for the Zhelobenko operators.

Chapter 5

Application of q-W–algebras to the description of the category of equivariant modules over a quantum group

In this rather short chapter we apply Proposition 4.6.7 to establish an equivalence between the category of finitely generated representations of a q-W–algebra and a category $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ of equivariant modules over a quantum group. Categories of this kind were denoted $A - \mathrm{mod}_B^\chi$ in the introduction. The structure of modules from the category $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ is similar to that of $\mathfrak{g} - K$–modules or of principal series representations over complex semisimple Lie algebras.

The proof of the main theorem of this chapter, Theorem 5.2.1, is based on Proposition 4.6.7. In this framework one can give precise values of ε for which the categorical equivalence holds. Remarkably, with slight modifications this method is also applicable to the study of the structure of finite–dimensional representations over quantum groups at roots of unity. This will be done in the next chapter.

5.1 A category of equivariant modules over a quantum group

In this section we define a category of equivariant representations over a quantum group.

Suppose that $\kappa = 1$ and let $\varepsilon \in \mathbb{C}$. Fix a root $\varepsilon^{\frac{1}{dr^2}}$ of ε of order $\frac{1}{dr^2}$. Let $U_\varepsilon^s(\mathfrak{g})$, $U_\varepsilon^s(\mathfrak{m}_-)$, $\mathbb{C}_\varepsilon^{loc}[G_*]$, $\mathbb{C}_\varepsilon[M_+]$, $\mathbb{C}_\varepsilon[B_+]$, $\mathbb{C}_\varepsilon[G^*]$, Q_ε^{loc}, $W_\varepsilon^{s,loc}(G)$, χ_ε^s, $\mathbb{C}_{\varepsilon_s}$, I_ε, ϕ_ε, $B_{n_1\ldots n_c}^\varepsilon$, $\mathbb{C}_1^{loc}[G]_\varepsilon$ be the natural specializations at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ of $U_\mathcal{A}^s(\mathfrak{g})$, $U_\mathcal{A}^s(\mathfrak{m}_-)$, $\mathbb{C}_\mathcal{B}^{loc}[G_*]$, $\mathbb{C}_\mathcal{B}[M_+]$, $\mathbb{C}_\mathcal{B}[B_+]$, $\mathbb{C}_\mathcal{B}[G^*]$, $Q_\mathcal{B}^{loc}$, $W_\mathcal{B}^{s,loc}(G)$, χ_q^s, $\mathcal{B}_{\varepsilon_s}$, $I_\mathcal{B}$, ϕ, $B_{n_1\ldots n_c}$, $\mathbb{C}_1^{loc}[G]$, respectively.

We shall always assume that $[n]_{\varepsilon_{\alpha_i}} \neq 0$ and $\varepsilon^{2d_i} \neq 1$ for $i = 1, \ldots, l$, $n \in \mathbb{N}$. Then $\mathbb{C}_\varepsilon[M_+] = U_\varepsilon^s(\mathfrak{m}_-) = U_\varepsilon^{s,res}(\mathfrak{m}_-)$, $U_\varepsilon^s(\mathfrak{g}) = U_\varepsilon^{s,res}(\mathfrak{g})$ and $\mathbb{C}_\varepsilon[G^*]$ is a subalgebra in $U_\varepsilon^{s,res}(\mathfrak{g})$ as $\mathbb{C}_\mathcal{B}[G^*]$ is a subalgebra in $U_\mathcal{B}^{s,res}(\mathfrak{g})$.

Let $J = \mathrm{Ker}\ \varepsilon_s|_{\mathbb{C}_\varepsilon[M_+]}$ be the augmentation ideal of $\mathbb{C}_\varepsilon[M_+]$ related to the counit ε_s, and $\mathbb{C}_{\varepsilon_s}$ the trivial representation of $\mathbb{C}_\varepsilon[M_+]$ given by the counit. Let V be a finitely generated $\mathbb{C}_\varepsilon^{loc}[G_*]$–module which satisfies the following conditions:

1. V is a right $\mathbb{C}_\varepsilon[M_+]$–module with respect to an action Ad such that the action of the augmentation ideal J on V is locally nilpotent.

2. The following compatibility condition holds for the two actions

$$\mathrm{Ad}x(yv) = \mathrm{Ad}x^1(y)\mathrm{Ad}x^2(v),\ x \in \mathbb{C}_\varepsilon[M_+],\ y \in \mathbb{C}_\varepsilon^{loc}[G_*],\ v \in V, \tag{5.1.1}$$

where $\Delta_s(x) = x^1 \otimes x^2$, $\mathrm{Ad}x^1(y)$ is the adjoint action of $x^1 \in \mathbb{C}_\varepsilon[B_+]$ on $y \in \mathbb{C}_\varepsilon^{loc}[G_*]$.

An element $v \in V$ is called a Whittaker vector if $\mathrm{Ad}xv = \varepsilon_s(x)v$ for any $x \in \mathbb{C}_\varepsilon[M_+]$. The space

$$\mathrm{Hom}_{\mathbb{C}_\varepsilon[M_+]}(\mathbb{C}_{\varepsilon_s}, V) = \mathrm{Wh}(V). \tag{5.1.2}$$

is called the space of Whittaker vectors of V.

Consider the induced $\mathbb{C}_\varepsilon[G^*]$–module $W = \mathbb{C}_\varepsilon[G^*] \otimes_{\mathbb{C}_\varepsilon^{loc}[G_*]} V$. Using the adjoint action of $\mathbb{C}_\varepsilon[G^*]$ on itself one can naturally extend the adjoint action of $\mathbb{C}_\varepsilon[M_+]$ from V to W in such a way that compatibility condition (5.1.1) is satisfied for the natural action of $\mathbb{C}_\varepsilon[G^*]$ and the adjoint action Ad of $\mathbb{C}_\varepsilon[M_+]$ on W. As we observed in Section 3.2 $\Delta_s(\mathbb{C}_\varepsilon[M_+]) \subset \mathbb{C}_\varepsilon[B_+] \otimes \mathbb{C}_\varepsilon[M_+]$.

We shall require that

3. For any $x \in \mathbb{C}_\varepsilon[M_+]$ the natural action of the element $(S_s \otimes \chi_\varepsilon^s)\Delta_s(x) \in \mathbb{C}_\varepsilon[G^*]$ on W coincides with the adjoint action Adx of x on W.

   As in the second part of the proof of Proposition 3.2.8 one can check that the last condition implies that for any $z \in \mathbb{C}_\varepsilon^{loc}[G_*] \cap I_\varepsilon$ and $v \in \mathrm{Wh}(V)$ $zv = 0$.

Denote by $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ the category of finitely generated $\mathbb{C}_\varepsilon^{loc}[G_*]$–modules which satisfy conditions (1)–(3) above. Morphisms in the category $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ are $\mathbb{C}_\varepsilon^{loc}[G_*]$- and $\mathbb{C}_\varepsilon[M_+]$–module homomorphisms. We call $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ the category of $(\mathbb{C}_\varepsilon[M_+], \chi_\varepsilon^s)$–equivariant modules over $\mathbb{C}_\varepsilon^{loc}[G_*]$.

Note that the algebra $W_{\mathcal{B}}^{s,loc}(G)$ naturally acts in the space of Whittaker vectors for any object V of the category $\mathbb{C}_\varepsilon^{loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$. Indeed, if $w, w' \in \mathbb{C}_\varepsilon^{loc}[G_*]$ are two representatives of an element from $W_{\mathcal{B}}^{s,loc}(G)$ then $w - w' \in \mathbb{C}_\varepsilon^{loc}[G_*] \cap I_\varepsilon$, and hence for any $v \in \mathrm{Wh}(V)$ $wv = w'v$. Moreover, by the definition of the algebra $W_{\mathcal{B}}^{s,loc}(G)$ and by condition (5.1.1) we have

$$\mathrm{Ad}x(wv) = \mathrm{Ad}x^1(w)\mathrm{Ad}x^2(v) = \mathrm{Ad}x^1(w)\varepsilon_s(x^2)v = \mathrm{Ad}x(w)v = \varepsilon_s(x)wv.$$

Therefore wv is a Whittaker vector independent of the choice of the representative w.

For any $\mathbb{C}$–module R we denote by $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+]), R)$ the subspace in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R)$ which consists of the linear maps vanishing on some power of the augmentation ideal $J = \mathrm{Ker}\ \varepsilon_s$ of $\mathbb{C}_\varepsilon[M_+]$, $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R) = \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R) : f(J^n) = 0 \text{ for some } n > 0\}$. Note that for every element f of $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R)$ one has $f(x) = 0$ if x does not belong to a finite–dimensional subspace of $\mathbb{C}_\varepsilon[M_+]$, and hence $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R) = \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathbb{C}) \otimes R$.

Equip the space $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R)$ with the right action of $\mathbb{C}_\varepsilon[M_+]$ induced by the multiplication in $\mathbb{C}_\varepsilon[M_+]$ from the left. To study the properties of this module we shall need a special filtration on the algebra $\mathbb{C}_\varepsilon[M_+] = U_\varepsilon^s(\mathfrak{m}_-)$.

Recall that the algebra $U_\varepsilon^s(\mathfrak{g})$ can be equipped with the DeConcini–Kac filtration such that the associated graded algebra is almost commutative. For $\mathbf{r},\ \mathbf{t} \in \mathbb{N}^D$ define the height of the element $u_{\mathbf{r},\mathbf{t},t} = e^{\mathbf{r}} t f^{\mathbf{t}}$, $t \in U_\varepsilon^s(\mathfrak{h})$ as follows ht $(u_{\mathbf{r},\mathbf{t},t}) = \sum_{i=1}^D (t_i + r_i)\mathrm{ht}\ \beta_i \in \mathbb{N}$, where ht β_i is the height of the root β_i. Introduce also the degree of $u_{\mathbf{r},\mathbf{t},t}$ by

$$d(u_{\mathbf{r},\mathbf{t},t}) = (r_1, \ldots, r_D, t_D, \ldots, t_1, \mathrm{ht}\ (u_{\mathbf{r},\mathbf{t},t})) \in \mathbb{N}^{2D+1}.$$

By Lemma 2.7.2 the elements $e^{\mathbf{r}} t f^{\mathbf{t}}$ span $U_\varepsilon^s(\mathfrak{g})$ as a linear space.

Equip $\mathbb{N}^{2D+1}$ with the total lexicographic order and denote by $(U_q^s(\mathfrak{g}))_k$ the span of elements $u_{\mathbf{r},\mathbf{t},t}$ with $d(u_{\mathbf{r},\mathbf{t},t}) \leq k$ in $U_q^s(\mathfrak{g})$. Then Proposition 1.7 in [20] implies that $(U_\varepsilon^s(\mathfrak{g}))_k$ is a filtration of $U_\varepsilon^s(\mathfrak{g})$ such that the associated graded algebra is the complex associative algebra with generators $e_\alpha,\ f_\alpha,\ \alpha \in \Delta_+,\ t_i^{\pm 1},\ i = 1, \ldots l$ subject to the relations

$$\begin{aligned}
&t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i e_\alpha t_i^{-1} = \varepsilon^{\frac{H_i(\alpha)}{d}} e_\alpha, \quad t_i f_\alpha t_i^{-1} = \varepsilon^{-\frac{H_i(\alpha)}{d}} f_j,\\
&e_\alpha f_\beta = \varepsilon^{(\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} f_\beta e_\alpha,\\
&e_\alpha e_\beta = \varepsilon^{(\alpha,\beta) + (\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} e_\beta e_\alpha, \ \alpha < \beta,\\
&f_\alpha f_\beta = \varepsilon^{(\alpha,\beta) + (\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} f_\beta f_\alpha, \ \alpha < \beta.
\end{aligned} \tag{5.1.3}$$

Such algebras are called semi–commutative.

Lemma 5.1.1. *Let* $J = \mathrm{Ker}\ \varepsilon_s$ *be the augmentation ideal of* $\mathbb{C}_\varepsilon[M_+]$*,* R *a* $\mathbb{C}$*–module,* $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R) = \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R) : f(J^p) = 0 \text{ for some } p > 0\}$. *Equip* $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R)$ *with the right action of* $\mathbb{C}_\varepsilon[M_+]$ *induced by multiplication on* $\mathbb{C}_\varepsilon[M_+]$ *from the left. Then the* $\mathbb{C}_\varepsilon[M_+]$*–module* $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], R)$ *is injective.*

Proof. First observe that the algebra $\mathbb{C}_\varepsilon[M_+] \simeq U_\varepsilon^s(\mathfrak{m}_-)$ is Notherian and ideal J satisfies the so-called weak Artin–Rees property, i.e. for every finitely generated left $U_\varepsilon^s(\mathfrak{m}_-)$–module M and its submodule N there exists an integer $n>0$ such that $J^nM\cap N\subset JN$.

Indeed, observe that the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ can be equipped with a filtration induced by the De Concini–Kac filtration on the algebra $U_\varepsilon^s(\mathfrak{g})$ in such a way that the associated graded algebra is finitely generated and semi–commutative (see (5.1.3)). The fact that $U_\varepsilon^s(\mathfrak{m}_-)$ is Notherian follows from the existence of the filtration on it for which the associated graded algebra is semi–commutative and from Theorem 4 in Ch. 5, §3 in [50] (compare also with Theorem 4.8 in [77]). The ideal J satisfies the weak Artin–Rees property because the subring $U_\varepsilon^s(\mathfrak{m}_-)+Jt+J^2t^2+\ldots\subset U_\varepsilon^s(\mathfrak{m}_-)[t]$, where t is a central indeterminate, is Notherian (see [79], Ch. 11, §2, Lemma 2.1). The last fact follows from the existence of a filtration on $U_\varepsilon^s(\mathfrak{m}_-)+Jt+J^2t^2+\ldots$ induced by the filtration on $U_\varepsilon^s(\mathfrak{m}_-)$ for which the associated graded algebra is semi–commutative and again from Theorem 4 in Ch. 5, §3 in [50].

Finally, the module $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],R)$ is obviously injective. By Lemma 3.2 in Ch. 3, [44] the module $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],R)=\{f\in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],R): f(J^p)=0 \text{ for some } p>0\}$ is also injective since the ideal J satisfies the weak Artin–Rees property.

□

5.2 Skryabin equivalence for equivariant modules over a quantum group

Now we can formulate the main theorem on the structure of the category $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ and on the properties of its objects.

Theorem 5.2.1. *If $[n]_{\varepsilon_{\alpha_i}}\neq 0$ and $\varepsilon^{2d_i}\neq 1$ for $i=1,\ldots,l$, $n\in\mathbb{N}$ then the functor $E\mapsto Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}E$, is an equivalence of the category of finitely generated left $W_\varepsilon^{s,loc}(G)$–modules and of the category $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$. The inverse equivalence is given by the functor $V\mapsto \mathrm{Wh}(V)$. In particular, the latter functor is exact.*

Every module $V\in\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ is isomorphic to $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})\otimes\mathrm{Wh}(V)$ as a right $\mathbb{C}_\varepsilon[M_+]$–module, where $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})$ is equipped with the right action of $\mathbb{C}_\varepsilon[M_+]$ induced by the multiplication in $\mathbb{C}_\varepsilon[M_+]$ from the left. Q_ε^{loc} is isomorphic to $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})\otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule, where the right $W_\varepsilon^{s,loc}(G)$–action is induced by the multiplication in $W_\varepsilon^{s,loc}(G)$ from the right. In particular, V is $\mathbb{C}_\varepsilon[M_+]$–injective, and $\mathrm{Ext}^\bullet_{\mathbb{C}_\varepsilon[M_+]}(\mathbb{C}_{\varepsilon_s},V)=\mathrm{Wh}(V)$.

Proof. Let E be a finitely generated $W_\varepsilon^{s,loc}(G)$–module. First we observe that by the definition of the algebra $W_\varepsilon^{s,loc}(G)$ we have $\mathrm{Wh}(Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}E)=E$. Therefore to prove the theorem it suffices to check that for any $V\in\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ the canonical map $f: Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}\mathrm{Wh}(V)\to V$ is an isomorphism.

First we prove that Q_ε^{loc} is an object in $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$. We shall prove that the adjoint action of the augmentation ideal J of $\mathbb{C}_\varepsilon[M_+]$ on Q_ε^{loc} is locally nilpotent. All the other properties of objects of the category $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ for Q_ε^{loc} were already established in Proposition 3.2.9.

Indeed, recalling the $\mathbb{C}_\varepsilon[M_+]$–module homomorphism of $\phi_\varepsilon:\mathbb{C}_1^{loc}[G]_\varepsilon\to Q_\varepsilon^{loc}$ and the definition of the adjoint action on $\mathbb{C}_1^{loc}[G]_\varepsilon$ in formula (4.5.13) we deduce that in order to show that the adjoint action of the augmentation ideal J of $\mathbb{C}_\varepsilon[M_+]$ on Q_ε^{loc} is locally nilpotent it suffices to show that the Ad_0–action of the augmentation ideal J of $\mathbb{C}_\varepsilon[M_+]$ on $\mathbb{C}_\varepsilon[G]$ is locally nilpotent. But the last fact is true as $\mathbb{C}_\varepsilon[G]=\bigoplus_{\lambda\in P_+}V_\lambda^*\otimes V_\lambda$, where V_λ is the finite–dimensional irreducible representation of $U_\varepsilon^s(\mathfrak{g})$ of highest weight λ, and the action of $\mathbb{C}_\varepsilon[M_+]$ on $V_\lambda^*\otimes V_\lambda$ induced by the adjoint action is locally nilpotent since the action of $\mathbb{C}_\varepsilon[M_+]$ on finite–dimensional irreducible representations is locally nilpotent.

Now let V be an object in the category $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$. Fix any linear map $\rho: V\to\mathrm{Wh}(V)$ the restriction of which to $\mathrm{Wh}(V)$ is the identity map, and let for any $v\in V$ $\sigma_\varepsilon(v):\mathbb{C}_\varepsilon[M_+]\to\mathrm{Wh}(V)$ be the $\mathbb{C}$–linear homomorphism given by $\sigma_\varepsilon(v)(x)=\rho(\mathrm{Ad}x(v))$. Since the adjoint action of J on V is locally nilpotent $\sigma_\varepsilon(v)\in\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathrm{Wh}(V))$, and we have a map $\sigma_\varepsilon: V\to\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathrm{Wh}(V))\simeq\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})\otimes\mathrm{Wh}(V)$.

By definition σ_ε is a homomorphism of right $\mathbb{C}_\varepsilon[M_+]$–modules, where the right action of $\mathbb{C}_\varepsilon[M_+]$ on

$$\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathrm{Wh}(V))$$

is induced by multiplication in $\mathbb{C}_\varepsilon[M_+]$ from the left. We claim that σ_ε is an isomorphism.

First we prove that σ_ε is injective. The proof will be based on the following lemma that will be also used later.

Lemma 5.2.2. *Let $\phi : X \to Y$ be a homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–modules. Denote by* Wh$(X)$ *the subspace of Whittaker vectors of X, i.e. the subspace of X which consists of elements v such that $xv = \varepsilon_s(x)v$, $x \in U_\varepsilon^s(\mathfrak{m}_-)$. Assume that the action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_-)$ on X is locally nilpotent and that the restriction of ϕ to the subspace of Whittaker vectors of X is injective. Then ϕ is injective.*

Proof. Let $Z \subset X$ be the kernel of ϕ. Assume that Z is not trivial. Observe that Z is invariant with respect to the action induced by the action of $U_\varepsilon^s(\mathfrak{m}_-)$ on X, and that the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_-)$ acts on X by locally nilpotent transformations. Therefore by Engel theorem Z must contain a nonzero $U_\varepsilon^s(\mathfrak{m}_-)$–invariant vector which is a Whittaker vector $v \in X$. But since the restriction of ϕ to the subspace of Whittaker vectors of X is injective $\phi(v) \neq 0$. Thus we arrive at a contradiction, and hence ϕ is injective.

□

Now recall that the action of J on V is locally nilpotent. All non–zero Whittaker vectors in V belong to Wh(V) and by the definition of σ_ε their images in $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ are non–zero homomorphisms non-vanishing at 1. Therefore by Lemma 5.2.2 σ_ε is injective.

Next we show that σ_ε is also surjective. Denote $x_{n_1\ldots n_c} = \prod_{p=1}^c \varepsilon_{\beta_p}^{-\frac{n_p(n_p-1)}{2}} \frac{1}{[n_p]_{\varepsilon_{\beta_p}}!}\phi_\varepsilon(B^\varepsilon_{n_1\ldots n_c})$. By Proposition 4.6.7

$$\mathrm{Ad}(f_{\beta_1}^{k_1}\ldots f_{\beta_d}^{k_c})(x_{n_1\ldots n_c}) = \left\{ \begin{array}{ll} 1 & \text{if} \quad n_p = k_p \quad \text{for} \quad p = 1, \ldots, c \\ 0 & \text{if} \quad k_i = n_i, i = 1, \ldots, p-1 \quad \text{and} \quad k_p > n_p \quad \text{for} \quad \text{some} \quad p \in \{1, \ldots, c\} \end{array} \right. . \qquad (5.2.1)$$

Since for any $v \in \mathrm{Wh}(V)$ and $z \in \mathbb{C}_\varepsilon^{loc}[G_*] \cap I_\varepsilon$ we have $zv = 0$, the elements $x_{n_1\ldots n_c}v$ are well–defined and satisfy $\mathrm{Ad}x(x_{n_1\ldots n_c}v) = \mathrm{Ad}x(x_{n_1\ldots n_c})v$, $x \in \mathbb{C}_\varepsilon[M_+]$. For the same reason formula (5.2.1) implies

$$\mathrm{Ad}(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c})(x_{n_1\ldots n_c}v) = \left\{ \begin{array}{ll} v & \text{if} \quad n_p = k_p \quad \text{for} \quad p = 1, \ldots, c \\ 0 & \text{if} \quad k_i = n_i, i = 1, \ldots, p-1 \quad \text{and} \quad k_p > n_p \quad \text{for} \quad \text{some} \quad p \in \{1, \ldots, c\} \end{array} \right. ,$$

and hence

$$\sigma_\varepsilon(x_{n_1\ldots n_c}v)(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}) = \left\{ \begin{array}{ll} v & \text{if} \quad n_p = k_p \quad \text{for} \quad p = 1, \ldots, c \\ 0 & \text{if} \quad k_i = n_i, i = 1, \ldots, p-1 \quad \text{and} \quad k_p > n_p \quad \text{for} \quad \text{some} \quad p \in \{1, \ldots, c\} \end{array} \right. . \qquad (5.2.2)$$

Observe that the elements $f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}$ form a linear basis of $\mathbb{C}_\varepsilon[M_+]$. Elements of this basis are labeled by elements of the set $\mathbb{N}^c$. Introduce the lexicographic order on this set, so that $(k_1, \ldots, k_c) > (n_1, \ldots, n_c)$ if $k_i = n_i$ for $i = 1, \ldots, p-1$ and $k_p > n_p$ for some $p \in \{1, \ldots, c\}$.

Now let $\mathbf{k} = (k_1, \ldots, k_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ and denote

$$\sigma_\varepsilon(x_{k_1\ldots k_c}v) = f_v^{\mathbf{k}}.$$

Since $f_v^{\mathbf{k}} \in \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ it does not vanish only on a finite number of the elements $f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}$ with $\mathbf{k} = (k_1, \ldots, k_c) > (n_1, \ldots, n_c) = \mathbf{n}$. Also by (6.3.8) $f_v^{\mathbf{k}}(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = 0$ for $\mathbf{k} < \mathbf{n} = (n_1, \ldots, n_c)$.

Let $\mathbf{n}^1 = (n_1^1, \ldots, n_c^1) \in \mathbb{N}^c$ be the largest element such that $\mathbf{k} > \mathbf{n}^1$ and

$$f_v^{\mathbf{k}}(f_{\beta_1}^{n_1^1}\ldots f_{\beta_c}^{n_c^1}) = v_1 \neq 0. \qquad (5.2.3)$$

Denote

$$f_{v_1}^{\mathbf{n}^1} = -\sigma_\varepsilon(x_{n_1^1\ldots n_c^1}v_1).$$

Then (5.2.2) and (5.2.3) imply that for $g_1 = f_v^{\mathbf{k}} + f_{v_1}^{\mathbf{n}^1} \in \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ one has

$$g_1(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \left\{ \begin{array}{l} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^1, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{array} \right. . \qquad (5.2.4)$$

Since $g_1 \in \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ it does not vanish on a finite number of the elements $f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}$ with $\mathbf{n}^1 > \mathbf{n} = (n_1, \ldots, n_c)$. Let $\mathbf{n}^2 = (n_1^2, \ldots, n_c^2) \in \mathbb{N}^c$ be the largest element such that $\mathbf{n}^1 > \mathbf{n}^2$ and

$$g_1(f_{\beta_1}^{n_1^2}\ldots f_{\beta_c}^{n_c^2}) = v_2 \neq 0. \qquad (5.2.5)$$

Denote

$$f_{v_2}^{\mathbf{n}^2} = -\sigma_\varepsilon(x_{n_1^2\ldots n_c^2}v_2).$$

Then (5.2.2) and (5.2.5) imply that for $g_2 = f_v^{\mathbf{k}} + f_{v_1}^{\mathbf{n}^1} + f_{v_2}^{\mathbf{n}^2} = g_1 + f_{v_2}^{\mathbf{n}^2} \in \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ one has

$$g_2(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^2, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}. \tag{5.2.6}$$

Iterating this procedure we obtain a sequence of elements $\mathbf{k} > \mathbf{n}^1 > \mathbf{n}^2 > \ldots > \mathbf{n}^i > \ldots$, $\mathbf{n}^i \in \mathbb{N}^c$ and a sequence of elements $g_i \in \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ such that

$$g_i(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^i, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}. \tag{5.2.7}$$

Since by Theorem 1.13 in [45] (see also Theorem 2.4.2 in [5]) the lexicographic order on $\mathbb{N}^c$ is a well–order the sequence $\mathbf{k} > \mathbf{n}^1 > \mathbf{n}^2 > \ldots > \mathbf{n}^i > \ldots$ must be finite, i.e. for some $i \in \mathbb{N}$

$$g_i(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^i, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}, \tag{5.2.8}$$

and there is no element $\mathbf{n} = (n_1, \ldots, n_c) < \mathbf{n}^i$ such that

$$g_i(f_{\beta_1}^{n_1^2}\ldots f_{\beta_c}^{n_c^2}) \neq 0. \tag{5.2.9}$$

Therefore $h_v^{\mathbf{k}} = g_i$ satisfies

$$h_v^{\mathbf{k}}(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \begin{cases} v & \text{if } (n_1, \ldots, n_c) = (k_1, \ldots, k_c), \\ 0 & \text{if } (n_1, \ldots, n_c) \neq (k_1, \ldots, k_c) \end{cases}.$$

This implies that the elements $h_v^{\mathbf{k}}$ with $(k_1, \ldots, k_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ generate $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$. We deduce that the elements $\sigma_\varepsilon(x_{n_1\ldots n_c}v)$ with $(n_1, \ldots, n_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ generate $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+], \mathrm{Wh}(V))$ as well. Therefore σ_ε is surjective.

To prove that f is an isomorphism we observe that $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ is an object of the category $\mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$ since Q_ε^{loc} is an object of this category.

The action of the augmentation ideal J on $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ is locally nilpotent. All Whittaker vectors of $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ belong to the subspace $1 \otimes \mathrm{Wh}(V)$, and the restriction of f to $1 \otimes \mathrm{Wh}(V)$ induces an isomorphism of the spaces of Whittaker vectors of $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ and of V. Therefore f is injective by Lemma 5.2.2.

In order to prove that f is surjective we shall need the following lemma.

Lemma 5.2.3. *Let $\phi : X \to Y$ be an injective homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–modules. Assume that ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y, and that $\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) = 0$, where $\mathbb{C}_{\varepsilon_s}$ is the trivial representation of $U_\varepsilon^s(\mathfrak{m}_-)$. Suppose also that the action of the augmentation ideal J of $U_\varepsilon^s(\mathfrak{m}_-)$ on the cokernel of ϕ is locally nilpotent. Then ϕ is surjective.*

Proof. Consider the exact sequence

$$0 \to X \to Y \to W' \to 0,$$

where W' is the cokernel of ϕ, and the corresponding long exact sequence of cohomology,

$$0 \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, Y) \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, W') \to$$

$$\to \mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) \to \ldots.$$

Since ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y, and $\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) = 0$, the initial part of the long exact cohomology sequence takes the form

$$0 \to \mathrm{Wh}(X) \to \mathrm{Wh}(Y) \to \mathrm{Wh}(W') \to 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $\mathrm{Wh}(W')=0$. But the augmentation ideal J acts on W' by locally nilpotent transformations. Therefore, by Engel theorem, if W' is not trivial there should exists a nonzero $U_\varepsilon^s(\mathfrak{m}_-)$–invariant vector in it. Thus we arrive at a contradiction, and $W'=0$. Therefore ϕ is surjective.

□

As we proved above $\sigma_\varepsilon : V \to \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathrm{Wh}(V)) \simeq \mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})\otimes \mathrm{Wh}(V)$ is an isomorphism of $\mathbb{C}_\varepsilon[M_+]$–modules for any module $V \in \mathbb{C}_\varepsilon^{loc}[G_*]-\mathrm{mod}_{\mathbb{C}_\varepsilon[M_+]}^{\chi_\varepsilon^s}$. By Lemma 5.1.1 $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C})\otimes \mathrm{Wh}(V)$ is injective over $\mathbb{C}_\varepsilon[M_+]$. Therefore V is injective as an $\mathbb{C}_\varepsilon[M_+]$–module with respect to the adjoint action. In particular, $Q_\varepsilon^{loc}\otimes_{W_{\varepsilon,\xi}^{s,loc}(G)}\mathrm{Wh}(V)$ is injective over $\mathbb{C}_\varepsilon[M_+]$, and hence $\mathrm{Ext}^1_{\mathbb{C}_\varepsilon[M_+]}(\mathbb{C}_{\varepsilon_s}, Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}\mathrm{Wh}(V))=0$.

Recall also that f induces an isomorphism of the spaces of Whittaker vectors of $Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}\mathrm{Wh}(V)$ and of V and that the adjoint action of J on V is locally nilpotent. Therefore f is surjective by Lemma 5.2.3 with $X=Q_\varepsilon^{loc}\otimes_{W_\varepsilon^{s,loc}(G)}\mathrm{Wh}(V)$, $Y=V$, $\phi=f$.

Note that by the definitions of the spaces $\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),W_\varepsilon^s(G))$ and $\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C})$ we also have an obvious right $U_\varepsilon^s(\mathfrak{m}_-)$–module isomorphism $\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-,W_\varepsilon^{s,loc}(G))=\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C})\otimes W_\varepsilon^{s,loc}(G)$.

Now consider the $U_\varepsilon^s(\mathfrak{m}_-)$–submodule $\sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))$ of Q_ε^{loc}, where

$$\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C})\subset \mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),W_\varepsilon^{s,loc}(G)).$$

Obviously $\sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\simeq \mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C})$ as a right $U_\varepsilon^s(\mathfrak{m}_-)$–module.

Let $\phi_\varepsilon : \sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G)\to Q_\varepsilon^{loc}$ be the map induced by the action of $W_\varepsilon^{s,loc}(G)$ on Q_ε^{loc}. Since this action commutes with the adjoint action of $U_\varepsilon^s(\mathfrak{m}_-)$ on Q_ε^{loc} we infer that ϕ_ε is a homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–$W_\varepsilon^{s,loc}(G)$–bimodules.

We claim that ϕ_ε is injective. This follows straightforwardly from Lemma 5.2.2 because all Whittaker vectors of $\sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G)$ belong to the subspace

$$1\otimes W_\varepsilon^{s,loc}(G)\subset \sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G),$$

and the restriction of ϕ_ε to this subspace is injective.

Now we show that ϕ_ε is surjective. By Lemma 5.1.1 one can immediately deduce that the right $U_\varepsilon^s(\mathfrak{m}_-)$–module $\sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G)\simeq \mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),W_\varepsilon^{s,loc}(G))$ is injective. In particular,

$$\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon,\sigma_\varepsilon^{-1}(\mathrm{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_-),\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G))=0.$$

One checks straightforwardly, similarly the case of the map σ_ε, that the other conditions of Lemma 5.2.3 for the map ϕ_ε are satisfied as well. Therefore ϕ_ε is surjective.

Thus Q_ε^{loc} is isomorphic to $\mathrm{hom}_{\mathbb{C}}(\mathbb{C}_\varepsilon[M_+],\mathbb{C}))\otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule. This completes the proof of the theorem.

□

5.3 Bibliographic comments

A categorical equivalence for Lie algebras, called the Skryabin equivalence, similar to that considered in this chapter was established in the Appendix to [82].

The main theorem of this chapter is an improvement of Theorem 7.7 in [101] where a similar equivalence was established in a quantum group case for q specialized to generic values $q=\varepsilon\in\mathbb{C}$. The proof of Theorem 7.7 in [101] relies on homological methods and arguments related to the properties of the quasiclassical limits $W^s(G)$ of q-W–algebras. In this book we use the approach similar to the original Skryabin's idea.

The definition of the category of equivariant representations over a quantum group given in Section 5.1 is a slight modification of a similar definition given in Section 7 in [101], minor changes being related to the fact that in Proposition 4.6.7, and more generally in the previous chapter, we dealt not with quantum groups themselves but with their localizations.

Chapter 6

Application of q-W–algebras to quantum groups at roots of unity and the proof of De Concini–Kac–Procesi conjecture

In this chapter we are going to use the elements $B_{n_1 \ldots n_c}$ introduced in Proposition 4.6.7 to study the structure of representations of quantum groups at roots of unity. De Concini and Kac observed that every irreducible representation of a quantum group $U_\varepsilon(\mathfrak{g})$ at a root of unity ε is in fact a representation of a finite–dimensional quotient $U_\eta(\mathfrak{g})$ of the quantum group, and hence every such representation is finite–dimensional itself. The quotient $U_\eta(\mathfrak{g})$ here depends on the representation. Later De Concini, Kac and Procesi also conjectured that the dimension of every such representation is divisible by a number b which depends on (an isomorphism class of) $U_\eta(\mathfrak{g})$; a precise definition of b will be given in Theorem 6.3.2.

Our main goal in this chapter is to prove this conjecture. We shall also obtain other related results on the structure of finite–dimensional representations of $U_\eta(\mathfrak{g})$. Firstly we are going to use an observation that every finite–dimensional representation of $U_\eta(\mathfrak{g})$ can be equipped with a second right action of a finite–dimensional subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ of the so–called small quantum group, and the dimension of this subalgebra is equal to b. The choice of the subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ depends on $U_\eta(\mathfrak{g})$ and the action of $U_{\eta_1}(\mathfrak{m}_-)$ satisfies a compatibility condition similar to condition (5.1.1) for equivariant modules over quantum groups at generic ε. Thus every finite–dimensional representation of $U_\eta(\mathfrak{g})$ is in fact an equivariant $U_\eta(\mathfrak{g}) - U_{\eta_1}(\mathfrak{m}_-)$–bimodule. Next we prove that every finite–dimensional representation of $U_\eta(\mathfrak{g})$ is cofree over the corresponding subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ which confirms, in particular, the De Concini–Kac–Procesi conjecture. Remarkably, to prove this statement one can apply almost verbatim the arguments from the proof of Theorem 5.2.1 on the quantum group version of the Skryabin equivalence for generic ε which overemphasizes again a striking similarity between the categories of finite–dimensional representations of algebras $U_\eta(\mathfrak{g})$ and the categories of equivariant modules over quantum groups introduced in Section 5.1.

The peculiarity of the quantum group case is that one can explicitly construct cofree bases of finite–dimensional $U_\eta(\mathfrak{g})$–modules using the elements $B_{n_1 \ldots n_c}$ from Proposition 4.6.7.

6.1 Quantum groups at roots of unity

In this section we recall some results on representation theory of quantum groups at roots of unity.

Let m be an odd positive integer number such that $m > d_i$ is coprime to all d_i for all i, ε a primitive m-th root of unity. An appropriate number d, which appears in the definiton of the algebras $U_\varepsilon(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$, can be found from the following proposition.

Proposition 6.1.1. *Let Δ be an irreducible root system, Δ_+^s the system of positive roots associated to the conjugacy class of a Weyl group element $s \in W$ in Theorem 1.5.2, $s = s_{\gamma_1} \ldots s_{\gamma_{l'}}$ representation (1.2.1) for s, $\alpha_1, \ldots, \alpha_l$ the system of simple roots in Δ_+^s. Then*

(i) if Δ is of exceptional type the lowest common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ is given in the tables in Appendix 2;

(ii) if Δ is of classical type then the conjugacy class of s corresponds to the sum of a number of blocks as in (1.4.5), (1.4.6), (1.4.9) or (1.4.11). To each block of type X we associate an integer $d_{ij}(X)$, $i,j=1,\ldots,l$ as follows:

if Δ is not of type A_l, D_l, an orbit with the smallest number of elements for the action of the group $\langle s\rangle$ on E corresponds to a block of type A_n and s does not fix any root from Δ then

for $\Delta = B_l$

$$d_{ij}(A_n)=\begin{cases} 2p+1 & \text{if } n=2p \text{ is even;}\\ p+1 & \text{if } n=2p+1,\ n\neq 4p-1 \text{ is odd;}\\ p & \text{if } n=4p-1 \text{ is odd and } i<j;\\ 2p & \text{if } n=4p-1 \text{ is odd and } i>j;\end{cases} \tag{6.1.1}$$

for $\Delta = C_l$

$$d_{ij}(A_n)=\begin{cases} 2p+1 & \text{if } n=2p \text{ is even;}\\ p+1 & \text{if } n=2p+1,\ n\neq 4p-1 \text{ is odd;}\\ 2p & \text{if } n=4p-1 \text{ is odd and } i<j;\\ p & \text{if } n=4p-1 \text{ is odd and } i>j;\end{cases} \tag{6.1.2}$$

for $\Delta = D_l$ if $A_{l-1}\subset D_l$ is the only nontrivial block of the conjugacy class of s then

$$d_{ij}(A_{l-1})=\begin{cases} 2p+1 & \text{if } l=2p+1 \text{ is odd;}\\ p+1 & \text{if } l=2p+2,\ l\neq 4p \text{ is even;}\\ p & \text{if } l=4p \text{ is even;}\end{cases} \tag{6.1.3}$$

for $\Delta = A_l$ if s is a representative in the Coxeter conjugacy class, i.e. the conjugacy class of s corresponds to the block of type A_l, then

$$d_{ij}(A_l)=1; \tag{6.1.4}$$

in all other cases

$$d_{ij}(A_k)=\begin{cases} k+1 & \text{if } k \text{ is even;}\\ \frac{k-1}{2}+1 & \text{if } k \text{ is odd;}\end{cases} \tag{6.1.5}$$

in all cases

$$d_{ij}(C_n)=d_{ij}(B_n)=d_{ij}(D_{v+w}(a_{w-1}))=1,$$

where, as before, we use the notation of [16], Section 7 for (blocks of) Weyl group conjugacy classes.

Then a common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)$, where $i,j=1,\ldots,l$ is the lowest common multiple of the numbers $d_{ij}(X)$, $i,j=1,\ldots,l$ for all blocks X of the conjugacy class of s.

If $\alpha_1',\ldots\alpha_l'$ is another system of simple roots then a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i',\alpha_j'\right)$ and vice versa.

Proof. First observe that if Δ_+' is another system of positive roots with the simple roots $\alpha_1',\ldots,\alpha_l'$ then $\alpha_i=\sum_{k=1}^l c_i^k\alpha_k'$, $\alpha_j^\vee=\sum_{k=1}^l b_i^k\alpha_k'^\vee$, where c_i^k, b_i^k are integer coefficients. Hence

$$\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)=\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j^\vee\right)=\sum_{k,p=1}^l c_i^k b_j^p\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_k',\alpha_p'^\vee\right)=\sum_{k,p=1}^l c_i^k b_j^p\frac{1}{d_p}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_k',\alpha_p'\right),$$

and a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i',\alpha_j'\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)$ and vice versa.

In case of classical irreducible root systems we shall compute a common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i',\alpha_j'\right)$, where Δ_+' is chosen in such a way that s is elliptic in a parabolic Weyl subgroup $W'\subset W$ generated by the simple reflections corresponding to roots from a subset of $\alpha_1',\ldots,\alpha_l'$ (for instance, one can take $\Delta_+'=\Delta_+^1$ from the proof of Theorem 1.5.2).

Since different blocks of the conjugacy class of s correspond to different disjoint mutually orthogonal subsets of simple roots in $\alpha_1',\ldots,\alpha_l'$ it suffices to consider the case when the conjugacy class of s corresponds to a diagram with a single nontrivial block. We shall compute d' in case when this block is of type A_k, $k>1$. Other cases can be considered in a similar way. Assume that the root system Δ is realized as in Section 1.5, where V is a real Euclidean n–dimensional vector space equipped with the standard scalar product, with an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. In that case simple roots are

$\mathbf{A_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n$;

$\mathbf{B_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i < n$, $\alpha'_n = \varepsilon_n$;

$\mathbf{C_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i < n$, $\alpha'_n = 2\varepsilon_n$;

$\mathbf{D_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i < n$, $\alpha'_n = \varepsilon_{n-1} + \varepsilon_n$;

Then s is of the form

$$s = s^1 s^2,\ s^1 = s_{\alpha'_{p+1}} s_{\alpha'_{p+3}} \ldots,\ s^2 = s_{\alpha'_{p+2}} s_{\alpha'_{p+4}} \ldots,$$

where in the formulas for $s^{1,2}$ the products are taken over mutually orthogonal simple roots labeled by indexes of the same parity; the last simple root which appears in those products is $\alpha'_{p+k} = \varepsilon_{p+k} - \varepsilon_{p+k+1}$, so $\gamma_1, \ldots, \gamma_k = \alpha'_{p+1}, \alpha'_{p+3}, \ldots, \alpha'_{p+2}, \alpha'_{p+4}, \ldots$.

We have to compute the numbers $\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j\right)$. We consider the case when $i < j$. The case when $i > j$ can be obtained from it by observing that

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j\right) = -\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_j, \alpha'^\vee_i\right) \frac{(\alpha'_i, \alpha'_i)}{(\alpha'_j, \alpha'_j)}. \tag{6.1.6}$$

First recall that by Lemma 2.7.1

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j\right) = \varepsilon_{ij} (\gamma_i, \gamma_j), \tag{6.1.7}$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

Let ω'_t be the fundamental weights of the root subsystem $A_k \subset \Delta$ with respect to the basis of simple roots α'_i, $i = p+1, \ldots, p+k$,

$$\omega'_t = \varepsilon_{p+1} + \ldots + \varepsilon_{p+t} - \frac{t}{k+1} \sum_{j=1}^{k+1} \varepsilon_{p+j}, t = 1, \ldots, k.$$

Since α'_{p+t}, $t = 1, \ldots, k$ form a linear basis of $\mathfrak{h}'^*$, and ω'_t, $t = 1, \ldots, k$ form the dual basis we have

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j\right) = \sum_{t,u=1}^{k} (\omega'^\vee_t, \alpha'_i) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_{p+t}, \alpha'^\vee_{p+u}\right) (\omega'_u, \alpha'^\vee_j).$$

Since the scalar product in V is normalized in such a way that $\alpha'^\vee_{p+u} = \alpha'_{p+u}$, $u = 1, \ldots, k$ we obtain using (6.1.7)

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j\right) = \sum_{t,u=1}^{k} (\omega'^\vee_t, \alpha'_i) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_{p+t}, \alpha'_{p+u}\right) (\omega'_u, \alpha'^\vee_j) = \tag{6.1.8}$$

$$= \sum_{t=1}^{k} (-1)^t (\omega'^\vee_t, \alpha'_i)(\omega'_{t-1} + \omega'_{t+1}, \alpha'^\vee_j),$$

where we assume that $\omega'_0 = \omega'_{k+1} = 0$.

Now one has to consider several cases.

If one of the roots α'_i, α'_j is orthogonal to $\mathfrak{h}'^*$ then the left hand side of the last equality is zero.

If $\alpha'_i, \alpha'_j \in \{\gamma_1, \ldots, \gamma_k\}$ then by (6.1.7) the left hand side of (6.1.8) is equal to ± 1.

If $\alpha'_i = \alpha'_{p+t}$, $1 < t < k$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^{\vee}_j\right) = (-1)^t(\omega'_{t-1} + \omega'_{t+1}, \alpha'^{\vee}_{p+k+1}) = (-1)^t(\vartheta - \delta\frac{2t}{k+1}), \tag{6.1.9}$$

where $\delta = 2$ if $\alpha'^{\vee}_j = 2\varepsilon_{p+k+1}$ or $\alpha'^{\vee}_j = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case. In all other cases $\vartheta = 0$ and $\delta = 1$. Note that $\delta \neq 1$ only in case when Δ is of type B_n or D_n; for arbitrary s this situation can only be realized if an orbit with the smallest number of elements for the action of the group $\langle s \rangle$ on E corresponds to a block of type A_k and s does not fix any root from Δ. The denominator r of the number in the right hand side of (6.1.9) is given by

$$r = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ p+1 & \text{if } k = 2p+1,\ k \neq 4p-1 \text{ is odd;} \\ \frac{2p}{\delta} & \text{if } k = 4p-1 \text{ is odd.} \end{cases} \tag{6.1.10}$$

If $\alpha'_i = \alpha'_{p+1}$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^{\vee}_j\right) = -(\omega'_2, \alpha'^{\vee}_{p+k+1}) = \delta\frac{2}{k+1} - \vartheta,$$

where $\delta = 2$ if $\alpha'^{\vee}_j = 2\varepsilon_{p+k+1}$ or $\alpha'^{\vee}_j = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case if $k = 2$. In all other cases $\vartheta = 0$ and $\delta = 1$. We again obtain (6.1.10).

If $\alpha'_i = \alpha'_{p+k}$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^{\vee}_j\right) = (-1)^k(\omega'_{k-1}, \alpha'^{\vee}_{p+k+1}) = -(-1)^k\delta\frac{k-1}{k+1},$$

and we obtain (6.1.10).

If $\alpha'_i = \alpha'_p$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^{\vee}_j\right) = \sum_{t=1}^{k}(-1)^t(\omega'_t, \alpha'^{\vee}_p)(\omega'_{t-1} + \omega'_{t+1}, \alpha'^{\vee}_{p+k+1}) =$$

$$= -\sum_{t=1}^{k-1}(-1)^t\left(-1 + \frac{t}{k+1}\right)\frac{2t\delta}{k+1} - (-1)^k\left(-1 + \frac{k}{k+1}\right)\frac{k-1}{k+1}\delta + (-1)^{k-1}\vartheta\left(-1 + \frac{k-1}{k+1}\right).$$

Using the fact that

$$\sum_{r=1}^{n}(-1)^{r+1}r^2 = (-1)^{n+1}\frac{n(n+1)}{2} \text{ and } \sum_{r=1}^{n}(-1)^{r+1}r = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

we obtain

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^{\vee}_j\right) = \begin{cases} -\frac{\delta}{k+1} + \vartheta\frac{2}{k+1} & \text{if } k \text{ is even;} \\ -\vartheta\frac{2}{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

The denominator r of the number in the right hand side of the last equality is given by

$$r = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ 1 & \text{if } k = 2p+1,\ n \text{ is odd and } \vartheta = 0; \\ p+1 & \text{if } k = 2p+1 \text{ is odd and } \vartheta = 1. \end{cases}$$

Summarizing all cases considered above and adding the case $i > j$ (see (6.1.6)) we arrive at (6.1.1), (6.1.2), (6.1.3), (6.1.4) and (6.1.5).

Other cases can be treated in a similar way.

□

Now one can choose $d = 2d'$, where d' is defined in the previous proiposition. However, we shall not always assume that d is chosen in this way and that the system of positive roots Δ_+^s will not be chosed as in Proposition 6.1.1 unless it is explicitly specified.

We shall always assume that d and m are coprime. This condition is equivalent to the existence of an integer n such that $\varepsilon^{nd-1} = 1$. From now on we shall also assume that $\kappa = nd$. With this choice of κ we have the following relation between the generators t_i and L_i of the quantum group $U_{\mathcal{A}}(\mathfrak{g})$, $t_i = L_i^n$. In particular, the specialization $U_\varepsilon(\mathfrak{g})$ of $U_{\mathcal{A}}(\mathfrak{g})$ coincides with the specialization of the simply connected form of the standard Drinfeld–Jimbo quantum group without generators t_i at $q = \varepsilon$.

Let Z_ε be the center of $U_\varepsilon(\mathfrak{g})$. In the following proposition we summarize the results on the structure of Z_ε. In particular, we recall that in case when ε is a root of unity Z_ε is much larger than in case of a generic ε. In fact in the former case Z_ε contains a remarkable subalgebra Z_0 the properties of which impose very strong restrictions on the structure of irreducible representations of $U_\varepsilon(\mathfrak{g})$.

Proposition 6.1.2. *Fix the normal ordering in the positive root system Δ_+ corresponding a reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ of the longest element $\overline{w}$ of the Weyl group W of $\mathfrak{g}$ and let $X_\alpha^\pm$ be the corresponding quantum root vectors in $U_\varepsilon(\mathfrak{g})$, and X_α the corresponding root vectors in $\mathfrak{g}$. Let $x_\alpha^- = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m (X_\alpha^-)^m$, $x_\alpha^+ = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m T_0(X_\alpha^-)^m$, where $T_0 = T_{i_1} \ldots T_{i_D}$, $\alpha \in \Delta_+$, and $l_i = L_i^m$, $i = 1, \ldots, l$.*

Then the following statements are true.

(i) The elements $x_\alpha^\pm$, $\alpha \in \Delta_+$, l_i, $i = 1, \ldots, l$ lie in Z_ε.

(ii) Let Z_0 ($Z_0^\pm$ and Z_0^0) be the subalgebras of Z_ε generated by the $x_\alpha^\pm$ and the $l_i^{\pm 1}$ (respectively by the $x_\alpha^\pm$ and by the $l_i^{\pm 1}$). Then $Z_0^\pm \subset U_\varepsilon(\mathfrak{n}_\pm)$, $Z_0^0 \subset U_\varepsilon(\mathfrak{h})$, $Z_0^\pm$ is the polynomial algebra with generators $x_\alpha^\pm$, Z_0^0 is the algebra of Laurent polynomials in the l_i, $Z_0^\pm = U_\varepsilon(\mathfrak{n}_\pm) \cap Z_0$, and multiplication defines an isomorphism of algebras

$$Z_0^- \otimes Z_0^0 \otimes Z_0^+ \to Z_0.$$

The subalgebra Z_0 is independent of the choice of the reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$.

(iii) $U_\varepsilon(\mathfrak{g})$ is a free Z_0–module with basis the set of monomials $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$, where for $\mathbf{s} = (s_1, \ldots s_l) \in \mathbb{Z}^l$,

$$L^{\mathbf{s}} = L_1^{s_1} \ldots L_l^{s_l}.$$

(iv) $\mathrm{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$ *is a complex affine space of dimension equal to* $\dim \mathfrak{g}$.

(v) The subalgebra Z_0 is preserved by the action of the braid group automorphisms T_i.

(vi) Let G be the connected simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ and G_0^* the solvable algebraic subgroup in $G \times G$ which consists of elements of the form $(L_+, L_-) \in G \times G$,

$$(L_+, L_-) = (t, t^{-1})(n_+, n_-),\ n_\pm \in N_\pm,\ t \in H.$$

Then $\mathrm{Spec}(Z_0^0)$ *can be naturally identified with the maximal torus H in G, and the map*

$$\widetilde{\pi} : \mathrm{Spec}(Z_0) = \mathrm{Spec}(Z_0^+) \times \mathrm{Spec}(Z_0^0) \times \mathrm{Spec}(Z_0^-) \to G_0^*,$$

$$\widetilde{\pi}(u_+, t, u_-) = (t\mathbf{X}^+(u_+), t^{-1}\mathbf{X}^-(u_-)^{-1}),\ u_\pm \in \mathrm{Spec}(Z_0^\pm),\ t \in \mathrm{Spec}(Z_0^0),$$

$$\mathbf{X}^\pm : \mathrm{Spec}(Z_0^\pm) \to N_\pm,$$

$$\mathbf{X}^- = \exp(x_{\beta_D}^- X_{-\beta_D}) \exp(x_{\beta_{D-1}}^- X_{-\beta_{D-1}}) \ldots \exp(x_{\beta_1}^- X_{-\beta_1}),$$

$$\mathbf{X}^+ = \exp(x_{\beta_D}^+ T_0(X_{-\beta_D})) \exp(x_{\beta_{D-1}}^+ T_0(X_{-\beta_{D-1}})) \ldots \exp(x_{\beta_1}^+ T_0(X_{-\beta_1})),$$

where $x_{\beta_i}^\pm$ should be regarded as complex-valued functions on $\mathrm{Spec}(Z_0)$, *is an isomorphism of varieties independent of the choice of reduced decomposition of $\overline{w}$.*

Parts (ii) and (iii) of Proposition 6.1.2 can also be reformulated in terms of the quantum root vectors e_α and f_α.

Proposition 6.1.3. *Let $s \in W$ be a Weyl group element, and e_α, f_α the quantum root vectors defined in Proposition 2.6.2. Then the following statements are true.*

(i) The subalgebra Z_0 is the tensor product of the polynomial algebra with generators e_α^m, f_α^m, $\alpha \in \Delta_+$ and of the algebra of Laurent polynomials in l_i, $i = 1, \ldots, l$.

(ii) $U_\varepsilon(\mathfrak{g})$ is a free Z_0–module with basis the set of monomials $f^{\mathbf{r}} L^{\mathbf{s}} e^{\mathbf{t}}$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$.

Let $\mathbf{K} : \mathrm{Spec}(Z_0^0) \to H$ be the map defined by $\mathbf{K}(h) = h^2$, $h \in H$.

Proposition 6.1.4. *Let $G^0 = N_-HN_+$ be the big cell in G. Then the map*

$$\pi = \mathbf{X}^-\mathbf{K}\mathbf{X}^+ : \mathrm{Spec}(Z_0) \to G^0$$

is independent of the choice of reduced decomposition of $\overline{w}$, and is an unramified covering of degree 2^l.

Denote by $\lambda_0 : G_0^* \to G^0$ the map defined by $\lambda_0(L_+, L_-) = L_-^{-1}L_+$. Then obviously $\pi = \lambda_0 \circ \widetilde{\pi}$.

Another important property of quantum groups at roots of unity, which distinguishes the root of unity case, is the existence of the so–called quantum coadjoint action which is an automorphism group action on an extension of $U_\varepsilon(\mathfrak{g})$. It is defined with the help of derivations $\underline{x}_i^\pm$ of $U_\mathcal{A}(\mathfrak{g})$ given by

$$\underline{x}_i^+(u) = \left[\frac{(X_i^+)^m}{[m]_{q_i}!}, u\right],\ \underline{x}_i^-(u) = T_0\underline{x}_i^+T_0^{-1}(u),\ i = 1, \ldots, l,\ u \in U_\mathcal{A}(\mathfrak{g}). \tag{6.1.11}$$

Let $\widehat{Z}_0$ be the algebra of formal power series in the $x_\alpha^\pm$, $\alpha \in \Delta_+$, and the $l_i^{\pm 1}$, $i = 1, \ldots, l$, which define holomorphic functions on $\mathrm{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$. Let

$$\widehat{U}_\varepsilon(\mathfrak{g}) = U_\varepsilon(\mathfrak{g}) \otimes_{Z_0} \widehat{Z}_0,\quad \widehat{Z}_\varepsilon = Z_\varepsilon \otimes_{Z_0} \widehat{Z}_0.$$

Proposition 6.1.5. *(i)On specializing to $q = \varepsilon$, (6.1.11) induces a well–defined derivation $\underline{x}_i^\pm$ of $U_\varepsilon(\mathfrak{g})$.*

(ii)The series

$$\exp(t\underline{x}_i^\pm) = \sum_{k=0}^\infty \frac{t^k}{k!}(\underline{x}_i^\pm)^k$$

converges for all $t \in \mathbb{C}$ to a well–defined automorphism of the algebra $\widehat{U}_\varepsilon(\mathfrak{g})$.

(iii)Let $\mathcal{G}$ be the group of automorphisms generated by the one–parameter groups $\exp(t\underline{x}_i^\pm)$, $i = 1, \ldots, l$. The action of $\mathcal{G}$ on $\widehat{U}_\varepsilon(\mathfrak{g})$ preserves the subalgebras $\widehat{Z}_\varepsilon$ and $\widehat{Z}_0$, and hence $\mathcal{G}$ acts by holomorphic automorphisms on the complex algebraic varieties $\mathrm{Spec}(Z_\varepsilon)$ and $\mathrm{Spec}(Z_0)$.

(iv)Let $\mathcal{O}$ be a conjugacy class in G. The intersection $\mathcal{O}^0 = \mathcal{O} \cap G^0$ is a smooth connected variety, and the variety $\pi^{-1}(\mathcal{O}^0)$ is a $\mathcal{G}$–orbit in $\mathrm{Spec}(Z_0)$.

Given a homomorphism $\eta : Z_0 \to \mathbb{C}$, let

$$U_\eta(\mathfrak{g}) = U_\varepsilon(\mathfrak{g})/I_\eta,$$

where I_η is the ideal in $U_\varepsilon(\mathfrak{g})$ generated by elements $z - \eta(z)$, $z \in Z_0$. By part (iii) of Proposition 6.1.2 $U_\eta(\mathfrak{g})$ is an algebra of dimension $m^{\dim\,\mathfrak{g}}$ with linear basis the set of monomials $(X^+)^{\mathbf{r}}L^{\mathbf{s}}(X^-)^{\mathbf{t}}$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$.

If $\widetilde{g} \in \mathcal{G}$ then for any $\eta \in \mathrm{Spec}(Z_0)$ we have $\widetilde{g}\eta \in \mathrm{Spec}(Z_0)$ by part (iii) of Proposition 6.1.5, and by part (ii) of the same proposition $\widetilde{g}$ induces an isomorphism of algebras,

$$\widetilde{g} : U_\eta(\mathfrak{g}) \to U_{\widetilde{g}\eta}(\mathfrak{g}). \tag{6.1.12}$$

Since on every irreducible representation of $U_\varepsilon(\mathfrak{g})$ the subalgebra Z_0 of the center Z_ε acts by a character $\eta : Z_0 \to \mathbb{C}$, every irreducible representation of $U_\varepsilon(\mathfrak{g})$ is a representation of some algebra $U_\eta(\mathfrak{g})$ for a unique η. This reduces the study of irreducible representations of $U_\varepsilon(\mathfrak{g})$ to the study of representations of finite–dimensional algebras $U_\eta(\mathfrak{g})$. Moreover, taking into account isomorphisms (6.1.12) it suffices to consider a representative in each isomorphism class of these algebras under the isomorphisms induced by the action of the elements of the group $\mathcal{G}$ on $\widehat{U}_\varepsilon(\mathfrak{g})$.

6.2 Whittaker vectors in modules over quantum groups at roots of unity

It turns out that any finite–dimensional representation V of $U_\eta(\mathfrak{g})$ can be equipped with another action of a subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ of a small quantum group which is a root of unity "truncated" version of the algebra $U_\mathcal{A}^s(\mathfrak{m}_-)$ for an appropriate s depending on η. The new action is compatible with the original action of $U_\eta(\mathfrak{g})$ in a certain equivariant way, and the dimension of $U_{\eta_1}(\mathfrak{m}_-)$ is equal to $b = m^{\frac{1}{2}\dim\,\mathcal{O}_{\pi\eta}} = m^{\dim\,\mathfrak{m}_-}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G$. The existence of the second action is crucial for the proof of the De Concini–Kac–Procesi conjecture and for the study of other properties of finite–dimensional representations of $U_\eta(\mathfrak{g})$.

In this section we define the algebras $U_{\eta_1}(\mathfrak{m}_-)$ and their actions on finite–dimensional representations of $U_\eta(\mathfrak{g})$. These definitions are related to the notion of Whittaker vectors for finite–dimensional $U_\eta(\mathfrak{g})$–modules which are defined using root of unity versions of characters χ_q^s. We start by reminding the definitions of these characters.

Firstly let us observe that $U_\varepsilon^s(\mathfrak{m}_-)$ can be regarded as a subalgebra in $U_\varepsilon(\mathfrak{g})$. Therefore for every character $\eta : Z_0 \to \mathbb{C}$ one can define the corresponding subalgebra in $U_\eta(\mathfrak{g})$ generated by f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$. We denote this subalgebra by $U_\eta(\mathfrak{m}_-)$. By part (ii) of Proposition 6.1.3 we have $\dim U_\eta(\mathfrak{m}_-) = m^{\dim\,\mathfrak{m}_-}$.

In order to define analogues of characters χ_q^s for quantum groups at roots of unity we shall need some properties of the finite dimensional algebras $U_\eta(\mathfrak{g})$ and $U_\eta(\mathfrak{m}_-)$ and auxiliary results on non–zero irreducible representations of the algebra $U_\eta(\mathfrak{m}_-)$.

Observe that by Proposition 1.6.1 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k\gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$, and hence from commutation relations (2.7.11) one can deduce that

$$f_\alpha f_\beta - \varepsilon^{(\alpha,\beta)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)} f_\beta f_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C'(k_1,\ldots,k_n) f_{\delta_n}^{k_n} f_{\delta_{n-1}}^{k_{n-1}} \ldots f_{\delta_1}^{k_1} \in \mathcal{J}, \tag{6.2.1}$$

where at least one of the roots δ_i in the right hand side of the last formula belongs to $\Theta = \{\alpha \in \Delta_{\mathfrak{m}_+} : \alpha \notin \{\gamma_1, \ldots, \gamma_{l'}\}\}$, $\mathcal{J}$ is the ideal in $U_\eta(\mathfrak{m}_-)$ generated by the elements $f_\beta \in U_\eta(\mathfrak{m}_-)$, $\beta \in \Theta$. Thus from part (ii) of Proposition 6.1.3 and commutation relations (6.2.1) it follows that if $\beta_1 < \beta_2 < \ldots < \beta_c$ are the roots in the segment $\Delta_{\mathfrak{m}_+}$, the elements

$$x_{k_1,\ldots,k_b} = f_{\beta_c}^{k_c} f_{\beta_{c-1}}^{k_{c-1}} \ldots f_{\beta_1}^{k_1} \tag{6.2.2}$$

for $k_i \in \mathbb{N}$, $k_i < m$ form a linear basis of $U_\eta(\mathfrak{m}_-)$, and elements (6.2.2) for $k_i \in \mathbb{N}$, $k_i < m$ and $k_i > 0$ for at least one $\beta_i \in \Theta$ form a linear basis of $\mathcal{J}$.

Lemma 6.2.1. *Let η be an element of* Spec(Z_0). *Assume that $\eta(f_{\gamma_i}^m) = a_i \neq 0$ for $i = 1, \ldots, l'$and that and $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, and hence $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i = 1, \ldots, l'$ and $f_\beta^m = 0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Then the ideal $\mathcal{J}$ is the Jacobson radical of $U_\eta(\mathfrak{m}_-)$ and $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra*

$$\mathbb{C}[f_{\gamma_1}, \ldots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = a_i\}_{i=1,\ldots,l'}$$

.

Proof. First we show that $\mathcal{J}$ is nilpotent.

Let i be the largest number such that $k_j = 0$ for $j > i$ in (6.2.2) and $k_i \neq 0$. Then we define the degree of $x_{k_1,\ldots,k_c}$ by

$$deg(x_{k_1,\ldots,k_c}) = (k_i, i) \in \{1, \ldots, m-1\} \times \{1, \ldots, c\}.$$

Equip $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$ with the order such that $(k, i) < (k', j)$ if $j > i$ or $j = i$ and $k' > k$.

For any given $(k, i) \in \{1, \ldots, m-1\} \times \{1, \ldots, c\}$ denote by $(U_\eta(\mathfrak{m}_-))_{(k,i)}$ the linear span of the elements $x_{k_1,\ldots,k_c}$ with $deg(x_{k_1,\ldots,k_c}) \leq (k, i)$ and define $\mathcal{J}_{(k,i)} = \mathcal{J} \cap (U_\eta(\mathfrak{m}_-))_{(k,i)}$. We also have $(U_\eta(\mathfrak{m}_-))_{(k,i)} \subset (U_\eta(\mathfrak{m}_-))_{(k',j)}$ and $\mathcal{J}_{(k,i)} \subset \mathcal{J}_{(k',j)}$ if $(k, i) < (k', j)$, and $\mathcal{J}_{(m-1,c)} = \mathcal{J}$. Note that for the first few i linear spaces $\mathcal{J}_{(k,i)}$ may be trivial, and these are all possibilities when those spaces can be trivial.

We shall prove that $\mathcal{J}$ is nilpotent by induction over the order in $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$. Let (k, i) be minimal possible such that $\mathcal{J}_{(k,i)}$ is not trivial. Then we must have $k = 1$. If $y \in \mathcal{J}_{(1,i)}$ then y must be of the form

$$y = f_\beta v, \tag{6.2.3}$$

where v is a linear combination of elements of the form $f_{\beta_{i_1}}^{k_1} \ldots f_{\beta_{i_r}}^{k_r}$ for $\beta_{i_1}, \ldots, \beta_{i_r} \in \{\gamma_1, \ldots, \gamma_n\}$, $\beta > \beta_{i_1}$, and β is the first root from the set Θ greater than γ_1 in the normal ordering of Δ_+ associated to s. Here it is assumed that $f_{\beta_{i_1}}^{k_1} \ldots f_{\beta_{i_r}}^{k_r} = 1$ if the set $\{\gamma_1, \ldots, \gamma_n\}$ is empty.

Now equation (6.2.1) implies that for any $f_{\beta_{i_j}}$ which appears in the expression for v one has

$$f_\beta f_{\beta_{i_j}} - \varepsilon^{(\beta,\beta_{i_j})+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\beta,\beta_{i_j})} f_{\beta_{i_j}} f_\beta \in \mathcal{J}_{(m-1,i-1)} = 0 \tag{6.2.4}$$

as by our choice of i $\mathcal{J}_{(m-1,i-1)}=0$.

Formula (6.2.4) implies that the product of m elements of type (6.2.3) can be represented in the form

$$f_\beta^m v',$$

where v' is of the same form as v. Since $f_\beta^m=0$ we deduce that $\mathcal{J}_{(1,i)}^m=0$.

Now assume that $\mathcal{J}_{(k,i)}^K=0$ for some $K>0$. Let (k',i') be the smallest element of $\{1,\ldots,m-1\}\times\{1,\ldots,c\}$ which satisfies $(k,i)<(k',i')$. Then by Propositions 2.6.2 and 6.1.3, and by (6.2.1), any element of $\mathcal{J}_{(k',i')}$ is of the form $f_{\beta_{i'}}u+u'$, where $u'\in\mathcal{J}_{(k,i)}$ and if $\beta_{i'}\in\Theta$ then $u\in(U_\eta(\mathfrak{m}_-))_{(k,i)}$; if $\beta_{i'}\notin\Theta$ then $u\in\mathcal{J}_{(k,i)}$.

Now equation (6.2.1) together with (2.7.11) imply that for any $u\in(U_\eta(\mathfrak{m}_-))_{(k,i)}$ one has

$$uf_{\beta_{i'}}=cf_{\beta_{i'}}u+w, \tag{6.2.5}$$

where c is a non–zero constant depending on u, and $w\in\mathcal{J}_{(k,i)}$. By formula (6.2.5) the product of m elements $f_{\beta_{i'}}u_p+u_p'$, $p=1,\ldots,m$ of the type described above can be represented in the form

$$\sum_{j=0}^{m} f_{\beta_{i'}}^j c_j, \tag{6.2.6}$$

where $c_j\in\mathcal{J}_{(k,i)}$ for $j=0,\ldots,m-1$ and if $\beta_{i'}\in\Theta$ then $c_m\in(U_\eta(\mathfrak{m}_-))_{(k,i)}$; if $\beta_{i'}\notin\Theta$ then $c_m\in\mathcal{J}_{(k,i)}$. In the former case $f_{\beta_{i'}}^m=0$, and the last term in sum (6.2.6) is zero; in the latter case $f_{\beta_{i'}}^m=\eta(f_{\beta_{i'}}^m)\neq 0$, and the last term in sum (6.2.6) is from $\mathcal{J}_{(k,i)}$. So we can combine it with the term corresponding to $j=0$. In both cases sum (6.2.6) takes the form

$$\sum_{j=0}^{m-1} f_{\beta_{i'}}^j c_j', \tag{6.2.7}$$

where $c_j'\in\mathcal{J}_{(k,i)}$. By (6.2.5) the product of K sums of type (6.2.7) is of the form

$$\sum_{j=0}^{(m-1)K} f_{\beta_{i'}}^j c_j'',$$

where each c_j'' is a linear combination of elements from $\mathcal{J}_{(k,i)}^K$. By our assumption $\mathcal{J}_{(k,i)}^K=0$, and hence the product of any mK elements of $\mathcal{J}_{(k',i')}$ is zero. This justifies the induction step and proves that $\mathcal{J}_{(m-1,c)}=\mathcal{J}$ is nilpotent. Hence $\mathcal{J}$ is contained in the Jacobson radical of $U_\eta(\mathfrak{m}_-)$.

Using commutation relations (2.7.11) we also have (see the proof of Proposition 3.2.4)

$$f_{\gamma_i}f_{\gamma_j}-f_{\gamma_j}f_{\gamma_i}\in\mathcal{J}.$$

Therefore the quotient algebra $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra

$$\mathbb{C}[f_{\gamma_1},\ldots,f_{\gamma_{l'}}]/\{f_{\gamma_i}^m=a_i\}_{i=1,\ldots,l'}$$

which is semisimple. Therefore $\mathcal{J}$ coincides with the Jacobson radical of $U_\eta(\mathfrak{m}_-)$. □

In Proposition 3.2.4 we constructed some characters of the algebra $U_\varepsilon^s(\mathfrak{m}_-)$. Now we show that the algebra $U_\eta(\mathfrak{m}_-)$ has a finite number of irreducible representations which are one–dimensional, and all those representations can be obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Proposition 6.2.2. *Let η be an element of* Spec(Z_0)*. Assume that $\eta(f_{\gamma_i}^m)=a_i\neq 0$ for $i=1,\ldots,l'$and that and $\eta(f_\beta^m)=0$ for $\beta\in\Delta_{\mathfrak{m}_+}$, $\beta\notin\{\gamma_1,\ldots,\gamma_{l'}\}$, and hence $f_{\gamma_i}^m=\eta(f_{\gamma_i}^m)=a_i\neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i=1,\ldots,l'$ and $f_\beta^m=0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta\in\Delta_{\mathfrak{m}_+}$, $\beta\notin\{\gamma_1,\ldots,\gamma_{l'}\}$. Then all non–zero irreducible representations of the algebra $U_\eta(\mathfrak{m}_-)$ are one–dimensional and have the form*

$$\chi(f_\beta)=\begin{cases}0 & \beta\notin\{\gamma_1,\ldots,\gamma_{l'}\}\\ c_i & \beta=\gamma_i,\ i=1,\ldots,l'\end{cases}, \tag{6.2.8}$$

where complex numbers c_i satisfy the conditions $c_i^m=a_i$, $i=1,\ldots,l'$. Moreover, all non–zero irreducible representations of $U_\eta(\mathfrak{m}_-)$ can be obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Proof. Let V be a non–zero finite–dimensional irreducible $U_\eta(\mathfrak{m}_-)$–module. By Corollary 54.13 in [27] elements of the ideal $\mathcal{J} \subset U_\eta(\mathfrak{m}_-)$ act by zero transformations on V. Hence V is in fact an irreducible representation of the algebra $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ which is isomorphic to the truncated polynomial algebra

$$\mathbb{C}[f_{\gamma_1}, \ldots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = a_i\}_{i=1,\ldots,l'}.$$

The last algebra is commutative and all its complex irreducible representations are one–dimensional. Therefore V is one–dimensional, and if v is a nonzero element of V then $f_{\gamma_i} v = c_i v$, for some $c_i \in \mathbb{C}$, $i = 1, \ldots, l'$. Note that $\eta(f_{\gamma_i}^m) = a_i \neq 0$, $i = 1, \ldots, l'$ and hence $c_i^m = a_i \neq 0$, $i = 1, \ldots, l'$. In particular, the elements f_{γ_i} act on V by semisimple automorphisms.

If we denote by $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ the character of $U_\eta(\mathfrak{m}_-)$ such that

$$\chi(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, \ i = 1, \ldots, l' \end{cases}$$

and by $\mathbb{C}_\chi$ the corresponding one–dimensional representation of $U_\eta(\mathfrak{m}_-)$ then we have $V = \mathbb{C}_\chi$.

Now we have to prove that the representations $\mathbb{C}_\chi$ for different characters χ are obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Since $c_i^m = a_i$, $i = 1, \ldots, l'$ there are only finitely many possible characters χ corresponding to the given η in the statement of this proposition. If χ and χ' are two such characters, $\chi(f_{\gamma_i}) = c_i$, $i = 1, \ldots, l'$ and $\chi'(f_{\gamma_i}) = c_i'$, $i = 1, \ldots, l'$ then the relations $c_i^m = {c_i'}^m = a_i$, $i = 1, \ldots, l'$ imply that $c_i' = \varepsilon^{m_i} c_i$, $0 \leq m_i \leq m-1$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, l'$.

Now observe that for any $h \in \mathfrak{h}$ the map defined by $f_\alpha \mapsto \varepsilon^{\alpha(h)} f_\alpha$, $\alpha \in \Delta_{\mathfrak{m}_+}$ is an automorphism of the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ generated by elements f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$ with defining relations (2.7.11). Here the principal branch of the analytic function ε^z is used to define $\varepsilon^{\alpha(h)}$, so that $\varepsilon^{\alpha(h)}\varepsilon^{\beta(h)} = \varepsilon^{(\alpha+\beta)(h)}$ for any $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$. If in addition $\varepsilon^{m\gamma_i(h)} = 1$, $i = 1, \ldots, l'$ the above defined map gives rise to an automorphism ς of $U_\eta(\mathfrak{m}_-)$. Indeed in that case $(\varepsilon^{\gamma_i(h)} f_{\gamma_i})^m = f_{\gamma_i}^m$, $i = 1, \ldots, l'$ and all the remaining defining relations $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$, $i = 1, \ldots, l'$, $f_\beta^m = \eta(f_\beta^m) = 0$, $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$ of the algebra $U_\eta(\mathfrak{m}_-)$ are preserved by the action of the above defined map ς.

Now fix $h \in \mathfrak{h}$ such that $\gamma_i(h) = m_i$, $i = 1, \ldots, l'$. Obviously we have $\varepsilon^{mm_i} = 1$, $i = 1, \ldots, l'$. We claim that the representation $\mathbb{C}_\chi$ twisted by the corresponding automorphism ς coincides with $\mathbb{C}_{\chi'}$. Indeed, we obtain

$$\chi(\varsigma f_{\gamma_i}) = \chi(\varepsilon^{m_i} f_{\gamma_i}) = \varepsilon^{m_i} c_i = c_i', \ i = 1, \ldots, l'.$$

This completes the proof of the proposition.

□

Now we can define the notion of Whittaker vectors. Let V be a $U_\eta(\mathfrak{g})$–module, where η is an element of $\mathrm{Spec}(Z_0)$ such that $\eta(f_{\gamma_i}^m) = a_i \neq 0$ for $i = 1, \ldots, l'$and that and $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Let $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ be a character defined in the Proposition 6.2.2, $\mathbb{C}_\chi$ the corresponding one–dimensional $U_\eta(\mathfrak{m}_-)$–module. Then the space $V_\chi = \mathrm{Hom}_{U_\eta(\mathfrak{m}_-)}(\mathbb{C}_\chi, V)$ is called the space of Whittaker vectors of V. Elements of V_χ are called Whittaker vectors.

Now we describe the space of Whittaker vectors in terms of a nilpotent action of the unital subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ generated by f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$ in the small quantum group $U_{\eta_1}(\mathfrak{g}) = U_\varepsilon^s(\mathfrak{g})/I_{\eta_1}$ corresponding to the trivial central character η_1 such that $\widetilde{\pi}(\eta_1) = 1 \in G_0^*$ and $\eta_1(x_\alpha^\pm) = 0$, $\alpha \in \Delta_+$, $\eta_1(l_i) = 1$, $i = 1, \ldots, l$.

Recall that $U_\varepsilon^s(\mathfrak{m}_-)$ is a right coideal in $U_\varepsilon^s(\mathfrak{g})$. One can also equip the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ with a character given by formula (3.2.12), where the numbers c_i are the same as in the definition of the character χ. We denote this character by the same letter, $\chi : U_\varepsilon^s(\mathfrak{m}_-) \to \mathbb{C}$.

Note that V can be regarded as a $U_\varepsilon(\mathfrak{g})$–module and a $U_\varepsilon^s(\mathfrak{g})$–module assuming that the ideal I_η acts on V in the trivial way. Now observe that $\Delta_s : U_\varepsilon^s(\mathfrak{m}_-) \to U_\varepsilon^s(\mathfrak{g}) \otimes U_\varepsilon^s(\mathfrak{m}_-)$ is a homomorphism of algebras. Composing it with the tensor product $S_s \otimes \chi$ of the anti–homomorphism S_s and of the character χ, which can be regarded as an anti–homomorphism as well, one can define an anti–homomorphism, $U_\varepsilon^s(\mathfrak{m}_-) \to U_\varepsilon^s(\mathfrak{g})$, $x \mapsto S_s(x_1)\chi(x_2)$, $\Delta_s(x) = x_1 \otimes x_2$, $x \in U_\varepsilon^s(\mathfrak{m}_-)$.

Using this anti–homomorphism one can introduce a right $U_\varepsilon^s(\mathfrak{m}_-)$–action on V which we call the adjoint action and denote it by Ad. It is given by the formula

$$\mathrm{Ad}\, xv = S_s(x_1)\chi(x_2)v, x \in U_\varepsilon^s(\mathfrak{m}_-), v \in V, \qquad (6.2.9)$$

where $\Delta_s(x) = x_1 \otimes x_2$.

Note that using the Swedler notation for the comultiplication, $(\Delta_s \otimes id \otimes id)(\Delta_s \otimes id)\Delta_s(x) = x_1 \otimes x_2 \otimes x_3 \otimes x_4$, the coassiciativity of the comultiplication and the definition of the antipode we have for any $x \in U_\varepsilon^s(\mathfrak{m}_-)$, $y \in U_\varepsilon^s(\mathfrak{g})$, $v \in V$ (compare with the proof of Lemma 2.2 in [53])

$$\mathrm{Ad}x(yv) = S_s(x_1)\chi(x_2)yv = S_s(x_1)yx_2S_s(x_3)\chi(x_4)v = \mathrm{Ad}x_1(y)\mathrm{Ad}x_2(v). \tag{6.2.10}$$

We shall need the following formula for the action of the comultiplication on the quantum root elements f_β, $\beta = \sum_{i=1}^l c_i\alpha_i \in \Delta_{\mathfrak{m}_+}$, $c_i \in \mathbb{N}$,

$$\begin{aligned}\Delta_s(f_\beta) = \prod_{i=1}^l K_i^{c_i} \prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j)c_i} \otimes f_\beta + f_\beta \otimes 1 + \\ +\sum_i y_i \otimes x_i,\ x_i \in U_{<\beta}, y_i \in U_{>\beta}U_\varepsilon^{s^{-1}}(\mathfrak{h}),\end{aligned} \tag{6.2.11}$$

where $U_{<\beta}$ is the subalgebra (without unit) in $U_\varepsilon^s(\mathfrak{m}_-)$ generated by f_α, $\alpha < \beta$ and $U_{>\beta}$ is the subalgebra (without unit) in $U_\varepsilon^s(\mathfrak{n}_-)$ generated by f_α, $\alpha > \beta$. Formula (6.2.11) is a straightforward consequence of (2.7.12).

Similarly to the Proposition in Section 5.6 in [22] we infer that Z_0 is a Hopf subalgebra in $U_\varepsilon^s(\mathfrak{g})$. Namely,

$$\Delta_s(f_i^m) = K_i^m \prod_{j=1}^l L_j^{-m\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j)} \otimes f_i^m + f_i^m \otimes 1,$$

$$\Delta_s(e_i^m) = e_i^m \otimes K_i^{-m} + \prod_{j=1}^l L_j^{m\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j)} \otimes e_i^m,$$

$$\Delta_s(L_i^m) = L_i^m \otimes L_i^m.$$

Therefore recalling that by the definition of χ for $x \in U_\varepsilon^s(\mathfrak{m}_-) \cap Z_0$ one has $\chi(x) = \eta(x)$ we deduce

$$\mathrm{Ad}\ xv = S_s(x_1)\chi(x_2)v = \eta(S_s(x_1)x_2)v = \varepsilon_s(x)v, v \in V,$$

where ε_s is the counit of $U_\varepsilon^s(\mathfrak{g})$. Note that by the definition of the ideal I_{η_1} the ideal $U_\varepsilon^s(\mathfrak{m}_-) \cap I_{\eta_1} \subset U_\varepsilon^s(\mathfrak{m}_-)$ is generated by the elements f_α^m, $\alpha \in \Delta_{\mathfrak{m}_+}$ and $\varepsilon_s(f_\alpha^m) = 0$ for $\alpha \in \Delta_{\mathfrak{m}_+}$ by the definition of ε_s. Hence the adjoint action of $U_\varepsilon^s(\mathfrak{m}_-)$ on V induces an action of the subalgebra $U_{\eta_0}(\mathfrak{m}_-)$ of the small quantum group $U_{\eta_0}(\mathfrak{g})$. We call this action the adjoint action as well.

Note that the small quantum group $U_{\eta_0}(\mathfrak{g})$ is a Hopf algebra with the comultiplication inherited from $U_\varepsilon^s(\mathfrak{g})$.

The space of Whittaker vectors V_χ can be characterized in terms of the adjoint action as follows.

Lemma 6.2.3. *The space of Whittaker vectors V_χ coincides with the space of $U_{\eta_1}(\mathfrak{m}_-)$–invariants for the adjoint action on V,*

$$V_\chi = \{v \in V : \mathrm{Ad}\ x(v) = \varepsilon_s(x)v \quad \forall x \in U_{\eta_1}(\mathfrak{m}_-)\}. \tag{6.2.12}$$

Proof. Indeed, denote by T_β the factor $\prod_{i=1}^l K_i^{c_i}\prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j)c_i}$ which appears in (6.2.11), $T_\beta = \prod_{i=1}^l K_i^{c_i}\prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j)c_i}$. Then similarly to (2.7.15) we obtain

$$S_s(f_\beta) = -S_s(T_\beta)f_\beta - \sum_i S_s(y_i)x_i. \tag{6.2.13}$$

Now for $\beta \in \Delta_{\mathfrak{m}_+}$, (6.2.11), (6.2.13) and definition (6.2.9) of the adjoint action imply

$$\begin{aligned}\mathrm{Ad}\ f_\beta v = T_\beta^{-1}\chi(f_\beta)v - T_\beta^{-1}f_\beta v - \sum_i S_s(y_i)x_iv + \sum_i S_s(y_i)\chi(x_i)v = \\ = T_\beta^{-1}(\chi(f_\beta) - f_\beta)v + \sum_i S_s(y_i)(\chi(x_i) - x_i)v,\ x_i \in U_{<\beta}, y_i \in U_{>\beta}U_\varepsilon^s(\mathfrak{h}).\end{aligned} \tag{6.2.14}$$

If $v \in V_\chi$ we immediately obtain from (6.2.14) that $\mathrm{Ad}\ f_\beta v = 0$ for any $\beta \in \Delta_{\mathfrak{m}_+}$, i.e. v belongs to the right hand side of (6.2.12).

Conversely, suppose that v belongs to the right hand side of (6.2.12). We shall show that $xv = \chi(x)v$ for any $x \in U_\varepsilon^s(\mathfrak{m}_-)$. Let $\overline{U}_{<\beta}$ be the subalgebra with unit generated by $U_{<\beta}$. We proceed by induction over the subalgebras $\overline{U}_{<\delta_k}$, $k = 1, \ldots b+1$, where as before $\delta_1 < \ldots < \delta_b$ is the normally ordered segment $\Delta_{\mathfrak{m}_+}$ and we define $\overline{U}_{<\delta_{b+1}}$ to be the subalgebra $U_\varepsilon^s(\mathfrak{m}_-)$.

Observe that δ_1 is a simple root and hence $U_{<\delta_1} = 0$. Therefore we deduce from (6.2.14) for $\beta = \delta_1$

$$\mathrm{Ad}\ f_{\delta_1} v = T_{\delta_1}^{-1}(\chi(f_{\delta_1}) - f_{\delta_1})v = 0.$$

Since $T_{\delta_1}^{-1}$ acts on V by an invertible transformation this implies $(\chi(f_{\delta_1}) - f_{\delta_1})v = 0$, and hence $xv = \chi(x)v$ for any $x \in \overline{U}_{<\delta_2}$ as $\overline{U}_{<\delta_2}$ is generated by f_{δ_1}.

Now assume that for some $k \leq b$ $xv = \chi(x)v$ for any $x \in \overline{U}_{<\delta_k}$. Then by (6.2.14)

$$\mathrm{Ad}\ f_{\delta_k} v = T_{\delta_k}^{-1}(\chi(f_{\delta_k}) - f_{\delta_k})v = 0.$$

As above this implies

$$(\chi(f_{\delta_k}) - f_{\delta_k})v = 0. \tag{6.2.15}$$

By Proposition 2.6.2 any element $y \in \overline{U}_{<\delta_{k+1}}$ can be uniquely represented in the form $y = f_{\delta_k}y' + y''$, where $y', y'' \in \overline{U}_{<\delta_k}$. Now by (6.2.15) and by the induction assumption

$$yv = (f_{\delta_k}y' + y'')v = \chi(f_{\delta_k})\chi(y')v + \chi(y'')v = \chi(y)v,$$

i.e. $yv = \chi(y)v$ for any $x \in \overline{U}_{<\delta_{k+1}}$. This establishes the induction step and completes the proof.

□

The following proposition is an analogue of the Engel theorem for quantum groups at roots of unity.

Proposition 6.2.4. *Let η be an element of* Spec(Z_0). *Assume that $\eta(f_{\gamma_i}^m) = a_i \neq 0$ for $i = 1, \ldots, l'$ and that $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, and hence $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i = 1, \ldots, l'$ and $f_\beta^m = 0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Let $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ be any character defined in Proposition 6.2.2. Then any non–zero finite–dimensional $U_\eta(\mathfrak{g})$–module contains a non–zero Whittaker vector.*

Proof. We begin the proof with the following lemma.

Lemma 6.2.5. *The augmentation ideal $\mathcal{J}^1$ of $U_{\eta_1}(\mathfrak{m}_-)$ coincides with its Jacobson radical which is nilpotent.*

Proof. The proof of this fact is similar to that of Lemma 6.2.1, and we shall keep the notation used in that proof.

We define $\mathcal{J}^1_{(k,i)} = \mathcal{J}^1 \cap (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$, so that $\mathcal{J}^1_{(k,i)} \subset \mathcal{J}^1_{(k',j)}$ if $(k,i) < (k',j)$, and $\mathcal{J}^1_{(m-1,c)} = \mathcal{J}^1$.

We shall prove that $\mathcal{J}^1$ is nilpotent by induction over the order in $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$. Note that $(k,i) = (1,1)$ is minimal possible such that $\mathcal{J}_{(k,i)}$ is not trivial. If $y \in \mathcal{J}_{(1,1)}$ then y must be of the form

$$y = af_{\beta_1}, a \in \mathbb{C}. \tag{6.2.16}$$

The product of m elements of type (6.2.16) is equal to zero,

$$f_{\beta_1}^m a^m = 0,$$

as $f_{\beta_1}^m = 0$ in $U_{\eta_1}(\mathfrak{m}_-)$. We deduce that $(\mathcal{J}^1_{(1,1)})^m = 0$.

Now assume that $(\mathcal{J}^1_{(k,i)})^K = 0$ for some $K > 0$. Let (k', i') be the smallest element of $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$ which satisfies $(k,i) < (k',i')$. Then by Propositions 2.6.2 and 6.1.3 any element of $\mathcal{J}^1_{(k',i')}$ is of the form $f_{\beta_{i'}}u + u'$, where $u' \in \mathcal{J}^1_{(k,i)}$ and $u \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$.

Now equation (2.7.11) implies that for any $u \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$ one has

$$uf_{\beta_{i'}} = bf_{\beta_{i'}}u + w, \tag{6.2.17}$$

where b is a non–zero constant depending on u, and $w \in \mathcal{J}^1_{(k,i)}$. By the formula (6.2.17) the product of m elements $f_{\beta_{i'}}u_p + u'_p$, $p = 1, \ldots, m$ of the type described above can be represented in the form

$$\sum_{j=0}^{m} f_{\beta_{i'}}^j c_j, \tag{6.2.18}$$

where $c_j \in \mathcal{J}^1_{(k,i)}$ for $j = 0, \ldots, m-1$ and $c_m \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$. Since $f^m_{\beta_{i'}} = 0$ the last term in sum (6.2.18) is zero. So sum (6.2.18) takes the form

$$\sum_{j=0}^{m-1} f^j_{\beta_{i'}} c'_j, \tag{6.2.19}$$

where $c'_j \in \mathcal{J}^1_{(k,i)}$. By (6.2.17) the product of K sums of type (6.2.19) is of the form

$$\sum_{j=0}^{(m-1)K} f^j_{\beta_{i'}} c''_j,$$

where each c''_j is a linear combination of elements from $(\mathcal{J}^1_{(k,i)})^K$. By our assumption $(\mathcal{J}^1_{(k,i)})^K = 0$, and hence the product of any mK elements of $\mathcal{J}^1_{(k',i')}$ is zero. This justifies the induction step and proves that $\mathcal{J}^1_{(m-1,c)} = \mathcal{J}^1$ is nilpotent. Hence $\mathcal{J}^1$ is contained in the Jacobson radical of $U_{\eta_1}(\mathfrak{m}_-)$.

The quotient algebra $U_{\eta_1}(\mathfrak{m}_-)/\mathcal{J}^1$ is isomorphic to $\mathbb{C}$. Therefore $\mathcal{J}^1$ coincides with the Jacobson radical of $U_{\eta_1}(\mathfrak{m}_-)$.

□

Now let V be a finite–dimensional $U_\eta(\mathfrak{g})$–module. Then V is also a finite–dimensional $U_{\eta_1}(\mathfrak{m}_-)$–module with respect to the adjoint action. Thus V must contain a non–trivial irreducible $U_{\eta_1}(\mathfrak{m}_-)$–submodule with respect to the adjoint action on which the Jacobson radical $\mathcal{J}^1$ must act trivially. From (6.2.12) it follows that this non–trivial irreducible submodule consists of Whittaker vectors. This completes the proof of the proposition.

□

Now we show that for any $\eta \in \mathrm{Spec}(Z_0)$ subalgebras and characters which appear in Propositions 6.2.2, 6.2.4 and in Lemma 6.2.3 indeed exist. Moreover, we shall see that to each $\eta \in \mathrm{Spec}(Z_0)$ one can associate a subalgebra of this type the dimension of which is equal to $m^{\frac{1}{2}\dim\, \mathcal{O}_{\pi\eta}}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_{\mathcal{C}}$.

Proposition 6.2.6. *Let*

$$G = \bigcup_{\mathcal{C}\in C(W)} G_{\mathcal{C}}.$$

be the Lusztig partition of G, $\eta \in \mathrm{Spec}(Z_0)$ be an element such that $\pi\eta \in G_{\mathcal{C}}$, $\mathcal{C} \in C(W)$ and $s^{-1} \in \mathcal{C}$. Let Δ^s_+ be the system of positive roots defined for s in Theorem 1.5.2, Δ_+ the corresponding system of positive roots associated to s, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime.

Then there exists a quantum coadjoint transformation $\widetilde{g}$ such that $\xi = \widetilde{g}\eta$ satisfies $\xi(f^m_{\gamma_i}) = a_i \neq 0$ for $i = 1, \ldots, l'$ and $\xi(f^m_\beta) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, where $f_\alpha \in U_\xi(\mathfrak{m}_-)$ are generators of the corresponding algebra $U_\xi(\mathfrak{m}_-) \subset U_\xi(\mathfrak{g})$. Let $\chi : U_\xi(\mathfrak{m}_-) \to \mathbb{C}$ be any character defined in Proposition 6.2.2. Then any finite–dimensional $U_\xi(\mathfrak{g})$–module contains a non–zero Whittaker vector with respect to the subalgebra $U_\xi(\mathfrak{m}_-)$ and the character χ, and any $U_\eta(\mathfrak{g})$–module contains a non–zero Whittaker vector with respect to the subalgebra $U^{\widetilde{g}}_\eta(\mathfrak{m}_-) = \tilde{g}^{-1}U_\xi(\mathfrak{m}_-)$ and the character $\chi^{\widetilde{g}}$ given by the composition of χ and $\widetilde{g}$, $\chi^{\widetilde{g}} = \chi \circ \widetilde{g} : U^{\widetilde{g}}_\eta(\mathfrak{m}_-) \to \mathbb{C}$.

Moreover, $\dim\, U_\xi(\mathfrak{m}_-) = \dim\, U^{\widetilde{g}}_\eta(\mathfrak{m}_-) = m^{\frac{1}{2}\dim\, \mathcal{O}_{\pi\eta}} = m^{\frac{1}{2}\dim\, \mathcal{O}_{\pi\xi}} = m^{\dim\, \mathfrak{m}_-}$*, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_{\mathcal{C}}$, and $\mathcal{O}_{\pi\xi}$ is the conjugacy class of $\pi\xi \in G_{\mathcal{C}}$.*

Proof. First observe that the system of positive roots Δ^s_+ satisfies the conditions of Theorem 1.5.2 when s is replaced with s^{-1}. Indeed, in the case of classical root systems its definition only depends on the spectral decomposition of $\mathfrak{h}$ under the action of s which is the same as the spectral decomposition of $\mathfrak{h}$ under the action of s^{-1}. In the case of exceptional root systems one has to note in addition that obviously $\dim\, \Sigma_s = \dim\, \Sigma_{s^{-1}}$, and hence all properties of Δ^s_+ used in the proof of Theorem 1.5.2 are satisfied if s is replaced with s^{-1} in the proof.

Let $\overline{N}_{s^{-1}} = \{\bar{n} \in \overline{N} : s^{-1}\bar{n}s \in N\}$. Applying Theorem 1.5.2 to s^{-1} and to the system of positive rootrs Δ^s_+ and swapping the roles of N and of $\overline{N}$ we deduce that all conjugacy classes in the stratum $G_{\mathcal{C}}$ intersect the variety $s^{-1}H_0\overline{N}_{s^{-1}}$ which is a subvariety of the transversal slice $\overline{\Sigma}_{s^{-1}} = s^{-1}Z\overline{N}_{s^{-1}}$ to the set of conjugacy classes in G. Note that $\overline{N}_{s^{-1}}$ is not a subgroup in N_+. But every element of $\bar{n}_{s^{-1}} \in \overline{N}_{s^{-1}}$ can be uniquely factorized as follows $\bar{n}_{s^{-1}} = \bar{n}^+_{s^{-1}}\bar{n}^-_{s^{-1}}$, $\bar{n}^\pm_{s^{-1}} \in \overline{N}_{s^{-1}} \cap N_\pm$, $\overline{N}_{s^{-1}} \cap N_- = \overline{N} \cap N_-$, $\overline{N}_{s^{-1}} \cap N_+ \subset M_+$, where $M_+ \subset N_+$ is the subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+$. Therefore every element $s^{-1}h_0\bar{n}_{s^{-1}} \in s^{-1}H_0\overline{N}_{s^{-1}}$ can be represented as follows $s^{-1}h_0\bar{n}_{s^{-1}} = s^{-1}h_0\bar{n}^+_{s^{-1}}\bar{n}^-_{s^{-1}}$, and conjugating by $\bar{n}^-_{s^{-1}}$ we obtain that $s^{-1}h_0\bar{n}_{s^{-1}}$ is conjugate to $\bar{n}^-_{s^{-1}}s^{-1}h_0\bar{n}^+_{s^{-1}}$.

Since the decomposition $s = s^1 s^2$ is reduced

$$s(\overline{N}_{s^{-1}} \cap N_-)s^{-1} = s(\overline{N} \cap N_-)s^{-1} \subset M_+.$$

Taking into account that H_0 normalizes M_+ we have $\bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} = s^{-1} h_0 h_0^{-1} s \bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} = s^{-1} h_0 m_s$, $m_s = h_0^{-1} s \bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} \in M_+$. We deduce that all conjugacy classes in the stratum $G_{\mathcal{C}}$ intersect the variety $s^{-1} H_0 M_+$.

Recall that by (3.4.25) with the roles of N_+ and N_- swapped we have $v = n' s m$ for some $n' \in N, m \in M_+$, where $v \in G$ is an element of the form

$$v = \prod_{i=1}^{l'} exp[t_i X_{-\gamma_i}],$$

$t_i \in \mathbb{C}$ are non–zero constants depending on the choice of the representative $s \in G$ and the product over roots is taken in the normal order (1.6.3) associated to s. We deduce that $s^{-1} = m v^{-1} n'$.

Now let $s^{-1} h_0 m_s, h_0 \in H_0, m_s \in M_+$ be an element of $s^{-1} H_0 M_+$. Using the previous expression for s^{-1} we can write $s^{-1} h_0 m_s = m u^{-1} n' h_0 m_s$. Conjugating this element by m^{-1} and recalling that H_0 normalizes N_+ we infer that $s^{-1} h_0 m_s$ is conjugate to

$$v^{-1} n' h_0 m_s m = v^{-1} h_0 n = \lambda_0 (h_0^{\frac{1}{2}} n, h_0^{-\frac{1}{2}} v),$$

where $n = h_0^{-1} n' m_s m \in N_+$, $h_0^{\frac{1}{2}} \in H_0$ is any element such that $h_0^{\frac{1}{2}} h_0^{\frac{1}{2}} = h_0$, λ_0 is defined immediately after Proposition 6.1.4. We deduce that all conjugacy classes in the stratum $G_{\mathcal{C}}$ intersect the variety $v^{-1} H_0 N_+$.

By part (iv) of Proposition 6.1.5 we conclude that if $\eta \in \mathrm{Spec}(Z_0)$ satisfies $\pi\eta \in G_{\mathcal{C}}$ then there is a quantum coadjoint transformation $\widetilde{g}$ such that $\widetilde{\pi}(\widetilde{g}\eta) = (h_0^{\frac{1}{2}} n, h_0^{-\frac{1}{2}} v)$ for some $n \in N_+$, $h_0^{\frac{1}{2}} \in H_0$.

Denote $\xi = \widetilde{g}\eta$. From the definition of the map $\widetilde{\pi}$ and of the element v it follows that

$$\exp(\xi(x^-_{\beta_D}) X_{-\beta_D}) \exp(\xi(x^-_{\beta_{D-1}}) X_{-\beta_{D-1}}) \ldots \exp(\xi(x^-_{\beta_1}) X_{-\beta_1}) = v^{-1}$$

which implies $\xi((X^-_{\gamma_i})^m) = -\frac{t_i}{(\varepsilon_{\gamma_i} - \varepsilon_{\gamma_i}^{-1})^m} \neq 0$ for $i = 1, \ldots, l'$ and that $\xi((X^-_\beta)^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$.

By the definition of the elements f_β with $\beta = \sum_{i=1}^l m_i \alpha_i$ we have $f_\beta = \prod_{i,j=1}^l L_j^{m_i n_{ij}} X^-_\beta$. Therefore the commutation relations between elements L_j and X^-_β imply that $f_\beta^m = c_\beta \prod_{i,j=1}^l L_j^{m m_i n_{ij}} (X^-_\beta)^m$, where c_β are non–zero constants, and hence $\xi(f_\beta^m) = c_\beta \prod_{i,j=1}^l \xi(L_j^m)^{m_i n_{ij}} \xi((X^-_\beta)^m)$. Since $\xi(L_j) \neq 0$ for $j = 1, \ldots, l$, $\xi((X^-_{\gamma_i})^m) = -\frac{t_i}{(\varepsilon_{\gamma_i} - \varepsilon_{\gamma_i}^{-1})^m} \neq 0$ for $i = 1, \ldots, l'$ and $\xi((X^-_\beta)^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$ we deduce $\xi(f^m_{\gamma_i}) = a_i \neq 0$ for $i = 1, \ldots, l'$ and $\xi(f^m_\beta) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Thus ξ satisfies the condition of Propositions 6.2.2 and 6.2.4. Let $U_{\widetilde{g}\eta}(\mathfrak{m}_-) = U_\xi(\mathfrak{m}_-)$ be the corresponding subalgebra in $U_\xi(\mathfrak{g})$.

Note that by Theorem 1.5.2 for any $g \in G_{\mathcal{C}}$ we have

$$\dim Z_G(g) = \dim \overline{\Sigma}_{s^{-1}},$$

where $Z_G(g)$ is the centralizer of g in G.

By the definition of $\overline{\Sigma}_{s^{-1}}$ we also have $\dim \overline{\Sigma}_{s^{-1}} = l(s) + 2D_0 + \dim \mathfrak{h}'^{\perp}$. Observe also that $\dim G = 2D + \dim \mathfrak{h}$ and $\dim \mathfrak{h} - \dim \mathfrak{h}'^{\perp} = \dim \mathfrak{h}' = l'$, and hence from (1.6.5) we deduce that $\dim \mathfrak{m}_- = D - D_0 - \frac{1}{2}(l(s) - l') = \frac{1}{2}(\dim G - \dim \overline{\Sigma}_{s^{-1}}) = \frac{1}{2}\dim \mathcal{O}_g$ and $\dim U_\eta^{\widetilde{g}}(\mathfrak{m}_-) = \dim U_{\widetilde{g}\eta}(\mathfrak{m}_-) = m^{\dim \mathfrak{m}_-} = m^{\frac{1}{2}\dim \mathcal{O}_g}$, where $\mathcal{O}_g$ is the conjugacy class of any $g \in G_{\mathcal{C}}$.

In particular, $\dim U_\eta^{\widetilde{g}}(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_{\mathcal{C}}$.

The remaining statements of this proposition are consequences of Proposition 6.2.4.

□

6.3 Skryabin equivalence for quantum groups at roots of unity and the proof of De Concini–Kac–Procesi conjecture

In this section we shall study the $U_{\eta_1}(\mathfrak{m}_-)$–action on finite–dimensional $U_\eta(\mathfrak{g})$–modules introduced in the previous section. We shall show that each such module is $U_{\eta_1}(\mathfrak{m}_-)$–cofree. Taking into account that $\dim U_{\eta_1}(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$ this will imply the De Concini–Kac–Procesi conjecture.

The main observation with the help of which we shall prove these statements is that the structure of $U_\eta(\mathfrak{g})$–modules is similar to that of a root of unity analogue Q_χ of the module $Q_{\mathcal{B}}^{loc}$, and the results of Propositions 4.6.1 and 4.6.7 can be specialized to $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ and transferred to Q_χ and, more generally, to any finite–dimensional $U_\eta(\mathfrak{g})$–module using root of unity analogues of the homomorphism ϕ. In particular, using images of the specializations $B^\varepsilon_{n_1\ldots n_c}$ of the elements $B_{n_1\ldots n_c}$ in finite–dimensional $U_\eta(\mathfrak{g})$–modules one can construct their $U_{\eta_1}(\mathfrak{m}_-)$–cofree bases.

To realize this program for any given $\eta \in \mathrm{Spec}(Z_0)$ we assume that Δ_+^s is the system of positive roots defined for s in Theorem 1.5.2, Δ_+ the corresponding system of positive roots associated to s, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime. Fix a quantum coadjoint transformation $\widetilde{g} \in \mathcal{G}$ as in Theorem 6.2.6 and denote $\xi = \widetilde{g}\eta \in \mathrm{Spec}(Z_0)$. Since according to (6.1.12) $\widetilde{g}$ gives rise to an isomorphism of the algebras $U_\eta(\mathfrak{g})$ and $U_\xi(\mathfrak{g})$ it suffices to consider the case of the algebra $U_\xi(\mathfrak{g})$.

Our first objective is to obtain root of unity analogues of Proposition 4.1.2 and Lemma 4.7.1. We start by introducing the notions required for the formulations of these statements. For indeterminate q these notions were introduced in Section 4.1.

Let χ be a character of $U_\xi(\mathfrak{m}_-)$ defined in Proposition 6.2.2, $\mathbb{C}_\chi$ the corresponding representation of $U_\xi(\mathfrak{m}_-)$. Denote by Q_χ the induced left $U_\xi(\mathfrak{g})$–module, $Q_\chi = U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{m}_-)} \mathbb{C}_\chi$. Q_χ can also be naturally regarded as a $U_\varepsilon^s(\mathfrak{g})$–module via the natural projection $U_\varepsilon^s(\mathfrak{g}) = U_\varepsilon(\mathfrak{g}) \to U_\xi(\mathfrak{g})$.

Let $\mathbb{C}_\varepsilon^{loc}[G]$, $\mathbb{C}_\varepsilon^{loc}[G_*]$, $\mathbb{C}_\varepsilon[G^*]$, $B^\varepsilon_{n_1\ldots n_c}$, $\mathbb{C}_1^{loc}[G]_\varepsilon$ be the natural specializations at $q^{\frac{1}{rd^2}} = \varepsilon^{\frac{1}{rd^2}}$ of $\mathbb{C}_{\mathcal{B}}^{loc}[G]$, $\mathbb{C}_{\mathcal{B}}^{loc}[G_*]$, $\mathbb{C}_{\mathcal{B}}[G^*]$, $B_{n_1\ldots n_c}$, $\mathbb{C}_1^{loc}[G]$, respectively.

Define the twisted adjoint action of $U_\varepsilon^{s,res}(\mathfrak{g})$ on $\mathbb{C}_\varepsilon[G]$ by

$$(\mathrm{Ad}_0 x f)(w) = f(\omega_0 S_s^{-1}(\mathrm{Ad}' x(S_s\omega_0 w))) = f((\omega_0 S_s^{-1})(x^1) w \omega_0(x^2)), f \in \mathbb{C}_\varepsilon[G], x, w \in U_\varepsilon^{s,res}(\mathfrak{g}). \quad (6.3.1)$$

Specializing isomorphism (3.2.19) at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ and twisting it by $\omega_0 S_s^{-1}$ we obtain a $U_\varepsilon^{s,res}(\mathfrak{g})$–module homomorphism

$$\varphi_\varepsilon : \mathbb{C}_\varepsilon[G] \to \mathbb{C}_\varepsilon[G_*], f \mapsto (id \otimes f)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s). \quad (6.3.2)$$

Observe that the subalgebra in $U_\varepsilon^{s,res}(\mathfrak{g})$ generated by $\tilde{f}_\beta$, $\beta \in \Delta_{\mathfrak{m}_+}$ is isomorphic to $U_{\eta_1}(\mathfrak{m}_-)$. Therefore composing homomorphism (6.3.2) with the natural projection $\mathbb{C}_\varepsilon[G_*] \subset U_\varepsilon^s(\mathfrak{g}) \to U_\xi(\mathfrak{g}) \to Q_\chi$ we obtain a homomorphism of $U_{\eta_1}(\mathfrak{m}_-)$–modules

$$\phi_\xi : \mathbb{C}_\varepsilon[G] \to Q_\chi, \phi_\xi(f) = \varphi_\varepsilon(f)1, \quad (6.3.3)$$

where $\mathbb{C}_\varepsilon[G]$ is equipped with the restriction of action (6.3.1) to $U_{\eta_1}(\mathfrak{m}_-)$ and Q_χ with the action induced by the adjoint action Ad of $U_{\eta_1}(\mathfrak{m}_-)$ and 1 is the image of $1 \in \mathbb{C}_\varepsilon[G_*]$ in Q_ξ under the map $\mathbb{C}_\varepsilon[G_*] \to Q_\xi$.

Similarly, for any finite–dimensional $U_\xi(\mathfrak{g})$–module V and any $w \in V_\chi$ one can define a $U_{\eta_1}(\mathfrak{m}_-)$–module homomorphism $\phi_\xi^w : \mathbb{C}_\varepsilon[G] \to V$ by

$$\phi_\xi^w : \mathbb{C}_\varepsilon[G] \to V, \phi_\xi(f) = \varphi_\varepsilon(f)w.$$

Proposition 6.3.1. *The specialization $\overline{I}_\varepsilon^1 \subset \mathbb{C}_\varepsilon[G]$ of the left ideal $\overline{I}_{\mathcal{B}}^1 \subset \mathbb{C}_{\mathcal{B}}[G]$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ lies in the kernel of ϕ_ξ^w.*

Moreover, if u is a highest weight vector in the specialization at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ of a finite rank indecomposable representation V_λ of $U_{\mathcal{B}}^{res}(\mathfrak{g})$ of highest weight λ and such that $(u,u) = 1$ then for any $f \in \mathbb{C}_\varepsilon[G]$

$$\phi_\xi^w(f \otimes (u, \cdot T_s^{-1} u)) = \varphi_\varepsilon(\mathrm{Ad}_0(\varepsilon^{-(nd\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))\phi_\xi^w((u, \cdot T_s^{-1}u)), \quad (6.3.4)$$

where $\mathrm{Ad}_0(\varepsilon^{-(nd\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f)$ is the adjoint action of the element

$$\varepsilon^{-(nd\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee} = q^{-(nd\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee} \pmod{(q^{\frac{1}{r^2d}} - \varepsilon^{\frac{1}{r^2d}})}, \varepsilon^{-(nd\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee} \in \mathbb{C}_\varepsilon[G^*]$$

on $f \in \mathbb{C}_\varepsilon[G]$.

One can define an action of an operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ on the image of $\mathbb{C}_\varepsilon[G]$ in V by the formula

$$\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\phi_\xi^w(f) = \varphi_\varepsilon(\mathrm{Ad}_0(\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f))\phi_\xi^w((u, \cdot T_s^{-1}u)), \quad (6.3.5)$$

where $\mathrm{Ad}_0(\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f)$ is the adjoint action of the element

$$\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} = q^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \pmod{(q^{\frac{1}{r^2d}} - \varepsilon^{\frac{1}{r^2d}})}, \varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in \mathbb{C}_\varepsilon[G^*]$$

on $f \in \mathbb{C}_\varepsilon[G]$.

Using this operator formula (6.3.4) can be rewritten as follows

$$\phi_\xi^w(f\otimes(u,\cdot T_s^{-1}u))=\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\phi_\xi^w(\mathrm{Ad}_0(\varepsilon^{(-nd\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f)). \tag{6.3.6}$$

The operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ *is invertible. More precisely, for some* $n_0\in\mathbb{N}$ *the action of* $(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0}$ *coincides with the action of the element*

$$\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}=q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\pmod{(q^{\frac{1}{r^2d}}-\varepsilon^{\frac{1}{r^2d}})},\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\in U_\varepsilon^s(\mathfrak{h}),$$

and hence $(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0m}=\xi(\varepsilon^{n_0m(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})=c_\lambda\in\mathbb{C}^*$.

ϕ_ξ^w *extends to a* $U_{\eta_1}(\mathfrak{m}_-)$*–module homomorphism* $\phi_\xi^w:\mathbb{C}_\varepsilon^{loc}[G]\to V$,

$$\phi_\xi^w(f\otimes\Delta_\lambda^{-1})=\varepsilon^{(\frac{(1-nd)s^{-1}+(1+nd)s-2}{1-s}P_{\mathfrak{h}'}\lambda^\vee,\lambda^\vee)}\left(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\right)^{-1}\phi_\xi^w(\mathrm{Ad}_0(\varepsilon^{(nd\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}-s^{-1})\lambda^\vee})(f)), \tag{6.3.7}$$

and the specialization $\overline{I}_\varepsilon^{1\,loc}\subset\mathbb{C}_\varepsilon^{loc}[G]$ *of* $\overline{I}_{\mathcal{B}}^{1\,loc}\subset\mathbb{C}_{\mathcal{B}}^{loc}[G]$ *at* $q^{\frac{1}{dr^2}}=\varepsilon^{\frac{1}{dr^2}}$ *belongs to the kernel of this homomorphism, so*

$$\phi_\xi^w:\mathbb{C}_1^{loc}[G]_\varepsilon\to V.$$

Proof. In order to prove this proposition one can apply Proposition 4.1.1 for $\kappa=nd$ and $q^{\frac{1}{dr^2}}$ specialized to $\varepsilon^{\frac{1}{dr^2}}$, and the appropriately modified arguments before Proposition 4.1.2 and from the proof of Lemma 4.7.1.

Indeed, by the definition of $I_{\mathcal{B}}^{\mathbf{k}}$ with $\kappa=nd$, $k_i=c_i$, $i=1,\ldots,l'$ for c_i, $i=1,\ldots,l'$ used in the definition of χ, the specialization of $I_{\mathcal{B}}^{\mathbf{k}}$ at $q^{\frac{1}{dr^2}}=\varepsilon^{\frac{1}{dr^2}}$ belongs to the annihilator of w, and hence by Proposition 4.1.1 the specialization $\overline{I}_\varepsilon^1\subset\mathbb{C}_\varepsilon[G]$ of the left ideal $\overline{I}_{\mathcal{B}}^1\subset\mathbb{C}_{\mathcal{B}}[G]$ at $q^{\frac{1}{dr^2}}=\varepsilon^{\frac{1}{dr^2}}$ lies in the kernel of ϕ_ξ^w.

Formulas (6.3.4), (6.3.5) and (6.3.6) are obtained by specializing the corresponding formulas in Proposition 4.1.1 at $q^{\frac{1}{dr^2}}=\varepsilon^{\frac{1}{dr^2}}$ with $\kappa=nd$.

The only essential difference is that the operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ is invertible and for some $n_0\in\mathbb{N}$ its action coincides with the action of the element $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\in U_\varepsilon^s(\mathfrak{h})$. This can be justified as follows.

Recall that elements $\gamma_i^\vee$, $i=1,\ldots,l'$ form a linear basis of $\mathfrak{h}'$. Let γ_i^*, $i=1,\ldots,l'$ be the basis of $\mathfrak{h}'$ dual to $\gamma_i^\vee$, $i=1,\ldots,l'$ with respect to the restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{h}'$. Since the numbers $(\gamma_i^\vee,\gamma_j^\vee)$ are integer each element γ_i^* has the form $\gamma_i^*=\sum_{j=1}^{l'}m_{ij}\gamma_j^\vee$, where $m_{ij}\in\mathbb{Q}$. Therefore $P_{\mathfrak{h}'}\lambda^\vee=\sum_{p=1}^{l'}(\lambda^\vee,\gamma_p^\vee)\gamma_p^*=\sum_{p,q=1}^{l'}(\lambda^\vee,\gamma_p^\vee)m_{pq}\gamma_q^\vee$ belongs to the rational span of the set of simple coweights Y_i, $i=1,\ldots,l$, and hence $(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee$ belongs to the rational span of the set of simple coweights Y_i, $i=1,\ldots,l$ as well. We conclude that there exists an integer $n_0\in\mathbb{N}$ such that $n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee$ belongs to the integer span of the set of simple coweights Y_i, $i=1,\ldots,l$, and $q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\in U_{\mathcal{B}}^s(\mathfrak{h})$. So if we define $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}=q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\mod(q^{\frac{1}{r^2d}}-\varepsilon^{\frac{1}{r^2d}})$ then $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\in U_\varepsilon^s(\mathfrak{h})$.

Now using (6.3.4) and (6.3.5) one immediately verifies that

$$(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0}\phi_\xi^w(f)=\varphi_\varepsilon(\mathrm{Ad}_0(\varepsilon^{-n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f))\phi_\xi^w((u,\cdot T_s^{-1}u)^{n_0})$$

and that

$$\phi_\xi^w((u,\cdot T_s^{-1}u)^{n_0})=\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}w.$$

Recalling the equivariance of φ_ε with respect to the action of $U_\varepsilon^{s,res}(\mathfrak{g})\supset U_\varepsilon^s(\mathfrak{h})$ we obtain from the last two identities that

$$(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0}\phi_\xi^w(f)=$$
$$=\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\varphi_\varepsilon(f)\varepsilon^{-n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}w=$$
$$=\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\varphi_\varepsilon(f)w=\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\phi_\xi^w(f).$$

Finally applying verbatim the arguments from the proof of Lemma 4.7.1 we deduce that ϕ_ξ^w extends to a $U_{\eta_1}(\mathfrak{m}_-)$–module homomorphism $\phi_\xi^w:\mathbb{C}_\varepsilon^{loc}[G]\to V$ in such a way that (6.3.7) holds, and the specialization $\overline{I}_\varepsilon^{1\,loc}\subset\mathbb{C}_\varepsilon^{loc}[G]$ of $\overline{I}_{\mathcal{B}}^{1\,loc}\subset\mathbb{C}_{\mathcal{B}}^{loc}[G]$ at $q^{\frac{1}{dr^2}}=\varepsilon^{\frac{1}{dr^2}}$ belongs to the kernel of this homomorphism, so

$$\phi_\xi^w:\mathbb{C}_1^{loc}[G]_\varepsilon\to V.$$

Note that due to invertibility of the operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ no localization of V, which appears in Lemma 4.7.1 for $Q_{\mathcal{B}}$ in the case of indeterminate q, is required.

□

Now we define root of unity counterparts of q-W–algebras. Let $W^s_{\varepsilon,\xi}(G) = \mathrm{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp}$ be the algebra of $U_\xi(\mathfrak{g})$–endomorphisms of Q_χ with the opposite multiplication. The algebra $W^s_{\varepsilon,\xi}(G)$ is also called a q-W–algebra associated to $s \in W$ and to $\xi \in \mathrm{Spec}(Z_0)$. Denote by $U_\xi(\mathfrak{g}) - \mathrm{mod}$ the category of finite–dimensional left $U_\xi(\mathfrak{g})$–modules and by $W^s_{\varepsilon,\xi}(G) - \mathrm{mod}$ the category of finite–dimensional left $W^s_{\varepsilon,\xi}(G)$–modules. Observe that if $V \in U_\xi(\mathfrak{g})-\mathrm{mod}$ then the algebra $W^s_{\varepsilon,\xi}(G)$ naturally acts on the finite–dimensional space $V_\chi = \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, V) = \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V)$ by compositions of homomorphisms.

The following theorem is a root of unity analogue of the Skryabin equivalence for equivariant modules over quantum groups. This theorem uncovers some striking similarity between the structure of the category of finite–dimensional representations of $U_\xi(\mathfrak{g})$ and of the category of equivariant modules over a quantum group for generic ε.

Theorem 6.3.2. *Every module $V \in U_\xi(\mathfrak{g})-\mathrm{mod}$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ as a right $U_{\eta_1}(\mathfrak{m}_-)$–module, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the left. Q_χ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-)) \otimes W^s_{\varepsilon,\xi}(G)$ as a $U_{\eta_1}(\mathfrak{m}_-)$–$W^s_{\varepsilon,\xi}(G)$–bimodule. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–injective, $\mathrm{Ext}^\bullet_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, V) = V_\chi$ and the dimension of V is divisible by $\dim U_\xi(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\xi}}$.*

The functor $E \mapsto Q_\chi \otimes_{W^s_{\varepsilon,\xi}(G)} E$ establishes an equivalence of the category of finite–dimensional left $W^s_{\varepsilon,\xi}(G)$–modules and the category $U_\xi(\mathfrak{g}) - \mathrm{mod}$. The inverse equivalence is given by the functor $V \mapsto V_\chi$. In particular, the latter functor is exact, and every finite–dimensional $U_\xi(\mathfrak{g})$–module is generated by Whittaker vectors.

Proof. Let V be an object in the category $U_\xi(\mathfrak{g}) - \mathrm{mod}$. Fix any linear map $\rho : V \to V_\chi$ the restriction of which to V_χ is the identity map, and let for any $v \in V$ $\sigma_\varepsilon(v) : U_{\eta_1}(\mathfrak{m}_-) \to V_\chi$ be the $\mathbb{C}$–linear homomorphism given by $\sigma_\varepsilon(v)(x) = \rho(\mathrm{Ad}x(v))$, and we have a map $\sigma_\varepsilon : V \to \mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$.

By definition σ_ε is a homomorphism of right $U_{\eta_1}(\mathfrak{m}_-)$–modules, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on

$$\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$$

is induced by multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the left.

We claim that σ_ε is an isomorphism. Firstly, σ_ε is injective for otherwise its kernel would contain a non–zero Whittaker vector. Indeed by Lemma 6.2.5 the augmentation ideal of $U_{\eta_1}(\mathfrak{m}_-)$ coincides with its Jacobson radical which is nilpotent. Therefore its action on the kernel of σ_ε is nilpotent, and hence the kernel, if it is non–trivial, must contain a non–zero Whittaker vector annihilated by the augmentation ideal of $U_{\eta_1}(\mathfrak{m}_-)$. But all non–zero Whittaker vectors in V belong to V_χ and by the definition of σ_ε their images in $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ are non–zero homomorphisms non-vanishing at 1.

Next we show that σ_ε is also surjective. Indeed, for $n_1, \ldots, n_c = 1, \ldots m-1$ and any $v \in V_\chi$ the elements $v^{n_1\ldots n_c} = \prod_{p=1}^c \varepsilon_{\beta_p}^{-\frac{n_p(n_p-1)}{2}} \frac{1}{[n_p]_{\varepsilon_{\beta_p}}!}\phi^v_\xi(B^\varepsilon_{n_1\ldots n_d})$ are well–defined and Proposition 4.6.7 for $k_1, \ldots, k_c = 1, \ldots m-1$ implies

$$\mathrm{Ad}(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c})v^{n_1\ldots n_c} = \begin{cases} v & \text{if} \quad n_p = k_p \quad \text{for} \quad p = 1, \ldots, c \\ 0 & \text{if} \quad k_i = n_i, i = 1, \ldots, p-1 \quad \text{and} \quad k_p > n_p \quad \text{for} \quad \text{some} \quad p \in \{1, \ldots, c\} \end{cases},$$

and hence

$$\sigma_\varepsilon(v^{n_1\ldots n_c})(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}) = \begin{cases} v & \text{if} \quad n_p = k_p \quad \text{for} \quad p = 1, \ldots, c \\ 0 & \text{if} \quad k_i = n_i, i = 1, \ldots, p-1 \quad \text{and} \quad k_p > n_p \quad \text{for} \quad \text{some} \quad p \in \{1, \ldots, c\} \end{cases}. \tag{6.3.8}$$

Observe that the elements $f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}$, $k_1, \ldots, k_c = 1, \ldots m-1$ form a linear basis of $U_{\eta_1}(\mathfrak{m}_-)$. Elements of this basis are labeled by elements of the set $\mathbb{N}_m^c$, where $\mathbb{N}_m = \{0, 1, \ldots, m-1\}$. Introduce the lexicographic order on this set, so that $(k_1, \ldots, k_c) > (n_1, \ldots, n_c)$ if $k_i = n_i$ for $i = 1, \ldots, p-1$ and $k_p > n_p$ for some $p \in \{1, \ldots, c\}$.

Note that for any $(k_1, \ldots, k_c) \in \mathbb{N}_m^c$ the number of elements $(n_1, \ldots, n_c) \in \mathbb{N}^c$ such that $(k_1, \ldots, k_c) > (n_1, \ldots, n_c)$ is finite.

Now let $(k_1, \ldots, k_c) \in \mathbb{N}_m^c$, $v \in V_\chi$. If for $(n_1, \ldots, n_c) \in \mathbb{N}_m^c$ such that $(k_1, \ldots, k_c) \geq (n_1, \ldots, n_c)$ we denote

$$\sigma_\varepsilon(v_{n_1\ldots n_c}^{n_1\ldots n_c}) = f_{n_1\ldots n_c},$$

where for $(k_1, \ldots, k_c) \geq (n_1, \ldots, n_c)$ $v_{n_1 \ldots n_c} \in V_\chi$ are defined by induction starting from $v_{k_1 \ldots k_c} = v$ as follows

$$v_{n_1 \ldots n_c} = - \sum_{(k_1,\ldots,k_c) \geq (n'_1,\ldots,n'_c) > (n_1,\ldots,n_c)} f_{n'_1 \ldots n'_c}(f_{\beta_1}^{n_1} \ldots f_{\beta_c}^{n_c}), \quad (6.3.9)$$

then using (6.3.8) one obtains

$$f_{n_1 \ldots n_c}(f_{\beta_1}^{n'_1} \ldots f_{\beta_c}^{n'_c}) = \begin{cases} v_{n_1 \ldots n_c} & \text{if } (n'_1, \ldots, n'_c) = (n_1, \ldots, n_c), \\ 0 & \text{if } (n'_1, \ldots, n'_c) > (n_1, \ldots, n_c) \end{cases} .$$

From this property and from (6.3.9) one immediately checks that if we define

$$f_v^{k_1 \ldots k_c} = \sum_{(k_1,\ldots,k_c) \geq (n_1,\ldots,n_c)} f_{n_1 \ldots n_c}$$

then for any $(n_1, \ldots, n_c) \in \mathbb{N}_m^c$

$$f_v^{k_1 \ldots k_c}(f_{\beta_1}^{n_1} \ldots f_{\beta_c}^{n_c}) = \begin{cases} v & \text{if } (n_1, \ldots, n_c) = (k_1, \ldots, k_c), \\ 0 & \text{if } (n_1, \ldots, n_c) \neq (k_1, \ldots, k_c) \end{cases} . \quad (6.3.10)$$

Since the elements $f_{\beta_1}^{k_1} \ldots f_{\beta_c}^{k_c}$, $k_1, \ldots, k_c = 1, \ldots m-1$ form a linear basis of $U_{\eta_1}(\mathfrak{m}_-)$ (6.3.10) implies that the elements $f_v^{k_1 \ldots k_c}$ with $(k_1, \ldots, k_c) \in \mathbb{N}_m^c$, $v \in V_\chi$ generate $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$, and hence the elements $\sigma_\varepsilon(v^{n_1 \ldots n_c})$ with $(n_1, \ldots, n_c) \in \mathbb{N}_m^c$, $v \in V_\chi$ generate $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ as well. Therefore σ_ε is surjective, and the first part of the theorem is proved.

Let E be a finite–dimensional $W_{\varepsilon,\xi}^s(G)$–module. On order to establish the second claim of the theorem we observe that by the definition of the algebra $W_{\varepsilon,\xi}^s(G)$ we have $W_{\varepsilon,\xi}^s(G) = \mathrm{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp} = \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, Q_\chi) = (Q_\chi)_\chi$ as a linear space, and hence $(Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} E)_\chi = E$. Therefore to prove the second statement of the theorem it suffices to check that for any $V \in U_\xi(\mathfrak{g}) -$ mod the canonical map $f : Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi \to V$ is an isomorphism.

Indeed, f is injective because otherwise by Proposition 6.2.4 its kernel would contain a non–zero Whittaker vector with respect to χ. But all Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ belong to the subspace $1 \otimes V_\chi$, and the restriction of f to $1 \otimes V_\chi$ induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ and of V.

In order to prove that f is surjective we consider the exact sequence

$$0 \to Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi \to V \to W \to 0,$$

where W is the cokernel of f, and the corresponding long exact sequence of cohomology,

$$0 \to \mathrm{Ext}^0_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) \to \mathrm{Ext}^0_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, V) \to \mathrm{Ext}^0_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, W) \to$$

$$\to \mathrm{Ext}^1_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) \to \ldots .$$

Now recall that f induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ and of V. As we proved above the finite–dimensional $U_\xi(\mathfrak{g})$–module $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ is injective over $U_{\eta_1}(\mathfrak{m}_-)$, and hence $\mathrm{Ext}^1_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) = 0$. Therefore the initial part of the long exact cohomology sequence takes the form

$$0 \to V_\chi \to V_\chi \to W_\chi \to 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $W_\chi = 0$. But if W were non–trivial it would contain a non–zero Whittaker vector by Proposition 6.2.4. Thus $W = 0$, and f is surjective.

The proof of the fact that Q_χ is isomorphic to $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-)) \otimes W_{\varepsilon,\xi}^s(G)$ as a $U_{\eta_1}(\mathfrak{m}_-)$–$W_{\varepsilon,\xi}^s(G)$–bimodule is similar to the proof of an analogous statement in the case of generic ε (see last part of the proof of Theorem 5.2.1). This completes the proof of the theorem.

□

By the previous theorem every module $V \in U_\xi(\mathfrak{g}) -$ mod is isomorphic to

$$\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi) = \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), \mathbb{C}) \otimes V_\chi$$

as a right $U_{\eta_1}(\mathfrak{m}_-)$–module. In fact, one can show that the algebra $U_{\eta_1}(\mathfrak{m}_-)$ is Frobenius, i.e. its left regular representation is isomorphic to the dual of the right regular representation and its right regular representation is isomorphic to the dual of the left regular representation. Thus as a right $U_{\eta_1}(\mathfrak{m}_-)$–module V is isomorphic to $U_{\eta_1}(\mathfrak{m}_-)\otimes V_\chi$, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $U_{\eta_1}(\mathfrak{m}_-)\otimes V_\chi$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the right. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–free.

More generally, we have the following proposition.

Proposition 6.3.3. *For any character $\eta : Z_0 \to \mathbb{C}$ the algebra $U_\eta(\mathfrak{g})$ and its subalgebra $U_\eta(\mathfrak{m}_-)$ are Frobenius algebras.*

Proof. The proof of this proposition is parallel to the proof of a similar statement for Lie algebras over fields of prime characteristic (see Proposition 1.2 in [35]) and for the restricted form of the quantum group in [62]. We shall only briefly outline the main steps of the proof for $U_\eta(\mathfrak{g})$. The proof for $U_\eta(\mathfrak{m}_-)$ is similar.

The key ingredient of the proof is the De Concini-Kac filtration on $U_\varepsilon(\mathfrak{g})\simeq U_\varepsilon^s(\mathfrak{g})$ defined as in Section 5.1. For $\mathbf{r},\ \mathbf{t}\in\mathbb{N}^D$ introduce the element $u_{\mathbf{r},\mathbf{t},t}=e^{\mathbf{r}}tf^{\mathbf{t}}$, $t\in U_\varepsilon(\mathfrak{h})$, where we use the notation of Lemma 2.7.2. Here the generators $f_\alpha, e_\alpha, \alpha\in\Delta_+$ and the ordered products of them are defined with the help of the normal ordering of Δ_+ associated to s. Define also the height of the element $u_{\mathbf{r},\mathbf{t},t}$ as follows ht $(u_{\mathbf{r},\mathbf{t},t})=\sum_{i=1}^D(t_i+r_i)\mathrm{ht}\ \beta_i\in\mathbb{N}$, where ht β_i is the height of the root β_i. Introduce also the degree of $u_{\mathbf{r},\mathbf{t},t}$ by

$$d(u_{\mathbf{r},\mathbf{t},t})=(r_1,\ldots,r_D,t_D,\ldots,t_1,\mathrm{ht}\ (u_{\mathbf{r},\mathbf{t},t}))\in\mathbb{N}^{2D+1}.$$

Equip $\mathbb{N}^{2D+1}$ with the total lexicographic order and for $k\in\mathbb{N}^{2D+1}$ denote by $(U_\varepsilon(\mathfrak{g}))_k$ the span of elements $u_{\mathbf{r},\mathbf{t},t}$ with $d(u_{\mathbf{r},\mathbf{t},t})\leq k$ in $U_\varepsilon(\mathfrak{g})$. Then Proposition 1.7 in [20] implies that $(U_\varepsilon(\mathfrak{g}))_k$ is a filtration of $U_\varepsilon(\mathfrak{g})$ such that the associated graded algebra is the associative algebra over $\mathbb{C}$ with generators $e_\alpha,\ f_\alpha,\ \alpha\in\Delta_+,\ L_i^{\pm1},\ i=1,\ldots l$ subject to the relations

$$\begin{aligned}
&L_iL_j=L_jL_i,\quad L_iL_i^{-1}=L_i^{-1}L_i=1,\quad L_ie_\alpha L_i^{-1}=\varepsilon^{\alpha(Y_i)}e_\alpha,\quad L_if_\alpha L_i^{-1}=\varepsilon^{-\alpha(Y_i)}f_\alpha,\\
&e_\alpha f_\beta=\varepsilon^{nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}f_\beta e_\alpha,\\
&e_\alpha e_\beta=\varepsilon^{(\alpha,\beta)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}e_\beta e_\alpha,\ \alpha<\beta,\\
&f_\alpha f_\beta=\varepsilon^{(\alpha,\beta)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}f_\beta f_\alpha,\ \alpha<\beta.
\end{aligned}\tag{6.3.11}$$

Such algebras are called semi–commutative.

By Theorem 61.3 in [27] it suffices to show that there is a non–degenerate bilinear form $B_\eta : U_\eta(\mathfrak{g})\times U_\eta(\mathfrak{g})\to\mathbb{C}$ which is associative in the sense that

$$B_\eta(ab,c)=B_\eta(a,bc),\quad a,b,c\in U_\eta(\mathfrak{g}).$$

Consider the free Z_0–basis of $U_\varepsilon(\mathfrak{g})$ introduced in part (ii) of Proposition 6.1.3. This basis consists of the monomials $x_I=f^{\mathbf{r}}L^{\mathbf{s}}e^{\mathbf{t}}$, $I=(r_1,\ldots,r_D,s_1,\ldots,s_l,t_1,\ldots,t_D)$ for which $0\leq r_k,t_k,s_i<m$ for $i=1,\ldots,l$, $k=1,\ldots,D$. Set $I'=(m-1-r_1,\ldots,m-1-r_D,m-1-s_1,\ldots,m-1-s_l,m-1-t_1,\ldots,m-1-t_D)$ and $P=(m-1,\ldots,m-1)$.

Let $\Phi : U_\varepsilon(\mathfrak{g})\to Z_0$ be the Z_0–linear map defined on the basis x_I of monomials by

$$\Phi(x_I)=\begin{cases}1 & I=P\\ 0 & \text{otherwise}\end{cases}.$$

Let $x=\sum_I c_Ix_I, c_I\in Z_0$ be an element of $U_\varepsilon(\mathfrak{g})$, and $c_K\neq0$ a coefficient such that $d(x_K)$ is maximal possible with $c_K\neq0$ in the sum defining x.

Using the definition of the De Concini–Kac filtration and commutation relations (6.3.11) one can check that $\Phi(xx_{K'})=a_xc_K$, where a_x is a nonzero complex number (see [62], proof of Theorem 2.2, Assertion I for details).

Therefore the bilinear form $B_\eta : U_\eta(\mathfrak{g})\times U_\eta(\mathfrak{g})\to\mathbb{C}$ associated to the associative Z_0–bilinear pairing B : $U_\varepsilon(\mathfrak{g})\otimes_{Z_0}U_\varepsilon(\mathfrak{g})\to Z_0$, $B(x,y)=\Phi(xy)$ is non–degenerate and associative. This completes the proof.

□

We restate the results of the discussion before the previous proposition as its corollary.

Corollary 6.3.4. *As a right $U_{\eta_1}(\mathfrak{m}_-)$–module, every module $V\in U_\xi(\mathfrak{g})-\mathrm{mod}$ is isomorphic to $U_{\eta_1}(\mathfrak{m}_-)\otimes V_\chi$, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $U_{\eta_1}(\mathfrak{m}_-)\otimes V_\chi$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the right. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–free.*

6.4 Properties of q-W–algebras associated to quantum groups at roots of unity

In conclusion we study some further properties of q-W–algebras at roots of unity and of the module Q_χ. We keep the notation introduced in the previous section. First we prove the following lemma.

Lemma 6.4.1. *The left $U_\xi(\mathfrak{g})$–module Q_χ is projective in the category $U_\xi(\mathfrak{g})-\mathrm{mod}$.*

Proof. We have to show that the functor $\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi,\cdot)$ is exact. Let $V^\bullet$ be an exact complex of finite–dimensional $U_\xi(\mathfrak{g})$–modules. Since by the previous theorem any object V of $U_\xi(\mathfrak{g})-\mathrm{mod}$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-),V_\chi)$ as a right $U_{\eta_1}(\mathfrak{m}_-)$–module we have

$$V^\bullet \simeq \mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-),\overline{V}^\bullet),$$

where $\overline{V}^\bullet$ is an exact complex of vector spaces and the action of $U_{\eta_1}(\mathfrak{m}_-)$ on $\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-),\overline{V}^\bullet)$ is induced by multiplication from the left on $U_{\eta_1}(\mathfrak{m}_-)$.

Now by the Frobenius reciprocity we have obvious isomorphisms of complexes,

$$\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi,V^\bullet)\simeq \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi,V^\bullet)=\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi,V^\bullet)\simeq$$
$$\simeq \mathrm{Hom}_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon,\mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-),\overline{V}^\bullet))\simeq \mathrm{Hom}_{\mathbb{C}}(U_{\eta_1}(\mathfrak{m}_-)\otimes_{U_{\eta_1}(\mathfrak{m}_-)}\mathbb{C}_\varepsilon,\overline{V}^\bullet)=\overline{V}^\bullet,$$

where the last complex is exact, and we used the fact that by Lemma 6.2.3 for any finite–dimensional $U_\xi(\mathfrak{g})$–module V one has $\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi,V)\simeq \mathrm{Hom}_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon,V)$. We conclude that the functor $\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(Q_\chi,\cdot)$ is exact. □

The properties of q-W–algebras at roots of unity are summarized in the following proposition.

Proposition 6.4.2. *Denote $b=m^{\dim\,\mathfrak{m}_-}=m^{\frac{1}{2}\dim\,\mathcal{O}_{\pi\xi}}$. Then $Q_\chi^b\simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$–modules, $U_\xi(\mathfrak{g})\simeq \mathrm{Mat}_b(W^s_{\varepsilon,\xi}(G))$ as algebras and $Q_\chi\simeq (W^s_{\varepsilon,\xi}(G)^{opp})^b$ as right $W^s_{\varepsilon,\xi}(G)$–modules.*

Proof. Let E_i, $i=1,\ldots,C$ be the simple finite–dimensional modules over the finite–dimensional algebra $U_\xi(\mathfrak{g})$. Denote by P_i the projective cover of E_i. Since by Theorem 6.3.2 the dimension of E_i is divisible by b we have $\dim\, E_i=br_i$, $r_i\in\mathbb{N}$, where r_i is the rank of E_i over $U_{\eta_1}(\mathfrak{m}_-)$ equal to the dimension of the space of Whittaker vectors in E_i. By Proposition 2.1 in [82]

$$U_\xi(\mathfrak{g})=\mathrm{Mat}_b(\mathrm{End}_{U_\xi(\mathfrak{g})}(P)^{opp}),$$

where $P=\bigoplus_{i=1}^C P_i^{r_i}$. Therefore to prove the second statement of the proposition it suffices to show that $P\simeq Q_\chi$. Since by the previous lemma Q_χ is projective we only need to verify that

$$r_i=\dim\,\mathrm{Hom}_{U_\xi(\mathfrak{g})}(P,E_i)=\dim\,\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi,E_i).$$

Indeed, by the Frobenius reciprocity we have

$$\dim\,\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi,E_i)=\dim\,\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi,E_i)=r_i.$$

This proves the second statement of the proposition. From Proposition 2.1 in [82] we also deduce that $P^b\simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$–modules. Together with the isomorphism $P\simeq Q_\chi$ this gives the first statement of the proposition.

Using results of Section 6.4 in [80] and the fact that Q_χ is projective one can find an idempotent $e\in U_\xi(\mathfrak{g})$ such that $Q_\chi\simeq U_\xi(\mathfrak{g})e$ as modules and $(W^s_{\varepsilon,\xi}(G))^{opp}\simeq eU_\xi(\mathfrak{g})e$ as algebras.

By the first two statements of this proposition one can also find idempotents $e=e_1,e_2,\ldots,e_b\in U_\xi(\mathfrak{g})$ such that $e_1+\ldots+e_b=1$, $e_ie_j=0$ if $i\neq j$ and $e_iU_\xi(\mathfrak{g})\simeq eU_\xi(\mathfrak{g})$ as right $U_\xi(\mathfrak{g})$–modules. Therefore $e_iU_\xi(\mathfrak{g})e\simeq eU_\xi(\mathfrak{g})e$ as right $eU_\xi(\mathfrak{g})e$–modules, and

$$Q_\chi\simeq U_\xi(\mathfrak{g})e=\bigoplus_{i=1}^b e_iU_\xi(\mathfrak{g})e\simeq (eU_\xi(\mathfrak{g})e)^b\simeq (W^s_{\varepsilon,\xi}(G)^{opp})^b$$

as right $W^s_{\varepsilon,\xi}(G)$–modules. This completes the proof of the proposition

□

Corollary 6.4.3. *The algebra $W^s_{\varepsilon,\xi}(G)$ is finite–dimensional, and* $\dim W^s_{\varepsilon,\xi}(G) = m^{\dim \Sigma_s}$.

Proof. By Proposition 3.4.5 $2\dim \mathfrak{m}_- + \dim \Sigma_s = \dim G$. Therefore by the definition of Q_χ we have $\dim Q_\chi = m^{\dim G - \dim \mathfrak{m}_-} = m^{\dim \mathfrak{m}_- + \dim \Sigma_s}$. Finally from the last statement of the previous theorem one obtains that $\dim W^s_{\varepsilon,\xi}(G) = \dim Q_\chi / m^{\dim \mathfrak{m}_-} = m^{\dim \Sigma_s}$.

□

Using Proposition 6.1.1 we deduce from Proposition 6.4.2 the following statement on the structure of the algebra $U_\eta(\mathfrak{g})$.

Corollary 6.4.4. *Let $\eta \in \mathrm{Spec}(Z_0)$ be an element such that $\pi\eta \in G_{\mathcal{C}}$, $\mathcal{C} \in C(W)$ and $s^{-1} \in \mathcal{C}$, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime.*

Then $U_\eta(\mathfrak{g}) \simeq \mathrm{Mat}_b(W^s_{\varepsilon,\xi}(G))$, where $\xi \in \mathrm{Spec}Z_0$ is chosen as in Proposition 6.4.2, and $b = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$.

Let $\mathcal{L}$ be a sheaf of algebras over $\mathrm{Spec}Z_0$ the stalk of which over $\eta \in \mathrm{Spec}Z_0$ is $U_\eta(\mathfrak{g})$. Assume that the conditions imposed on m are satisfied for all Weyl group conjugacy classes in $C(W)$. Then the sheaf $\mathcal{L}$ is isomorphic to a sheaf the stalk of which over any $\eta \in \mathrm{Spec}Z_0$ with $\pi\eta \in G^0 \cap G_{\mathcal{C}}$, $\mathcal{C} \in C(W)$ is $\mathrm{Mat}_b(W^s_{\varepsilon,\xi}(G))$, where $\xi \in \mathrm{Spec}Z_0$ is chosen as in Proposition 6.4.2, $s^{-1} \in \mathcal{C}$, $b = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$.

6.5 Bibliographic comments

The study of representations of quantum groups at roots of unity was initiated in [20], where the quantum coadjoint action was defined as well. This action was studied in detail in [22] where the De Concini–Kac–Procesi conjecture on the dimensions of irreducible representations of quantum groups at roots of unity was formulated.

The results on quantum groups at roots of unity stated in Section 6.1 can be found in [20] and [22]. Proposition 6.1.1 first appeared in Appendix A to [103]. The statements of Proposition 6.1.2 can be found in [20], Corollary 3.3, [22], Theorems 3.5, 7.6 and Proposition 4.5. Proposition 6.1.4 is Corollary 4.7 in [22], and the statements of Proposition 6.1.5 appear in Propositions 3.4, 3.5, [20], and in Proposition 6.1 and Theorem 6.6 in [22]. Finite–dimensional quotients $U_\eta(\mathfrak{g})$ were introduced in [21].

The notions of Whittaker vectors for representations of quantum groups at roots of unity, of the algebras $U_{\eta_1}(\mathfrak{m}_-)$, and of their actions on finite–dimensional representations of $U_\eta(\mathfrak{g})$ were introduced in [102], and the exposition in Section 6.2 follows [102] as well.

In the representation theory of Lie algebras in prime characteristic there is a conjecture similar to the De Concini–Kac–Procesi conjecture. It is called the Kac–Weisfeiler conjecture. Our proof of the the De Concini–Kac–Procesi conjecture is conceptually similar to the proof of the Kac–Weisfeiler conjecture given in [105] which is in turn a straightforward prime characteristic generalization of the proof of the Skryabin equivalence for reductive Lie algebras over algebraically closed fields of zero characteristic suggested in the Appendix to [81]. All these proofs go back to the original Kostant's idea on the proof of the classification theorem for Whittaker representations of complex semisimple Lie algebras in [60], the proof of the Skryabin equivalence in [81] being a significantly refined and simplified version of the proof of the main Theorem 3.3 in [60].

The properties of q-W–algeras at roots of unity are similar to those of W–algebras associated to semisimple Lie algebras in prime characteristic proved in [82], Proposition 6.4.2 being an analogue of Theorem 2.3 in [82].

Appendix

Appendix 1. Normal orderings of root systems compatible with involutions in Weyl groups

By Theorem A in [86] every involution w in the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ is the longest element of the Weyl group of a Levi subalgebra in $\mathfrak{g}$ with respect to some system of positive roots, and w acts by multiplication by -1 in the Cartan subalgebra $\mathfrak{h}_w \subset \mathfrak{h}$ of the semisimple part $\mathfrak{m}_w$ of that Levi subalgebra. By Lemma 5 in [16] the involution w can also be expressed as a product of dim $\mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal roots, $w = s_{\gamma_1} \ldots s_{\gamma_n}$, and the roots $\gamma_1, \ldots, \gamma_n$ span the subalgebra $\mathfrak{h}_w$.

If w is the longest element in the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$ with respect to some system of positive roots, where $\mathfrak{m}_w$ is a simple Lie algebra and $\mathfrak{h}_w$ is a Cartan subalgebra of $\mathfrak{m}_w$, then w is an involution acting by multiplication by -1 in $\mathfrak{h}_w$ if and only if $\mathfrak{m}_w$ is of one of the following types: $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$.

Fix a system of positive roots $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$. Let $w = s_{\gamma_1} \ldots s_{\gamma_n}$ be a representation of w as a product of dim $\mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal positive roots. A normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ is called compatible with the decomposition $w = s_{\gamma_1} \ldots s_{\gamma_n}$ if it is of the following form

$$\beta_1, \ldots, \beta_{\frac{p-n}{2}}, \gamma_1, \beta^1_{\frac{p-n}{2}+2}, \ldots, \beta^1_{\frac{p-n}{2}+n_1}, \gamma_2, \beta^1_{\frac{p-n}{2}+n_1+2} \ldots, \beta^1_{\frac{p-n}{2}+n_2}, \gamma_3, \ldots, \gamma_n,$$

where p is the number of positive roots, and for any two positive roots $\alpha, \beta \in \Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ such that $\gamma_1 \leq \alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$.

Existence of such compatible normal orderings is checked straightforwardly for all simple Lie algebras of types $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4$ and G_2. In case A_1 this is obvious since there is only one positive root. In the other cases normal orderings defined by the properties described below for each of the types $B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$ exist and are compatible with decompositions of nontrivial involutions in Weyl group. We use the Bourbaki notation for the systems of positive and simple roots (see [9]).

- B_l

  Dynkin diagram:

  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$.

  Positive roots: ε_i $(1 \leq i \leq l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$.

  The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1} \ldots s_{\varepsilon_l}$.

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1} \ldots s_{\varepsilon_l}$:

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_l,$$

  where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(B_l)$ are situated to the left from ε_1, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from ε_1.

- C_l

  Dynkin diagram:

  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$.

  Positive roots: $2\varepsilon_i$ $(1 \leq i \leq l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$.

  The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$.

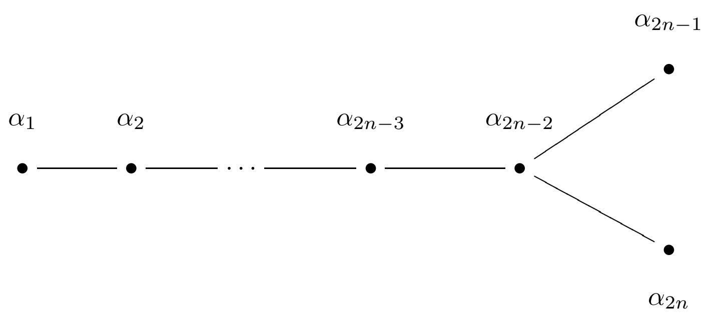
  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$:

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_1, \ldots, 2\varepsilon_2, \ldots, 2\varepsilon_l,$$

  where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(C_l)$ are situated to the left from $2\varepsilon_1$, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from $2\varepsilon_1$.

- D_{2n}

  Dynkin diagram:



  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{2n-1} = \varepsilon_{2n-1} - \varepsilon_{2n}, \alpha_{2n} = \varepsilon_{2n-1} + \varepsilon_{2n}$.

  Positive roots: $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 2n)$.

  The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}}.$$

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

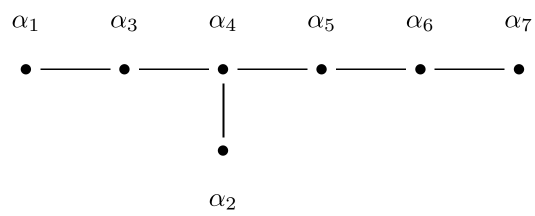
$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}} :$$

$$\varepsilon_2 - \varepsilon_3, \varepsilon_4 - \varepsilon_5, \ldots, \varepsilon_{2n-2} - \varepsilon_{2n-1}, \ldots, \varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_{2n-1} - \varepsilon_{2n-2},$$
$$\varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{2n-1} + \varepsilon_{2n},$$

  where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(C_l)$ are situated to the left from $\varepsilon_1 + \varepsilon_2$, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from $\varepsilon_1 + \varepsilon_2$.

- E_7

  Dynkin diagram:



Simple roots: $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5$.

Positive roots: $\pm\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 6)$, $\varepsilon_8 - \varepsilon_7$, $\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^{6}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{6}\nu(i)$ odd.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

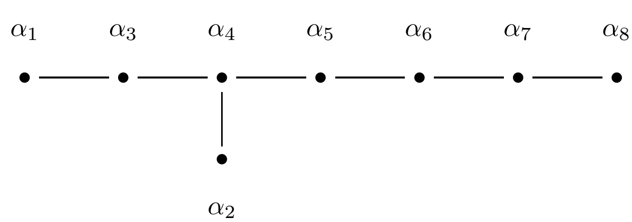
$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7} :$$

$$\alpha_1, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \ldots, \varepsilon_8 - \varepsilon_7, \ldots, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \ldots,$$
$$\varepsilon_6 + \varepsilon_5, \ldots, \varepsilon_4 + \varepsilon_3, \ldots, \varepsilon_2 + \varepsilon_1,$$

where the roots $\pm\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 6)$ forming the subsystem $\Delta_+(D_6) \subset \Delta_+(E_7)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_6)$, the only roots from the subsystem $\Delta_+(A_5) \subset \Delta_+(D_6)$ situated to the right from the maximal root $\varepsilon_8 - \varepsilon_7$ are $\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5$, the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 6)$ are situated to the right from $\varepsilon_6 + \varepsilon_5$, and a half of the positive roots which do not belong to the subsystem $\Delta_+(D_6) \subset \Delta_+(E_7)$ are situated to the left from $\varepsilon_8 - \varepsilon_7$ and the other half of those roots are situated to the right from $\varepsilon_8 - \varepsilon_7$.

- E_8

Dynkin diagram:



Simple roots: $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5, \alpha_8 = \varepsilon_7 - \varepsilon_6$.

Positive roots: $\pm\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 8)$, $\frac{1}{2}(\varepsilon_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{7}\nu(i)$ even.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}s_{\varepsilon_8+\varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

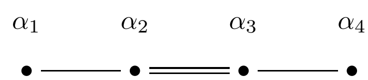
$$w = s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_2+\varepsilon_1}s_{\varepsilon_4-\varepsilon_3}s_{\varepsilon_4+\varepsilon_3}s_{\varepsilon_6-\varepsilon_5}s_{\varepsilon_6+\varepsilon_5}s_{\varepsilon_8-\varepsilon_7}s_{\varepsilon_8-\varepsilon_7} :$$

$$\alpha_1, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6, \ldots, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5,$$
$$\varepsilon_8 - \varepsilon_7, \ldots, \varepsilon_8 + \varepsilon_7, \ldots, \varepsilon_6 + \varepsilon_5, \ldots, \varepsilon_4 + \varepsilon_3, \ldots, \varepsilon_2 + \varepsilon_1,$$

where the roots $\pm\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 8)$ forming the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_8)$, the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 8)$ are situated to the right from $\varepsilon_8 + \varepsilon_7$; the positive roots which do not belong to the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ can be split into two groups: the roots from the first group contain $\frac{1}{2}(\varepsilon_8 + \varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$, and the roots from the second group contain $\frac{1}{2}(\varepsilon_8 - \varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$; a half of the roots from the first group are situated to the left from $\varepsilon_2 - \varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8 + \varepsilon_7$; a half of the roots from the second group are situated to the left from $\varepsilon_2 - \varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8 - \varepsilon_7$.

- F_4

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$.

Positive roots: ε_i $(1 \leq i \leq 4)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 4)$, $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$.

The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1} s_{\varepsilon_2} s_{\varepsilon_3} s_{\varepsilon_4}$.

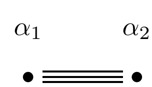
Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1} s_{\varepsilon_2} s_{\varepsilon_3} s_{\varepsilon_4}$:

$$\alpha_4, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_4,$$

where the roots $\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated as in case of B_4, and a half of the positive roots which do not belong to the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated to the left from ε_1 and the other half of those roots are situated to the right from ε_1.

- G_2

Dynkin diagram:



Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Positive roots: $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2$.

The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\alpha_1} s_{3\alpha_1 + 2\alpha_2}$.

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\alpha_1} s_{3\alpha_1 + 2\alpha_2}$:

$$\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, \alpha_1.$$

Appendix 2. Transversal slices for simple exceptional algebraic groups.

In this appendix, for simple exceptional algebraic groups we present the data related to the varieties $\Sigma_{s,\mathbf{k}}$ defined in Theorem 1.5.2. Let $G_{\mathbf{k}}$ be a connected simple algebraic group of an exceptional type over an algebraically closed field $\mathbf{k}$ of characteristic good for $G_{\mathbf{k}}$, and $\mathcal{O} \in \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$.

Let Δ be the root system of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$ and Δ_+^s the system of positive roots in Δ associated to s and defined in Section 1.2 with the help of decomposition (1.2.14) where the subspaces $\mathfrak{h}_i$ are ordered in such a way that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s, the one–dimensional subspaces $\mathfrak{h}_i$ on which s^1 acts by multiplication by -1 are immediately preceding $\mathfrak{h}_0$ in (1.2.14), and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$. We also use a decomposition $s = s^1 s^2$ for which the direct sum $\bigoplus_{k=0, i_k>0}^{r} \mathfrak{h}_{i_k}$ of the one–dimensional subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 is trivial. Such decomposition always exists. As a consequence condition (1.6.2) is satisfied. Let $\Sigma_{s,\mathbf{k}}$ be the corresponding variety defined in the beginning of Section 1.5.

Then straightforward calculation shows that

$$\dim\, Z_{G_p}(n) = \dim\, \Sigma_{s,\mathbf{k}}$$

for any $n \in \mathcal{O} \in \underline{\mathcal{N}}(G_p) \subset \widehat{\underline{\mathcal{N}}}(G)$. The numbers $\dim\, Z_{G_p}(n)$ can be found in [63], Chapter 22 (note, however, that the notation in [63] for some classes is different from ours; we follow [75, 108]). The numbers $\dim\, \Sigma_{s,\mathbf{k}}$ are contained in the tables below. These two numbers coincide in all cases. The tables below contain also the following information for each $\mathcal{O} \in \widehat{\underline{\mathcal{N}}}(G_{\mathbf{k}})$:
– The Weyl group conjugacy class $\Psi^W(\mathcal{O})$ which can be found in [75];
– The two involutions s^1 and s^2 in the decomposition $s = s^1 s^2 \in \Psi^W(\mathcal{O})$; they are represented by sets of natural numbers which are the numbers of roots appearing in decompositions $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, where the system of positive roots Δ_+^s is chosen as in Theorem 1.5.2, and the numeration of positive roots is given in Appendix 3;
– The dimension of the fixed point space $\mathfrak{h}_0$ for the action of s on $\mathfrak{h}$;
– The number $|\Delta_0|$ of roots fixed by s;
– The type of the root system Δ_0 fixed by s;
– The Dynkin diagram Γ_0^s of Δ_0, where the numbers at the vertices of Γ_0^s are the numbers of simple roots in Δ_+^s which appear in Γ_0^s; the numeration of simple roots is given in Appendix 3;
– The length $l(s)$ of s with respect to the system of simple roots in Δ_+^s;
– $\dim\, \Sigma_{s,\mathbf{k}} = \dim\, \mathfrak{h}_0 + |\Delta_0| + l(s)$;
– The lowest common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$;

$\mathbf{G_2}$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	$\dim\, \mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	$\dim\, \Sigma_{s,\mathbf{k}}$	d'
A_1	A_1	–	6	1	2	A_1	$\overset{1}{\bullet}$	5	8	1
$(\tilde{A}_1)_3$	$\tilde{A}_1$	–	4	1	2	A_1	$\overset{2}{\bullet}$	5	8	1
$\tilde{A}_1$	$A_1 + \tilde{A}_1$	–	1 6	0	0	–	–	6	6	1
$G_2(a_1)$	A_2	5	2	0	0	–	–	4	4	3
G_2	G_2	1	2	0	0	–	–	2	2	1

$\mathbf{F_4}$.

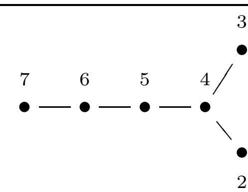
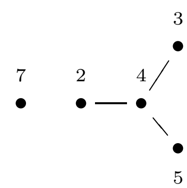
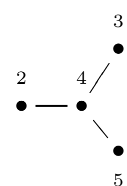
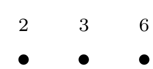
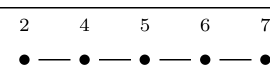
$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_1	A_1	–	24	3	18	C_3	$\overset{4}{\bullet}-\overset{3}{\bullet}=\overset{2}{\bullet}$	15	36	1
$(\tilde{A}_1)_2$	$\tilde{A}_1$	–	21	3	18	B_3	$\overset{1}{\bullet}-\overset{2}{\bullet}=\overset{3}{\bullet}$	15	36	1
$\tilde{A}_1$	$2A_1$	–	16 24	2	8	B_2	$\overset{2}{\bullet}=\overset{3}{\bullet}$	20	30	1
$A_1+\tilde{A}_1$	$4A_1$	–	5 11 18 23	0	0	–	–	24	24	1
A_2	A_2	23	1	2	6	A_2	$\overset{3}{\bullet}-\overset{4}{\bullet}$	14	22	3
$\tilde{A}_2$	$\tilde{A}_2$	19	4	2	6	A_2	$\overset{1}{\bullet}-\overset{2}{\bullet}$	14	22	3
$(B_2)_2$	B_2	16	8	2	8	B_2	$\overset{2}{\bullet}=\overset{3}{\bullet}$	10	20	2
$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	23	1 7	1	0	–	–	17	18	3
$(\tilde{A}_2+A_1)_2$	$\tilde{A}_2+A_1$	19	4 5	1	0	–	–	17	18	3
$\tilde{A}_2+A_1$	$A_2+\tilde{A}_2$	23 3	1 4	0	0	–	–	16	16	3
B_2	A_3	16	1 14	1	2	A_1	$\overset{3}{\bullet}$	13	16	2
$(C_3(a_1))_2$	B_2+A_1	16	8 9	1	2	A_1	$\overset{2}{\bullet}$	13	16	2
$C_3(a_1)$	$A_3+\tilde{A}_1$	16	5 6 11	0	0	–	–	14	14	2
$F_4(a_3)$	$D_4(a_1)$	16 2	5 11	0	0	–	–	12	12	1
B_3	D_4	1	16 9 2	0	0	–	–	10	10	1
C_3	C_3+A_1	4	1 3 14	0	0	–	–	10	10	1
$F_4(a_2)$	$F_4(a_1)$	1 3	9 10	0	0	–	–	8	8	1
$F_4(a_1)$	B_4	9 2	1 4	0	0	–	–	6	6	1
F_4	F_4	1 3	2 4	0	0	–	–	4	4	1

$\mathbf{E_6}$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_1	A_1	–	36	5	30	A_5	1 — 3 — 4 — 5 — 6	21	56	1
$2A_1$	$2A_1$	–	23 36	4	12	A_3	3 — 4 — 5	30	46	1
$3A_1$	$4A_1$	–	8 19 27 35	2	0	–	–	36	38	1
A_2	A_2	35	2	4	12	$2A_2$	1 — 3 5 — 6	20	36	3
A_2+A_1	A_2+A_1	35	2 7	3	6	A_2	5 — 6	23	32	3
$2A_2$	$2A_2$	1 35	2 3	2	6	A_2	5 — 6	22	30	3
A_2+2A_1	A_2+2A_1	35	2 7 11	2	0	–	–	26	28	3
A_3	A_3	23	2 24	3	4	$2A_1$	3 5	19	26	2
$2A_2+A_1$	$3A_2$	1 6 35	2 3 5	0	0	–	–	24	24	3
A_3+A_1	A_3+2A_1	23	2 3 5 24	1	0	–	–	21	22	2
$D_4(a_1)$	$D_4(a_1)$	4 23	8 19	2	0	–	–	18	20	2
A_4	A_4	21 24	1 2	2	2	A_1	5	14	18	5
D_4	D_4	2	23 4 15	2	0	–	–	16	18	1
A_4+A_1	A_4+A_1	21 24	1 2 5	1	0	–	–	15	16	5
A_5	A_5+A_1	1 6	8 9 10 19	0	0	–	–	14	14	1
$D_5(a_1)$	$D_5(a_1)$	2 7	15 12 16	1	0	–	–	13	14	2
A_5+A_1	$E_6(a_2)$	1 2 6	9 10 19	0	0	–	–	12	12	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
D_5	D_5	4 15	1 2 6	1	0	–	–	9	10	2
$E_6(a_1)$	$E_6(a_1)$	6 8 9	1 2 5	0	0	–	–	8	8	1
E_6	E_6	1 4 6	2 3 5	0	0	–	–	6	6	1

$\mathbf{E_7}$.

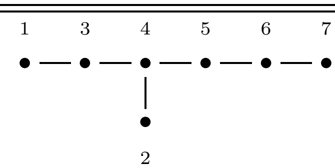
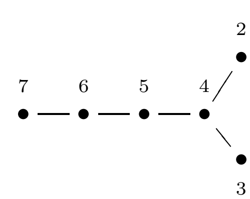
$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_1	A_1	–	63	6	60	D_6		33	99	1
$2A_1$	$2A_1$	–	49 63	5	26	$A_1 + D_4$		50	81	1
$(3A_1)''$	$(3A_1)'$	–	7 49 63	4	24	D_4		51	79	1
$(3A_1)'$	$(4A_1)''$	–	19 40 41 63	3	6	$3A_1$		60	69	1
A_2	A_2	62	1	5	30	A_5		32	67	3
$4A_1$	$7A_1$	–	21 62 33 44 18 19 16	0	0	–	–	63	63	1
$A_2 + A_1$	$A_2 + A_1$	62	1 30	4	12	A_3		41	57	3
$A_2 + 2A_1$	$A_2 + 2A_1$	62	1 18 30	3	2	A_1		46	51	3

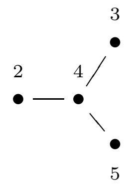
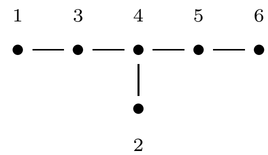
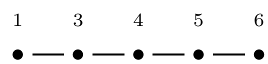
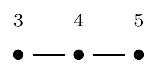
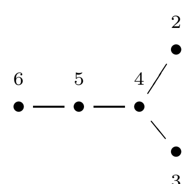
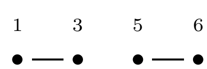
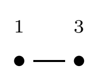
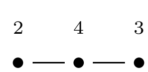
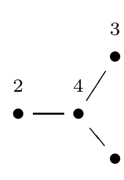
$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_2+3A_1	A_2+3A_1	62	1 5 18 30	2	0	–	–	47	49	3
$2A_2$	$2A_2$	23 62	1 7	3	6	A_2	4 — 5	40	49	3
A_3	A_3	49	1 37	4	14	A_1+A_3	7 2 — 4 — 5	31	49	2
$(A_3+A_1)''$	$(A_3+A_1)'$	49	1 7 37	3	12	A_3	2 — 4 — 5	32	47	2
$2A_2+A_1$	$3A_2$	5 25 62	1 2 6	1	0	–	–	42	43	3
$(A_3+A_1)'$	$(A_3+2A_1)''$	49	7 14 26 28	2	2	A_1	3	37	41	2
A_3+2A_1	A_3+3A_1	49	1 4 7 16 37	1	0	–	–	38	39	2
$D_4(a_1)$	$D_4(a_1)$	3 49	8 32	3	6	$3A_1$	2 5 7	30	39	2
$D_4(a_1)+A_1$	$D_4(a_1)+A_1$	3 49	7 8 32	2	4	$2A_1$	2 5	31	37	2
D_4	D_4	1	3 28 49	3	6	$3A_1$	2 5 7	28	37	1
$(A_3+A_2)_2$	A_3+A_2	22 49	4 20 21	2	2	A_1	7	33	37	6
A_3+A_2	$D_4(a_1)+2A_1$	3 49	2 7 8 32	1	2	A_1	5	32	35	2
$A_3+A_2+A_1$	$2A_3+A_1$	3 49	7 9 11 14 26	0	0	–	–	33	33	2
A_4	A_4	37 45	1 6	3	6	A_2	2 — 4	24	33	5
A_5''	A_5'	6 40	7 20 21	2	6	A_2	3 — 4	23	31	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
$D_4 + A_1$	$D_4 + 3A_1$	1	5 18 28 29 30 31	0	0	–	–	31	31	1
$A_4 + A_1$	$A_4 + A_1$	37 45	1 6 9	2	0	–	–	27	29	5
$A_4 + A_2$	$A_4 + A_2$	4 37 45	1 2 6	1	0	–	–	26	27	15
$D_5(a_1)$	$D_5(a_1)$	1 18	24 28 36	2	2	A_1	5 •	23	27	2
$(A_5 + A_1)''$	$A_5 + A_2$	22 23 24	4 7 20 21	0	0	–	–	25	25	3
A_5'	$(A_5 + A_1)''$	6 19	8 15 17 32	1	0	–	–	24	25	3
$D_5(a_1) + A_1$	$D_5(a_1) + A_1$	1 18	5 24 28 36	1	0	–	–	24	25	2
$D_6(a_2)$	$D_6(a_2) + A_1$	1 2	4 6 15 31 40	0	0	–	–	23	23	1
$(A_5 + A_1)'$	$E_6(a_2)$	1 4 16	28 29 31	1	0	–	–	22	23	3
D_5	D_5	3 28	1 6 19	2	2	A_1	2 •	17	21	2
$D_6(a_2) + A_1$	$E_7(a_4)$	1 4 7	12 22 31 35	0	0	–	–	21	21	1
$D_5 + A_1$	$D_5 + A_1$	3 28	1 2 6 19	1	0	–	–	18	19	2
A_6	A_6	11 19 26	6 9 10	1	0	–	–	18	19	7
$D_6(a_1)$	$D_6(a_1)$	3 5 28	1 12 13	1	2	A_1	2 •	16	19	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
$D_6(a_1)+A_1$	A_7	1 12 13	9 10 11 22	0	0	–	–	17	17	1
D_6	D_6+A_1	1 6	2 3 5 7 28	0	0	–	–	15	15	1
$E_6(a_1)$	$E_6(a_1)$	8 19 22	1 4 6	1	0	–	–	14	15	3
E_6	E_6	3 6 19	1 9 11	1	0	–	–	12	13	3
D_6+A_1	$E_7(a_3)$	1 2 6	7 10 11 22	0	0	–	–	13	13	1
$E_7(a_2)$	$E_7(a_2)$	1 4 16	2 3 12 13	0	0	–	–	11	11	1
$E_7(a_1)$	$E_7(a_1)$	6 9 10	1 2 5 7	0	0	–	–	9	9	1
E_7	E_7	1 4 6	2 3 5 7	0	0	–	–	7	7	1

$\mathbf{E_8}$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_1	A_1	–	120	7	126	E_7		57	190	1
$2A_1$	$2A_1$	–	97 120	6	60	D_6		90	156	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
$3A_1$	$(4A_1)'$	–	7 61 97 120	4	24	D_4		108	136	1
A_2	A_2	119	8	6	72	E_6		56	134	3
$4A_1$	$8A_1$	–	9 13 19 50 67 69 83 119	0	0	–	–	120	120	1
A_2+A_1	A_2+A_1	119	8 69	5	30	A_5		77	112	3
A_2+2A_1	A_2+2A_1	119	8 31 69	4	12	A_3		86	102	3
A_3	A_3	97	8 74	5	40	D_5		55	100	2
A_2+3A_1	A_2+4A_1	119	2 8 32 45 57	2	0	–	–	92	94	3
$2A_2$	$2A_2$	63 119	2 8	4	12	$2A_2$		76	92	3
$2A_2+A_1$	$3A_2$	6 63 119	2 5 8	2	6	A_2		78	86	3
A_3+A_1	$(A_3+2A_1)'$	97	7 22 61 62	3	12	A_3		69	84	2
$D_4(a_1)$	$D_4(a_1)$	7 97	15 68	4	24	D_4		54	82	2

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
D_4	D_4	8	7 61 97	4	24	D_4	2 — 4 — 3, 4 — 5	52	80	1
$2A_2+2A_1$	$4A_2$	1 6 63 119	2 3 5 8	0	0	–	–	80	80	3
A_3+2A_1	A_3+4A_1	97	5 7 32 36 50 61	1	0	–	–	75	76	2
$D_4(a_1)+A_1$	$D_4(a_1)+A_1$	7 97	15 32 68	3	6	$3A_1$	2 3 5	63	72	2
$(A_3+A_2)_2$	A_3+A_2	55 97	6 29 56	3	4	$2A_1$	2 3	65	72	6
A_3+A_2	$(2A_3)'$	7 97	13 22 40 62	2	4	$2A_1$	2 3	64	70	2
A_4	A_4	74 93	1 8	4	20	A_4	2 — 4 — 5 — 6	44	68	5
$A_3+A_2+A_1$	$2A_3+2A_1$	7 97	5 26 27 32 36 50	0	0	–	–	66	66	2
$D_4(a_1)+A_2$	$D_4(a_1)+A_2$	7 25 97	4 15 68	2	0	–	–	62	64	6
D_4+A_1	D_4+4A_1	8	9 13 25 35 59 63 80	0	0	–	–	64	64	1
$2A_3$	$2D_4(a_1)$	2 3 7 97	11 12 15 68	0	0	–	–	60	60	1
A_4+A_1	A_4+A_1	74 93	1 8 26	3	6	A_2	4 — 5	51	60	5

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
$D_5(a_1)$	$D_5(a_1)$	8 31	39 61 75	3	12	A_3	•3 — •4 — •5	43	58	2
$(D_4+A_2)_2$	D_4+A_2	8 63	2 49 59 71	2	0	–	–	54	56	3
A_4+2A_1	A_4+2A_1	74 93	1 8 12 26	2	0	–	–	54	56	5
A_4+A_2	A_4+A_2	18 74 93	1 6 8	2	2	A_1	•4	50	54	15
$A_4+A_2+A_1$	$A_4+A_2+A_1$	18 74 93	1 4 6 8	1	0	–	–	51	52	15
A_5	$(A_5+A_1)'$	1 44	15 34 35 68	2	6	A_2	•4 — •5	44	52	3
$D_5(a_1)+A_1$	$D_5(a_1)+A_1$	8 31	19 39 61 75	2	2	A_1	•4	48	52	2
D_4+A_2	D_4+A_3	8 31	2 32 53 61 64	1	0	–	–	49	50	2
$(A_5+A_1)''$	$E_6(a_2)$	1 8 44	34 35 68	2	6	A_2	•4 — •5	42	50	3
D_5	D_5	7 61	1 8 44	3	12	A_3	•2 — •4 — •5	33	48	2
A_4+A_3	$2A_4$	2 5 74 93	1 4 6 8	0	0	–	–	48	48	5
$D_5(a_1)+A_2$	$D_5(a_1)+A_3$	4 8 31	3 5 39 61 75	0	0	–	–	46	46	2
$(A_5+A_1)'$	$A_5+A_2+A_1$	23 24 25	1 15 34 35 68	0	0	–	–	46	46	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
$D_6(a_2)$	$2D_4$	2 8	11 12 31 53 61 64	0	0	–	–	44	44	1
A_5+2A_1	$E_6(a_2)+A_2$	1 5 8 44	4 34 35 68	0	0	–	–	44	44	3
A_5+A_2	$E_7(a_4)+A_1$	2 5 8	9 19 41 59 76	0	0	–	–	42	42	1
D_5+A_1	D_5+2A_1	7 61	3 8 16 30 32	1	0	–	–	39	40	2
$2A_4$	$E_8(a_8)$	1 2 6 8	19 41 59 76	0	0	–	–	40	40	1
$D_6(a_1)$	$D_6(a_1)$	3 7 61	8 9 37	2	4	$2A_1$	•2 •5	32	38	2
A_6	A_6	40 44 62	1 13 14	2	2	A_1	•2	34	38	7
A_6+A_1	A_6+A_1	40 44 62	1 2 13 14	1	0	–	–	35	36	7
$D_6(a_1)+A_1$	A_7'	8 9 37	20 33 34 35	1	2	A_1	•5	33	36	2
$(D_5+A_2)_2$	D_5+A_2	7 25 61	4 8 23 24	1	0	–	–	35	36	6
D_5+A_2	A_7+A_1	8 9 37	5 20 33 34 35	0	0	–	–	34	34	2
$E_6(a_1)$	$E_6(a_1)$	15 44 55	1 6 8	2	6	A_2	•2 — •4	26	34	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\lvert\Delta_0\rvert$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
D_6	$D_6 + 2A_1$	1 8	4 14 17 26 27 55	0	0	–	–	32	32	1
$D_7(a_2)$	$D_7(a_2)$	4 7 61	3 8 16 30	1	0	–	–	31	32	2
E_6	E_6	1 7 44	8 26 27	2	6	A_2	$\overset{4}{\bullet} - \overset{5}{\bullet}$	24	32	3
$(A_7)_3$	A_7''	24 38 48	10 11 14 15	1	0	–	–	31	32	4
A_7	$D_8(a_3)$	3 5 7 61	4 8 23 24	0	0	–	–	30	30	1
$E_6(a_1) + A_1$	$E_6(a_1) + A_1$	15 44 55	1 6 8 10	1	0	–	–	29	30	3
$D_8(a_3)$	A_8	22 23 26 31	7 11 12 25	0	0	–	–	28	28	3
$D_6 + A_1$	$E_7(a_3)$	1 2 8	27 28 32 41	1	2	A_1	$\overset{5}{\bullet}$	25	28	1
$(D_7(a_1))_2$	$D_7(a_1)$	1 8 12	19 21 33 49	1	0	–	–	27	28	2
$D_7(a_1)$	$D_8(a_2)$	1 6 8	12 21 25 27 49	0	0	–	–	26	26	1
$E_6 + A_1$	$E_6 + A_2$	1 4 7 44	5 8 26 27	0	0	–	–	26	26	3
$E_7(a_2)$	$E_7(a_2) + A_1$	6 8 48	7 10 12 16 30	0	0	–	–	24	24	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$\|\Delta_0\|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{s,\mathbf{k}}$	d'
A_8	$E_8(a_6)$	1 2 5 8	21 25 27 49	0	0	–	–	24	24	1
D_7	$D_8(a_1)$	10 11 15 25	1 20 21 22	0	0	–	–	22	22	1
$E_7(a_2)+A_1$	$E_8(a_7)$	4 7 23 24	5 8 20 33	0	0	–	–	22	22	1
$E_7(a_1)$	$E_7(a_1)$	1 13 14	3 5 8 32	1	2	A_1	$\overset{2}{\bullet}$	17	20	1
$D_8(a_1)$	$E_8(a_3)$	3 7 23 24	4 8 26 27	0	0	–	–	20	20	1
$E_7(a_1)+A_1$	D_8	1 13 14	4 8 17 18 19	0	0	–	–	18	18	1
D_8	$E_8(a_5)$	10 16 20 22	2 3 5 7	0	0	–	–	16	16	1
E_7	E_7+A_1	1 6 8	2 3 5 7 32	0	0	–	–	16	16	1
E_7+A_1	$E_8(a_4)$	7 11 12 25	1 2 6 8	0	0	–	–	14	14	1
$E_8(a_2)$	$E_8(a_2)$	2 3 5 7	1 8 10 20	0	0	–	–	12	12	1
$E_8(a_1)$	$E_8(a_1)$	6 8 10 11	1 2 5 7	0	0	–	–	10	10	1
E_8	E_8	1 4 6 8	2 3 5 7	0	0	–	–	8	8	1

Appendix 3. Irreducible root systems of exceptional types.

In this Appendix we give the lists of positive roots in irreducible root systems of exceptional types. All simple roots are numbered as shown at the Dynkin diagrams. The other roots in each list are given in terms of their coordinates with respect to the basis of simple roots. The coordinates are indicated in the brackets ( ). Each set of coordinates is preceded by the number of the corresponding root. These numbers are used to indicate roots which appear in the columns s^1, s^2 and Γ_0^s in the tables in Appendix 2.

$\mathbf{G_2}$.

1 (1 0)

2 (0 1)

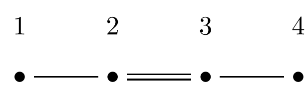
3 (1 1)

4 (2 1)

5 (3 1)

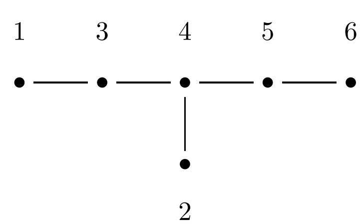
6 (3 2)

F4.



1	(1 0 0 0)	9	(0 1 2 0)	17	(1 2 2 1)
2	(0 1 0 0)	10	(0 1 1 1)	18	(1 1 2 2)
3	(0 0 1 0)	11	(1 1 2 0)	19	(1 2 3 1)
4	(0 0 0 1)	12	(1 1 1 1)	20	(1 2 2 2)
5	(1 1 0 0)	13	(0 1 2 1)	21	(1 2 3 2)
6	(0 1 1 0)	14	(1 2 2 0)	22	(1 2 4 2)
7	(0 0 1 1)	15	(1 1 2 1)	23	(1 3 4 2)
8	(1 1 1 0)	16	(0 1 2 2)	24	(2 3 4 2)

$\mathbf{E_6}$.



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2	(0 1 0 0 0 0)	14	(0 1 0 1 1 0)	26	(1 1 1 2 1 0)
3	(0 0 1 0 0 0)	15	(0 0 1 1 1 0)	27	(1 1 1 1 1 1)
4	(0 0 0 1 0 0)	16	(0 0 0 1 1 1)	28	(0 1 1 2 1 1)
5	(0 0 0 0 1 0)	17	(1 1 1 1 0 0)	29	(1 1 2 2 1 0)
6	(0 0 0 0 0 1)	18	(1 0 1 1 1 0)	30	(1 1 1 2 1 1)
7	(1 0 1 0 0 0)	19	(0 1 1 1 1 0)	31	(0 1 1 2 2 1)
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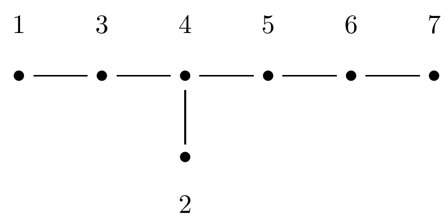
11	(0 0 0 0 1 1)	23	(1 0 1 1 1 1)	35	(1 1 2 3 2 1)
12	(1 0 1 1 0 0)	24	(0 1 1 2 1 0)	36	(1 2 2 3 2 1)

$\mathbf{E_7}$.



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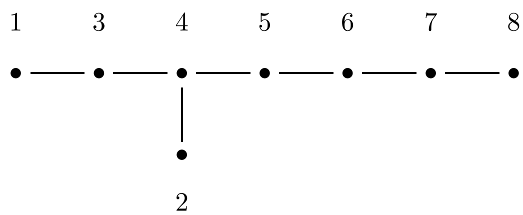
$\mathbf{E_8}$.



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