Invariant torus and its destruction for an oscillator with dry friction

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Abstract

Mechanical systems with dry friction are typical Filippov systems. Such class of systems have complicated dynamical behaviors due to the existence of sliding motion. In this work, we consider a one-degree-of-freedom oscillator with dry friction force. The phase map is derived to reduce the system to a circle map, and then the existence of forward invariant torus is proved under suitable assumptions. Moreover, the typical resonance phenomenon and the grazing bifurcation of invariant torus are discussed. We find that the destruction of invariant tori is due to a loss of transversality for sufficiently large perturbation, which is different from the usual smooth tori dynamics.

Keywords: dry friction oscillator, phase map, invariant torus, grazing-sliding bifurcation

1. Introduction

The mechanical systems with dry friction are typical Filippov systems [1, 2], which have many important realistic applications in the field of engineering, such as drill strings, rotor models, railway wheels and so on. Moreover, the dynamical behaviors of such systems are quite different from the smooth ones, see, for example [2–8]. Hence, the study of systems with dry friction has gained considerable ground in recent years due to the theoretical and applications interests.

One of the main characters of Filippov systems is that the orbits can slide along the separatrix surface of the piecewise smooth vector field. This fact introduces the non-smoothness of dynamics for Filippov systems. The real dimensions of such systems are also decreased due to the existence of sliding motion. For example, it is known that one of the Floquet multipliers of a sliding periodic orbit is zero, see [9]. When the sliding motion is involved, we can reduce the dimensions of Filioppov systems by two by taking suitable Poincaré sections. The Poincaré map, called phase map (see Section 2 for the definition), which is a circle map first introduced in [10] for the oscillator with dry friction. Using the phase map, the calculation of Lyapunov exponents for dry friction system is discussed in [10]. Kunze and Küpper [3] studied the non-smooth bifurcations for a dry friction system

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through the phase map, and some resonance phenomena were shown by computing the rotation number of the phase map. Galvanetto [11] discussed the numerical algorithm for the stable set of an attractive sliding periodic orbit based on the phase map. If the degree of freedom of the system is larger than one, the reduction of dimension can be larger than two. For example, Galvanetto and Knudsen [12] considered two-degree-freedom systems with dry friction and one dimension maps were derived at the intersection of separatrix surfaces of vector fields.

Since most of the known results are numerical, we consider, in this work, a dimension reduction method analytically for an oscillator with dry friction. Under suitable conditions, we show that the phase map is well defined and is a small perturbation of a rotation on the circle. So it is natural to ask whether there exists invariant torus for the original system. For smooth systems, one usually consider the existence of invariant tori under some smooth angular structure in the phase space, such as the neighborhoods of periodic orbits, or the neighborhoods of elliptic fixed points and tori, see [13–16] and references therein. Since the system in this work is non-smooth, the traditional methods cannot be applied. So some new methods are introduced to overcome this difficulty. Due to the piecewise smoothness property of the vector field, we glue the surfaces generated by smooth vector fields to show the existence of invariant torus (called the sliding invariant torus). The intersection of such torus with the separatrix surface is a cylinder. In addition, since only positive flow is well defined for sliding orbits, we need to verify the forward invariance of the torus. We also explain the relationship between the the flow on the sliding invariant torus and the dynamics of the phase map by the KAM theory on circle maps.

The existence of grazing motion is based on the complexity of the piecewise smooth systems. The bifurcations of grazing periodic orbits are well studied for impact systems and Filippov systems [17–21]. The grazing bifurcations of invariant tori for impact systems are studied in [22, 23]. In this paper, we discuss the grazing bifurcation of invariant tori and its destruction for the Filippov systems. For smooth systems, the destruction of invariant tori are usually due to too large perturbations, which make the tori to lose smoothness and lead to their destructions, see [24, 25]. The case is different for the Filippov system in this work. The invariant torus first lose the transversality instead of smoothness due to large perturbation. Some orbits in the invariant torus tend to grazing ones when the parameter is varied. It turns out that the torus breaks down but there still exists an invariant surface. Such phenomenon is the so called *grazing-sliding bifurcation of invariant torus*, which is similar to *the grazing-sliding bifurcation of periodic orbit* [20].

The remaining of this paper is organised as follows. In Section 2, we introduce the model and give the definition of the phase map. The main theorems of this work and illustrations of the phase map are also given. We discuss the resonance phenomenon and present the grazing-sliding bifurcation of invariant torus numerically in Section 3. The proofs of the theorems of Section 2 are presented in Section 4. The relationship between the flow on invariant torus and the dynamics of phase map are discussed in Section 5. We draw the conclusions and discuss some extensions in Section 6.

2. The oscillator with dry friction and the phase map

We consider the following equations that describe the mechanical system shown in Figure 1:

$$\begin{aligned} \dot{x} =& y, \\ \dot{y} =& -cy - kx + F(y - v_s) + \epsilon f(t), \end{aligned} \tag{1}$$

where $(x, y) \in \mathbb{R}^2$, $k, c > 0$ are constants, f is a C^r $(r \geq 1)$ periodic function with period

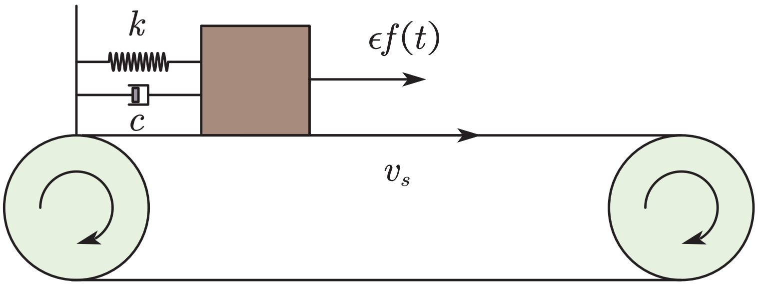


Figure 1: The mechanical model of dry friction oscillator.

$T > 0$, ϵ is a small constant and $v_s > 0$ is the velocity of belt. The friction function $F : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is defined as

$$F(x) = -A\ \mathrm{Sign}(x)(\frac{\alpha}{1 + \gamma|x|} + \lambda + \eta x^2),$$

where $\alpha, \gamma, \lambda, \eta, A > 0$ are constants. Let

$$\pm F_0 = \lim_{x \to 0^{\mp}} F(x) = \pm A(\alpha + \lambda).$$

See Figure 2 for an illustration of F (the parameter values are given at the end of this section), and see [26] and reference therein for the physical meaning of the parameters in the formula of F.

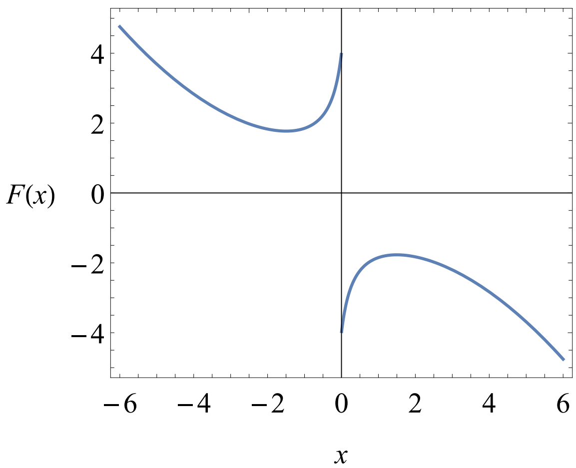


Figure 2: The dry friction function F.

Firstly we extend the equation (1) on the extended phase space $\mathbb{R}^2 \times S^1$:

$$\begin{aligned} \dot{x} =& y, \\ \dot{y} =& -cy - kx + F(y - v_s) + \epsilon f(\theta), \\ \dot{\theta} =& 1, \end{aligned} \tag{2}$$

where S^1 is the circle with period T. Set $X = (x, y, \theta)$. Since v_s is a fixed constant, we identify $\bar{X} = (x, \theta)$ and (x, v_s, θ) in the remaining of this paper. Let

$$H(X) = y - v_s$$

The surface $\{X : H = 0\}$ is called the sliding surface on the extended phase space. The vector field on the sliding surface $\{X : y = v_s\}$ is

$$\begin{aligned} \dot{x} &= v_s \\ \dot{\theta} &= 1. \end{aligned} \tag{3}$$

Let

$$M(X) = -kx - cv_s + F_0 + \epsilon f(\theta).$$

Note that $\{\bar{X} : M = 0\}$ is the curve on the sliding surface, which describes the condition that the orbit can exit the sliding surface and get into the region $\{X : H < 0\}$. Since $M = 0$, it implies that

$$x = \frac{1}{k}(-cv_s + F_0 + \epsilon f(\theta)),$$

and $\{\bar{X} : M = 0\}$ is a graph of a function from S^1 to $\mathbb{R}$. It follows that $\{\bar{X} : M = 0\}$ is homeomorphic to S^1 and we can choose θ as its global coordinate. Denote $\{\bar{X} : M(x, \theta, \epsilon) = 0\}$ by S_ϵ. Choose θ as the global coordinate of S_ϵ in the remaining of the paper. We define the Poincaré map (called the phase map)

$$\mathfrak{R}_\epsilon : S_\epsilon \to S_\epsilon$$

which is the usual first return map of the flow in the extended phase space. Roughly speaking, the orbits of the phase map are the time revolution of the orbits exiting the sliding surface and getting into the region $\{X : H < 0\}$.

In general, the phase map $\mathfrak{R}_\epsilon$ may be discontinuous or even be not defined. We restrict ourselves to some relevant cases. A periodic solution of (1) is called a sliding periodic solution if it intersects $\{(x, y) : y = v_s\}$ with at least a line segment. We need the following assumptions:

H. *There exists a sliding periodic orbit of* (1) *on* $\{(x, y) : y - v_s \leq 0\}$ *when* $\epsilon = 0$, *and the following transversality condition holds:*

$$0 < -cv_s - kx_1 + F_0 < 2F_0, \tag{4}$$

where x_1 *is the position coordinate at where the periodic orbit enters into sliding surface.*

The assumption **H** holds for the usual parameter values of (1), see the end of this section. The condition (4) is to insure that the orbit can get into the sliding surface from the region $\{X : H < 0\}$ transversally and it will not pass through it. When the system is unperturbed, the phase map turns out to be a rigid rotation on a circle under the assumption **H**. For small ϵ, we will prove that the phase map is a smooth circle homeomorphism perturbed by a rigid rotation.

Here we list some notations that will be used in what follows. Denote $\phi(t, X, \epsilon)$ by the solution of the vector field on $\{X : H < 0\}$ with initial conditions

$$x(0) = x, y(0) = y, \theta(0) = \theta.$$

Denote $\phi_s(t, \bar{X})$ by the solution of the vector field on the sliding surface with initial conditions

$$x(0) = x, \theta(0) = \theta.$$

Let $h : S^1 \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ be defined as

$$h(\theta,\epsilon) = (\frac{1}{k}(-cv_s + F_0 + \epsilon f(\theta)), v_s, \theta).$$

Note that $\{h(\theta,\epsilon) : \theta \in S^1\} = S_\epsilon$, and an orbit leaving from the sliding surface to the region $\{X : H < 0\}$ satisfies

$$(\bar{x}, v_s, \theta) = h(\theta,\epsilon),$$

where $(\bar{x}, v_s, \theta)$ is the state such that orbit exits sliding surface.

Under assumption **H**, let $t^1(\theta,\epsilon)$ be the time such that the orbit first enters into the sliding surface, i.e.,

$$\phi(t^{(1)}(\theta,\epsilon), h(\theta,\epsilon), \epsilon) = (x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)).$$

And there exists the unique time $t^{(2)}(\theta,\epsilon)$ such that

$$\phi_s(t^{(2)}(\theta,\epsilon), x^{(1)}(\theta,\epsilon), \theta + t^{(1)}(\theta,\epsilon)) \in S_\epsilon,$$

See Figure 3 for the meaning of notations and Section 4 for the existence of $t^{(1)}(\theta,\epsilon)$ and

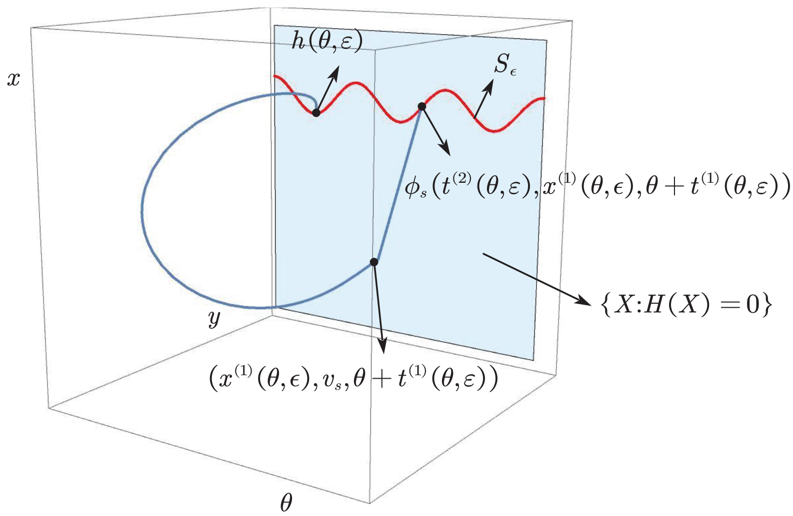


Figure 3: The main notations.

$t^{(2)}(\theta,\epsilon)$. Define the following sets:

$$\mathfrak{T}_\epsilon^{(1)} := \bigcup_{\theta\in S^1} \bigcup_{\tau\in[0,t^{(1)}(\theta,\epsilon)]} \phi(\tau, h(\theta,\epsilon), \epsilon),$$

$$\mathfrak{T}_\epsilon^{(2)} := \bigcup_{\theta\in S^1} \bigcup_{\tau\in[0,t^{(2)}(\theta,\epsilon)]} \phi_s(\tau, x^{(1)}(\theta,\epsilon), \theta + t^{(1)}(\theta,\epsilon)),$$

$$\mathfrak{T}_\epsilon := \mathfrak{T}_\epsilon^{(1)} \cup \mathfrak{T}_\epsilon^{(2)}.$$

The following theorems are the main theoretical results of our work.

Theorem 2.1. *Under assumption* **H**, *there exists* $\epsilon_0 > 0$ *such that* $\mathfrak{R}_\epsilon$ *is well defined when* $|\epsilon| < \epsilon_0$. *Moreover, we have*

$$\mathfrak{R}_\epsilon(\theta) = \theta + T_0 + \epsilon g(\theta) + o(\epsilon), \ (mod\ T), \tag{5}$$

where T_0 *is the period of the unperturbed periodic orbit and* g *is a* C^{r-1} *function with periodic* T.

Theorem 2.2. *Under assumption* **H**, *let* $\mathcal{O}$ *be the sliding periodic orbit of unperturbed system* (1) *on the plane* (x, y). $\mathfrak{T}_\epsilon$ *is homeomorphic to the topological torus* $\mathfrak{T}_0 = \mathcal{O} \times S^1$ *when* $|\epsilon|$ *is sufficiently small. Moreover,* $\mathfrak{T}_\epsilon$ *tends to* $\mathfrak{T}_0$ *as* ϵ *tends to zero.*

See Section 4 for the proofs.

In the remaining of this section we give some numerical illustrations about the phase map. The following parameters are taken from [11], and are fixed throughout this paper:

$$k = 1, \ c = 0.2, \ v_s = 1,$$
$$A = 10, \ \alpha = 0.3, \ \gamma = 3, \ \lambda = 0.1, \ \eta = 0.01.$$

Figure 4 shows that the system (1) has a sliding periodic solution when $\epsilon = 0$.

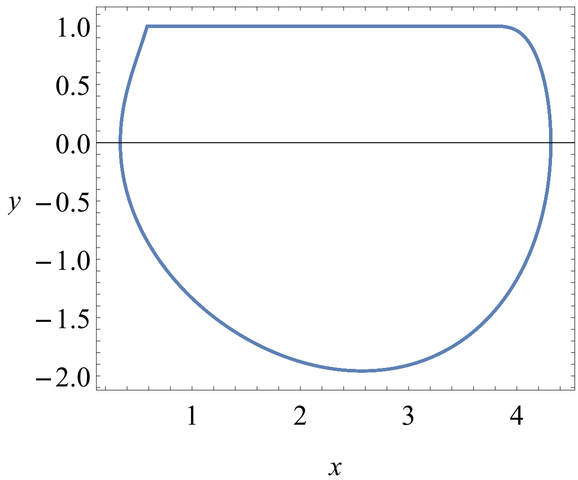


Figure 4: The sliding periodic orbit of equation (1) when $\epsilon = 0$.

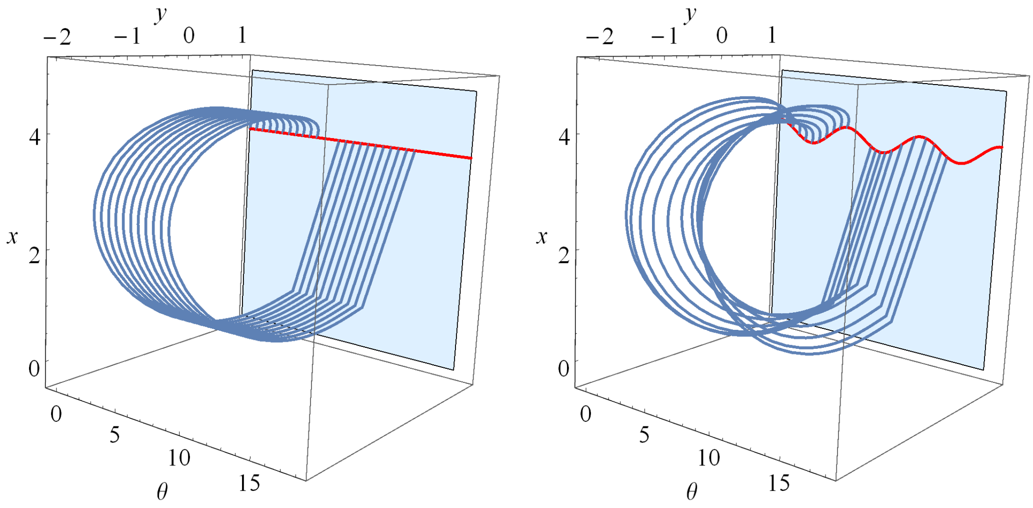
When $M(x, \theta, 0) = 0$, x is a constant independent of θ. Now we have $\mathfrak{R}_0(\theta) = \theta + T_0(\text{mod } T)$, where T_0 is the period of the sliding periodic solution. Take $f(t) = \cos(\omega t)$ and $\omega = 1.067$. The phase maps for $\epsilon = 0$ and $\epsilon = 0.2$ in the extended phase space are illustrated in Figure 5. We plot the graph of $\mathfrak{R}_\epsilon$ numerically in Figure 6.

3. Typical resonance phenomena and the destruction of invariant torus

By the formula (5) in Theorem 2.1, the phase map is expected to show typical resonance phenomena such as the devil staircase when the frequencies of f changes. We verify this fact by numerical simulation.

Let $f(t) = \cos(\omega t)$. The phase map is defined on the circle S_ϵ with period $2\pi/\omega$. To fix the parametrization of S_ϵ, we rescale the phase map by defining the following transformation:

$$\bar{\mathfrak{R}}_{\epsilon,\omega}(\theta) = \omega \mathfrak{R}_\epsilon(\frac{\theta}{\omega}).$$



(a) $\epsilon = 0$

(b) $\epsilon = 0.2$

Figure 5: The phase map in the extended phase space for (a) $\epsilon = 0$ and (b) $\epsilon = 0.2$.

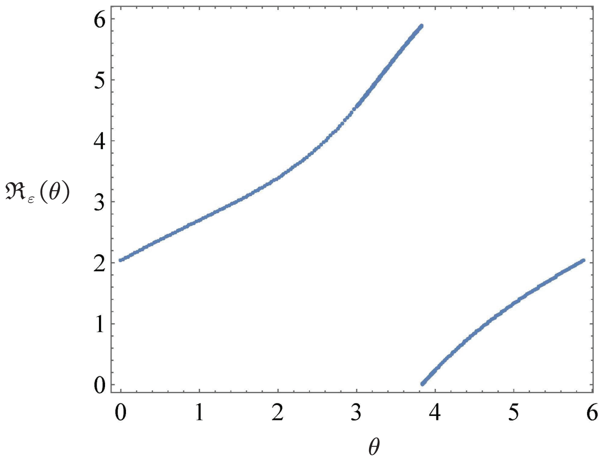


Figure 6: Graph of $\mathfrak{R}_\epsilon$ for $\epsilon = 0.2$.

Hence, $\bar{\mathfrak{R}}_{\epsilon,\omega}$ is defined on the circle with period 2π. Let $\epsilon = 0.35$ be fixed. Denote the rotation number of $\bar{\mathfrak{R}}_{\epsilon,\omega}$ by $\rho(\omega)$. The graph of ρ for $\omega \in [2, 2.8]$ is shown in Figure 7, which shows typical resonance phenomena.

According to Theorem 2.2, there exists an invariant torus when $|\epsilon|$ is small. The destruction of invariant torus in here is different from the usual smooth tori. For sufficiently large ϵ, the invariant torus first lose transversality instead of losing smoothness. Some orbits starting at $h(\theta, \epsilon)$ still intersect the sliding surface transversally. While some orbits take more time in the region $\{X : H < 0\}$ but they still get into the sliding surface eventually. Moreover, there exist grazing orbits such that they and the sliding surface are tangent at S_ϵ, and the phase map turns out to be discontinuous. In such a case, we can still define a positive invariant surface, though it is not homeomorphic to a torus in general. It is expected that such invariant surface is still preserved under small perturbations since both the grazing orbits and the regular orbits usually do not disappear.

Let ϵ be the control parameter and let $f = \cos(\omega t)$, where $\omega = 1.067$ is fixed in the

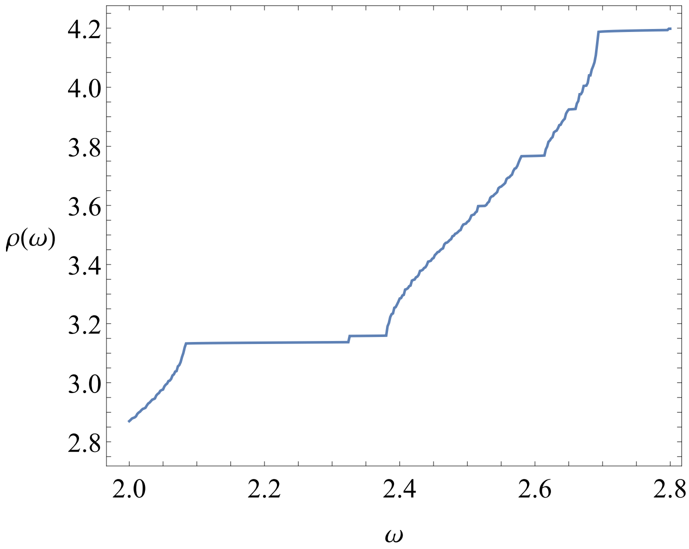


Figure 7: The rotation numbers $\rho(\omega)$ for $\epsilon = 0.35$ and $\omega \in [2, 2.8]$.

following numerical studies. When $0 \leq \epsilon \leq 0.5$, the phase map is a homeomorphism (see Figure 8). The invariant torus in cylindrical coordinates is shown in Figure 9, where $X_1 = (x+2)\cos(\omega\theta), Y_1 = (x+2)\sin(\omega\theta), Z_1 = y$.

The bifurcation portrait of the phase map is presented in Figure 10. It can be seen that with $\epsilon \approx 0.514$, the torus breaks down and the portrait is interwoven with attractive periodic orbits and complicated invariant sets. When $\epsilon \in [0.57, 0.585]$, the period-doublings and period-halvings from a 4-periodic orbit to 8-periodic orbit and back occur, compared to [3].

To show the grazing effect for the invariant torus as ϵ varies, it is convenient to plot the set $\{(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$, i.e., the points that the orbits first arrive at $\{X : H(X) = 0\}$, starting from $h(\theta,\epsilon)$, see Section 2 for the notations. It is shown that $\{(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ is a smooth curve, being a graph of S^1 (see the proof of Lemma 4.3) under the graph of $h(\theta,\epsilon)$ when ϵ is small. When $\epsilon = 0.51403518$, the set is not a graph of S^1 any longer and there is a cusp on S_ϵ which is corresponding to the grazing orbit (the orbit and the sliding surface are tangent at S_ϵ), see Figure 11 (a). When $\epsilon = 0.52$, the set $\{(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ is not a continuous curve anymore, see Figure 11 (b). The intersections of S_ϵ and $\{(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ are two points that correspond to the grazing orbits and such intersections are preserved under small perturbations. The torus is broken down but the phase map is still well defined though it is not continuous anymore (see Figure 12). One of the grazing orbit is shown in Figure 13.

4. Proofs of theorems in Section 2

With the notations in Section 2, now we prove Theorem 2.1.

Proof of Theorem 2.1. We choose the initial position $h(\theta,\epsilon)$, which implies that the orbit starts at S_ϵ. Since the time it takes for the periodic orbit to arrive at the sliding surface is independent of θ when $\epsilon = 0$, there exists $t_1 > 0$ such that $H(\phi(t_1, \frac{1}{k}(-cv_s +$

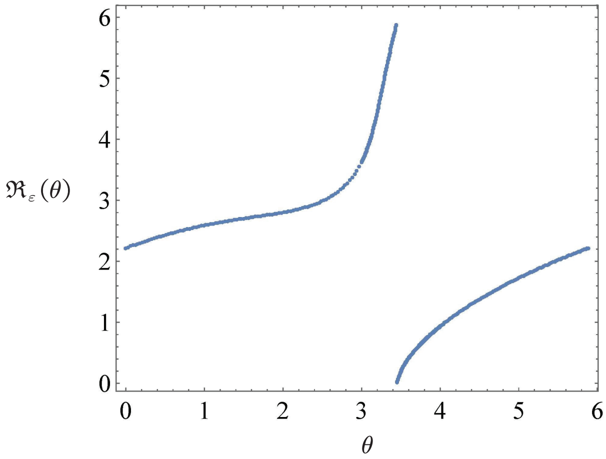


Figure 8: The graph of the map $\mathfrak{R}_\epsilon$ for $\epsilon = 0.5$.

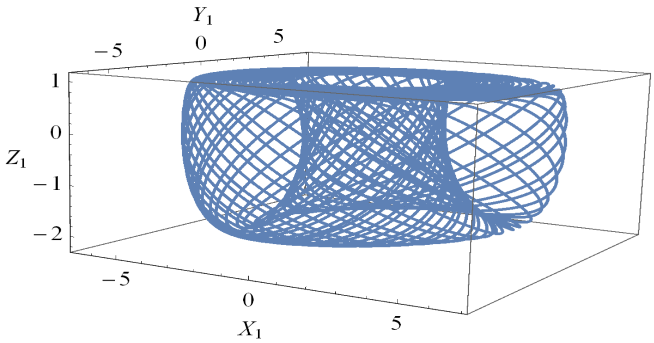


Figure 9: Invariant torus in cylinder coordinate for $\epsilon = 0.2$.

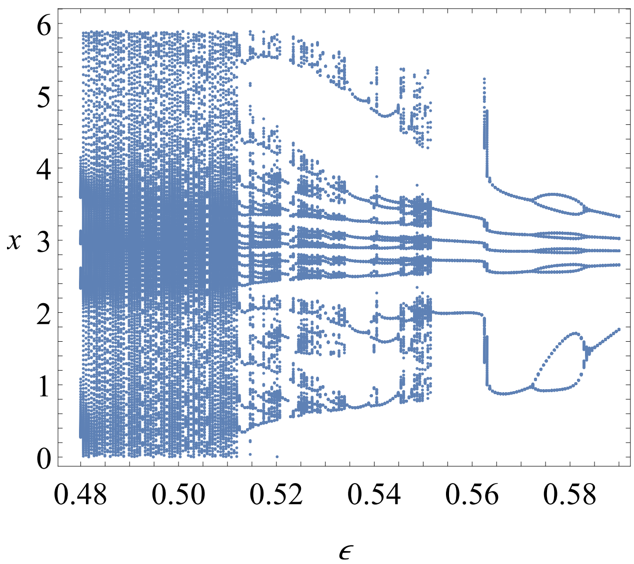


Figure 10: Bifurcation diagram of the phase map for $\epsilon \in [0.48, 0.59]$.

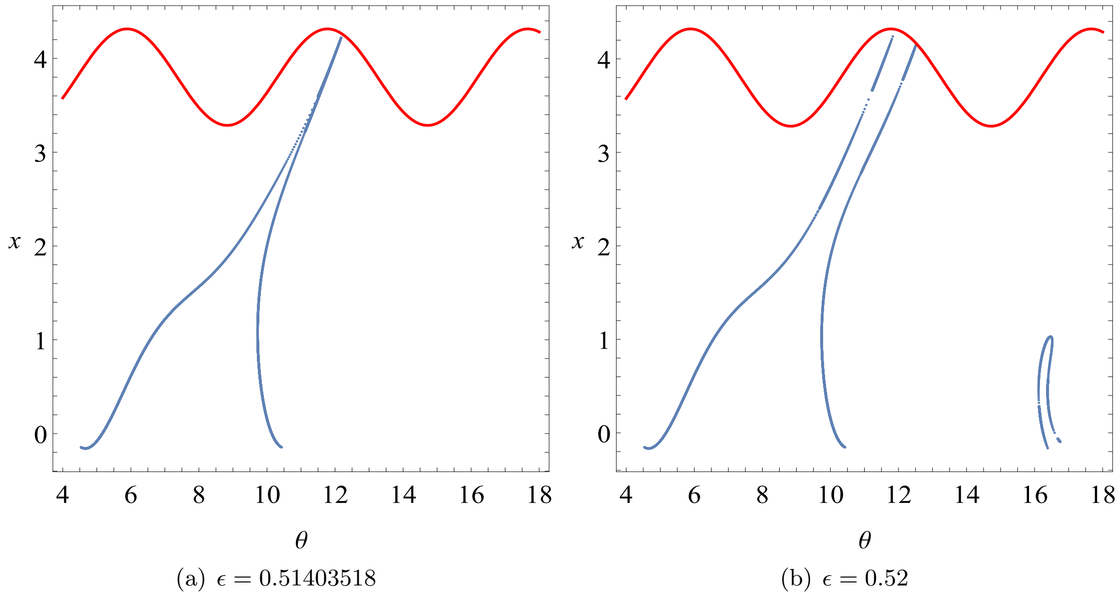


Figure 11: The sets $\{(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ and S_ϵ for (a) $\epsilon = 0.51403518$ and (b) $\epsilon = 0.52$.

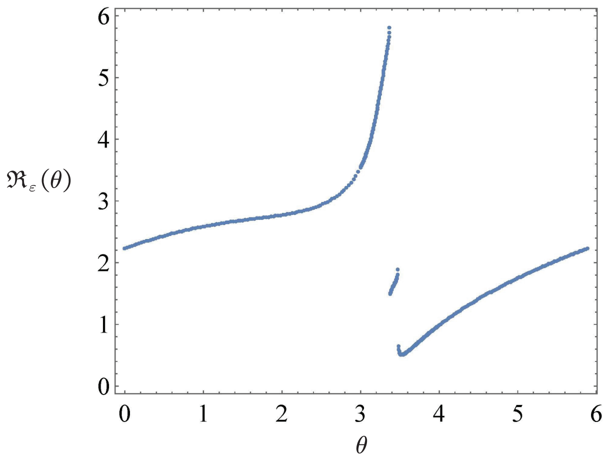


Figure 12: The graph of $\mathfrak{R}_\epsilon$ for $\epsilon = 0.52$.

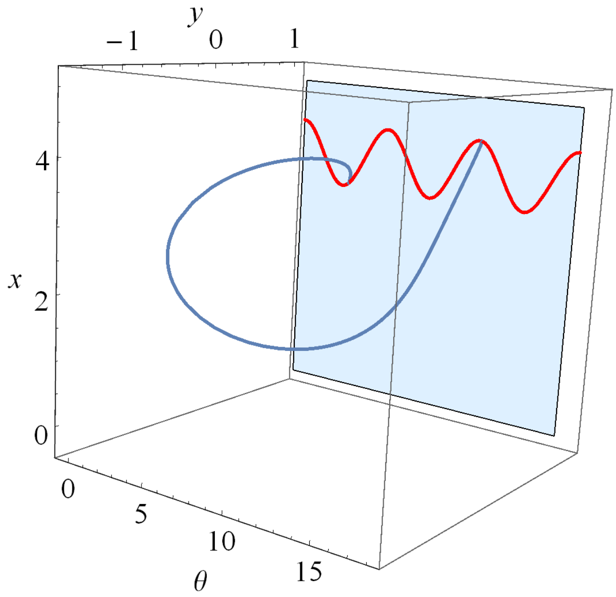


Figure 13: The grazing orbit start from $h(3.37317889, 0.52)$.

$F_0), v_s, \theta, 0)) = 0$ for any $\theta \in S^1$, i.e.,

$$H(\phi(t, h(\theta, \epsilon), \epsilon))|_{\epsilon=0,t=t_1} = 0.$$

Moreover, by assumption **H** we have

$$\begin{aligned}\frac{\partial}{\partial t}H(\phi(t, h(\theta, \epsilon), \epsilon))|_{\epsilon=0,t=t_1} =& \langle \nabla H(\phi(t_1, h(\theta, 0), 0)), \frac{\partial}{\partial t}\phi(t_1, h(\theta, 0), 0)\rangle \\ =& - cv_s - kx_1 + F_0 \neq 0, \ \forall \theta \in S^1.\end{aligned}$$

By the implicit function theorem, there exists a function $t^{(1)} : S^1 \times (-\tilde{\epsilon_0}, \tilde{\epsilon_0}) \to \mathbb{R}$ such that

$$t^{(1)}(\theta, 0) = t_1 \text{ and } H(\phi(t^{(1)}(\theta, \epsilon), h(\theta, \epsilon), \epsilon)) = 0$$

for $(\theta, \epsilon) \in S^1 \times (-\tilde{\epsilon_0}, \tilde{\epsilon_0})$.

The deviation of the perturbed orbits $\phi(t, h(\theta, \epsilon), \epsilon)$ and the unperturbed orbits $\phi(t, h(\theta, 0), 0)$ at the times arrive at the sliding surface is

$$\begin{aligned}\phi(t^{(1)}(\theta, \epsilon), h(\theta, \epsilon), \epsilon) - \phi(t^{(1)}(\theta, 0), h(\theta, 0), 0) =& \epsilon\frac{\partial \phi}{\partial t}(t_1, h(\theta, 0), 0)\frac{\partial t}{\partial \epsilon}(\theta, 0) + o(\epsilon) \\ &+ \epsilon\frac{\partial \phi}{\partial X}(t_1, h(\theta, 0), 0)\frac{\partial h}{\partial \epsilon}(\theta, 0) + o(\epsilon) + \epsilon\frac{\partial \phi}{\partial \epsilon}(t_1, h(\theta, 0), 0) + o(\epsilon). \quad (6)\end{aligned}$$

By (4) and (6), we have

$$0 < -kx^{(1)}(\theta, \epsilon) - cv_s + F_0 + \epsilon f(t^{(1)}(\theta, \epsilon)) < 2F_0$$

for $|\epsilon|$ sufficiently small, where $x^{(1)}(\theta, \epsilon)$ is the second coordinate of $\phi(t^{(1)}(\theta, \epsilon), h(\theta, \epsilon), \epsilon)$ and $x^{(1)}(\theta, 0) = x_1$. Therefore, the orbits cannot pass though the sliding surface for small $|\epsilon|$.

Let $\phi(t^{(1)}(\theta,0),h(\theta,0),0)=(x_1,v_s,\theta_1)$. Note that $\theta_1=\theta+t^{(1)}(\theta,0)=\theta+t_1 \pmod T$. Select the projection $\Pi(x,y,\theta)=(x,\theta)$. Then we have

$$\Pi(\phi(t^{(1)}(\theta,\epsilon),h(\theta,\epsilon),\epsilon))=(x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon)).$$

Let t_2 be the time such that

$$M(\phi_s(t_2,x_1,\theta_1),0)=0,\ \forall\theta\in S^1,$$

i.e.,

$$M(\phi_s(t,x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon)),\epsilon)|_{t=t_2,\epsilon=0}=0,\ \forall\theta\in S^1.$$

By assumption **H**, we have

$$t_1+t_2=T_0 \text{ and } \phi_s(t_2,x^{(1)}(\theta,0),\theta+t^{(1)}(\theta,0))=(\frac{1}{k}(-cv_s+F_0),\theta+T_0). \tag{7}$$

A direct computation yields

$$\begin{aligned}&\frac{\partial}{\partial t}M(\phi_s(t,x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon)),\epsilon))),\epsilon)|_{t=t_2,\epsilon=0}=\\&\qquad\frac{\partial M}{\partial\bar X}(\frac{1}{k}(-cv_s+F_0),\theta+T_0,0)\frac{\partial\phi_s}{\partial t}(t_2,x_1,\theta+t_1)=(-k,0)\cdot(v_s,1)=-kv_s\neq 0.\end{aligned}$$

Hence, by the implicit function theorem, there exists a function $t^{(2)}:S^1\times(-\epsilon_0,\epsilon_0)\to\mathbb{R}$ such that

$$M(\phi_s(t^{(2)}(\theta,\epsilon),x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon)),\epsilon))),\epsilon)=0$$

for $(\theta,\epsilon)\in S^1\times(-\epsilon_0,\epsilon_0)$, where $t^{(2)}(\theta,\epsilon)$ is the time spent on the sliding surface for the orbit which starts from $h(\theta,\epsilon)$ to S_ϵ. Therefore, the phase map $\mathfrak{R}_\epsilon$ is well defined when $|\epsilon|<\epsilon_0$.

We now estimate the distance between the starting point and the end point on S_ϵ. Since we have

$$\begin{aligned}&\phi_s(t^{(2)}(\theta,\epsilon),x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon))-\phi_s(t^{(2)}(\theta,0),x^{(1)}(\theta,0),\theta+t^{(1)}(\theta,0))\\&=\epsilon(\frac{\partial\phi_s}{\partial t}(t_2,x_1,\theta+t_1)\frac{\partial t^{(2)}}{\partial\epsilon}(\theta,0)+\frac{\partial\phi_s}{\partial x}(t_2,x_1,\theta+t_1)\frac{\partial x^{(1)}}{\partial\epsilon}(\theta,0)+\\&\quad\frac{\partial\phi_s}{\partial\theta}(t_2,x_1,\theta+t_1)\frac{\partial t^{(1)}}{\partial\epsilon}(\theta,0))+o(\epsilon)\\&=\epsilon\left(\begin{pmatrix}v_s\\1\end{pmatrix}\frac{\partial t^{(2)}}{\partial\epsilon}(\theta,0)+\begin{pmatrix}1\\0\end{pmatrix}\frac{\partial x^{(1)}}{\partial\epsilon}(\theta,0)+\begin{pmatrix}0\\1\end{pmatrix}\frac{\partial t^{(1)}}{\partial\epsilon}(\theta,0)\right)+o(\epsilon)\\&=(\epsilon v_s\frac{\partial t^{(2)}}{\partial\epsilon}(\theta,0)+\epsilon\frac{\partial x^{(1)}}{\partial\epsilon}(\theta,0),\epsilon\frac{\partial t^{(2)}}{\partial\epsilon}(\theta,0)+\epsilon\frac{\partial t^{(1)}}{\partial\epsilon}(\theta,0))^{\text{tr}}+o(\epsilon),\end{aligned}\tag{8}$$

where tr denotes the transpose of the vector. Note that the second coordinate of

$$\phi_s(t^{(2)}(\theta,\epsilon),x^{(1)}(\theta,\epsilon),\theta+t^{(1)}(\theta,\epsilon))$$

is $\mathfrak{R}_\epsilon(\theta)$ (we use the coordinate on S_ϵ). Hence, (8) and (7) give that

$$\mathfrak{R}_\epsilon(\theta)=\theta+T_0+\epsilon(\frac{\partial t^{(2)}}{\partial\epsilon}(\theta,0)+\frac{\partial t^{(1)}}{\partial\epsilon}(\theta,0))+o(\epsilon).$$

By the implicit function theorem, $\mathfrak{R}_\epsilon, t^{(1)}, t^{(2)}$ are C^r functions. By setting

$$g(\theta) = \frac{\partial t^{(2)}}{\partial \epsilon}(\theta, 0) + \frac{\partial t^{(1)}}{\partial \epsilon}(\theta, 0),$$

the proof of theorem is completed. □

Remark 1. *There are no large difficulties to extend Theorem 2.1 to more complicated and unperturbed sliding periodic solutions as long as we assume that the transversality condition holds when the orbit gets into the sliding surface. We deal with the simplest case here to avoid heavy notations.*

In general, the sliding orbits cannot be reversed. Therefore, we can only define positive invariant sets when sliding orbits are involved. Due to transversality, we can still define unique positive orbits. We have shown that there exists a circle on the extended phase space whose first return map is well defined when ϵ is small. It is natural to ask if there exists an invariant torus (at least positive invariant) on the extended phase space.

The surfaces $\mathfrak{T}_\epsilon^{(1)}$ and $\mathfrak{T}_\epsilon^{(2)}$ (see Section 2 for definitions) are cylinders. In fact, we can parametrize $\mathfrak{T}_\epsilon^{(1)}$ and $\mathfrak{T}_\epsilon^{(2)}$. The parametric map $P_\epsilon^{(1)} : S^1 \times [0, t^{(1)}(\theta, \epsilon)] \to \mathfrak{T}_\epsilon^{(1)}$ is naturally defined by

$$P_\epsilon^{(1)}(\theta, \tau) \to \phi(\tau, h(\theta, \epsilon), \epsilon).$$

One can show that $P_\epsilon^{(1)}$ is a diffeomorphsim. $P_\epsilon^{(2)} : S^1 \times [0, t^{(2)}(\theta, \epsilon)] \to \mathfrak{T}_\epsilon^{(2)}$ can be defined in a similar way. Hence, both $\mathfrak{T}_\epsilon^{(1)}$ and $\mathfrak{T}_\epsilon^{(2)}$ are cylinders in the extended phase space.

Now we show that $\mathfrak{T}_\epsilon$ is homeomorphic to a torus when $|\epsilon|$ is small. First we need the following glue lemma.

Lemma 4.1. *Let X and Y be metric spaces and $X = A \cup B$, $Y = C \cup D$, where A, B, C and D are closed subsets. Moreover, we assume*

$$A \cap B = \partial A = \partial B, \ C \cap D = \partial C = \partial D.$$

$G_1 : A \to C$ and $G_2 : B \to D$ are homeomorphisms such that

$$G_1|_{\partial A} = G_2|_{\partial B}.$$

Let G be defined by

$$G(x) = \begin{cases} G_1(x) & \text{if } x \in A \setminus \partial A, \\ G_2(x) & \text{if } x \in B \setminus \partial B, \\ G_1(x)(G_2(x)) & \text{if } x \in \partial A(\partial B). \end{cases}$$

Then $G : X \to Y$ is a homeomorphism.

Proof. Since G is well defined and it is a bijection, it suffices to show that G (resp. G^{-1}) is continuous on the boundary of A (resp. C). Take $x \in \partial A$. For any $\epsilon > 0$, there exists $\delta_1 > 0$ (resp. $\delta_2 > 0$) such that

$$d_Y(G_1(x), G_1(y)) < \epsilon \text{ (resp. } d_Y(G_2(x), G_2(y)) < \epsilon) \text{ if } d_X(x, y) < \delta_1 (\text{resp. } d_X(x, y) < \delta_2)$$

for $y \in A$ (resp. $y \in B$), where $d_X(\cdot, \cdot)$ (resp. $d_Y(\cdot, \cdot)$) is the metric of X (resp. Y). By taking $0 < \delta < \min(\delta_1, \delta_2)$ we have shown that G is continuous on ∂A. Similarly, G^{-1} is continuous on ∂C. The proof is complete. □

The proof of the following lemma is straightforward.

Lemma 4.2. *Suppose that* $g_1, g_2, m : \mathbb{R} \to \mathbb{R}$ *are continuous functions such that* $g_2(x) > g_1(x)$ *and* m *is strictly increasing. Then all the straight line segments with endpoints* $(x, g_1(x))$ *and* $(m(x), g_2(m(x)))$ *do not intersect each other.*

Recall that we identify (x, v_s, θ) and (x, θ).

Lemma 4.3. *There exists a vector field* $(v_s+O(\epsilon), 1+O(\epsilon))$ *between the curves* $\{(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ *and* $\{h(\theta,\epsilon) : \theta \in S^1\}$ *such that*

$$\phi_{s,\epsilon}(t^{(2)}(\theta,\epsilon), x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) = h(\theta+T_0,\epsilon),$$

where $\phi_{s,\epsilon}$ *is the flow generated by* $(v_s + O(\epsilon), 1 + O(\epsilon))$*. And the following identity holds:*

$$\begin{aligned}\mathfrak{T}_\epsilon^{(2)} &= \bigcup_{\theta\in S^1} \bigcup_{\tau\in[0,t^{(2)}(\theta,\epsilon)]} \phi_s(\tau, x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) \\ &= \bigcup_{\theta\in S^1} \bigcup_{\tau\in[0,t^{(2)}(\theta,\epsilon)]} \phi_{s,\epsilon}(\tau, x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)). \qquad (9)\end{aligned}$$

Proof. To define the vector field between $\{(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ and $\{h(\theta,\epsilon) : \theta \in S^1\}$, we connect the points

$$(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) \text{ and } \ h(\theta+T_0,\epsilon)$$

by a straight line segment. Let

$$\bar{\theta}(\theta) = \theta + t^{(1)}(\theta,\epsilon) = \theta + t_1 + O(\epsilon).$$

Denote the inverse of $\bar{\theta}$ by $\theta(\cdot)$. Note that

$$x^{(1)}(\theta(\bar{\theta})) < \frac{1}{k}(-cv_s + F_0 + \epsilon f(\theta(\bar{\theta})))$$

and that $\theta(\bar{\theta}) + T_0$ is strictly increasing for small $|\epsilon|$. By Lemma 4.2, all straight line segments connected by $(x^{(1)}(\theta(\bar{\theta}),\epsilon), \bar{\theta})$ and $h(\theta(\bar{\theta})+T_0,\epsilon)$ do not intersect each other.

We associate a vector at the point $(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon))$ with

$$\frac{1}{t^{(2)}(\theta,\epsilon)}(x^{(2)}(\theta+T_0,\epsilon) - x^{(1)}(\theta,\epsilon), \theta+T_0-(\theta+t^{(1)}(\theta,\epsilon))) = (v_s+O(\epsilon), 1+O(\epsilon)),$$

where $x^{(2)}(\theta+T_0,\epsilon)$ is the first coordinate of $h(\theta+T_0,\epsilon)$. For points on the straight line segments with endpoints $(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon))$ and $h(\theta+T_0,\epsilon)$, we associate the same vector to $(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon))$. Since all the straight line segments do not intersect, the vector field is well defined and we have

$$\phi_{s,\epsilon}(t^{(2)}(\theta,\epsilon), x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) = h(\theta+T_0,\epsilon).$$

The identity (9) holds since we simply change the vector field slightly between the curves $\{(x^{(1)}(\theta,\epsilon), \theta+t^{(1)}(\theta,\epsilon)) : \theta \in S^1\}$ and $\{h(\theta,\epsilon) : \theta \in S^1\}$, and the curves remain the same. □

Proof of Theorem 2.2. Let $I_\epsilon^{(1)} : \mathfrak{T}_\epsilon^{(1)} \to \mathfrak{T}_0^{(1)}$ be defined by

$$\phi(\tau, h(\theta,\epsilon), \epsilon) \to \phi(\frac{t_1}{t^{(1)}(\theta,\epsilon)}\tau, h(\theta,0), 0),\ \tau \in [0, t^{(1)}(\theta,\epsilon)].$$

Note that $t^{(1)}(\theta,\epsilon)$ is nearly t_1, which is nonzero when $|\epsilon|$ is small. Therefore, $I_\epsilon^{(1)}$ is well defined. Then we have

$$\begin{aligned}\phi(\tau, h(\theta,\epsilon), \epsilon) - \phi(\frac{t_1}{t^{(1)}(\theta,\epsilon)}\tau, h(\theta,0), 0) &= \frac{\partial \phi}{\partial t}(\tau, h(\theta,0))(\tau - \tau(1+\epsilon\frac{1}{t_1}\frac{\partial t}{\partial \epsilon}(\theta,0) + o(\epsilon))) \\ &+ \epsilon\frac{\partial \phi}{\partial X}(\tau, h(\theta,0),0)\frac{\partial h}{\partial \epsilon}(\theta,0) + \epsilon\frac{\partial \phi}{\partial \epsilon}(\tau, h(\theta,0),0) + o(\epsilon).\end{aligned}$$

Recall that $X = (x, y, \theta)$ and $\bar{X} = (x, \theta)$. Since $\tau \in [0, t^{(1)}(\theta,\epsilon)]$ is bounded, $I_\epsilon^{(1)}(X) = X + O(\epsilon)$. Let $I_\epsilon^{(2)} : \mathfrak{T}_\epsilon^{(2)} \to \mathfrak{T}_0^{(2)}$ be defined by

$$\phi_{s,\epsilon}(\tau, x^{(1)}(\theta,\epsilon), \theta + t^{(1)}(\theta,\epsilon)) \to \phi_s(\frac{t_2}{t^{(2)}(\theta,\epsilon)}\tau, x^{(1)}(\theta,0), \theta + t^{(1)}(\theta,0)),\ \tau \in [0, t^{(2)}(\theta,\epsilon)),$$

where $\phi_{s,\epsilon}$ is the flow in Lemma 4.3. Similarly, one can show that $I_\epsilon^{(2)}(\bar{X}) = \bar{X} + O(\epsilon)$.

By the definitions of $\mathfrak{T}_\epsilon^{(1)}$ and $\mathfrak{T}_\epsilon^{(2)}$, we have $\partial\mathfrak{T}_\epsilon^{(1)} = \partial\mathfrak{T}_\epsilon^{(2)}$ and $\partial\mathfrak{T}_0^{(1)} = \partial\mathfrak{T}_0^{(2)}$. Moreover, $I_\epsilon^{(1)}$ and $I_\epsilon^{(2)}$ are coincident on $\partial\mathfrak{T}_\epsilon^{(1)} = \partial\mathfrak{T}_\epsilon^{(2)}$. In fact, by definitions of $I_\epsilon^{(1)}$, $I_\epsilon^{(2)}$ and Lemma 4.3, we have

$$\begin{aligned}I_\epsilon^{(1)}(h(\theta,\epsilon)) = h(\theta,0),\ I_\epsilon^{(1)}(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) = (x_1, v_s, \theta + t_1), \\ I_\epsilon^{(2)}(h(\theta+T_0,\epsilon)) = h(\theta+T_0,0),\ I_\epsilon^{(2)}(x^{(1)}(\theta,\epsilon), v_s, \theta + t^{(1)}(\theta,\epsilon)) = (x_1, v_s, \theta + t_1).\end{aligned}$$

Hence, by Lemma 4.1, the map $I_\epsilon : \mathfrak{T}_\epsilon \to \mathfrak{T}_0$ given by

$$I_\epsilon(X) = \begin{cases} I_\epsilon^{(1)}(X) & \text{if } X \in \mathfrak{T}_\epsilon^{(1)} \setminus \partial\mathfrak{T}_\epsilon^{(1)} \\ I_\epsilon^{(2)}(X) & \text{if } X \in \mathfrak{T}_\epsilon^{(2)} \setminus \partial\mathfrak{T}_\epsilon^{(2)} \\ I_\epsilon^{(1)}(X) = I_\epsilon^{(2)}(X) & \text{if } X \in \partial\mathfrak{T}_\epsilon^{(1)} = \partial\mathfrak{T}_\epsilon^{(2)} \end{cases}$$

is a homeomorphism.

$\mathfrak{T}_\epsilon$ tends to $\mathfrak{T}_0$ is followed by the fact that $I_\epsilon^{(1)}$ and $I_\epsilon^{(2)}$ tends to the identity map as ϵ tends to zero. □

5. The dynamics on the invariant torus

We are concerned with the dynamics on the invariant torus. If the rotation number of $\mathfrak{R}_\epsilon$ is rational, then the existence of periodic points of $\mathfrak{R}_\epsilon$ implying the existence of periodic orbits for system (2). We are thus concerned whether every orbit is dense on the invariant torus when the rotation number of $\mathfrak{R}_\epsilon$ is irrational and $\mathfrak{R}_\epsilon$ is topologically conjugate to a rigid rotation. The following Proposition 5.1 shows that the answer is positive.

Lemma 5.1. *When $|\epsilon|$ is sufficiently small, the flows restricted to the torus $\mathfrak{T}_\epsilon$ are complete and depend continuously on the initial conditions.*

Proof. Denote the flow on $\mathfrak{T}_\epsilon$ by $\Phi(t, X, \epsilon)$. Since the return map $\mathfrak{R}_\epsilon$ is well defined, so is the positive flow. We can reverse $\mathfrak{R}_\epsilon$ so that the negative flow is also well defined. For example, if $X \in \mathfrak{T}_\epsilon^{(1)}$ and $X = \phi(\tau, h(\theta, \epsilon), \epsilon)$, then we can define the negative flow as follows:

$$\begin{aligned}
&\Phi(-t, X, \epsilon) = \phi(\tau - t, h(\theta, \epsilon), \epsilon) \text{ for } 0 \le t \le \tau;\\
&\Phi(-t, X, \epsilon) = \phi_s(t^{(2)}(\theta, \epsilon) - t + \tau, \Pi(\phi(t^{(1)}(\mathfrak{R}_\epsilon^{-1}(\theta), \epsilon), h(\mathfrak{R}_\epsilon^{-1}(\theta), \epsilon), \epsilon)))\\
&\qquad\qquad\qquad \text{for } \tau < t \le t^{(2)}(\theta, \epsilon) + \tau;\\
&\Phi(-t, X, \epsilon) = \phi(t^{(1)}(\theta, \epsilon) + \tau + t^{(2)}(\theta, \epsilon) - t, h(\mathfrak{R}_\epsilon^{-1}(\theta), \epsilon), \epsilon)\\
&\qquad\qquad\qquad \text{for } t^{(2)}(\theta, \epsilon) + \tau < t \le t^{(2)}(\theta, \epsilon) + \tau + t^{(1)}(\theta, \epsilon);\\
&\qquad\qquad \vdots
\end{aligned}$$

Since $\mathfrak{R}_\epsilon$ is a homeomorphism, $\Phi(-t, X, \epsilon)$ is unique. Similarly, we can define $\Phi(-t, X, \epsilon)$ for $X \in \mathfrak{T}_\epsilon^{(2)}$. Hence, $\Phi(t, X, \epsilon)$ is well defined and $\Phi(t, X, \epsilon) \in \mathfrak{T}_\epsilon$ for any $X \in \mathfrak{T}_\epsilon, t \in (-\infty, \infty)$. The continuous dependence of initial conditions directly follows from the continuousness of $\mathfrak{R}_\epsilon$ and piecewise smoothness property of the vector field. □

Proposition 5.1. *Under the assumption* **H***, every orbit of* (2) *is dense on the torus* $\mathfrak{T}_\epsilon$ *if* $\mathfrak{R}_\epsilon$ *is topologically conjugate to an irrational rigid rotation.*

Proof. Suppose it is not true. Let $\Phi(t, X, \epsilon)$ be the flow on $\mathfrak{T}_\epsilon$. We can find a nonempty open set U and an orbit γ in $\mathfrak{T}_\epsilon$ such that $U \cap \gamma$ is empty. Let $X_1 \in U$ and $X_1 \notin S_\epsilon$. Then we may choose a point $\bar{X}_1 \in S_\epsilon$ such that $\Phi(s, \bar{X}_1, \epsilon) = X_1$ and $\Phi(\tau, \bar{X}_1, \epsilon) \notin S_\epsilon$ for $\tau \in (0, s]$. Since $\mathfrak{R}_\epsilon$ is topologically conjugate to an irrational rigid rotation, any orbit in $\mathfrak{T}_\epsilon$ has dense intersection with S_ϵ. For any $\delta > 0$, we can choose $X_2 \in S_\epsilon \cap \gamma$ such that $|\bar{X}_1 - X_2| < \delta$. While

$$|\Phi(s, \bar{X}_1, \epsilon) - \Phi(s, \bar{X}_1, \epsilon)| \ge K,$$

where K is the distance between X_1 and the boundary of U, which contradicts to the fact that δ is arbitrarily small by Lemma 5.1. □

However, we cannot assert that the phase map $\mathfrak{R}_\epsilon$ is topologically conjugate to an irrational rigid rotation, even the ratio of the period of f to the period of unperturbed sliding periodic solution is an irrational. The following Moser's theorem [27], together with Proposition 5.1, at least allows us to inform that every orbit of (2) is dense on $\mathfrak{T}_\epsilon$ for most frequencies of f when $|\epsilon|$ is sufficiently small.

Theorem 5.1. *For sufficiently smooth map*

$$g = \theta + a + \delta\alpha(\theta), \ \ mod\ 1,$$

the proportion of $(a, \delta) \in [0, 1] \times [0, \delta_0]$*, such that* g *cannot be smoothly conjugated to an irrational rigid rotation, tends to zero as* δ_0 *tends to zero, where* $\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ *is a smooth function.*

6. Concluding remarks

In this work, we consider the dynamics in the neighborhood of the sliding periodic solution for periodic perturbation of an autonomous one-degree-of-freedom oscillator with dry friction. The system is reduced to a circle map and the existence of invariant tori is proved by gluing the cylinders generated by smooth vector fields. Typical resonance phenomenon of a circle map appears as the frequency of periodic excitation varies. We show that the destruction of invariant tori is due to the loss of transversality when the perturbation gets large, which is different from the usual smooth tori break down.

The results in Section 4 may have some reasonable generalizations. The integral manifold theory [14] insures that an hyperbolic periodic solution of time-independent systems become an invariant torus under a small periodic perturbation. The sliding periodic orbits in time-independent systems have two Floquet multipliers, 0 and 1, under some transversality conditions. It is natural to ask if the sliding periodic orbit becomes to an invariant torus under small periodic perturbation when the norms of other Floquet multipliers are not 1. Moreover, the stability of invariant torus is expected to coincide with that of the unperturbed periodic orbit when the perturbation is small.

It is more involved to prove that the flow restricted to the sliding invariant torus is conjugate to a linear flow, i.e., that there exists a torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that

$$g(\Phi(t, X, \epsilon)) = \varphi(t, g(X)),$$

where $g : \mathfrak{T}_\epsilon \to T^2$ is a homomorphism, $\varphi(t, \cdot)$ is the flow generated by

$$\dot{\theta}_1 = \omega_1, \text{ mod } 1,$$
$$\dot{\theta}_2 = \omega_2, \text{ mod } 1,$$

ω_1 and ω_2 being constants. The piecewise smoothness property of $\mathfrak{T}_\epsilon$ and the flow on it cause new difficulty since the usual coordinate transformation method cannot apply directly, and the cylindrical coordinate near the unperturbed periodic orbit is also not available. Moreover, we have to smooth both $\mathfrak{T}_\epsilon$ and the flow restricted to it.

The existence of an $n+1$ dimensional invariant torus is expected to be established if the perturbation of the system is a quasi-periodic function with n non-resonance frequencies.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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