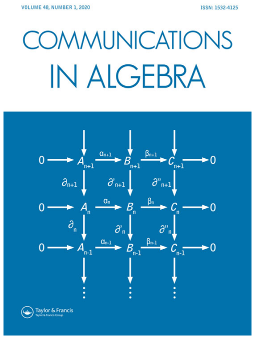

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Principal series for general linear groups over finite commutative rings

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ABSTRACT

We construct, for any finite commutative ring R, a family of representations of the general linear group $\mathrm{GL}_n(R)$ whose intertwining properties mirror those of the principal series for GL_n over a finite field.

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1. Introduction

Among the irreducible, complex representations of reductive groups over finite fields, the simplest to construct and to classify are the *principal series*: those obtained by Harish-Chandra induction from a minimal Levi subgroup; see, for instance, [13]. In this paper we use a generalization of Harish-Chandra induction to construct a "principal series" of representations of the group $\mathrm{GL}_n(R)$, where R is any finite commutative ring with identity. Our main results assert that the well-known intertwining relations among the principal series for GL_n over a finite field also hold for the representations that we construct.

The study of the principal series for reductive groups over finite fields can be viewed as the first step in the program to understand all irreducible complex representations of such groups in terms of what Harish-Chandra called the 'philosophy of cusp forms' [10, 20]. This program has met with considerable success. The basic ideas appear already in Green's determination [8] of the irreducible characters of $\mathrm{GL}_n(\Bbbk)$, where $\Bbbk$ is a finite field, and these ideas have since been developed and generalized to a very great extent; see [7] for an overview.

The theory for groups over finite rings is in a far less advanced state. Most efforts so far have been directed toward groups over principal ideal rings: see for instance [21] and references therein. By contrast, the results presented below are valid for all finite rings, with the essential jump in generality being from principal ideal rings to local rings. Moreover, our results depend

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on the algebraic properties of the base ring in only a very limited way: for instance, we give a uniform construction of a family of irreducible representations of $\mathrm{GL}_n(R)$ for all finite local rings R, and to our knowledge these are the first results obtained in this degree of generality.

The present paper is part of a project whose aim is to extend the philosophy of cusp forms to reductive groups over finite rings. Our construction, which is a special case of a general induction procedure developed in [3], extends in a natural way to produce more general 'Harish-Chandra series'. The analysis of the intertwining properties of these more general series seems, however, to be substantially more involved than the results for the principal series presented here. See [3, Section 5] and [4] for some partial results in this more general setting.

1.1. Notation and definitions

Let R be a finite commutative ring with 1. Let $G = \mathrm{GL}_n(R)$, let $L \cong (R^\times)^n$ be the subgroup of diagonal matrices in G, and let U and V be the upper-unipotent subgroup and the lower-unipotent subgroup, respectively, in G. Let $B = LU$ be the subgroup of upper-triangular matrices. We write $G(R)$, $L(R)$, etc., when it is necessary to specify R.

The ring R decomposes as a direct product of local rings: $R \cong R_1 \times \cdots \times R_m$, and this decomposition is unique up to permuting the factors [17, Theorem VI.2]. There is a corresponding decomposition $G(R) \cong G(R_1) \times \cdots \times G(R_m)$, and similarly for L, U, and V. If R is a local ring then we let $N(R)$ be the subgroup of monomial matrices in $G(R)$, that is, products of permutation matrices with diagonal matrices. If R is not local then we define $N(R) = N(R_1) \times \cdots \times N(R_m)$, where the R_i are the local factors of R as above. Let $W(R) = N(R)/L(R)$. It will be convenient to realize $W(R)$ as a subgroup of $G(R)$, as follows: if R is local, then we identify $W(R)$ with the group of permutation matrices; and in the general case we identify $W(R)$ with the product of the permutation subgroups in $G(R) \cong G(R_1) \times \cdots \times G(R_m)$. Note that following Lemma 4, we will be able to assume without loss of generality that R is a local ring.

If $\chi : L \to \mathrm{GL}(X)$ is a representation of L on a complex vector space X, and if $w \in W$, then we let $w^*\chi$ denote the representation $\chi \circ \mathrm{Ad}_w^{-1} : L \to \mathrm{GL}(X)$. We let $W_\chi = \{w \in W | w^*\chi \cong \chi\}$.

For each subgroup $H \subseteq G$ we let e_H denote the idempotent in the complex group ring $\mathbb{C}[G]$ corresponding to the trivial character of H: $e_H = |H|^{-1} \sum_{h\in H} h$. Since L normalizes U and V, the idempotents e_U and e_V commute with $\mathbb{C}[L]$ inside $\mathbb{C}[G]$.

We consider the functors

$$\begin{aligned} \mathrm{i} : \mathrm{Rep}(L) \to \mathrm{Rep}(G) \qquad & X \mapsto \mathbb{C}[G]e_U e_V \otimes_{\mathbb{C}[L]} X \\ \mathrm{r} : \mathrm{Rep}(G) \to \mathrm{Rep}(L) \qquad & Y \mapsto e_U e_V \mathbb{C}[G] \otimes_{\mathbb{C}[G]} Y, \end{aligned}$$

where $\mathrm{Rep}(G)$ denotes the category of complex representations, identified in the usual way with the category of left $\mathbb{C}[G]$-modules. This is a special case of the construction defined in [3, Section 2], which generalizes a definition due to Dat [6]. The functors i and r are two-sided adjoints to one another; see [3, Theorem 2.15] for a proof of this and other basic properties.

Definition. Let us say that an irreducible representation of G is in the *principal series* if it is isomorphic to a subrepresentation of $\mathrm{i}\chi$ for some representation χ of L.

Example. For each representation $\chi : L \to \mathrm{GL}(X)$ of L, the representation $\mathrm{i}\chi = \mathbb{C}[G]e_U e_V \otimes_{\mathbb{C}[L]} X$ of G is a nonzero quotient of the representation $\mathbb{C}[G]e_U \otimes_{\mathbb{C}[L]} X$, the latter being the representation of G obtained by first extending χ from L to LU by letting U act trivially on X, and then inducing from LU to G. If this representation $\mathbb{C}[G]e_U \otimes_{\mathbb{C}[L]} X$ is irreducible, then it must equal $\mathrm{i}\chi$.

If R is a field, then the map $\mathbb{C}[G]e_U \xrightarrow{f \mapsto f e_V} \mathbb{C}[G]e_V$ is known to be an isomorphism of $\mathbb{C}[G]$-$\mathbb{C}[L]$ bimodules; see [15, Theorem 2.4]. It follows that in this case the functors i and r are naturally isomorphic to the familiar functors of *Harish-Chandra induction and restriction*, i.e., the functors of tensor product with the bimodules $\mathbb{C}[G]e_U$ and $e_U\mathbb{C}[G]$, respectively. The same is not true if R is not a product of fields, as the following example illustrates.

Example. Let 1_L denote the trivial representation of L. Then we have $\mathbb{C}[G]e_U \otimes_{\mathbb{C}[L]} 1_L \cong \mathbb{C}[G/LU]$, with G acting by permutations of G/LU; and likewise $\mathbb{C}[G]e_V \otimes_{\mathbb{C}[L]} 1_L \cong \mathbb{C}[G/LV]$. Let $w_0 \in G$ be the permutation matrix that conjugates U into V, and vice versa; then the map $gLV \mapsto gw_0LU$ induces a G-equivariant isomorphism $\mathbb{C}[G/LV] \xrightarrow{\cong} \mathbb{C}[G/LU]$. Making these identifications, the map

$$\mathbb{C}[G]e_U \otimes_{\mathbb{C}[L]} 1_L \xrightarrow{f\otimes 1 \mapsto fe_V \otimes 1} \mathbb{C}[G]e_V \otimes_{\mathbb{C}[L]} 1_L \tag{*}$$

becomes, up to a nonzero scalar multiple, the map $\mathbb{C}[G/LU] \to \mathbb{C}[G/LU]$ of multiplication on the right by the characteristic function of the double coset LUw_0LU. If R is a field, then the latter map is well-known to be invertible (as are all of the standard generators of the Iwahori-Hecke algebra $\mathbb{C}[LU\ G/LU]$; see for instance [5, §67 A]).

By contrast, suppose now that R is not a field. Let $\mathfrak{m}$ be a maximal ideal of R, and let V_0 be the subgroup of V comprising those lower-unipotent matrices over R that reduce, modulo $\mathfrak{m}$, to the identity matrix. The product $I = LUV_0$ is a subgroup of G (namely, the group of upper-triangular-modulo-$\mathfrak{m}$ matrices). Since V_0 is a subgroup of V we have $e_V = e_{V_0}e_V$, and so the map (*) factors through the map

$$\mathbb{C}[G]e_U \otimes_{\mathbb{C}[L]} 1_L \xrightarrow{f\otimes 1 \mapsto fe_{V_0} \otimes 1} \mathbb{C}[G]e_{V_0} \otimes_{\mathbb{C}[L]} 1_L,$$

whose image is isomorphic to the permutation module $\mathbb{C}[G/I]$. The latter has strictly smaller dimension than $\mathbb{C}[G/LU]$, and so (*) cannot be an isomorphism.

For general rings, the permutation module $\mathbb{C}[G/LU]$ can be quite complicated. For instance, for $R = \mathbb{Z}/p^k\mathbb{Z}$ (with p a prime and k a positive integer), the results of [18] show that the intertwining algebra of this representation depends both on p and on k. By contrast, it follows from Theorem 2 below that for any R the intertwining algebra of $\mathrm{i}1_L$ is isomorphic to the tensor product $\mathbb{C}[S_n]^{\otimes m}$, where $G = \mathrm{GL}_n(R)$ and where m is the number of maximal ideals in R.

Example. Suppose that R is a finite discrete valuation ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk$, and let r be the largest integer such that $\mathfrak{m}^r \neq 0$. Reduction modulo $\mathfrak{m}^r$ gives rise to a group extension

$$0 \to G_r \cong (M_n(\Bbbk), +) \to G(R) \to G(R/\mathfrak{m}^r) \to 0,$$

which one can use to study the representations of $G(R)$ via Clifford theory; see [11], for example. In [12], Hill identified a class of representations that are particularly amenable to this approach: an irreducible representation π of $G(R)$ is called *regular* if its restriction to G_r contains a character whose stabilizer under the adjoint action of $G(\Bbbk)$ is an abelian group (see [12, Theorem 3.6] for details and alternative characterizations of regularity). Explicit constructions of all such representations are given in [16, 22].

An application of [3, Theorem 3.4] gives the following criterion for regularity of the induced representations $\mathrm{i}\chi$: if χ is an irreducible representation of $L(R)$, then $\mathrm{i}\chi$ is regular if and only if the restriction of χ to the subgroup $L(R) \cap G_r \cong \Bbbk^n$ has trivial stabilizer under the permutation action of S_n. Moreover, the representations $\mathrm{i}\chi$, for χ satisfying the above condition, account for all of the regular representations associated to the split semisimple classes in $M_n(\Bbbk)$.

For $n = 2$, all of the principal series representations of $G(R) = \mathrm{GL}_2(R)$ can be described in terms of regular representations, as follows. Let $\chi : L \to \mathbb{C}^\times$ be an irreducible representation of L. If $\mathrm{i}\chi$ is irreducible, then there is a character $\tau : R^\times \to \mathbb{C}^\times$, an integer k, and a regular representation π of $G(R/\mathfrak{m}^k)$ associated to a split semisimple class in $M_2(\Bbbk)$ such that $\mathrm{i}\chi$ is isomorphic to the representation $(\tau \circ \det) \otimes \pi$, where π is pulled back to a representation of $G(R)$. If $\mathrm{i}\chi$ is not irreducible, then there is a character $\tau : R^\times \to \mathbb{C}^\times$ such that $\mathrm{i}\chi$ is isomorphic to the representation $(\tau \circ \det) \otimes (1_G \oplus \mathrm{St})$, where 1_G is the trivial representation, and St is the Steinberg representation of $G(\Bbbk)$ pulled back to $G(R)$.

To prove these assertions, we use the obvious isomorphism $L \cong R^\times \times R^\times$ to write χ as a product $\chi_1 \otimes \chi_2$. The criterion for regularity given above shows that if $\mathrm{i}\chi$ is not itself regular, then χ_1 and χ_2 agree on $1 + \mathfrak{m}^r$. Supposing this to be the case, we use Lemma 14 (below) to write $\mathrm{i}\chi \cong (\chi_1 \circ \det) \otimes \mathrm{i}(1 \otimes \chi_1^{-1}\chi_2)$, where the character $1 \otimes \chi_1^{-1}\chi_2$ is trivial on $L \cap G_r$ and is therefore pulled back from a character χ' of $L(R/\mathfrak{m}^r)$. Now [3, Theorem 3.4] implies that $\mathrm{i}(1 \otimes \chi_1^{-1}\chi_2)$ is the pullback to $G(R)$ of the representation $\mathrm{i}\chi'$ of $G(R/\mathfrak{m}^r)$. If $\mathrm{i}\chi'$ is not regular then we can repeat the above procedure, as many times as necessary. In the case where $\mathrm{i}\chi$ is not irreducible we have $\chi_1 = \chi_2$, by Theorem 1 (below), and then Lemma 14 gives $\mathrm{i}\chi \cong (\chi_1 \circ \det) \otimes \mathrm{i}1_L$, where $\mathrm{i}1_L$ is the pullback to $G(R)$ of the representation $\mathrm{i}1_{L(\Bbbk)}$ (by [3, Theorem 3.4]). The latter representation is, as is well known, isomorphic to sum of the trivial representation and the Steinberg representation.

For $n \geq 3$ the relationship between the principal series and the regular representations becomes more complicated.

2. Main results

We will show that the following well-known properties of the Harish-Chandra functors are shared by the functors i and r for R an arbitrary finite commutative ring.

Theorem 1. *There is a natural isomorphism* $\mathrm{ri} \cong \bigoplus_{w \in W} w^*$ *of functors on* $\mathrm{Rep}(L)$*. Consequently, if* χ *and* σ *are irreducible representations of L, then*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(\mathrm{i}\chi, \mathrm{i}\sigma)) = \#\{w \in W | w^*\chi = \sigma\}.$$

When $\sigma = \chi$, we have the following more precise statement:

Theorem 2. *For each irreducible representation* χ *of L one has* $\mathrm{End}_G(\mathrm{i}\chi) \cong \mathbb{C}[W_\chi]$ *as algebras.*

Theorems 1 and 2 readily imply the following combinatorial formula for the number of principal series representations. Following [1], we let $P_k(n)$ denote the number of multipartitions of n with k parts: i.e., the number of k-tuples $(\lambda^{(1)}, \ldots, \lambda^{(k)})$, where each $\lambda^{(i)}$ is a partition of some non-negative integer n_i, and $\sum_i n_i = n$.

Corollary 3. *If R is isomorphic to a product* $R_1 \times \cdots \times R_m$ *of finite local rings, and for each j we set* $k_j = |R_j^\times|$*, then the principal series of* $\mathrm{GL}_n(R)$ *contains precisely* $\prod_j P_{k_j}(n)$ *distinct isomorphism classes of irreducible representations.*

Remarks.

- In the case where R is a field, Theorems 1 and 2 are essentially due to Green [8]; see [23] for the case $\chi = 1_L$, and see [19] for an exposition. Both of these results have been generalized to arbitrary Harish-Chandra series for arbitrary reductive groups: see [10] and [14], respectively.
- Theorems 1 and 2 can be extended, using [3, Theorem 2.15(5)], to the setting of smooth representations of the profinite groups $G(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in a nonarchimedean local field.
- Some of our results apply beyond the case of GL_n. For instance, an analogue of Theorem 1 holds whenever G is a split classical group: indeed, such groups are easily seen to satisfy properties (a)–(f) in Proposition 5 below, and our proof of Theorem 1 relies only on those properties. We have restricted our attention here to GL_n, both in order to simplify the exposition, and because that is the case in which we use these results in [4].

- On the other hand, adapting our proof of Theorem 1 to the case where L is replaced by a larger Levi subgroup does not seem to be so straightforward. For one thing, the failure of Proposition 5(d) in this more general setting greatly complicates matters.

3. Proofs

The first step in the proof of the main results is to reduce to the case of local rings.

Lemma 4. *If Theorems 1 and 2 and Corollary 3 are true for all finite commutative local rings, then they are true for all finite commutative rings.*

Proof. Let R be a finite commutative ring, and write R as a product of local rings $R_1 \times \cdots \times R_m$. All of the groups and the representation categories in Theorems 1 and 2 and in Corollary 3 then decompose into products accordingly: $G(R) \cong G(R_1) \times \cdots \times G(R_m), \mathrm{Rep}(G(R)) \cong \mathrm{Rep}(G(R_1)) \times \cdots \times \mathrm{Rep}(G(R_m))$, and so on. The bimodule $\mathbb{C}[G(R)]e_{U(R)}e_{V(R)}$ decomposes as the tensor product of the bimodules $\mathbb{C}[G(R_j)]e_{U(R_j)}e_{V(R_j)}$, and likewise for $e_{U(R)}e_{V(R)}\mathbb{C}[G(R)]$, so the functors i and r are compatible with the above decompositions. By definition, the group W also decomposes compatibly. Thus Theorems 1 and 2 and Corollary 3 over R follow immediately from the corresponding results over the local factors R_j. □

Assume from now on that R is a finite commutative local ring Let $\mathfrak{m}$ denote the maximal ideal of R, and let $\mathbb{k}$ denote the residue field $R/\mathfrak{m}$. Recall that $W \cong S_n$ is then the group of permutation matrices in G. We write ℓ for the word-length function on W with respect to the standard generating set $S = \{(12), ..., (n-1n)\}$.

The following proposition collects the group-theoretical ingredients of the proof of Theorem 1.

Proposition 5.

a. *The multiplication map $U \times L \times V \to G$ is injective.*
b. *The reduction-mod-$\mathfrak{m}$ map $G(R) \to G(\mathbb{k})$ is surjective.*
c. *For each subgroup H of G, let H_0 denote the intersection of H with the kernel G_0 of the above reduction homomorphism. Then the multiplication map $U_0 \times L_0 \times V_0 \to G_0$ is a bijection, and the same is true for any ordering of the three factors.*
d. *For each $w \in W$ the multiplication maps*

$$(U \cap U^w) \times (U \cap V^w) \to U \qquad \text{and} \qquad (V \cap U^w) \times (V \cap V^w) \to V$$

are bijections, where $U^w = w^{-1}Uw$, etc.

e. *G is the disjoint union $G = \sqcup_{w\in W} G_w$, where $G_w = VwLUG_0$.*
f. *For each $r, t \in W$ with $\ell(t) \leq \ell(r)$ and $t \neq r$ one has $ULV \cap t^{-1}Ur = \emptyset$.*

Proof. Parts (a), (b), and (d) are well-known and easily verified.

For part (c), the map $U_0 \times L_0 \times V_0 \to G_0$ is injective by part (a). Now the ideal $\mathfrak{m}$ is nilpotent, so every matrix of the form $1 + x$ with $x \in M_n(\mathfrak{m})$ is invertible, and thus $G_0 = \{1 + x | x \in M_n(\mathfrak{m})\}$, while L_0, U_0, and V_0 are the subgroups in which x is, respectively, diagonal, strictly upper-triangular, or strictly lower-triangular. Counting matrix entries then shows that the finite sets $U_0 \times L_0 \times V_0$ and G_0 have equal cardinality, and so the injective multiplication map is bijective.

Part (e) follows immediately from the Bruhat decomposition of $G(\mathbb{k})$ [5, (65.4)].

In part (f) we may assume without loss of generality that R is a field, since $ULV \cap t^{-1}Ur$ is empty if its reduction modulo $\mathfrak{m}$ is empty. This assumption implies that (B, N, W, S) is a *BN-pair* in G, where we are writing B for the upper-triangular subgroup LU of G; see, e.g., [5, (65.10)].

Let w_0 denote the longest element $(1,2,...,n)\mapsto(n,...,2,1)$ of W. It follows from [2, Ch. IV §2 Lemme 1] that, under the stated assumptions on t and r, we have $tBw_0B\cap Brw_0B=\emptyset$. Since $ULVw_0=ULw_0U=Bw_0B$, while $t^{-1}Urw_0\subseteq t^{-1}Brw_0B$, we conclude that $ULV\cap t^{-1}Ur=\emptyset$. □

We equip $\mathbb{C}[G]$ with the Hermitian inner product $\langle|\rangle$ for which the group elements $g\in G$ constitute an orthonormal basis; and with the conjugate-linear involution $*$ defined on basis elements by $g^*=g^{-1}$. The two structures are related by the identity $\langle abc|d\rangle=\langle b|a^*dc^*\rangle$ for all $a,b,c,d\in\mathbb{C}[G]$. An element $a\in\mathbb{C}[G]$ is called self-adjoint if $a=a^*$.

Lemma 6. *There is a self-adjoint, invertible element $z\in\mathbb{C}[G]$ that commutes with e_U, e_V, and $\mathbb{C}[L]$, and that satisfies $z(e_Ue_V)^2=e_Ue_V$ and $z(e_Ve_U)^2=e_Ve_U$.*

Proof. This follows from a general fact about pairs of orthogonal projections on a finite-dimensional Hilbert space: see [9, Theorem 2], for example. □

Remark. If R is a field then [15, Theorem 2.4] implies that there is a *unique* element z as in Lemma 6. This is not the case over a general ring.

Lemma 7. *For each $w\in W$ we have $e_Ve_{U^w}e_{V^w}=e_Ve_Ue_{V^w}$.*

Proof. It is clear that $e_V=e_Ve_{(V\cap U^w)}$ and similarly that $e_U=e_{(U\cap U^w)}e_U$. Proposition 5(d) gives $e_{(V\cap U^w)}e_{(U\cap U^w)}=e_{U^w}$, and it follows that $e_Ve_U=e_Ve_{U^w}e_U$. The same reasoning gives $e_{U^w}e_{V^w}=e_{U^w}e_Ue_{V^w}$, and so $e_Ve_Ue_{V^w}=e_Ve_{U^w}e_Ue_{V^w}=e_Ve_{U^w}e_{V^w}$. □

Lemma 8. *For each $w\in W$ the map*

$$\varphi_w:e_{U^w}e_{V^w}\mathbb{C}[G]\overset{x\mapsto e_Vx}{\longrightarrow}e_Ve_U\mathbb{C}[G]$$

is an isomorphism of $\mathbb{C}[L]$-$\mathbb{C}[G]$ bimodules.

Proof. The following argument is taken from [6, Lemme 2.9]. The map φ_w is well-defined, because

$$e_Ve_{U^w}e_{V^w}\mathbb{C}[G]=e_Ve_Ue_{V^w}\mathbb{C}[G]\subseteq e_Ve_U\mathbb{C}[G]$$

by Lemma 7. The map φ_w is injective, because for each $f\in\mathbb{C}[G]$ we have

$$w^{-1}zwe_{U^w}e_{V^w}(e_Ve_{U^w}e_{V^w}f)=z^w(e_{U^w}e_{V^w})^2f=e_{U^w}e_{V^w}f$$

where z is as in Lemma 6, and in the first equality we used that $V=(V\cap V^w)(V\cap U^w)$. The domain and target of φ_w are isomorphic as vector spaces: indeed, $e_Ve_U\mathbb{C}[G]=w_0we_{U^w}e_{V^w}\mathbb{C}[G]$, where w_0 is the longest element of W. Since φ_w is injective it is thus also an isomorphism. □

For each subset $K\subseteq G$, we let $\mathbb{C}[K]$ denote the vector subspace of $\mathbb{C}[G]$ spanned by K.

Proposition 9. *For each $w\in W$ the map*

$$\Phi:\mathbb{C}[wL]\to e_Ue_V\mathbb{C}[G_w]e_Ue_V\qquad wl\mapsto e_Ue_Vwle_Ue_V$$

is an isomorphism of $\mathbb{C}[L]$-bimodules.

Here the sets wL and G_w are invariant under multiplication by L, on either side, and we are using these multiplication actions to view $\mathbb{C}[wL]$ and $\mathbb{C}[G_w]$ as $\mathbb{C}[L]$-bimodules.

Proof. Φ is clearly a bimodule map. Let us show that it is injective. For $h \in \mathbb{C}[L]$ we have

$$\Phi(wh) = e_U e_V e_{U^{w^{-1}}} e_{V^{w^{-1}}} wh.$$

The maps

$$e_{U^{w^{-1}}} e_{V^{w^{-1}}} \mathbb{C}[G] \xrightarrow{x \mapsto e_V x} e_V e_U \mathbb{C}[G]$$

and

$$e_V e_U \mathbb{C}[G] \xrightarrow{x \mapsto e_U x} e_U e_V \mathbb{C}[G]$$

are isomorphisms by Lemma 8, so we are left to prove that the map

$$wh \mapsto e_{U^{w^{-1}}} e_{V^{w^{-1}}} wh = we_U h e_V$$

is injective on $\mathbb{C}[wL]$. It is, because Proposition 5(a) implies that the cosets UlV are all disjoint as l ranges over L. Thus Φ is injective.

To prove that Φ is surjective, first note that $G_w = VwLG_0U$ because G_0 is normal in G. Since $e_V v = e_V$ and $ue_U = e_U$ for all $v \in V$ and $u \in U$, we find that $e_U e_V \mathbb{C}[G_w] e_U e_V$ is spanned by elements of the form $e_U e_V wlg e_U e_V$, where $l \in L$ and $g \in G_0$. We will show that each element of this form is in the image of Φ.

For each $x \in V^w$ we have

$$gx = x(x^{-1}gx) \in V^w G_0 = V^w (V_0^w L_0 U_0^w) = V^w L_0 U_0^w$$

by Proposition 5(c). Let $\alpha : V^w \to V^w, \beta : V^w \to L_0$ and $\gamma : V^w \to U_0^w$ be the (unique) functions satisfying $gx = \alpha(x)\beta(x)\gamma(x)$ for all $x \in V^w$. Writing $e_U = e_{U \cap V^w} e_{U \cap U^w}$ and $e_V = e_{V \cap U^w} e_{V \cap V^w}$, we then have

$$\begin{aligned}
e_V wlg e_U e_V &= e_V wlg e_{U \cap V^w} e_{U \cap U^w} e_{V \cap U^w} e_{V \cap V^w} \\
&= e_V wlg \left(|U \cap V^w|^{-1} \sum_{x \in U \cap V^w} x \right) e_{U^w} e_{V \cap V^w} \\
&= |U \cap V^w|^{-1} \sum_{x \in U \cap V^w} e_V wl\alpha(x)\beta(x)\gamma(x) e_{U^w} e_{V \cap V^w}.
\end{aligned}$$

Since $\gamma(x) \in U^w$ we have $\gamma(x) e_{U^w} = e_{U^w}$ for each $x \in U \cap V^w$. Since $\alpha(x) \in V^w$ we have $wl\alpha(x)l^{-1}w^{-1} \in V$, and consequently $e_V wl\alpha(x) = e_V wl$ for each x. Continuing the computation with the space-saving notation $h = |U \cap V^w|^{-1} \sum_{x \in U \cap V^w} l\beta(x) \in \mathbb{C}[L]$, we find that

$$\begin{aligned}
e_V wlg e_U e_V &= e_V wh e_{U^w} e_{V \cap V^w} = e_V e_{U^{w^{-1}} \cap V} wh e_{U^w} e_{V \cap V^w} \\
&= e_V wh e_{U \cap V^w} e_{U \cap U^w} e_{V \cap U^w} e_{V \cap V^w} = e_V wh e_U e_V,
\end{aligned}$$

and so $e_U e_V wlv e_U e_V = \Phi(wh)$. □

Proposition 10. *The set $\{e_U e_V wl e_U e_V \in \mathbb{C}[G] | w \in W, l \in L\}$ is linearly independent.*

Proof. We know from Proposition 9 that for each $w \in W$ the set $\{e_U e_V wl e_U e_V | l \in L\}$ is linearly independent. We must show that for different choices of w these sets are independent from one another.

Suppose we had elements $h_w \in \mathbb{C}[L]$, not all zero, with $\sum_{w \in W} e_U e_V wh_w e_U e_V = 0$. Let $t \in W$ be an element of minimal length such that h_t is nonzero. To compactify the notation we shall write $y = t^{-1}$.

Let z be as in Lemma 6, and write $\zeta = y^{-1}zy$. Thus ζ is a self-adjoint, invertible element of $\mathbb{C}[G]$ which commutes with e_{U^y} and e_{V^y} and which satisfies $\zeta(e_{U^y} e_{V^y})^2 = e_{U^y} e_{V^y}$. For each $r \in W$ with $r \neq t$ such that $h_r \neq 0$ we have

$$\begin{aligned}\langle\zeta^2 e_{U^y}(e_U e_V t h_t e_U e_V)|e_U e_V r h_r e_U e_V\rangle &= \langle\zeta^2 e_{U^y} e_U e_V e_{U^y} e_{V^y} t h_t|r h_r e_{U^r} e_{V^r} e_U e_V\rangle \\ &= \langle\zeta^2 e_{U^y} e_U e_V e_{U^y} e_{V^y} t h_t e_V e_U e_{V^r} e_{U^r} h_r^*|r\rangle = \langle\zeta^2 e_{U^y} e_U e_V e_{U^y} e_{V^y} e_{U^y} e_{V^{ry}} e_{U^{ry}} t h_t h_r^*|r\rangle \\ &= \langle\zeta^2 e_{U^y} e_U e_{V^y} e_{U^y} e_{V^y} e_{U^y} e_{V^y} e_{U^{ry}} t h_t h_r^*|r\rangle = \langle\zeta^2 (e_{U^y} e_{V^y})^3 e_{U^{ry}} t h_t h_r^*|r\rangle \\ &= \langle e_{U^y} e_{V^y} e_{U^{ry}} t h_t h_r^*|r\rangle = \langle t e_U h_t h_r^* e_V e_{U^r}|r\rangle = \langle e_U h_t h_r^* e_V|t^{-1} e_U r\rangle = 0.\end{aligned}$$

Here we have repeatedly used the equality $\langle abc|d\rangle = \langle b|a^* d c^*\rangle$; in the fourth step we used Lemma 7 to replace $e_U e_V e_{U^y}$ with $e_U e_{V^y} e_{U^y}$ and to replace $e_{U^y} e_{V^{ry}} e_{U^{ry}}$ with $e_{U^y} e_{V^y} e_{U^{ry}}$; in the fifth step we used Proposition 5(d) to write $e_{U^y} e_U e_{V^y} = e_{U^y} e_{V^y}$; and in the final equality we used Proposition 5(f), which applies because of the minimality of $\ell(t)$, and which implies that the functions $e_U h_t h_r^* e_V$ and $t^{-1} e_U r$ are supported on disjoint subsets of G and are therefore orthogonal.

It follows from this that

$$\begin{aligned}0 &= \langle\zeta^2 e_{U^y} e_U e_V t h_t e_U e_V \mid \sum_{w\in W} e_U e_V w h_w e_U e_V\rangle \\ &= \langle\zeta^2 e_{U^y} e_U e_V t h_t e_U e_V \mid e_U e_V t h_t e_U e_V\rangle \\ &= \langle\zeta e_{U^y} e_U e_V t h_t e_U e_V \mid \zeta e_{U^y} e_U e_V t h_t e_U e_V\rangle,\end{aligned}$$

where the last equality holds because ζ is self-adjoint, e_{U^y} is a self-adjoint idempotent, and ζ and e_{U^y} commute. Thus $\zeta e_{U^y} e_U e_V t h_t e_U e_V = 0$. Since ζ is invertible, and left multiplication by e_{U^y} is injective on $e_U e_V \mathbb{C}[G]$ (Lemma 8), we conclude that $e_U e_V t h_t e_U e_V = 0$. By Proposition 9 this implies that $h_t = 0$, contradicting our choice of t and completing the proof of the proposition. □

Proof of Theorem 1. The functor ri is naturally isomorphic to the functor of tensor product (over $\mathbb{C}[L]$) with the $\mathbb{C}[L]$-bimodule $e_U e_V \mathbb{C}[G] e_U e_V$, while the functor $\oplus_{w\in W} w^*$ is naturally isomorphic to the tensor product with the bimodule $\mathbb{C}[W \ltimes L]$. Since $G = \sqcup G_w$ we have

$$e_U e_V \mathbb{C}[G] e_U e_V = \sum_{w\in W} e_U e_V \mathbb{C}[G_w] e_U e_V.$$

Proposition 9 thus implies that the $\mathbb{C}[L]$-bimodule map

$$\mathbb{C}[W \ltimes L] = \bigoplus_{w\in W} \mathbb{C}[wL] \xrightarrow{h\mapsto e_U e_V h e_U e_V} e_U e_V \mathbb{C}[G] e_U e_V$$

is surjective. Proposition 10 implies that this map is injective, so it is an isomorphism of bimodules, and induces a natural isomorphism of functors ri $\cong \oplus w^*$. The formula for the intertwining number follows from this isomorphism and from the fact that i and r are adjoints. □

We now turn to the proof of Theorem 2. Every irreducible representation χ of the abelian group $L \cong (R^\times)^n$ has the form

$$\chi_1 \otimes \cdots \otimes \chi_n : \mathrm{diag}(r_1, ..., r_n) \mapsto \chi_1(r_1)\cdots\chi_n(r_n)$$

where each χ_i is a linear character $R^\times \to \mathbb{C}^\times$. For each such χ we let $e_\chi = |L|^{-1}\sum_{l\in L}\chi(l)^{-1} l$ be the corresponding primitive central idempotent in $\mathbb{C}[L]$.

Lemma 11. *The algebra* $\mathrm{End}_G(\mathrm{i}\chi)$ *is isomorphic to the subalgebra* $e_\chi e_U e_V \mathbb{C}[G] e_U e_V e_\chi$ *of* $\mathbb{C}[G]$.

Proof. We have

$$\mathrm{i}\chi \cong \mathbb{C}[G] e_U e_V \otimes_{\mathbb{C}[L]} \mathbb{C}[L] e_\chi \cong \mathbb{C}[G] e_U e_V e_\chi = \mathbb{C}[G] z e_U e_V e_\chi$$

where z is as in Lemma 6. Since $z e_U e_V$ and e_χ are commuting idempotents in $\mathbb{C}[G]$, their product $E = z e_U e_V e_\chi$ is an idempotent and we have $\mathrm{End}_G(\mathbb{C}[G]E) \cong (E\mathbb{C}[G]E)^{\mathrm{opp}}$ via the action of

$E\mathbb{C}[G]E$ on $\mathbb{C}[G]E$ by right multiplication. Now $E\mathbb{C}[G]E$ is a finite-dimensional complex semisimple algebra, so it is isomorphic to its opposite, and we have $E\mathbb{C}[G]E = e_\chi e_U e_V \mathbb{C}[G] e_U e_V e_\chi$. □

Lemma 12. *For the trivial representation 1_L of L we have $\mathrm{End}_G(\mathrm{i}1_L) \cong \mathbb{C}[W]$ as algebras.*

Proof. First suppose that R is a field, so that the functor i is isomorphic to the functor of Harish-Chandra induction. Then, as we noted above, $\mathrm{i}1_L$ is isomorphic to the permutation representation on $\mathbb{C}[G/LU]$, and the isomorphism $\mathrm{End}_G(\mathrm{i}1_L) \cong \mathbb{C}[W]$ is a special case of well-known results of Iwahori-Matsumoto and Tits (see [5, §68] for an exposition).

Now let R be a local ring with residue field $\Bbbk$. The quotient map $R \to \Bbbk$ induces a surjective map of algebras

$$e_{L(R)}e_{U(R)}e_{V(R)}\mathbb{C}[G(R)]e_{U(R)}e_{V(R)}e_{L(R)} \to e_{L(\Bbbk)}e_{U(\Bbbk)}e_{V(\Bbbk)}\mathbb{C}[G(\Bbbk)]e_{U(\Bbbk)}e_{V(\Bbbk)}e_{L(\Bbbk)}. \tag{13}$$

Theorem 1 implies that the domain of (13) is isomorphic as a vector space to $\mathbb{C}[W]$, while we have just seen that the range of (13) is isomorphic as an algebra to $\mathbb{C}[W]$. Since (13) is surjective, it is an algebra isomorphism. □

Remark. The isomorphism in Lemma 12 is not canonical. One can trace through the various maps appearing in the proof to construct a set of Iwahori-Hecke generators of $e_L e_U e_V \mathbb{C}[G] e_U e_V e_L$, although this will depend on the choice of an element z as in Lemma 6.

Lemma 14. *Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n$ be an irreducible representation of L, let $\tau : R^\times \to \mathbb{C}^\times$ be a character of $R^\times$, and let $\chi' = \tau\chi_1 \otimes \cdots \otimes \tau\chi_n$. Then $\mathrm{i}\chi' \cong (\tau \circ \det) \otimes \mathrm{i}\chi$.*

Proof. The algebra automorphism

$$\mathbb{C}[G] \to \mathbb{C}[G], \qquad g \mapsto \tau(\det g) g \tag{15}$$

sends $e_{\chi'}$ to e_χ, and fixes e_U and e_V. Thus (15) induces an isomorphism of $\mathbb{C}[G]$-modules

$$\mathbb{C}[G]e_U e_V \otimes_{\mathbb{C}[L]} \mathbb{C}_{\chi'} \xrightarrow{\cong} (\tau \circ \det) \otimes \mathbb{C}[G] e_U e_V \otimes_{\mathbb{C}[L]} \mathbb{C}_\chi.$$

□

Lemma 16. *If $\chi = \chi_1^n$ is a tensor-multiple of a single character of $R^\times$, then $\mathrm{End}_G(\mathrm{i}\chi) \cong \mathrm{End}_G(\mathrm{i}1_L)$ as algebras.*

Proof. Lemma 14 ensures that $\mathrm{i}\chi \cong (\chi_1 \circ \det) \otimes \mathrm{i}1_L$. □

Lemma 17. *For each $w \in W$ there is a natural isomorphism of functors $\mathrm{i} \circ w^* \cong \mathrm{i}$.*

Proof. The functor $\mathrm{i} \circ w^*$ is given by tensor product with the $\mathbb{C}[G]$-$\mathbb{C}[L]$ bimodule $\mathbb{C}[G]e_U e_V w = \mathbb{C}[G]e_{U^w}e_{V^w}$, while the functor i is given by tensor product with $\mathbb{C}[G]e_U e_V$. These two bimodules are isomorphic, by Lemma 8. □

Lemma 17 implies that in order to compute the intertwining algebra $\mathrm{End}_G(\mathrm{i}\chi)$ for an arbitrary character χ of L we may permute the factors χ_i so that χ takes the form

$$\chi = \chi_1^{n_1} \otimes \chi_2^{n_2} \otimes \cdots \otimes \chi_k^{n_k} \qquad \text{where} \qquad \chi_j \neq \chi_i \text{ unless } j = i. \tag{18}$$

(The exponents indicate tensor powers.) We then have $W_\chi \cong S_{n_1} \times \cdots \times S_{n_k}$.

In the next lemma we shall consider general linear groups of different sizes, and we shall accordingly embellish the notation with subscripts to indicate the size of the matrices involved:

so, for example, L_a denotes the diagonal subgroup in $G_a = \mathrm{GL}_a(R)$, and i_a is a functor from $\mathrm{Rep}(L_a)$ to $\mathrm{Rep}(G_a)$.

Lemma 19. *If χ is as in (18) then $\mathrm{End}_{G_n}(\mathrm{i}_n\chi) \cong \otimes_{j=1}^k \mathrm{End}_{G_{n_j}}(\mathrm{i}_{n_j}(\chi_j^{n_j}))$ as algebras.*

Proof. Let us write L' for the block-diagonal subgroup $G_{n_1} \times \cdots \times G_{n_k} \subseteq G_n$, which contains as subgroups the groups $U' = U_{n_1} \times \cdots \times U_{n_k}$ and $V' = V_{n_1} \times \cdots \times V_{n_k}$. Let U'' be the subgroup of block-upper-unipotent matrices

$$U'' = \left\{ \begin{bmatrix} 1_{n_1 \times n_1} & & * \\ & \ddots & \\ 0 & & 1_{n_k \times n_k} \end{bmatrix} \in G_n \right\},$$

and let $V'' = (U\prime\prime)^{\mathrm{t}}$ be the corresponding group of block-lower-unipotent matrices.

Let $\mathrm{i}' : \mathrm{Rep}(L') \to \mathrm{Rep}(G_n)$ be the functor of tensor product with the $\mathbb{C}[G_n]$-$\mathbb{C}[L']$ bimodule $\mathbb{C}[G_n]e_{U''}e_{V''}$. The semidirect product decompositions $U = U_n = U' \ltimes U''$ and $V = V_n = V' \ltimes V''$ give equalities $e_U = e_{U'}e_{U''}$ and $e_V = e_{V'}e_{V''}$, and hence an isomorphism of $\mathbb{C}[G_n]$-$\mathbb{C}[L_n]$ bimodules

$$\mathbb{C}[G_n]e_{U_n}e_{V_n} \cong \mathbb{C}[G_n]e_{U''}e_{V''} \otimes_{\mathbb{C}[L']} \mathbb{C}[L']e_{U'}e_{V'}.$$

It follows that

$$\mathrm{i}_n\chi \cong \mathrm{i}'\left(\otimes_{j=1}^k \mathrm{i}_{n_j}(\chi_j^{n_j})\right).$$

Since i' is a functor, we obtain from this isomorphism a map of algebras

$$\mathrm{i}' : \bigotimes_{j=1}^{k} \mathrm{End}_{G_{n_j}}(\mathrm{i}_{n_j}(\chi_n^{n_j})) \to \mathrm{End}_{G_n}(\mathrm{i}_n\chi). \tag{20}$$

Now, the $\mathbb{C}[L']$-bimodule map

$$\mathbb{C}[L'] \to \mathbb{C}[G_n]e_{U''}e_{V''}, \qquad h \mapsto he_{U''}e_{V''}$$

is injective, because the multiplication map $L' \times U'' \times V'' \to G_n$ is one-to-one. It follows from this that the identity functor on $\mathrm{Rep}(L')$ is a subfunctor of $\mathrm{Res}_{L'}^{G_n} \circ \mathrm{i}'$. Thus i' is a faithful functor, and in particular the map (20) is injective. Since the domain and the range of this map have the same dimension as complex vector spaces, by Theorem 1, we conclude that (20) is an algebra isomorphism. □

Proof of Theorem 2. Lemma 17 allows us to assume that χ has the form (18), and in this case we have algebra isomorphisms

$$\begin{aligned} \mathrm{End}_G(\mathrm{i}\chi) &\xrightarrow[\cong]{\text{Lem. 19}} \bigotimes_j \mathrm{End}_{G_{n_j}}(\mathrm{i}_{n_j}(\chi_j^{n_j})) \\ &\xrightarrow[\cong]{\text{Lem. 16}} \bigotimes_j \mathrm{End}_{G_{n_j}}(\mathrm{i}_{n_j} 1_{L_{n_j}}) \\ &\xrightarrow[\cong]{\text{Lem. 12}} \bigotimes_j \mathbb{C}\left[S_{n_j}\right] \cong \mathbb{C}[W_\chi]. \end{aligned}$$ □

Proof of Corollary 3. Choose an ordering $\{\chi_1, \dots, \chi_k\}$ of the character group $\hat{R^\times}$. Lemma 17 and the intertwining number formula in Theorem 1 imply that for each principal series representation π of $\mathrm{GL}_n(R)$ there is a unique k-tuple of non-negative integers $n_1, \dots, n_k$ having $\sum_i n_i = n$, such that π embeds in $\mathrm{i}(\chi_1^{n_1} \otimes \cdots \otimes \chi_k^{n_k})$.

Theorem 2 implies that the number of distinct irreducible subrepresentations of $\mathrm{i}(\chi_1^{n_1} \otimes \cdots \otimes \chi_k^{n_k})$ is equal to the number of distinct irreducible representations of $S_{n_1} \times \cdots \times S_{n_k}$. The latter number is equal to the number of k-tuples $(\lambda^{(1)}, \dots, \lambda^{(k)})$, where each $\lambda^{(i)}$ is a partition of n_i. Allowing the exponents n_i to vary shows that the total number of principal series representations is equal to $P_k(n)$, as claimed. □

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