Education in the North

UNIVERSITY OF ABERDEEN

ARTICLE

Curriculum for Excellence: A cross-sector investigation of Scottish teachers' mathematical beliefs of problem solving.

Paul Argyle McDonald p.mcdonald.1@research.gla.ac.uk

University of Glasgow

DOI: https://doi.org/10.26203/k4rj-mc56

To cite this article: [MCDONALD, P.A., (2016). Curriculum for Excellence: A cross-sector investigation of Scottish teachers' mathematical beliefs of problem solving. *Education in the North*, **23**(2), pp.24-52]

Curriculum for Excellence: A cross-sector investigation of Scottish teachers' mathematical beliefs of problem solving.

Paul Argyle McDonald

University of Glasgow

(Received May, 2016)

Abstract

This paper explores Scottish primary and secondary teachers' mathematical beliefs of problem solving. Curriculum for Excellence advocates the centrality of problem solving within the nucleus of learning and teaching of mathematics. This transformational change has reconceptualised problem solving as the panacea to raising attainment at all levels of school mathematics. Teachers' beliefs are seen as pivotal to the successful enactment of any curricular reform. It has been widely assumed that cross-sector mathematical perspectives are compatible with problem solving. A mixed methods explanatory sequential design study was conducted in which online questionnaires and semi-structured interviews were used to collect data from in-service teachers. More specifically, it explored beliefs regarding the nature of mathematics, the learning of mathematics and the teaching of mathematics. Factor analysis identified three belief systems reliable with a social constructivist, problem solving and transmissionist orientation. It found that teachers' deep-rooted mathematical beliefs are not aligned to one particular group of belief systems but are embedded mutually within a cluster and that inconsistencies exist between beliefs and reported practice. Statistically significant differences exist between primary and secondary sectors. Factors impacting on teachers' mathematical beliefs are identified.

Keywords: curriculum for excellence; teachers' beliefs; primary/secondary; problem solving.

Introduction

There is a general paucity of empirical research focussing specifically on Scottish teachers' mathematical beliefs. So far, there have been no attempts to examine the dissonance between in-service Scottish primary and secondary teachers' mathematical beliefs of problem solving. This paper reports on a study which attempted to fill this knowledge gap. In particular, it will evaluate, compare and consider the impact of selected variables on teachers' mathematical beliefs of problem solving.

Within Curriculum for Excellence (CfE), both primary and secondary sectors share a mutual responsibility for the learning and teaching of mathematics respectively and have to ensure continuity and progression across this continuum. Similarly, both sectors are required to develop succinct learning experiences and expected outcomes through each of the different stages, along with advancing pupils' cognitive skills. Entrenched within the fabric of this domain, problem solving is advocated as an integral component within all levels of mathematics education. More accurately, one of the dominant manifestations of CfE is the emancipation of problem solving. Prior to the implementation of CfE, the Scottish Government (2009, p.2) opined that 'Problem solving will be at the heart of all our learning and teaching'. This assurance echoed with a previous declaration by the Scottish Executive (2006, p.20):

> To emphasise that problem solving is fundamental to good learning and teaching in all aspects of mathematics and its applications, problem solving will be addressed within all lines of development rather than appearing as a separate element.

On reforming national guidelines, the architects of CfE have delivered an unambiguous message that problem solving is to be embedded within the daily machinations of mathematical instruction throughout primary and secondary education. However, neglected by all of the existing resource documentation is a theoretical framework underpinning their espoused philosophy. Inescapably absent is the conceptualisation of problem solving. Simultaneously, little is known about individual teachers' established mathematical beliefs including how such beliefs impact on professional practice and arguably, more importantly, is the status of compatible beliefs with making problem solving central to the learning and teaching of mathematics.

In this paper, problem solving and teachers' mathematical beliefs are viewed through the lens of a constructivist perspective. As a learning theory, constructivism is championed by CfE and promotes a student-centred pedagogy illustrated by children taking an active role in their learning. Other associated classroom characteristics include active engagement, problem posing and collaboration with others (Scottish Government, 2010a).

Theoretical Context

I shall describe the theoretical context within which I base my argument, with a brief review of problem solving and teachers' beliefs. Prior to this, I would like to supplement my justification for incorporating both sectors in this paper. As a secondary practitioner, I have often felt disconnected from primary colleagues, due to possessing no knowledge of their pedagogical structures, practices and values. Outwith transition arrangements, few, if any, opportunities exist to share expertise or discuss the conceptualisation of learning approaches. In a study which set out to determine Scottish teachers' understanding of CfE, Priestley and Minty (2013) note tension between viewpoints of primary and secondary teachers. It is expected that by researching educators from both sectors, valuable empirical evidence will be gathered that may serve to increase awareness of respective contributions and strengthen future partnerships. More prosaically, understanding what the nuances are will help to harmonise didactical methods of working together in order to maintain continuity between primary and secondary education.

Problem solving

It is recognised that pupils need support in improving their problem solving competencies. Shortly after the implementation of CfE, evidence provided by Education Scotland (2012, p.10) acknowledged that "it is evident from children and young people's responses, that there is a need to strengthen their capacity to solve problems". Teachers have a shared responsibility to ensure they address this pedagogical requirement. Halmos (1980, p.523) reminds us that "it is the duty of all teachers, and of course teachers of mathematics in particular, to expose their students to problems much more so than facts". Problem solving is pivotal to doing, learning and teaching mathematics (Schoenfeld, 1992) and is a central goal of CfE. Emphatically, Her Majesty's Inspectorate of Education (2010, p.8) promulgates that a "problem solving approach is at the heart of effective learning and teaching of mathematics". Furthermore, guidelines offered to practitioners (Scottish Government, 2010b, p.8) explicitly instruct teachers to "embody problem solving as an intrinsic element of mathematical approaches". Halmos (1985, p.322) warns that:

> A teacher who is not always thinking about solving problems – ones he does not know the answer to -is psychologically simply not prepared to teach problem solving to his students.

In spite of a vast wealth of international educational research and academic publications being readily available for the learning and teaching of problem solving (e.g. Polya, 1957; Schoenfeld, 1985, 1992; Silver, 1985; Stanic and Kilpatrick, 1989; Lesh & Zawojewski, 2007), there is no theoretical articulation of this influence within the mathematical framework of CfE. The Scottish Government (2016) makes no attempt to include the role of problem solving as an essential skill for employment.

Fundamentally, it is essential for teachers to recognise what constitutes a mathematical problem, in order to make a distinction from typical routine algorithmic exercises located at the end of a textbook chapter. According to Posamentier and Krulik (2008, p.1), "a problem is a situation that confronts a person, that requires resolution, and for which the path to the solution is not immediately known". Likewise, Cai and Lester (2005, p.221), note that a mathematical problem will require an "individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine". Cai and Nie (2007, p.471) argue that problem solving activities are viewed as a goal to achieve and as an instructional approach to foster critical thinking: "The purpose of teaching problem solving in the classroom is to develop students' problem solving skills, help them acquire ways of thinking, form habits of persistence, and build their confidence with dealing with unfamiliar situations".

It is evident from the literature (e.g. Schoenfeld, 1985, 1992) that a mathematical problem is not a routine computational task enveloped in a blanket of words but a cognitively demanding non-routine undertaking. Naturally, it requires higher-level thinking (i.e. hierarchical learning skills such as analysing, synthesising, evaluating, see for example Krathwohl's (2002) overview of Bloom's Revised Taxonomy) and provides a suitable intellectual challenge that will foster the conceptual development of pupils' mathematical needs, consistent with CfE. Intrinsic features of such problems automatically encourage pupil engagement and may contain various answers or allow the construction of multiple solutions.

In their seminal text, Schroeder and Lester (1989) describe a mathematical framework outlining three distinct instructional approaches to support the implementation of problem solving in any classroom:

- Teaching for problem solving
- Teaching about problem solving and
- Teaching through problem solving

(i) Teaching for problem solving

Schroeder and Lester (1989, p.32) express that "the teacher concentrates on ways in which the mathematics being taught can be applied in the solution of both routine and non-routine problems" and that "students are given many instances of the mathematical concepts and structures they are studying and many opportunities to apply that mathematics in solving problems". In other words, problem solving is undertaken after new concepts and procedures have been illuminated. For example, in calculus pupils learn the rule for differentiation and then once mastered, apply this technique to solve optimisation problems.

Although, this method is engrained as the conventional mathematical teaching approach to problem solving (Van de Walle, Karp and Bay-Williams, 2014), it requires that all learners have the necessary previous knowledge to understand the new concept being introduced. Typically, it involves the teacher presenting one tactic to perform a procedure which may disadvantage learners who possess alternative solutions. Van de Walle, Karp and Bay-Williams (2014, p.55) advise that this one dimensional approach "can communicate that there is only one way to solve the problem, a message that misrepresents the rule of mathematics and disempowers students who naturally may want to try to do it their own way".

Another concern from my own experience is that pupils may be afforded a large amount of help to solve challenging problems. Whilst this might appear to be an effective strategy, it eliminates a fertile opportunity for learners to 'struggle' (Hiebert and Grouws, 2007) and persevere which in turn leads to pupils attempting to succeed by memorising canonical algorithms. Nevertheless, it may be claimed that this method still has merit and with the colossal pressure to prepare pupils for national examinations will probably ensure that this approach will continue for a long time. It is inferred that teaching for problem solving is the preferred approach within CfE. For example, the Scottish Government (2009, p.2) state that "Mathematics is at its most powerful when the knowledge and understanding that have been developed are used to solve problems". Moreover, the Scottish Government (2011, p.4) emphasise "development of higher-order thinking skills that enable the learner to identify which particular mathematical techniques can be appropriately applied in order to progress towards a solution to a problem". One major drawback of this approach is that it fails to position problem solving at the heart of mathematics. Siemon (1986, p.35) cautions: "To spend the majority of one's time doing mathematics as it has always been done, with "problem solving" added on as an interesting appendage, actively acts against encouraging a problem-solving approach". This perspective is shared with Cai (2010) who warns that separating learning skills and concepts from problem solving does not contribute to improving pupil learning.

(ii) Teaching about problem solving

This should be embedded within the curriculum but requires significant time to demonstrate to pupils the processes and strategies involved in solving problems. For example, learners are taught heuristics such as draw a picture, make a table, organise a list, look for a pattern, write an equation, etc. As an instructional approach, this will seek to develop and encourage an awareness of mechanisms that will allow pupils to access a range of appropriate methods to attempt to solve problems, at the expense of learning mathematics (English, Lesh and Fennewald, 2008). It requires pupils to link their conceptual and procedural knowledge to a cycle of thinking and asking questions, as a way to augment their generic ability. However, there has been little empirical evidence that teaching heuristic strategies improves pupils' ability to solve general mathematical problems (Schoenfeld, 1992; Lester and Kehle, 2003). Equally, Schroeder and Lester (1989, p.34) warn that its main limitation is "instead of problem solving serving as a context in which mathematics is learned and applied, it may become just another topic, taught in isolation from the content and relationships of mathematics". Although this approach does not ensure that children are able to solve all types of mathematical problems, it provides an eclectic mix of ideas that help to bolster a teacher's knowledge trajectory.

(iii) Teaching through problem solving

Schroeder and Lester (1989, p.33) avow that with this approach, “problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems”. Similarly, Lambdin (2003, p.7) proclaims that, “A primary tenet of teaching through problem solving is that individuals confronted with honest-to-goodness problems are forced into a state of needing to connect what they know with the problem at hand”. In this way, the interplay between problem solving and concept learning is symbiotic. For example, in exploring the vertex of a quadratic function, pupils are led to discover the procedure for completing the square and how to identify the axis of symmetry.

Widespread support exists to endorse this approach as an important linkage between theoretical research and effective practice which fosters learners’ problem solving, reasoning skills and mathematical conceptual understanding. Lester and Lambdin (2004, p.196) draw a laudable parallel with constructivism when they maintain pupils “become active participants in the creation of knowledge rather than passive receivers of rules and procedures”. Schoen (2003) argues that this method is an effective process for acquiring new mathematical knowledge.

From a pedagogical prospective, teaching through problem solving requires a paradigm shift in the philosophical role of the teacher (Van de Walle, Karp and Bay-Williams, 2014). Enhanced responsibility to select appropriate quality tasks that develop mathematical knowledge blended with strategic questioning and an effective understanding of when to extend and formalise pupil thinking, will place an increase on the demand of the teacher (Van de Walle, Karp and Bay-Williams, 2014). Orchestrating classroom discourse is particularly complex and requires high cognitive levels while pupils are learning and validating mathematical concepts (Smith, Hughes and Engle, 2009; Kilic et al., 2010) In his discussion on the Japanese school approach to teaching mathematics through problem solving, Shimizu (2009, p.100) concludes that, ‘In order to be successful, teachers have to understand well the relationship between the mathematics content to be taught and students’ thinking about the problem to be posed. Anticipating students’ responses to the problem is the critical aspect of lesson planning’. Much encouragement and support is required for teachers to learn this role which cannot be easily accomplished through attendance at training courses but predominantly through interactions with colleagues and professional learning. In a study of Swedish primary teachers, van Bommel and Palmer (2015) report that a collaborative professional development initiative influenced participants’ awareness of problem solving, evidenced by the quality of produced lesson plans.

Teachers’ beliefs

Imperative to successful implementation of any education reform is the teacher. Spillane (1999, p.144) contends “teachers are the key agents when it comes to changing practice: They are the final policy brokers. Local enactment depends in great part on the capacity and will of the teachers”. Teachers’ beliefs about mathematics and mathematics pedagogy have long been detected as one of the overarching obstacles to educational reform (Pajares, 1992; Handel and Herrington, 2003). For example, Ernest (1989) upholds that adopting a problem solving approach to the teaching of mathematics depends on institutional reform but more essentially on individual teachers changing their engrained philosophy to the learning and teaching of mathematics. He opines that:

> It depends fundamentally on the teacher’s system of beliefs, and in particular on the teacher’s conception of the nature of mathematics and mental models of teaching and learning of mathematics. Teaching reforms cannot take place unless teachers’ deeply held beliefs about mathematics and its teaching and learning change (Ernest, 1989 p.249).

Likewise, Göldin, Rosken and Töner (2009, p.7) support this viewpoint by warning that “Prevailing beliefs have been perceived as impediments to problem-solving based reforms of the mathematics curriculum and of classroom teaching methods”.

In Scotland, both primary and secondary colleagues are jointly responsible for integrating problem solving into the learning and teaching of mathematics. Concurrently interwoven into the educational theoretical fabric is the challenge of facilitating learning from a constructivist prospective. The shift from employing a transmission approach where mathematics is viewed as a collection of facts, procedures and examples presents multiple pedagogical dilemmas for teachers due to the transformational change encouraged by CfE. Due to its complexity, the implementation of a problem solving approach demands extensive preparation and may be, for some, probably an unachievable outcome since beliefs tend to be highly resistant to change. Priestley (2005, p. 36) advises that “Teachers must clearly understand reform and have the pre-requisite skills to put it in place, if they are to enact it successfully”. Although, paradoxically, ensuring that policies are coherent and grounded in research does not guarantee their ready adoption in practice (Hayward, Priestley and Young, 2004). However, such explanations tend to overlook the fact that teachers’ beliefs are thought of as powerful indicators of human behaviour (Thompson, 1992). In their cross-cultural study of primary teachers’ beliefs, Perry et al. (2002) concluded that beliefs make a difference to what transpires in practice and is an instrumental factor in shaping pupils learning of mathematics.

Interestingly, Donaldson (2011, p.70) in his vigorous review of Scottish teacher education maintains: “If we are to achieve the aspiration of teachers being leaders of educational improvement, they need to develop expertise in using research, inquiry and reflection as part of their daily skill set". Consequently, the latest restructuring of national standards by the General Teaching Council for Scotland (2012, p.8) prescribe that practitioners are expected to develop and apply their knowledge, skills and expertise through enquiry and sustained professional learning to 'critically engage with a range of educational literature, research and policy to make meaningful links to inform and change practice'. In spite of this ambitious doctrine, it is questionable if a suitable framework exists to allow teachers to operationalise successfully research literature to help execute this didactical requirement, forcing them to rely on their own, unexplored and possibly limited past experiences (Ellis, 2010).

Teachers' beliefs have long been acknowledged as vital to the reform of mathematics education (Cooney and Shealy, 1997). Subsequently, in an autonomous environment with no prescriptive curriculum, the enactment of problem solving is predominantly influenced and shaped by theoretical conceptualisations of practitioners' mathematical beliefs. Logically, awareness of such perspectives can only help to enhance our knowledge of teachers and the terrain where they exercise professional duties. Aguirre and Speer (1999, p.327) assert that "being able to identify and describe the mechanism underlying the influence of beliefs on instructional interactions would deepen and enrich our understanding of the teaching process".

Two research questions are addressed within this paper:

1. Are there any differences between the mathematical beliefs of Scottish primary and secondary mathematics teachers?
2. What factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers?

Methodology

A mixed methods explanatory sequential design study (Creswell and Plano Clark, 2011) was utilised to test the hypothesis that Scottish primary and secondary mathematics teachers hold similar mathematical beliefs and evaluate the impact of various associated factors. In the first phase, quantitative data were collected and analysed. The quantitative results influenced the design of the second phase, in which qualitative data was collected and analysed to help explain distinctive quantitative results derived from the first phase.

Participants

In the first phase, I draw upon quantitative data gathered from 478 teachers (229 primary and 249 secondary mathematics teachers respectively) employed within 21 local education authorities in Scotland. The generation of data followed the successful implementation of an earlier pilot study conducted with a sample of teachers from both sectors (n = 31) excluded from the main study. Demographic information from the main study included characteristics such as gender, sector, age, experience, employment type and highest level of qualification in the field of education. The group consisted of 148 males and 330 females. The estimated overall mean age was 42.8 years and estimated overall mean length of teaching experience calculated to be 17.2 years. Almost one third of the participants from each sector indicated are holding a promoted post. A graph showing the distribution of grades per sector is displayed in Figure 1.

Figure 1 DISTRIBUTION OF PARTICIPANTS GRADE BY SECTOR

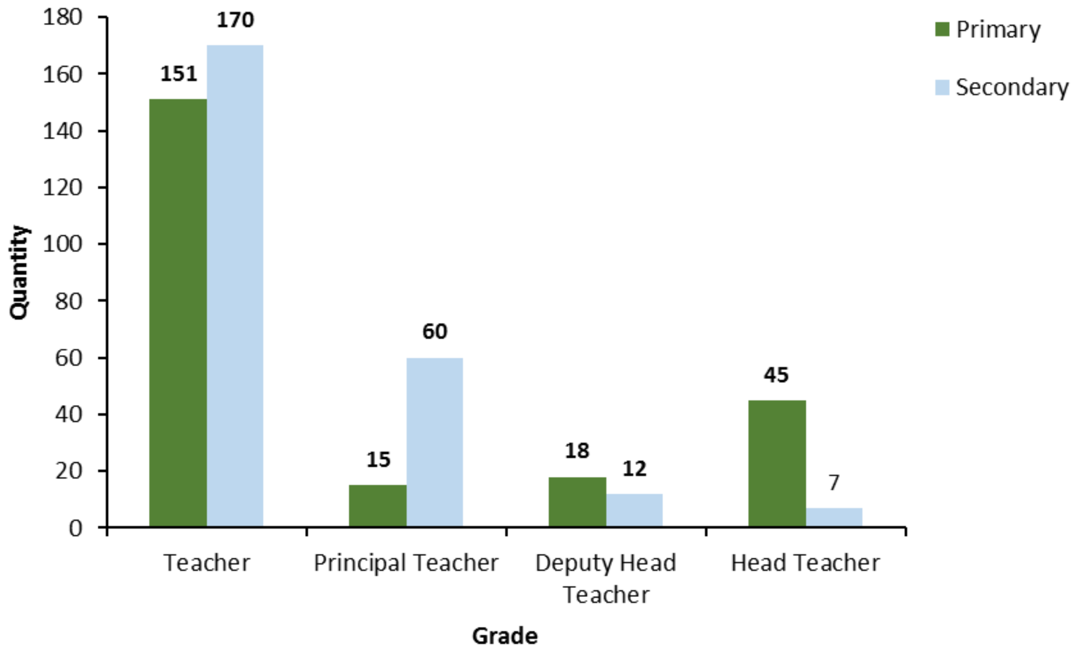
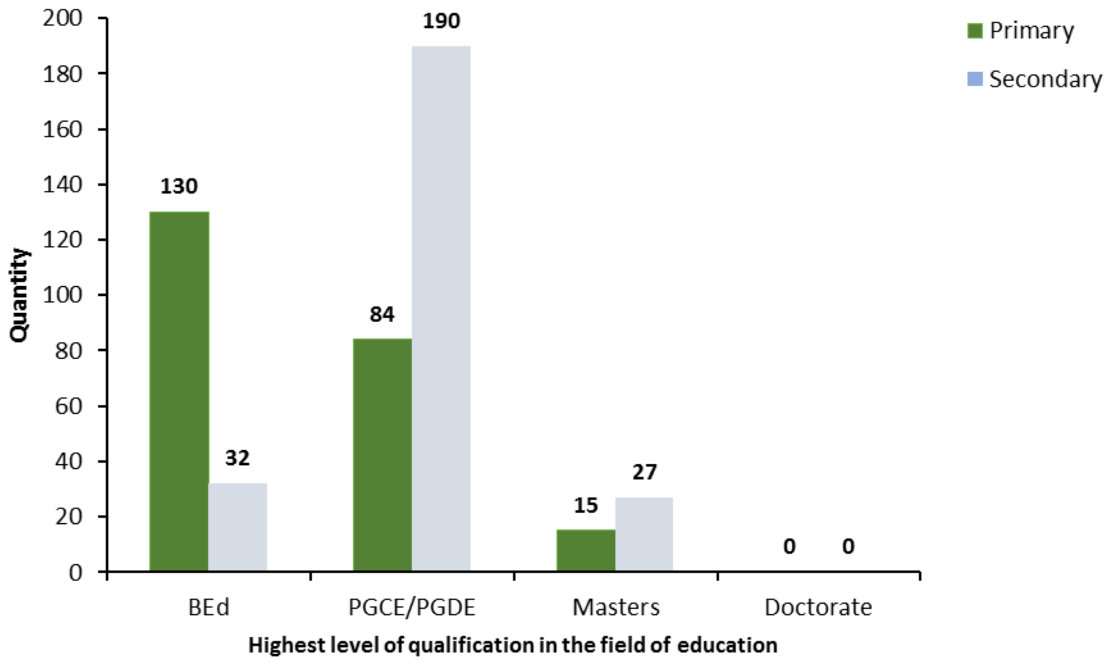


Figure 2 DISTRIBUTION OF PARTICIPANTS HIGHEST LEVEL OF QUALIFICATION IN THE FIELD OF EDUCATION BY SECTOR



Whilst this study did not feature any individuals holding a doctorate, it contains data from 42 participants possessing a Master's qualification in the field of education (exact details known only in two cases due to interview). A graph showing the distribution of highest level of qualification in the field of education per sector is displayed in Figure 2.

Instruments

Two instruments were employed in this study. In the first phase, an online questionnaire consisting of 39 belief statements (22 positive and 17 negative items) was used as the quantitative component. The design of the questionnaire incorporated numerous elements from previous field studies including references to the literature (e.g. Ernest, 1989; Perry, Howard and Tracey, 1999; Anderson, White and Sullivan, 2005; Barkatsas and Malone, 2005; Lester, 2013). A five-point Likert scale ranging from 1 (Strongly Agree), 2 (Agree), 3 (Undecided), 4 (Disagree) to 5 (Strongly Disagree) was selected as response options. The belief statements were compartmentalized using five distinct classifications as follows:

- Factor 1 A social constructivist orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics (7 positive items).

- Factor 2 A dynamic problem solving orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics (10 positive items).

- Factor 3 A static transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics (9 negative items).

- Factor 4 A mechanistic transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics (8 negative items).

- Factor 5 A collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics (5 positive items).

In the second phase, semi-structured interviews were exercised as the qualitative mechanism. Each interview was audio-recorded at each of the participants' place of work and lasted on average forty minutes in duration. The format comprised of several identical questions followed by miscellaneous questions based on individual responses to the survey. All of the interviews were professionally transcribed 'intelligent verbatim' by an independent company.

Sampling

It is normal for mixed methods research to employ more than one type of probability sampling strategy (Cohen, Manion and Morrison, 2011). Teddie and Yu (2007, p.80) provide an illustration of this when "schools (the clusters) are randomly selected and then teachers (the units of interest) in those schools are randomly sampled". However, in the case of the online questionnaire, the sample was obtained without the employment of a recognised technique due to the idiosyncratic nature of educational research in Scotland. Once formal approval is granted to approach a school establishment, the individual Head Teacher assumes the role of gatekeeper and determines the destiny of each research request. Access to the whole population is not permitted by any local education authority (LEA). With this in mind, I contacted all 32 local education authorities. The outcome of which is as follows:

- Pilot study access granted by LEA (1)
- Access granted by LEA and number of participants obtained over 25 (7)
- Access granted by LEA and number of participants obtained between 10 and 25 (12)

- Access granted by LEA and number of participants obtained 0 (2)
- Restricted access granted to a single school establishment selected by LEA (1)
- Restricted access granted to a single participant selected by LEA (1)
- Access refused by LEA (3)
- No engagement by LEA (4)

The questionnaire harvested 63 volunteers which allowed purposeful sampling to select a sample of 11 interview participants. Pseudonyms were used throughout to ensure anonymity.

Analysis of the questionnaires

After cleaning procedures were applied to ensure accuracy (Pallant, 2013), the online data was directly imported into SPSS version 22 to allow statistical analysis to be performed. To reduce response bias, all negatively worded items were reversed to allow computation of a total mathematics belief score. This produced a theoretical range from 39 (most favourable) to 195 (least favourable). In order to check for scale reliability of the instrument, Cronbach's alpha coefficient was calculated as 0.884, indicating high levels of internal consistency (Pallant, 2013). The suitability of the data for factor analysis was assessed and deemed to be acceptable. For example, the Kaiser-Meyer-Olkin measure of sampling statistic was 0.903 and Bartlett's Test of the Sphericity reached statistical significance [χ^2 (741) = 6057.958, $p <$ .001] supporting the factorability of the correlation matrix. Further examination revealed the presence of three distinct mathematical belief systems as follows:

- A social constructivist, dynamic problem solving and collaborative orientation.
- A social constructivist and dynamic problem solving orientation.
- A static transmission and mechanistic transmission orientation.

Descriptive statistical analysis was executed on the data using the summation of each score and categorised according to sector. The mean, standard deviation, median and interquartile range were then calculated for each total mathematics belief score. Parametric testing was performed where the data was investigated using an independent samples t-test and Analysis of variance (ANOVA).

Analysis of the Interviews

Thematic analysis has been described as a qualitative method for "identifying, analysing, and reporting patterns (themes) within data. It minimally organises and describes your data set in rich detail. However, it also often goes further than this, and interprets various aspects of the research topic" (Braun and Clark, 2006, p.79). Consequently, I used thematic analysis to search for themes through careful reading and re-reading of the data. During this overall procedure, the data was interrogated for similarities and inconsistencies (Cohen, Manion and Morrison, 2011).

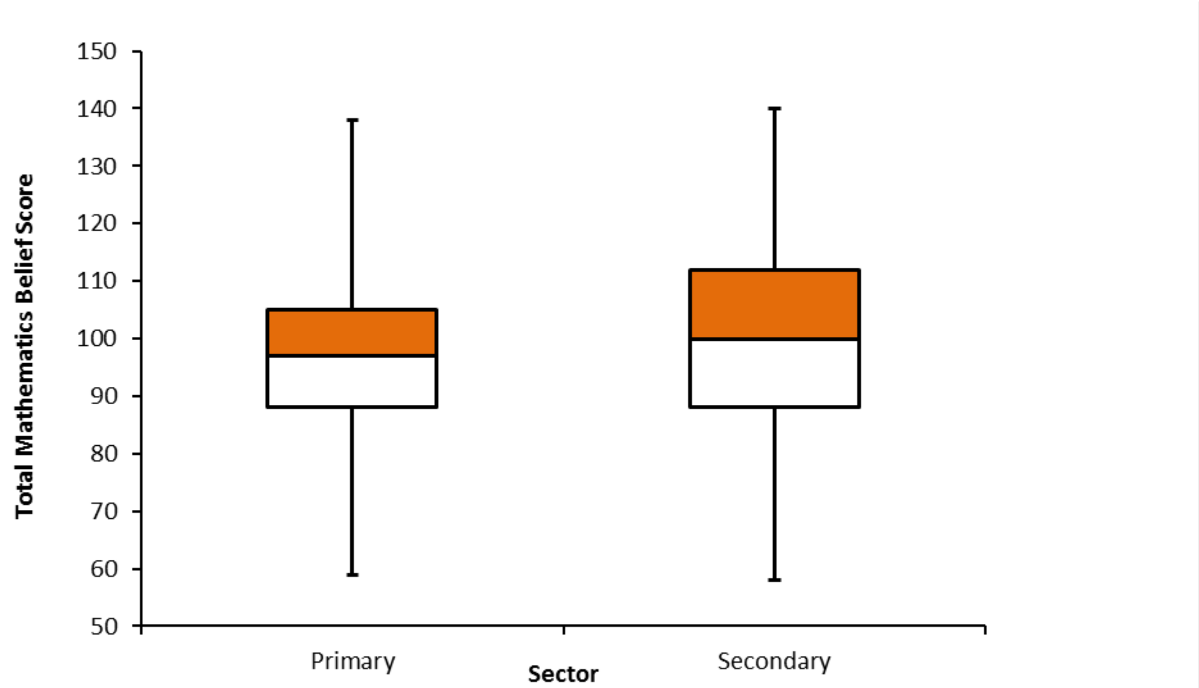
Findings

The findings are organised in terms of the research questions and priority (Creswell and Clark, 2011) has been given to the qualitative data from the semi-structured interviews.

1. **Are there any differences between the mathematical beliefs of Scottish primary and secondary mathematics teachers?**

An independent samples t-test was conducted to compare the total mathematical beliefs scores for the sectors. There was a statistically significant difference in scores for primary teachers (M = 97.71, SD = 14.60) and secondary mathematics teachers (M = 100.63, SD = 16.92); t (474.098) = -2.13, p = .043, two-tailed). The magnitude of the differences in the means (mean difference = 2.92, 95% *CI:* -5.76 to -.09) was small (eta squared = .01). Therefore, I rejected the null hypothesis that there is no difference in total mathematical beliefs scores between primary and secondary mathematics teachers. A boxplot of each sector is shown in Figure 3, from where it can be observed that the distribution of data is less dispersed for primary teachers.

Figure 3 BOXPLOTS OF TEACHERS' TOTAL MATHEMATICAL BELIEFS SCORES



Unexpectedly, it was found that 32 of the 39 belief statements produced statistically significant results. Due to space restrictions, I will describe the responses to two specimen items as follows:

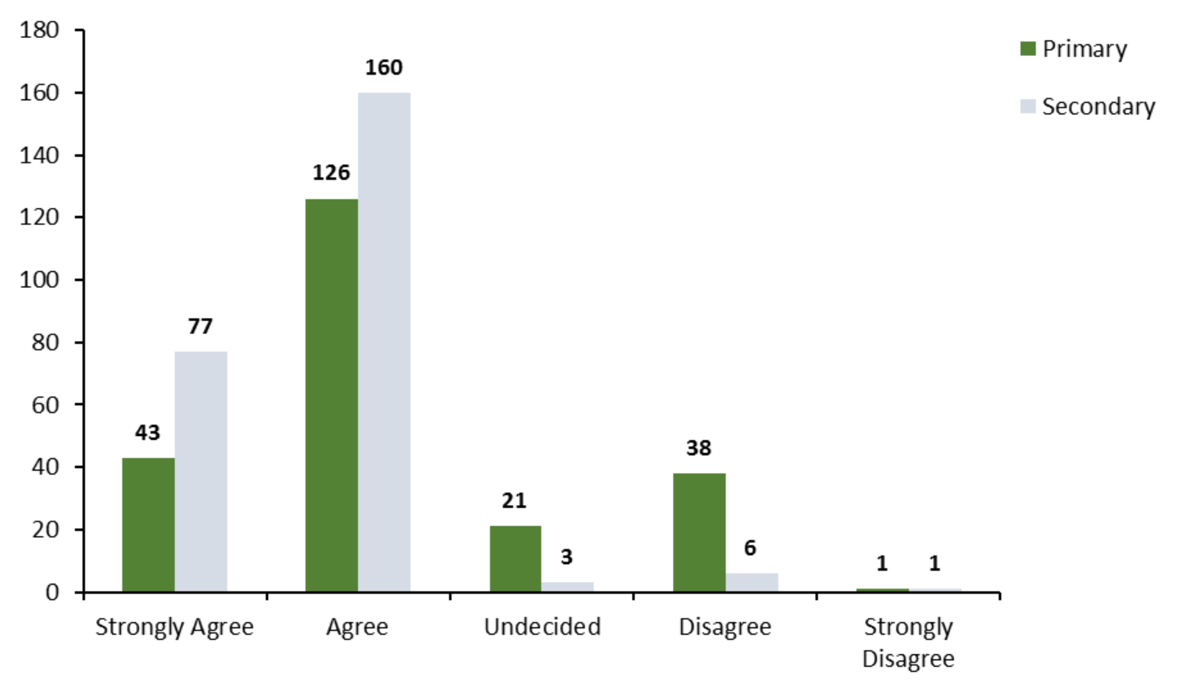
Teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves (positive item)

This dynamic problem solving belief statement is based on the theoretical work of Lester (2013) who maintains that teachers themselves need to be expert problem solvers, but have proficiency in teaching pupils how to solve mathematical problems. The vast majority (95%) of secondary mathematics teachers strongly agreed or agreed with this item opposed to less than three quarters (74%) of primary teachers (Figure 4). With more than a quarter (26%) of primary teachers undecided or disagreeing with this statement, it is suggested that this may be a topic for closer investigation. Some practitioners lack enthusiasm and confidence in teaching problem solving. This theme of mathematical self-efficacy emerged during an interview with Isabella, a 22 year old primary probationer, who said:

> I didn't do well in maths at school because I always struggled to understand what was being asked. I failed Higher [Mathematics] because it contained lots of wordy questions... I think to be honest this is why I don't know how to teach problem solving... Sorry to say this but at the end of the day, English [Higher] is much more important as you cannot get into teacher training in Scotland without it [laughter].

Thompson (1985, p.288) reminds us about Jeanne [one of her teacher participants] skipping some pages in a textbook containing story problems involving rates and proportions: "She then indicated that the reason for her skipping the pages involving problems was that the students did not enjoy working them and that problems caused them to experience a great deal of frustration with mathematics". This experience fuelled her argument for teachers 'to experience mathematical problem solving from the perspective of the problem solver before they can adequately deal with its teaching' (ibid p.292).

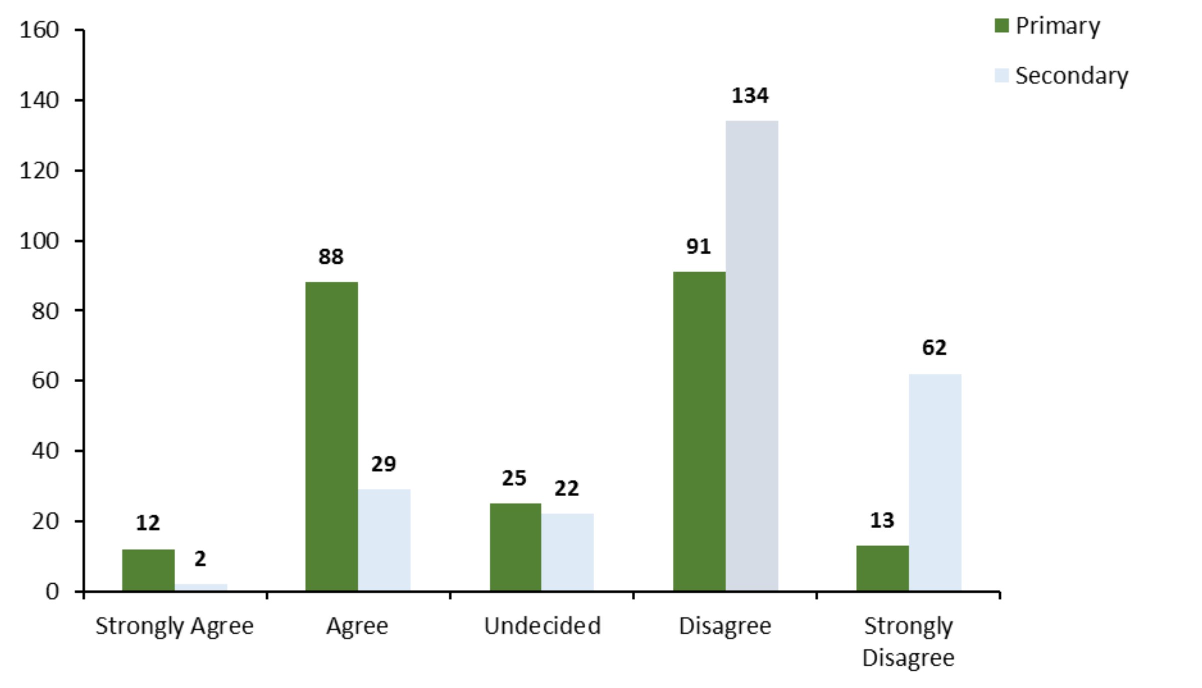
Figure 4 TEACHERS SHOULD BE EXPERIENCED PROBLEM SOLVERS AND SHOULD HAVE A FIRM GRASP OF WHAT SUCCESSFUL PROBLEM SOLVING INVOLVES



Mathematical problems can only have one final correct answer (negative item)

This static transmission belief statement is a common mathematical misconception among pupils and has been previously highlighted by several researchers including Schoenfeld (1992) who recorded it in his appraisal of the literature. In this study, almost fourth fifths (79%) of secondary mathematics teachers strongly disagreed or disagreed with this item compared to less than half (45%) of primary teachers (Figure 5). During an interview with Grace, a primary teacher with over 20 years of experience, justified her decision to strongly agree with this item by relating her belief about the nature of mathematics as “an exact subject than can only have a right answer and lots of wrong answers”. In contrast, Alasdair, a mature secondary mathematics teacher, strongly disagreed because as a pupil at school, he had encountered problems containing various answers and stated that “it’s a matter of training the mind to be open to more than one response”.

Figure 5 MATHEMATICS PROBLEMS CAN ONLY HAVE ONE FINAL CORRECT ANSWER



Whilst it may be reasoned that secondary mathematics teachers are more naturally placed to refute this epistemological view, the sector divide is considerable given the critical nature of this belief statement within the operationalisation of problem solving. Another obvious concern is that over one fifth (21%) of mathematics teachers were undecided or agreed with this line of thinking. Based on this apparent lack of pedagogical content knowledge, I wonder how such individuals instil the applications of quadratics, trigonometric functions, the binomial theorem or optimisation problems in calculus.

Finally, an independent samples t-test was conducted for each of the five belief categories. There was a significant difference between the groups for two results (Table 1).

Table 1 RESULTS OF INDEPENDENT SAMPLES T-TESTS BY BELIEF FACTORS

Scale	Primary		Secondary		Statistic			
	M	SD	M	SD	df	t	p	Cd
Belief Factor 1	16.08	2.91	16.82	3.63	467.589	-2.484	**.013**	-.230
Belief Factor 5	10.44	2.35	12.61	3.24	451.793	-8.432	**< .001**	-.793

2. What factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers?

Primary

A one-way between-groups ANOVA was conducted to explore age, experience, grade and highest qualification in the field of education. No significant differences were found between age and experience.

Participants were divided into four groups according to their grade (Group 1: Teacher; Group 2: Principal Teacher; Group 3: Deputy Head Teacher: Group 4: Head Teacher). There was a statistically significant difference at the $p < .05$ level for total mathematics beliefs scores for the four groups, [F (3, 225) = 3.401, p = .019]. The effect size, calculated using eta squared, was .045. Post-hoc comparisons using the Tukey HSD test indicated that the mean score for Group 1 (M = 99.75, SD = 15.80) was significantly different from Group 3 (M = 90.44, SD = 13.96). Group 2 (M = 95.00, SD = 8.58) and Group 4 (M = 94.67, SD = 10.37) did not differ significantly from either Group 1 or 3.

Participants were divided into three groups according to their highest qualification in the field of education (Group 1: BEd; Group 2: PGCE/PGDE; Group 3: Master's). There was a statistically significant difference at the $p < .05$ level in total mathematics beliefs scores for the three groups, [F (2, 226) = 4.862, p = .009]. The effect size, calculated using eta squared, was .043. Post-hoc comparisons using the Tukey HSD test indicated that the mean score for Group 1 (M = 99.47, SD = 14.81) was significantly different from Group 3 (M = 87.60, SD = 15.50). Group 2 (M = 96.80, SD = 13.42) did not differ significantly from either Group 1 or 3.

In order to assess gender, a two-way between-groups ANOVA was conducted to explore the impact of age, experience, grade and highest qualification in the field of education. No significant differences for gender were found.

The effect of grade and highest qualification in the field of education was aptly illustrated in the narrative of Fraser, a science graduate and vastly experienced primary Deputy Head Teacher (he also holds the Scottish Qualification for Headship). His responses to the questionnaire were unambiguously consistent with a dynamic problem solving view of mathematics. For example, he strongly agreed or agreed with the following belief statements:

- The priority in teaching mathematics is to ensure students develop confidence in problem posing and problem solving.

- Teaching mathematics through problem solving is the best method to help students learn.
- Mathematics is a continually expanding field of human creation and invention.

Consistent with these views, Fraser also strongly disagreed or disagreed with the following belief statements:

- Mathematics is an accumulation of facts, rules and skills.
- You explain in detail what the students have to do to solve problems.
- Mathematics is a collection of procedures and rules that specify how to solve problems.

Fraser noted his enthusiasm for a problem solving philosophy of mathematics was fuelled during time spent as a former development officer for science. He dismissed the notion that the terrain of mathematics is all about numbers, but that it can be appreciated by considering a myriad of scientific relationships. In particular, Fraser expressed a desire to understand why a given formula is true, in order that the formula can be internalised without memorisation. He declared:

> Although I learned maths at school the traditional way by following procedures, I could still pass exams but never really understood what I was doing... When I had to plan science lessons, I realised that a problem solving approach could be applied to other areas... To me, understanding is the key opposed to reciting facts or formulas... Maths is an incredibly useful tool in the real world, it's at the core of the curriculum and without it, we can't solve problems. The fact that it can be adapted to so many different applications, not just science, means that it's flexible and essential for everyone to learn.

Fraser underlines a false dichotomy confronted by many practitioners; the execution of basic skills versus the development of conceptual understanding. In truth, skills and understanding are completely intertwined (Wu, 1999).

Secondary

A one-way between-groups ANOVA was conducted to explore age, experience, grade and highest qualification in the field of education. No significant differences were found between age, experience and grade.

Participants were divided into three groups according to their highest qualification in the field of education (Group 1: BEd; Group 2: PGCE/PGDE; Group 3: Master's). There was a statistically significant difference at the $p < .05$ level in total mathematics beliefs scores for the three groups, [F (2, 246) = 8.949, $p < .001$]. The effect size, calculated using eta squared, was .073. Post-hoc comparisons using the Tukey HSD test indicated that the mean score for Group 1 (M = 102.72, SD = 16.38) and Group 2 (M = 102.07, SD = 16.70) was significantly different from Group 3 (M = 88.04, SD = 14.17).

In order to assess gender, a two-way between-groups ANOVA was conducted to explore the impact of age, experience, grade and highest qualification in the field of education. No significant differences for gender were found.

Discussion

The features of this paper are contained within a larger research project conducted as part of my doctoral studies within the School of Education, University of Glasgow.

The first research question sought to determine the differences between the mathematical beliefs of primary and secondary mathematics teachers. The results of this study indicate that statistically significant differences exist between the sectors. Nevertheless, the interpretation of these variations has to be grounded in a meaningful context. When comparing a core curriculum subject delivered by non-mathematical specialists and mathematics teachers, it may be natural to assume that diverse mathematical belief systems feature intrinsically; however, theoretically this is rejected by the linear structure of CfE which advocates that both sectors are expected to operationalise mathematical problem solving throughout their practice.

Perhaps the most compelling finding was that primary teachers hold stronger (i.e. statistically significant) mathematical beliefs regarding a social constructivist and collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics. A possible explanation for this might be that this result is skewed by the superior number of primary Head Teachers. Another possible reason may be that secondary mathematics teachers are resilient to change. For example, Thompson (1989, p.234) on the theme of problem solving notes:

> While secondary teachers tend to be stronger than elementary [primary] teachers in their knowledge of the subject matter, I have found secondary teachers generally more resistant to introducing changes into their teaching. Elementary teachers, for the most part, tend to act more enthusiastically to new techniques, but their generally weaker mathematics background, and feelings of inadequacy to handle mathematical problem solving, become serious obstacles.

A common view amongst the primary participants was that mathematics teaching is fixated on nurturing ‘competence’ and ‘confidence’. This was manifested through their emphasis on the development of basic social arithmetical fluency as a key practical life skill in order to prepare learners for a world beyond school, and a truncated provision for the employment of non-routine mathematical problems. Dominated throughout both sectors is an attainment perspective driven by external forces, that all children must be seen as progressing irrespective of ability. This was typically accentuated during an interview with Cormac, an inexperienced secondary mathematics teacher who expressed:

> I feel under intense pressure to teach not my subject [mathematics] but the exam techniques that the SQA are looking for... In fact, everything is geared towards the final exam, so much so that my pupils dinnae really care about learning content as long as I show them how to pass the exam.

What appears to be deep-rooted is a culture where the conceptualisation of mathematical problem solving is not fully understood and which, as an essential pedagogical requirement, does not resonate with both sectors. Moreover, teachers’ mathematical beliefs do not perpetuate concurrence with regular engagement of investigations aimed at enriching critical thinking, possibly due to other curriculum priorities. One possible factor presented to explain this result is mathematical self-efficacy, which may help to clarify why some primary teachers do not possess a working knowledge of problem solving strategies such as working backwards, finding a pattern, logical reasoning, etc. Reflecting on the issue of ‘academic entry standard’ raised by Isabella, recent empirical evidence suggests that students in primary initial teacher education with Higher Mathematics do not perform significantly better than students without Higher Mathematics (McKechan and Day, 2015).

This study found that among secondary mathematics teachers, there is a dominant viewpoint that as a curricula priority, all pupils should possess strong numeracy skills in order to survive in the real world. Misconceptions exist with the conceptualisation of mathematical problem solving, which has triggered a lack of engagement for teaching mathematics through problem solving. Whilst it is acknowledged by many that this method can release the power of mathematics, such an approach appears to be rarely demonstrated in practice.

Concurrently, in terms of a salient learning focus, little support exists for empowering learners to develop critical thinking skills at the expense of passing national examinations (in my view it is important to deliberate why some teachers deem these themes to be mutually exclusive). A recurrent theme in the interviews was a sense among participants that there is insufficient time to engage with problem solving due to excessive workload demands. It can thus be suggested that the vast majority of primary and secondary mathematics teachers are manipulated by a national assessment instrument which shapes their professional practice. It is of considerable interest that problem solving is not amalgamated within the current assessment model for mathematics in Scotland. In his review of Australian education, Clarke (1987, p.9) noted that “schools continue to succeed in the teaching of routine computation and to fail in the teaching of such skills as problem solving... the maintenance of current assessment procedures serves only to maintain the illusion that significant learning is taking place.”

In general, the perpetual resonance between both sectors seems to posit that many of the influences in the formulation of mathematical beliefs that underpin their practice are implicit rather than explicit. On the basis of the findings presented here, both primary and secondary mathematics teachers’ beliefs about learning and teaching of mathematics are influenced by their beliefs about the nature of mathematics and prior mathematical learning developed from experiences as a pupil. In this respect, this finding is consistent with previous research (Hudson, Henderson and Hudson, 2015).

The second research question was aimed at identifying factors impacting on teachers’ mathematical beliefs. The results of this study show that gender, age and experience have no effect on both sectors.

Regarding primary, it is suggested that Deputy Head Teachers hold significantly stronger mathematical beliefs compared with classroom teachers. What is surprising is that Head Teachers’ mathematical beliefs did not differ significantly from any other position. The most compelling finding involved the highest level of qualification in the field of education, in which participants possessing a Master’s hold significantly stronger mathematical beliefs than participants with a BEd. What is curious about this result is that there was no difference between participants with a PGCE/PGDE and a Master’s.

Regarding secondary, it was unanticipated that grade had no impact. This is a troubling finding given that senior management are expected to drive reform agendas and uphold professional standards. It is difficult to explain this result, but it might be related to the view that teachers’ mathematical beliefs are shaped profoundly by early life experiences (e.g. their own schooling) that have continued to influence classroom practice and, as previously intimated, do not vary over time. Pajares (1992, p.324-325) notes: “Beliefs are formed early” and “the earlier a belief is incorporated into the belief structure, the more difficult it is to alter”. The results indicate that participants possessing a Master’s hold significantly stronger mathematical beliefs than participants with a BEd or PGCE/PGDE. Hence, it could conceivably be hypothesised that the ‘professional gap’ between secondary grades is not distinguishable by theoretical pedagogical knowledge.

Collectively, the results from both sectors tentatively support the argument that a Master’s qualification in the field of education is positively linked to more robust mathematical beliefs, although, there is already a wealth of anecdotal evidence that undertaking postgraduate research and Master’s study modifies how teachers view aspects of learning and teaching. Nevertheless, this finding is consistent with previous research. For example, in a study of South Korean primary teachers, Kim, Sihn and Mitchell (2014) discovered that practitioners holding a Master’s in mathematics education had significantly stronger mathematical self-efficacy beliefs than colleagues with a Bachelor degree.

Conclusions

This study set out to investigate Scottish teachers’ mathematical beliefs of problem solving. In general, both sectors exhibited a range of positive and negative mathematical beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics. In particular, the associated philosophy of mathematical beliefs regarding the nature of mathematics aligns with Ernest (1989) whose conceptualised views were designated as instrumentalist, Platonist and problem solving.

The findings of this research suggest that limited support for teaching mathematics through problem solving exists and a widespread belief that problems can be solved if you know the right steps to follow. Teachers from both sectors appear to conceive problem solving as an irregular follow on step after learners have acquired the mastery of basic numerical and computational skills coupled with procedural understanding. Modest encouragement is extant for the promotion of multiple solutions. Overall, it was found that, when judged against the mathematical philosophy as championed by CfE, primary teachers hold significantly stronger positive mathematical beliefs than secondary mathematics teachers.

Although the current study is based on a moderate sample of participants, the findings suggest that the conceptualisation and operationalisation of mathematical problem solving is restricted in practice. A number of variables impinge on the process including a lack of a shared understanding of what constitutes a mathematical problem. Inconsistencies between teachers' espoused beliefs and reported practices appear to be mainly attributed to an over dominant national assessment orthodoxy, which fosters the attainment of targets opposed to the attainment of knowledge. Given the international dimension and theoretical significance of problem solving, it is a major concern that Scottish teachers' (secondary mathematics practitioners in particular) entrenched beliefs may impede the natural mathematical development and creativity of young people. Neglecting the influence of teachers' beliefs will only serve to diminish the implementation of curricula change and ensure that modifications to teaching practices are cosmetic (Handel and Harrington, 2003). According to Raymond (1997, p.574), "early and continued reflection about mathematics beliefs and practices, beginning in teacher preparation, may be the key to improving the quality of mathematics instruction and minimizing inconsistencies between beliefs and practices."

Finally, from a personal perspective, this research experience has been enriched by the decision to engage in a cross-sector study. It furnished me with increased professional knowledge, coupled with a wider appreciation of the diversity that exists across classrooms and schools. If we are to ameliorate the quality of mathematics education in Scotland, we must first expand our fragmented knowledge of teachers' mathematical beliefs within both sectors. Future studies should attempt to identify additional evidence that Curriculum for Excellence requires further conceptual clarity and detailed guidance on the effective mobilisation of problem solving.

References

AGUIRRE, J. and SPEER, N.M. (1999). Examining the relationship between beliefs and goals in practice. *The Journal of Mathematical Behavior,* **18**(3), pp.327-356.

ANDERSON, J., WHITE, P. and SULLIVAN, P. (2005). Using a schematic model to represent influences on and relationships between teachers' problem-solving beliefs and practices. *Mathematical Education Research Journal*, **17**(2), pp.9-38.

BARKATSAS, A. and MALONE, J. (2005). A typology of mathematics teachers' beliefs about teaching and learning mathematics and instructional practices. *Mathematics Education Research Journal*, **17**(2), pp.69-90.

BRAUN, V. and CLARKE, V. (2006). Using thematic analysis in psychology*. Qualitative Research in Psychology*, **3**(2), pp.77-101.

CAI, J. (2010). Evaluation of mathematics education programs. In P. Peterson, E. Baker and B. McGraw (Eds.). *International Encyclopaedia of Education.* Oxford: Elsevier, 3, pp.653-659.

CAI, J. and LESTER, F.K. (2005). Solution representations and pedagogical representations in Chinese and U.S. classrooms. *Journal of Mathematical Behavior*, **24**, pp.221–237.

CAI, J. and NIE, J. (2007). Problem solving in Chinese mathematics education: research and practice. *ZDM Mathematics Education*, **39,** pp.459-473.

CLARKE, D.J. (1987). A rational for assessment alternatives in mathematics. *The Australian Mathematics Teacher*, **46**(3), pp.8-10.

COHEN, L., MANION, L. and MORRISON, K. (2011). *Research methods in education*, 7th edn, Oxon: Routledge.

COONEY, T.J. and SHEALY, B.E. (1997). On understanding the structure of teachers' beliefs and their relationship to change. In E. Fennema and B. Nelson (Eds.) *Mathematics teachers in transmission,* Mahwah: Lawrence Erlbaum, pp.87-109.

CRESWELL, J. W. and PLANO CLARK, V.L. (2011). *Designing and conducting mixed methods research*, 2nd edn, Thousand Oaks, SAGE.

DONALDSON, G. (2011). *Teaching Scotland's future. Report of a review of teacher education in Scotland.* Edinburgh. The Scottish Government.

EDUCATION SCOTLAND (2012). *Scottish survey of literacy and numeracy (SSLN) 2011.* Professional Learning Resource. Livingston.

ELLIS, S. (2010). The Scottish context for the curriculum. In J.Arthur and T.Cremin (Eds). *Learning to teach in the Primary school,* 2nd edn, London, Routledge, pp.221-223.

ENGLISH, L.D., LESH, R. and FENNEWALD, T. (2008). *Future directions and perspectives for problem solving and curriculum development.* Paper presented at ICME 11, Monterey, Mexico. Retrieved on 05 March 2016 from http://tsg.icme11.org/document/get/458

ERNEST, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.) *Mathematics teaching: The state of the Art*. London: Falmer Press, pp.249-253.

GENERAL TEACHING COUNCIL FOR SCOTLAND (2012). *The standard for career-long professional learning: supporting the development of teacher professional learning*. Retrieved on 28 February 2016 from http://www.gtcs.org.uk/web/FILES/the-standards/standard-for-career-long-professional-learning-1212.pdf

GÖLDIN, G., ROSKEN, B. and TÖRNER, G. (2009). Beliefs - No longer a hidden variable in mathematical teaching and learning processes. In J. Maaß and W. Schlöglmann (Eds.), *Beliefs and attitudes in Mathematics Education*, Sense Publishers, pp.1-18.

HALMOS, P. (1980). The heart of mathematics. *The American Mathematical Monthly*, **87**(7), pp.519-24.

HALMOS, P. (1985). *I want to be a mathematician. An automathography*. New York, Springer-Verlag.

HANDEL, B. and HERRINGTON, A. (2003). Mathematics teacher’s beliefs and curriculum reform. *Mathematics Education Research Journal*, **15**(1), pp.59-69.

HAYWARD, L., PRIESTLEY, M. & YOUNG, M. (2004). Ruffling the calm of the ocean floor: merging practice, policy and researching assessment in Scotland. *Oxford Review of Education*, **30**(3), pp.397-415.

HER MAJESTY’S INSPECTORATE OF EDUCATION (2010). Learning together: Mathematics. Livingston: HMIE.

HIEBERT, J. and GROUWS, D.A. (2007). The effects of classroom mathematics teaching on students’ learning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning.* Charlotte, NC: Information Age, pp.371–404.

HUDSON, B., HENDERSON, S. and HUDSON, A. (2015). Developing mathematical thinking in the primary classroom: liberating students and teachers as learners of mathematics. *Journal of Curriculum Studies*, **47**(3), pp.374-398.

KILIC, H., CROSS, D.I., ERSOZ, F.A., MEWBORN, D.S., SWANAGON, D. and KIM, J. (2010). Techniques for small group discourse. *Teaching children mathematics*, **16**(6), pp.350-357.

KIM, R., SIHN, H.G. and MITCHELL, R. (2014). South Korean elementary teachers’ mathematics teaching efficacy beliefs. *Mathematics Education Trends and Research*, 1-17.

KRATHWOHL, D.R. (2002). A Revision of Bloom's Taxonomy: An Overview. Theory Into Practice, 41(4), 212-218, DOI: 10.1207/s15430421tip4104_2

LAMBDIN, D.V. (2003). Benefits of teaching through problem solving. In F.K. Lester and R.I. Charles (Eds). *Teaching mathematics through problem solving: Prekindergarten – grade 6*, Reston, VA: National Council of Teachers of Mathematics, pp.3-13.

LESH, R. and ZAWOJEWSKI, J.S. (2007). Problem-solving and modelling. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* Reston, VA: NCTM, pp.763-804.

LESTER, F.K. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, **10**(1-2), pp.245-278.

LESTER, F.K. and KEHLE, P.E. (2003). From problem solving to modelling: The evaluation of thinking about research on complex mathematical activity. In R.A. Lesh and H.M. doer (Eds.) Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning and teaching, Mahwah, NJ: Erlbaum, pp.501-518.

LESTER, F.K. and LAMBDIN, D.V. (2004). Teaching mathematics through problem solving. In B.Clarke et al. (Eds). Proceedings of the Midsummer World Mathematics Education Conference, Göteborg, Sweden: National Centre for Mathematics Education (NCM), *International Perspectives on Learning and Teaching Mathematics.* Göteborg: NCM, pp.189–204.

McKECHAN, S. and DAY, S. (2015). Do advanced qualifications equate to better mathematical knowledge for primary teaching? *Scottish Educational Review*, **47**(2), pp.78-99.

PAJARES, M.F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, **62**, pp.307–332.

PALLANT, J. (2013). *SPSS survival manual; a step by step guide to data analysis*, 5th edn, Berkshire: Open University Press.

PERRY, B., HOWARD, P. and TRACEY, D. (1999). Head mathematics teachers' beliefs about the learning and teaching of mathematics. *Mathematics Educational Research Journal,* **11**(1), pp.39-53.

PERRY, B., VISTRO-YU, C., HOWARD, P., WONG, N.Y and KEONG, F.H. (2002). Beliefs of primary teachers about mathematics and its teaching and learning: Views from Singapore, Philippines, Mainland China, Hong Kong, Taiwan and Australia. In B. Baton, K.C. Irwin, M. Pfannkuch and M.O.J. Thomas (Eds.) *Mathematics Education in the South pacific* (Proceedings of the 25th annual conference of MERGA), Sydney, pp.551-558.

POLYA, G. (1957). *How to solve it,* 2nd edn, New York: Doubleday Anchor.

POSAMENTIER, A.S. and KRULIK, S. (2008). *Problem-solving strategies for efficient and elegant solutions grades 6-12*. Thousand Oakes: Corwin Press.

PRIESTLEY, M. (2005). Making the most of the curriculum review: Some reflections on supporting and sustaining change in schools. *Scottish Educational Review*, **37**(1), pp.29-38.

PRIESTLEY, M. and MINTY, S. (2013). Curriculum for Excellence: 'A brilliant idea, but...' *Scottish Educational Review*, **45**(1), pp.39-52.

RAYMOND, A.M. (1997). Inconsistencies between a beginning elementary school teacher's mathematical beliefs and teaching practices. *Research in Mathematics Education*, **28**(5), pp.550-576.

SCHOEN, H.L. (Ed.). (2003). *Teaching mathematics through problem solving: Grades 6–12*. Reston, VA: National Council of Teachers of Mathematics.

SCHOENFELD, A.H. (1985). *Mathematical problem solving.* New York: Academic Press.

SCHOENFELD, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D.A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning*, New York: Macmillan, pp.334-370.

SCHROEDER, T.L. and LESTER, F.K. (1989). Developing understanding in mathematics via problem solving. In P.R. Trafton (Ed.) *New directions for elementary school mathematics* Reston, VA: NCTM, pp.31-42.

SCOTTISH EXECUTIVE (2006). *A Curriculum for Excellence. Building the Curriculum 1: the contribution of curriculum areas*. Edinburgh.

SCOTTISH GOVERNMENT (2009). *Curriculum for Excellence: Mathematics principles and practice*. Learning and Teaching Scotland.

SCOTTISH GOVERNMENT (2010a). *Curriculum for Excellence building the curriculum 2 – active learning: A guide to developing professional practice*. Edinburgh.

SCOTTISH GOVERNMENT (2010b). *Curriculum for Excellence building the curriculum 1 – the contribution of curriculum areas: A guide to developing professional practice*. Edinburgh.

SCOTTISH GOVERNMENT (2011). *Excellence in mathematics. Report from the Maths Excellence Group*.

SCOTTISH GOVERNMENT (2016). *Transforming Scotland into a maths positive nation. The final report of the making maths count group.* Edinburgh.

SHIMIZU, Y. (2009). Japanese approach to teaching mathematics via problem solving. In B. Kaur, Y.B. Har and M. Kapur (Eds.), *Mathematical problem solving: Yearbook 2009, Association of Mathematics Educators.* Singapore: World Scientific, pp.89–101.

SIEMON, D.E. (1986). *The role of metacognition in children's mathematical problem solving – A report of research in progress*. In Proceedings of the Tenth International Conference for the Psychology of Mathematics Education, London.

SILVER, E.A. (1985). *Teaching and learning mathematical problem solving.* London: Erlbaum.

SMITH, M.S., HUGHES, E.K. and ENGLE, R.A. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School*, **14**(9), pp.548-556.

SPILLANE, J. (1999). External reform efforts and teachers' initiatives to reconstruct their practice: The medicating role of teachers zones of enactment. *Journal of Curriculum Studies,* **31**, 143-175.

STANIC, G. and KILPATRICK, J. (1989). Historical perspectives on problem solving in the mathematics curriculum. In R.I Charles and E.A. Silver (Eds). *The teaching and assessing of mathematical problem solving.* Reston. VA: Lawrence Erlbaum, pp.1-22.

TEDDIE, C. and YU, F. (2007). Mixed methods sampling: A typology with examples. *Journal of Mixed Methods Research*, **1**, pp.77-100.

THOMPSON, A.G. (1985). Teacher's conceptions of mathematics and the teaching of problem solving. In E.A. Silver (Ed.). *Teaching and learning mathematical problem solving: Multiple research perspectives*, Hillsdale: Lawrence Erlbaum, pp.281-294.

THOMPSON, A.G. (1989). Learning to teach mathematical problem solving: Changes in teachers' conceptions and beliefs. In R.I. Charles and E.A. Silver (Eds.). *The teaching and assessing of mathematical problem solving.* Reston, VA: Lawrence Erlbaum, pp.232-243.

THOMPSON, A.G. (1992). Teacher's beliefs and conceptions: A synthesis of research. In D. Grouws (Ed.). *Handbook of research on mathematics teaching and learning*, New York: Macmillan, pp.127-146.

VAN DE WALLE, J.A., KARP, K.S. and BAY-WILLIAMS, J.M. (2014). *Elementary and middle school mathematics: Teaching developmentally.* New international edition, Essex: Pearson Education Ltd.

VAN BOMMEL, J. and PALMER, H. (2015). From solving problems to problem solving – Primary school teachers developing their mathematics teaching through collaborative professional development. *KAPET*, **11**(1), pp.72-89.

WU, H. (1999). Basic skills versus conceptual understanding – A Bogus Dichotomy. *American Educator*, Fall 1999.