

SEGAL'S CONJECTURE AND THE BURNSIDE RINGS OF FUSION SYSTEMS

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ABSTRACT. For a given saturated fusion system \mathcal{F} we define the ring $A(S)^{\mathcal{F}}$ of the \mathcal{F} -invariants of the Burnside ring functor. We show how this ring is related to the Burnside ring of the fusion system \mathcal{F} and how it appears naturally in the analogue of Segal's conjecture for the classifying spectrum $\mathbb{B}\mathcal{F}$. We give an explicit description of $A(S)^{\mathcal{F}}$ and we prove it is a local ring.

1. INTRODUCTION

A saturated fusion system \mathcal{F} on a finite p -group S is a small category whose objects are the subgroups of S . Its morphism sets $\mathcal{F}(P, Q)$ consist of group monomorphisms $P \rightarrow Q$, where $P, Q \leq S$, which are subject to a certain set of axioms listed in §2. Isomorphic objects in \mathcal{F} are called \mathcal{F} -conjugate. Puig was the first to define these objects but in this note we will use the formulation of Broto-Levi-Oliver in [2]. The model is the category $\mathcal{F}_S(G)$ associated to a Sylow p -subgroup S of a finite group G . In this case the objects are the subgroups of S and the morphisms in $\mathcal{F}_S(G)$ are those monomorphisms $P \rightarrow Q$ which are restrictions of inner automorphisms of G .

For any contravariant functor $H : \mathcal{F} \rightarrow \mathcal{C}$ we can consider the inverse limit $\varprojlim_{\mathcal{F}} H$. We call this limit the \mathcal{F} -invariants of $H(S)$ because, as it is easy to see, it consists of the elements $x \in H(S)$ such that $\varphi(x) = \psi(x)$ for every $\varphi, \psi \in \mathcal{F}(P, S)$ and every subgroup P of S . We will use the suggestive notation $H(S)^{\mathcal{F}}$ to denote this inverse limit. For example, by [2, Theorem 5.8], the cohomology of the classifying space of a p -local finite group is isomorphic to $H^*(S; \mathbb{F}_p)^{\mathcal{F}}$ for the obvious functor $P \mapsto H^*(P; \mathbb{F}_p)$ which assigns to $P \leq S$ its mod- p cohomology. In this paper we study the \mathcal{F} -invariants $A(S)^{\mathcal{F}}$ of the functor $A : \mathcal{F} \rightarrow \mathbf{Rings}$ which maps $P \leq S$ to its Burnside ring $A(P)$. We call this ring “the ring of \mathcal{F} -invariant virtual S -sets”. Clearly $A(S)^{\mathcal{F}}$ is a subring of $A(S)$ which contains the identity and therefore the standard augmentation map $\epsilon : A(S) \rightarrow \mathbb{Z}$ restricts to an augmentation epimorphism $\epsilon : A(S)^{\mathcal{F}} \rightarrow \mathbb{Z}$ whose kernel is denoted $I(S)^{\mathcal{F}}$. Proposition 3.6 and Lemma 4.2 give an alternative description of $A(S)^{\mathcal{F}}$ and $I(S)^{\mathcal{F}}$.

Recall that the Burnside ring $A(G)$ of a finite group G is the Grothendieck group of the monoid $B(G)$ of the isomorphism classes of finite G -sets. In symbols, $A(G) = Gr(B(G))$. The multiplication in this ring is induced by cartesian product of G -sets. As an abelian group $A(G)$ is free with one basis element for each conjugacy class of subgroups of G . In [3] we construct the Burnside ring $A(\mathcal{F})$ of saturated

Date: June 19, 2009.

2000 Mathematics Subject Classification. 55Q55, 19A22, 20C20 .

Key words and phrases. Fusion systems, Burnside ring.

The authors were supported by an EPSRC grant EP/D506484/1.

fusion system \mathcal{F} . As in the case of finite groups, where $A(G)$ is the Grothendieck group of the monoid of the isomorphism classes of finite G -sets, $A(\mathcal{F})$ is defined as the Grothendieck group of the monoid of the isomorphism classes of objects in a small category which is derived from \mathcal{F} and has finite coproducts and products. See §2 for details. The additive group of $A(\mathcal{F})$ is free with one basis element for every \mathcal{F} -conjugacy class of \mathcal{F} -centric subgroups of S , see [2, Def. 1.6].

Our first result shows the close relation between the rings $A(\mathcal{F})$ and $A(S)^\mathcal{F}$. We will write $A_{(p)}(S)^\mathcal{F}$ for the ring $\mathbb{Z}_{(p)} \otimes A(S)^\mathcal{F}$ and similarly $A_{(p)}(\mathcal{F})$ for $\mathbb{Z}_{(p)} \otimes A(\mathcal{F})$.

Theorem A. *The underlying group of $A(S)^\mathcal{F}$ is free with one generator for each \mathcal{F} -conjugacy class of subgroups of S . The $\mathbb{Z}_{(p)}$ -submodule of $A_{(p)}(S)^\mathcal{F}$ generated by the \mathcal{F} -invariant S -sets all of whose isotropy groups are non- \mathcal{F} -centric subgroups of S , forms an ideal N . There is a ring isomorphism $A_{(p)}(\mathcal{F}) \cong A_{(p)}(S)^\mathcal{F}/N$.*

The product in the ring $A_{(p)}(\mathcal{F})$ is described on basis elements explicitly in [3, Theorem 4.6]. Thus we have a good understanding of a quotient ring of $A_{(p)}(S)^\mathcal{F}$. The desire to understand this ring springs from Segal's conjecture. This conjecture was proven by Carlsson and it asserts that for any finite group G the stable cohomotopy group $\pi^0(BG_+)$ is isomorphic to the I -adic completion $A(G)_I^\wedge$, where I is the augmentation ideal of the Burnside ring $A(G)$, $\pi^*(-)$ denotes stable cohomotopy groups and the subscript $+$ means adding a disjoint base-point.

A variation of Segal's conjecture at the prime p is proven by Ragnarsson in [8]. He extends results of May-McClure and describes $\tilde{\pi}^0((BG_p^\wedge)_+)$ where G is a finite group and $\tilde{\pi}^*(-)$ denotes the reduced stable cohomotopy groups. He shows that this group is isomorphic to the p -adic completion of a quotient of the submodule of $A(G)$ generated by the G -sets with p -power isotropy. In this paper we prove an analogue of Segal's conjecture for saturated fusion systems at the prime p .

Any saturated fusion system \mathcal{F} on a p -group S has a natural classifying spectrum $\mathbb{B}\mathcal{F}$ constructed by Ragnarsson in [9]. It is equipped with a structure map $\sigma_\mathcal{F}: \mathbb{B}S \rightarrow \mathbb{B}\mathcal{F}$ where $\mathbb{B}S$ is the suspension spectrum of BS and \mathcal{F} can be recovered from the pair $(\sigma_\mathcal{F}, \mathbb{B}\mathcal{F})$ [9, Theorem A]. When \mathcal{F} has a classifying space, i.e., if there is a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ (see [2]), then $\mathbb{B}\mathcal{F}$ is the suspension spectrum of the p -completed classifying space $|\mathcal{L}|_p^\wedge$. In particular, when $\mathcal{F} = \mathcal{F}_S(G)$ is the saturated fusion system induced by a finite group G , then there is a homotopy equivalence $\mathbb{B}\mathcal{F} \simeq \Sigma^\infty(BG)_p^\wedge$ and $\sigma_\mathcal{F}$ is induced from the inclusion $BS \rightarrow BG$. Conjecturally, $\mathbb{B}\mathcal{F}$ is always a suspension spectrum of some space that one can associate to the saturated fusion system \mathcal{F} . Throughout the symbol \mathbb{S} will denote the sphere spectrum. The next result gives a description of the cohomotopy group $\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S})$ as the completion of $A(S)^\mathcal{F}$ by its augmentation ideal. Recall that $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong A(S)_{I(S)}^\wedge$ (see Remark 4.6.)

Theorem B. *Let \mathcal{F} be a saturated fusion system over a p -group S .*

- (1) *The structure map $\sigma_\mathcal{F}$ induces an isomorphism $\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \cong (A(S)^\mathcal{F})_I^\wedge$, where I is the augmentation ideal of $A(S)^\mathcal{F}$.*
- (2) *As abelian groups $(A(S)^\mathcal{F})_I^\wedge \cong \mathbb{Z} \oplus (I \otimes \mathbb{Z}_p^\wedge)$ and $I \otimes \mathbb{Z}_p^\wedge$ is a free \mathbb{Z}_p^\wedge -module with one generator for every \mathcal{F} -conjugacy class of subgroups $H \leq S$ with $H \neq S$.*

As the referee pointed out to us, it is probably more natural to consider $\mathbb{B}\mathcal{F}_+$ in Theorem B than $\mathbb{B}\mathcal{F} \vee \mathbb{S}$. However, in order to do this, one has to rewrite

Ragnarsson's results in [9] and to redefine the structure map $\sigma_{\mathcal{F}}$ as a map $\mathbb{B}S_+ \rightarrow \mathbb{B}\mathcal{F}_+$ rather than $\mathbb{B}S \rightarrow \mathbb{B}\mathcal{F}$. To avoid this and use Ragnarsson's work in its present form we chose to work with $\mathbb{B}\mathcal{F} \vee \mathbb{S}$.

Corollary C. *If $(S, \mathcal{F}, \mathcal{L})$ is a p -local finite group then there is an isomorphism of rings $\pi^0((\mathcal{L}|_p^\wedge)_+) \cong (A(S)^{\mathcal{F}})_I^\wedge$. In particular, if G is a finite group then*

$$\pi^0((BG_p^\wedge)_+) \cong (A(S)^{\mathcal{F}_S(G)})_I^\wedge \cong \mathbb{Z} \oplus (\mathbb{Z}_p^\wedge \otimes I)$$

where I is the augmentation ideal of $A(S)^{\mathcal{F}_S(G)}$.

There is some overlap between this note and Ragnarsson's results in [8]. The authors would like to thank him for helpful discussions during a week long visit to Aberdeen in 2008.

We start in Section 2 with preliminaries about fusion systems, Burnside rings and completions. Then we devote Section 3 to prove Theorem A. Finally, in Section 4 we prove Theorem B and in Proposition 4.12 we describe the spectrum of the prime ideals of $\mathbb{Z}_{(p)} \otimes (A(S)^{\mathcal{F}})_I^\wedge$.

Notation: We denote the trivial subgroup of a groups G by e . If X is a G -set and $H \leq G$, we write X^H for the set of points of X fixed by H and $|X^H|$ for its cardinality. We will identify a G -set X with its isomorphism class in the monoid $B(G)$. Throughout $\pi^*(-)$ will always mean stable cohomotopy groups of spectra or of pointed spaces. If G is a finite group $\mathbb{B}G$ is the suspension spectrum of BG with some basepoint chosen. If X is any CW-complex (pointed or not) X_+ is obtained by adding a disjoint new basepoint to X . Finally, we reserve the letter \mathbb{k} to denote a commutative ring, frequently it is torsion-free.

2. PRELIMINARIES: FUSION SYSTEMS, BURNSIDE RINGS AND COMPLETIONS

Saturated fusion systems and their classifying spectrum. Let S be a finite p -group. A fusion system over S is a small category whose objects are the subgroups of S . The morphism sets $\mathcal{F}(P, Q)$ where $P, Q \leq S$ consists of group monomorphisms $P \rightarrow Q$ such that

- (a) The set $\text{Hom}_S(P, Q)$ of the morphisms $P \rightarrow Q$ obtained by conjugation in S , is contained in $\mathcal{F}(P, Q)$.
- (b) Every morphism in \mathcal{F} factors as an isomorphism in \mathcal{F} followed by an inclusion of groups.

For example, if S is a Sylow p -subgroup of a finite group G then there results a fusion system $\mathcal{F}_S(G)$ where $\mathcal{F}(P, Q) = \text{Hom}_G(P, Q)$ is the set of the homomorphisms $c_g: P \xrightarrow{x \mapsto gxg^{-1}} Q$ obtained by conjugation in G . This fusion system has a rigid structure which was first recognized and axiomatised by Puig and later by others. The set of axioms we will use are due to Broto-Levi-Oliver in [2].

Isomorphic objects in a fusion system \mathcal{F} are called \mathcal{F} -conjugate and we will write $P \simeq_{\mathcal{F}} P'$. A subgroup $P \leq S$ is called *fully \mathcal{F} -centralised* if $|C_S(P)| \geq |C_S(P')|$ for any P' which is \mathcal{F} -conjugate to P . Similarly, P is called *fully \mathcal{F} -normalised* if $|N_S(P)| \geq |N_S(P')|$ for any P' which is \mathcal{F} -conjugate to P .

2.1. Definition. (See [2, Def. 1.2]) A fusion system \mathcal{F} over S is called *saturated* if

- (I) Every fully \mathcal{F} -normalised $P \leq S$ is also fully \mathcal{F} -centralised and $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.

(II) For every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ set

$$N_\varphi = \{g \in N_S P : \varphi \circ c_g \circ \varphi^{-1} \in \text{Aut}_S(\varphi(P))\}.$$

If $\varphi(P)$ is fully \mathcal{F} -centralised then there is $\bar{\varphi} \in \mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

A subgroup $P \leq S$ is called \mathcal{F} -centric if P and all its \mathcal{F} -conjugates contain their S -centraliser, see [2, Def. 1.6]. It is clear that if P is \mathcal{F} -centric and $P \leq Q$ then Q is also \mathcal{F} -centric. We will write \mathcal{F}^c for the full subcategory of \mathcal{F} generated by the \mathcal{F} -centric subgroups of S .

Ragnarsson constructed the classifying spectrum $\mathbb{B}\mathcal{F}$ in [9, Section 7]. It is equipped with a structure map $\sigma_{\mathcal{F}}: \mathbb{B}S \rightarrow \mathbb{B}\mathcal{F}$ and a transfer map $t_{\mathcal{F}}: \mathbb{B}\mathcal{F} \rightarrow \mathbb{B}S$. The fusion system is completely determined by its classifying spectrum $\mathbb{B}\mathcal{F}$ and the map $\sigma_{\mathcal{F}}$ in the sense that \mathcal{F} can be recovered from this data by means of the set of stable maps $\{\mathbb{B}P, \mathbb{B}\mathcal{F}\}$ where $P \leq S$. The composite $t_{\mathcal{F}} \circ \sigma_{\mathcal{F}}$ is homotopic to the stable characteristic idempotent $\tilde{\omega}_{\mathcal{F}} \in \{\mathbb{B}S, \mathbb{B}S\}$ and $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq \text{id}_{\mathbb{B}\mathcal{F}}$. In particular $\mathbb{B}\mathcal{F}$ splits off $\mathbb{B}S$.

The Burnside ring. The *orbit category* of a fusion system \mathcal{F} is the small category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S and $\mathcal{O}(\mathcal{F})(P, Q) = \mathcal{F}(P, Q)/\text{Inn}(Q)$; See [2, Def. 2.1]. We write $\mathcal{O}(\mathcal{F}^c)$ for the full subcategory generated by the \mathcal{F} -centric subgroups of S .

In [3, §4] we consider the category $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ of the finite collections in $\mathcal{O}(\mathcal{F}^c)$. To make the construction precise one looks at the category of the contravariant functors $\mathcal{O}(\mathcal{F}^c) \rightarrow \mathbf{Sets}$ which are isomorphic to $\prod_{i=1}^n \mathcal{O}(\mathcal{F}^c)(-, P_i)$ where $n < \infty$ and P_i are objects in $\mathcal{O}(\mathcal{F}^c)$. By construction, $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ is closed to finite coproducts and by Yoneda's Lemma it contains $\mathcal{O}(\mathcal{F}^c)$ as a full subcategory. The surprise is that $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ has finite products [7], [3, Theorem 1.2]. The product distributes over the coproduct and we define the Burnside ring $A(\mathcal{F})$ as a special case of the following general construction applied to $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$. See [5].

Consider an essentially small category \mathcal{C} with finite coproducts and products and assume that the product distributes over the coproduct. Define $A(\mathcal{C})$ as the free abelian group generated by the isomorphism classes $[C]$ of the objects $C \in \mathcal{C}$ subject to the relation $[C_1 \amalg C_2] = [C_1] + [C_2]$. The product in \mathcal{C} gives $A(\mathcal{C})$ the structure of a ring where $[C_1] \cdot [C_2] = [C_1 \times C_2]$. If the morphism sets of \mathcal{C} are finite then for every isomorphism class $[C]$, the ring $A(\mathcal{C})$ is equipped with a ring homomorphism

$$\chi_C: A(\mathcal{C}) \rightarrow \mathbb{Z}, \quad (X \mapsto |\mathcal{C}(X, C)|).$$

For any commutative ring \mathbb{k} we define $A_{\mathbb{k}}(\mathcal{C}) := \mathbb{k} \otimes_{\mathbb{Z}} A(\mathcal{C})$. We will denote

$$A_{(p)}(\mathcal{C}) := A_{\mathbb{Z}_{(p)}}(\mathcal{C}) \quad \text{and} \quad A_{\hat{p}}(\mathcal{C}) := A_{\mathbb{Z}_{\hat{p}}}(\mathcal{C}).$$

If C is an object of \mathcal{C} we obtain, by tensoring with \mathbb{k} , another ring homomorphism

$$(2.2) \quad \mathbb{k} \otimes \chi_C: A_{\mathbb{k}}(\mathcal{C}) \rightarrow \mathbb{k},$$

that we denote also by χ_C . It would be clear from the context which homomorphism we mean.

2.3. Example. The Burnside ring $A(G)$ of a finite group G is $A(\{\text{finite } G\text{-sets}\})$. It is a free group generated by the G -conjugacy classes of the transitive G -sets, each of which has as representative G/H for some $H \leq G$. The homomorphism (2.2) for the subgroup $H \leq G$ assigns to any G -set X the cardinality of X^H , namely the subset of X fixed by H .

Given a prime p we let $A(G; p)$ be the subring of the finite G -sets whose isotropy groups are p -subgroups of G . This is the Burnside ring of the category of the finite G -sets whose isotropy groups are p -groups. From the homomorphisms (2.2), we obtain ring homomorphisms

$$(2.4) \quad \chi: A(G) \rightarrow \prod_{\text{ccs}(G)} \mathbb{k} \quad \text{and} \quad \chi: A(G; p) \rightarrow \prod_{\text{ccs}_p(G)} \mathbb{k},$$

where $\text{ccs}(G)$ and $\text{ccs}_p(G)$ denote the set of conjugacy classes of the subgroups of G and the set of the conjugacy classes of the p -subgroups of G . These homomorphisms are, in fact, ring monomorphisms provided \mathbb{k} is torsion free. The trivial G -set $*$ is the identity element in $A(G)$ and more generally in $A_{\mathbb{k}}(G)$. We will use the integer n to denote $n \cdot *$.

The augmentation homomorphism $\chi_{(1)}: A_{\mathbb{k}}(G) \rightarrow \mathbb{k}$ is of particular importance. It is usually denoted by ϵ and it sends a finite G -set X to its cardinality $|X|$. Its kernel is the augmentation ideal $I_{\mathbb{k}}(G)$. If \mathbb{k} is torsion free then $I_{\mathbb{k}}(G)$ has an additive basis $G/H - |G/H|$ with H running over the G -conjugacy classes of subgroups H of G where $H \neq G$.

Going back to saturated fusion systems, we define $A(\mathcal{F})$ as $A(\mathcal{O}(\mathcal{F}^c)_{\square})$. As an abelian group it is free with one basis element for each \mathcal{F} -conjugacy class of \mathcal{F} -centric subgroups of S [3, Proposition 4.10]. We obtain a ring homomorphism

$$(2.5) \quad \Phi: A(\mathcal{F}) \rightarrow \prod_{[P] \in \text{ccs}(\mathcal{F}^c)} \mathbb{Z}, \quad [Q] \mapsto |\mathcal{O}(\mathcal{F}^c)(P, Q)|,$$

where $\text{ccs}(\mathcal{F}^c)$ stands for the set of the \mathcal{F} -conjugacy classes of the \mathcal{F} -centric subgroups of S . This is, in fact, a monomorphism by [3, Theorem 5.3] and therefore $\Phi_{(p)} = \mathbb{Z}_{(p)} \otimes \Phi$ is also a ring monomorphism.

In [3, Theorem 5.4] we show that (y_P) is in the image of $\Phi_{(p)}$ if and only if for any fully \mathcal{F} -normalised $Q \leq S$ the following congruence holds

$$(2.6) \quad \sum_{P \in [\mathcal{F}^c]} n(Q, P) \cdot y_P \equiv 0 \pmod{(|\text{Out}_S(Q)|)},$$

where P runs over a set of representatives for $\text{ccs}(\mathcal{F}^c)$, where $(|\text{Out}_S(Q)|)$ is the ideal in $\mathbb{Z}_{(p)}$ generated by $|\text{Out}_S(Q)|$ and where

$$n(Q, P) = |\{c_s \in \text{Out}_S(Q) : \langle s, Q \rangle \simeq_{\mathcal{F}} P\}|.$$

Completion of rings. Let I be an ideal in a commutative ring R . Then the ideals I^n form a set of neighbourhoods for $0 \in R$ which generates the I -adic topology on R and on any R -module M . The I -adic completion of an R -module M is

$$M_I^\wedge := \varprojlim_n M/I^n M.$$

It is clear from the construction that M_I^\wedge is an R_I^\wedge -module. Also note that $M \rightarrow M_I^\wedge$ is injective if and only if the I -adic topology on M is Hausdorff, namely $\bigcap_n I^n M = 0$.

When R is Noetherian one has more control on the I -completion. We recall below some basic results. Throughout, we write \hat{M} for M_I^\wedge and $\hat{R} = R_I^\wedge$ etc.

2.7. Theorem. ([4, Theorem 7.1].) *If R is Noetherian then \hat{R} is Noetherian. Moreover, as an R -module, \hat{R} is I -complete.*

2.8. Theorem. (Artin-Rees Lemma, [4, Theorem 7.2].) *If R is Noetherian and M is a finitely generated R -module then $\hat{M} = \hat{R} \otimes_R M$. Moreover \hat{R} is a flat R -module.*

The I -adic topology on R/I is discrete and therefore $\widehat{R/I} = R/I$. If R is Noetherian then Theorem 2.8 gives rise to a short exact sequence $0 \rightarrow \hat{I} \rightarrow \hat{R} \rightarrow R/I \rightarrow 0$.

2.9. Theorem. ([4, Corollary 7.13].) *If R is Noetherian then $I\hat{R} = \hat{I}$. In particular \hat{R} is \hat{I} -complete.*

3. \mathcal{F} -INVARIANT SETS

In this section we prove Theorem A in the introduction. We begin studying and describing \mathcal{F} -invariant sets.

For any finite group G denote by $B(G)$ the monoid, via disjoint unions, of the isomorphism classes of the finite G -sets, and by $A(G) = \text{Gr}(B(G))$ its Grothendieck group, i.e. the Burnside ring of G . Recall that the product in this ring is induced by the product of G -sets. A homomorphism $\varphi : H \rightarrow G$ gives rise to a morphism of monoids $\varphi_* : B(G) \rightarrow B(H)$ and a homomorphism of rings $\varphi^* : A(G) \rightarrow A(H)$. If φ is the inclusion of H into G we write res_H^G instead of φ^* .

Hence, if \mathcal{F} is a saturated fusion system we have functors $B : \mathcal{F} \rightarrow \mathbf{Monoids}$ and $A : \mathcal{F} \rightarrow \mathbf{Rings}$ whose values on the subgroup P is $B(P)$ and $A(P)$ respectively. Moreover, for any torsion-free commutative ring \mathbb{k} we consider also the functor $A_{\mathbb{k}} = \mathbb{k} \otimes A$.

3.1. Definition. Let \mathcal{F} be a saturated fusion system over S and let \mathbb{k} be a torsion-free commutative ring. We define the \mathcal{F} -invariant S -sets as $B(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} B$ and $A(S; \mathcal{F}) := \text{Gr}(B(S)^{\mathcal{F}})$. We also define $A_{\mathbb{k}}(S; \mathcal{F}) = \mathbb{k} \otimes A(S; \mathcal{F})$.

It is easy to see that $A(S; \mathcal{F})$ is a subalgebra of $A(S)$ and therefore $A_{\mathbb{k}}(S; \mathcal{F})$ is a \mathbb{k} -subalgebra of $A_{\mathbb{k}}(S)$ provided \mathbb{k} is torsion-free.

3.2. Definition. Let \mathcal{F} be a saturated fusion system over S and let \mathbb{k} be a torsion-free commutative ring. We define the ring of \mathcal{F} -invariant virtual S -sets as $A(S)^{\mathcal{F}} := \varprojlim_{\mathcal{F}} A$. We also define $A_{\mathbb{k}}(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} A_{\mathbb{k}}$.

It is clear that $A_{\mathbb{k}}(S)^{\mathcal{F}}$ is a \mathbb{k} -subalgebra of $A_{\mathbb{k}}(S)$. The next goal is to prove that $A_{\mathbb{k}}(S; \mathcal{F}) = A_{\mathbb{k}}(S)^{\mathcal{F}}$ provided \mathbb{k} is a torsion-free commutative ring.

By definition, an S -set X , which we identify with its isomorphism class in $B(S)$, is \mathcal{F} -invariant if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ the P -sets $\text{res}_P^S(X)$ and $\varphi^*(X)$ are isomorphic. It is easy to see that X is \mathcal{F} -invariant if and only if $|X^P| = |X^{P'}|$ whenever P and P' are \mathcal{F} -conjugate subgroups of S .

3.3. Proposition. *For any fully \mathcal{F} -normalised $P \leq S$ there exists an \mathcal{F} -invariant S -set Ω_P such that*

- (1) *All the isotropy groups of Ω_P are \mathcal{F} -conjugate to subgroups of P .*
- (2) *Ω_P contains exactly one orbit isomorphic to S/P , and hence $|(\Omega_P)^P| = \frac{|N_S P|}{|P|}$.*

Proof. Let $(P)_{\mathcal{F}}$ denote the \mathcal{F} -conjugacy class of P . Let P_1, \dots, P_k be representatives for the S -conjugacy classes in $(P)_{\mathcal{F}}$. We may assume that $P_1 = P$. Then $|N_S P_1| \geq |N_S P_i|$ for all i and we consider the S -set

$$\Omega_0 = \coprod_i \frac{|N_S P_1|}{|N_S P_i|} \cdot S/P_i.$$

Observe that for any i we have

$$|\Omega_0^{P_i}| = \frac{|N_S P_1|}{|N_S P_i|} \cdot |N_S P_i / P_i| = \frac{|N_S P|}{|P|}.$$

Let \mathcal{H} be the collection of all the subgroup $Q \leq S$ which are \mathcal{F} -conjugate to a proper subgroup of P . If $Q \notin \mathcal{H}$ then either Q is not \mathcal{F} -conjugate to a subgroup of P or it is \mathcal{F} -conjugate to P . In the first case it is clear that $\Omega_0^Q = \emptyset = \Omega_0^{Q'}$ for any Q' which is \mathcal{F} -conjugate to Q . In the second case Q is S -conjugate to one of the P_i 's and therefore $|\Omega_0^Q| = |N_S P / P|$. We can now apply [2, Lemma 5.4] and deduce that there exists an \mathcal{F} -invariant S -set Ω which contains Ω_0 and which also satisfies $|\Omega^Q| = |\Omega_0^Q|$ for all $Q \notin \mathcal{H}$. In particular it follows that all the isotropy groups of Ω belong to $\mathcal{H} \cup \{(P)_{\mathcal{F}}\}$ and that $|\Omega^P| = |N_S P / P|$. \square

Since $A_{\mathbb{k}}(S)^{\mathcal{F}}$ is a \mathbb{k} -subalgebra of $A_{\mathbb{k}}(S)$ the statement of the following lemma makes sense.

3.4. Lemma. *For any $u \in A_{\mathbb{k}}(S)^{\mathcal{F}}$ and any two \mathcal{F} -conjugate subgroups $Q, Q' \leq S$, $\chi_Q(u) = \chi_{Q'}(u)$ where χ_Q is defined in (2.2) (see also (2.4)).*

Proof. Note that $\chi_Q(u)$ is the coefficient of the trivial Q -set Q/Q in $\text{res}_Q^S(u) \in A_{\mathbb{k}}(Q)$. Similarly $\chi_{Q'}(u)$ is the coefficient of the trivial Q' -set in $\text{res}_{Q'}^S(u)$. In turn, this is the coefficient of the trivial Q -set Q/Q in $\varphi^*(u) \in A_{\mathbb{k}}(Q)$ where $\varphi \in \mathcal{F}(Q, S)$ is an isomorphism of Q onto Q' . The result follows since $u \in A_{\mathbb{k}}(S)^{\mathcal{F}}$ whence $\text{res}_Q^S(u) = \varphi^*(u)$. \square

3.5. Remark. Consider the restriction of (2.4) to $\chi : A_{\mathbb{k}}(S)^{\mathcal{F}} \rightarrow \prod_{\text{ccs}(S)} \mathbb{k}$. From Lemma 3.4 it is clear that χ factors through an injective ring homomorphism $\chi : A_{\mathbb{k}}(S)^{\mathcal{F}} \rightarrow \prod_{\text{ccs}(\mathcal{F})} \mathbb{k}$, where the product runs through the \mathcal{F} -conjugacy classes of the subgroups of S .

If X is an \mathcal{F} -invariant S -set then it represents an element in $A_{\mathbb{k}}(S)^{\mathcal{F}}$. Thus, $A_{\mathbb{k}}(S; \mathcal{F}) \subseteq A_{\mathbb{k}}(S)^{\mathcal{F}}$, where we regard both as \mathbb{k} -subalgebras of $A_{\mathbb{k}}(S)$. Now we are ready to prove that this inclusion is an equality and to describe a basis for $A_{\mathbb{k}}(S; \mathcal{F})$.

3.6. Proposition. *Let \mathcal{F} be a saturated fusion system over S and let \mathbb{k} be a torsion-free commutative ring. Let Q_1, \dots, Q_r be representatives for the \mathcal{F} -conjugacy classes of the subgroups of S such that the Q_i 's are fully \mathcal{F} -normalised. Then*

- (1) $A_{\mathbb{k}}(S; \mathcal{F}) = A_{\mathbb{k}}(S)^{\mathcal{F}}$.
- (2) $A_{\mathbb{k}}(S; \mathcal{F})$ is a free \mathbb{k} -module with basis $\Omega_{Q_1}, \dots, \Omega_{Q_r}$ where Ω_{Q_i} are defined in Proposition 3.3.

Proof. We arrange the Q_i 's so that $|Q_i| \geq |Q_{i+1}|$. Thus, if $j > i$ then Q_i is not \mathcal{F} -conjugate to a subgroup of Q_j and in particular $(S/P)^{Q_i} = \emptyset$ if P is \mathcal{F} -conjugate to Q_j . From Remark 3.5 and Example 2.3, there is a ring monomorphism

$$\chi : A_{\mathbb{k}}(S)^{\mathcal{F}} \rightarrow \prod_{i=1}^r \mathbb{k}, \quad \chi = (\chi_{Q_1}, \dots, \chi_{Q_r}).$$

Since $(\Omega_{Q_i})^{Q_j} = \emptyset$ if $j < i$ we see from Proposition 3.3 that

$$\chi(\Omega_{Q_i}) = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, \frac{|N_S(Q_i)|}{|Q_i|}, \dots)$$

where $\frac{|N_S(Q_i)|}{|Q_i|} \neq 0$ because \mathbb{k} is torsion-free. Thus, $\chi(\Omega_{Q_1}), \dots, \chi(\Omega_{Q_r})$ are \mathbb{k} -linearly independent in \mathbb{k}^r and therefore $\Omega_{Q_1}, \dots, \Omega_{Q_r}$ form a basis for the \mathbb{k} -submodule $B \leq A_{\mathbb{k}}(S)$ it generates. We now have

$$B \leq A_{\mathbb{k}}(S; \mathcal{F}) \leq A_{\mathbb{k}}(S)^{\mathcal{F}} \xrightarrow{\chi} \prod_{j=1}^r \mathbb{k}.$$

It remains to prove that $B = A_{\mathbb{k}}(S)^{\mathcal{F}}$.

Assume to the contrary that this is not the case and let m be the maximal integer with the property that there exists some $u \in A_{\mathbb{k}}(S)^{\mathcal{F}} \setminus B$ such that $\chi_{Q_i}(u) = 0$ for all $i = 1, \dots, m-1$. Clearly $m \leq r$ or else $u = 0$ because χ is injective. Also $\chi_{Q_m}(u) \neq 0$. Express u in terms of the basis of $A_{\mathbb{k}}(S)$, namely

$$u = \sum_{(Q) \in \text{ccs}(S)} \alpha_Q \cdot S/Q$$

and consider the smallest integer $j = 1, \dots, r$ for which there is some $R \simeq_{\mathcal{F}} Q_j$ such that $\alpha_R \neq 0$. Thus, if $\alpha_Q \neq 0$ then $|R| \geq |Q|$. Since \mathbb{k} is torsion-free, by Lemma 3.4 and Proposition 3.3 we have

$$\chi_{Q_j}(u) = \chi_R(u) = \sum_{(Q) \in \text{ccs}(S)} \alpha_Q \cdot \chi_R(S/Q) = \alpha_R \cdot |N_S R/R| \neq 0.$$

It follows from the definition of m that $j \geq m$. Now, if $\alpha_Q \neq 0$ then Q is \mathcal{F} -conjugate to Q_i where $i \geq j$, and therefore $i \geq m$. We deduce that $|Q_m| \geq |Q|$ so $(S/Q)^{Q_m} = \emptyset$ unless Q is S -conjugate to Q_m . It follows that

$$(3.7) \quad \chi_{Q_m}(u) = \alpha_{Q_m} \cdot |N_S Q_m/Q_m|,$$

which implies that $\alpha_{Q_m} \neq 0$ because $\chi_{Q_m}(u) \neq 0$. From the minimality of j we deduce that $j \leq m$ and therefore $j = m$.

Consider $v := u - \alpha_{Q_m} \Omega_{Q_m}$. Clearly $v \in A_{\mathbb{k}}(S)^{\mathcal{F}} \setminus B$. Moreover, if $i < m$ then $(\Omega_{Q_m})^{Q_i} = \emptyset$ by Proposition 3.3 so

$$\chi_{Q_i}(v) = \chi_{Q_i}(u) - \chi_{Q_i}(\alpha_{Q_m} \Omega_{Q_m}) = 0, \quad \text{if } i < m.$$

From (3.7) we also deduce that

$$\chi_{Q_m}(v) = \chi_{Q_m}(u) - \chi_{Q_m}(\alpha_{Q_m} \Omega_{Q_m}) = \chi_{Q_m}(u) - \alpha_{Q_m} \frac{|N_S Q_m|}{|Q_m|} = 0.$$

This contradicts the maximality of m and therefore $B = A_{\mathbb{k}}(S)^{\mathcal{F}}$. \square

3.8. Remark. It follows from Proposition 3.6 that if \mathbb{k} is torsion free then

$$A_{\mathbb{k}}(S; \mathcal{F}) = \mathbb{k} \otimes A(S; \mathcal{F}) = \mathbb{k} \otimes A(S)^{\mathcal{F}} = A_{\mathbb{k}}(S)^{\mathcal{F}} \leq A_{\mathbb{k}}(S).$$

3.9. Example. Let S be a Sylow p -subgroup of a finite group G . If X is a finite G -set then $\text{res}_S^G(X)$ is an $\mathcal{F}_S(G)$ -invariant S -set. This gives a ring homomorphism $\text{res}: A(G; p) \rightarrow A(S; \mathcal{F})$, see Example 2.3. Observe that a basis for $A(G; p)$ is the set $\{G/Q_i\}_{i=1}^r$ and a basis for $A(S; \mathcal{F}_S(G))$ is the set $\{\Omega_{Q_i}\}_{i=1}^r$ where $Q_i \leq S$ are chosen as in Proposition 3.6. If $P = Q_i$ then

$$\text{res}_S^G(G/P) = \sum_{g \in S \backslash G/P} S/S \cap {}^g P = \frac{|N_G P|}{|N_S P|} \cdot S/P + Y$$

where Y is a finite S -set whose isotropy groups are conjugate in G to subgroups $Q \leq P$ and Q is not S -conjugate to P itself. Thus, by Propositions 3.3 and 3.6,

the matrix which represents $\text{res}: A(G; p) \rightarrow A(S; \mathcal{F})$ with respect to the bases described above, is upper triangular with diagonal entries $|\frac{N_G Q_i}{N_S Q_i}|$. Since $N_S(Q_i)$ is a Sylow p -subgroup of $N_G Q_i$, this matrix is invertible in $\mathbb{Z}_{(p)}$. Hence $\mathbb{Z}_{(p)} \otimes \text{res}$ is an isomorphism $A_{(p)}(S; \mathcal{F}) \cong A_{(p)}(G; p)$.

Proof of Theorem A. In light of Proposition 3.6 and remark 3.8, we may replace $A_{(p)}(S)^\mathcal{F}$ with $A_{(p)}(S; \mathcal{F})$. Let Q_1, Q_2, \dots, Q_r be fully \mathcal{F} -normalised representatives for the \mathcal{F} -conjugacy classes of the subgroups of S . We order them in such a way that Q_1, \dots, Q_s are \mathcal{F} -centric, Q_{s+1}, \dots, Q_r are not \mathcal{F} -centric and $|Q_1| \geq \dots \geq |Q_s|$. In Proposition 3.6 we showed that $A_{(p)}(S; \mathcal{F})$ is a free $\mathbb{Z}_{(p)}$ -module generated by the S -sets Ω_{Q_i} .

Consider the restriction $\chi: A_{(p)}(S; \mathcal{F}) \rightarrow \prod_{j=1}^r \mathbb{Z}_{(p)}$ of the ring homomorphism (2.4) for $\mathbb{k} = \mathbb{Z}_{(p)}$. See also Remark 3.5. Then by Proposition 3.3, $\chi_{Q_j}(\Omega_{Q_i}) = 0$ if $j < i$ and $\chi_{Q_i}(\Omega_{Q_i}) = |\frac{N_S Q_i}{Q_i}| \neq 0$. From this it easily follows that N is the kernel of the composite homomorphism

$$A_{(p)}(S; \mathcal{F}) \xrightarrow{\chi} \prod_{\text{ccs}(\mathcal{F})} \mathbb{Z}_{(p)} \xrightarrow{\text{proj}} \prod_{\text{ccs}(\mathcal{F}^c)} \mathbb{Z}_{(p)}$$

because every $u \in N$ must be a linear combination of those Ω_{Q_i} 's such that Q_i is not \mathcal{F} -centric. Here, $\text{ccs}(\mathcal{F})$ is the set of the \mathcal{F} -conjugacy classes of the subgroups of S and $\text{ccs}(\mathcal{F}^c)$ is the set of the \mathcal{F} -conjugacy classes of the \mathcal{F} -centric subgroups of S . We deduce that there is a ring monomorphism

$$\bar{\chi}: A_{(p)}(S; \mathcal{F})/N \rightarrow \prod_{\text{ccs}(\mathcal{F}^c)} \mathbb{Z}_{(p)}, \quad \bar{\chi} = (\chi_{Q_1}, \dots, \chi_{Q_s}).$$

In light of (2.5), it remains to show that $\text{Im}(\bar{\chi}) = \text{Im}(\Phi_{(p)})$. First we claim that $\text{Im}(\bar{\chi}) \subseteq \text{Im}(\Phi_{(p)})$. To see this we choose some \mathcal{F} -invariant S -set X . Then for every $i \leq s$ we have $\bar{\chi}_{Q_i}(X) = |X^{Q_i}|$ and to see that $\bar{\chi}(X)$ belongs to $\text{Im}(\Phi_{(p)})$ we use (2.6). By Cauchy-Frobenius formula, for any fully \mathcal{F} -normalised $Q \in \mathcal{F}^c$,

$$\begin{aligned} \sum_{P \in \text{ccs}(\mathcal{F}^c)} n(Q, P) \cdot |X^P| &= \sum_{P \in \text{ccs}(\mathcal{F}^c)} \left(\sum_{s \in \frac{N_S Q}{Q}, \langle Q, s \rangle \simeq \mathcal{F}P} |X^P| \right) = \\ &= \sum_{s \in \frac{N_S Q}{Q}} |(X^Q)^s| = |\frac{N_S Q}{Q}| \cdot \#\{\text{orbits of } \frac{N_S Q}{Q} \text{ on } X^Q\} = 0 \pmod{(|\frac{N_S Q}{Q}|)}. \end{aligned}$$

Since Q is \mathcal{F} -centric and fully \mathcal{F} -normalised, $|\text{Out}_S(Q)| = |\frac{N_S Q}{Q}|$, hence $\bar{\chi}(X) \in \text{Im}(\Phi_{(p)})$.

Now, a basis for $\text{Im}(\bar{\chi})$ is the set $\{\bar{\chi}(\Omega_{Q_i})\}_{i=1}^s$ and a basis for $\text{Im}(\Phi_{(p)})$ is the set $\{\Phi_{(p)}([Q_i])\}_{i=1}^s$. Note that for every i we have $\bar{\chi}_{Q_j}(\Omega_i) = \Phi_{Q_j}([Q_i]) = 0$ if $j < i$ by Proposition 3.3 and (2.5). Furthermore, $\Phi_{Q_i}([Q_i]) = |\text{Out}_{\mathcal{F}}(Q_i)|$ and $\bar{\chi}_{Q_i}(\Omega_{Q_i}) = |\frac{N_S Q_i}{Q_i}|$ differ by a unit in $\mathbb{Z}_{(p)}$ because $\text{Out}_S(Q_i)$ is a Sylow p -subgroup of $\text{Out}_{\mathcal{F}}(Q_i)$. It is now an easy exercise in linear algebra to see that $\text{Im}(\bar{\chi}) = \text{Im}(\Phi_{(p)})$. \square

4. SEGAL'S CONJECTURE

The aim of this section is to prove Theorem B of the introduction. We start describing the augmentation ideal $I_{\mathbb{k}}(S; \mathcal{F})$ of $A_{\mathbb{k}}(S; \mathcal{F})$ and modules for which the $I_{\mathbb{k}}(S; \mathcal{F})$ -adic topology is equivalent to the p -adic topology. To avoid triviality we will assume throughout this section that the saturated fusion system \mathcal{F} are

defined over a non trivial p -group $S \neq 1$. We recall from Proposition 3.6 that $A_{\mathbb{k}}(S; \mathcal{F}) = A_{\mathbb{k}}(S)^{\mathcal{F}}$ if \mathbb{k} is torsion-free.

Recall from Remark 3.8 that if \mathbb{k} is torsion-free then $A_{\mathbb{k}}(S; \mathcal{F})$ is a \mathbb{k} -subalgebra of $A_{\mathbb{k}}(S)$.

4.1. Definition. There is a standard augmentation map $\epsilon: A_{\mathbb{k}}(S) \rightarrow \mathbb{k}$ sending an S -set X to its cardinality. Let $I_{\mathbb{k}}(S; \mathcal{F})$ denote the kernel of $\epsilon: A_{\mathbb{k}}(S; \mathcal{F}) \rightarrow \mathbb{k}$.

Recall that the trivial S -set is the unit in $A_{\mathbb{k}}(S)$. We will therefore use integers to represent the corresponding elements in $A_{\mathbb{k}}(S)$.

4.2. Lemma. *With the notation and hypotheses of Proposition 3.6, $I_{\mathbb{k}}(S; \mathcal{F})$ is a free \mathbb{k} -module of rank $r-1$ with basis $\{\Omega_{Q_i} - |\Omega_{Q_i}|\}$ where i runs through the indices such that $Q_i \neq S$.*

Proof. We may assume that $Q_1 = S$ and that $\Omega_{Q_1} = *$. It is clear from Proposition 3.6 that $\Omega_{Q_2} - |\Omega_{Q_2}|, \dots, \Omega_{Q_r} - |\Omega_{Q_r}|$ are \mathbb{k} -linearly independent in $A_{\mathbb{k}}(S; \mathcal{F})$. The result follows from the fact that the Ω_{Q_i} 's form a basis for $A_{\mathbb{k}}(S; \mathcal{F})$. \square

4.3. Proposition. *Assume that \mathbb{k} is torsion-free. Then the $I_{\mathbb{k}}(S; \mathcal{F})$ -adic topology on $I_{\mathbb{k}}(S; \mathcal{F})$ is equivalent to the p -adic topology.*

Proof. May and McClure show in [6, p. 212] that $I(S)^{n+1} \subseteq p \cdot I(S)$ where $|S| = p^n$. Therefore, after tensoring with \mathbb{k} and using Lemma 4.2, we see that $I_{\mathbb{k}}(S)^{n+1} \subseteq p \cdot I_{\mathbb{k}}(S)$. In addition,

$$I_{\mathbb{k}}(S; \mathcal{F})^{n+1} \subseteq A_{\mathbb{k}}(S; \mathcal{F}) \cap I_{\mathbb{k}}(S)^{n+1} \subseteq A_{\mathbb{k}}(S; \mathcal{F}) \cap p \cdot I_{\mathbb{k}}(S) = p \cdot I_{\mathbb{k}}(S; \mathcal{F}),$$

where for the last equality we argue as follows. First, the inclusion \supseteq is obvious. Consider $u \in A_{\mathbb{k}}(S; \mathcal{F}) \cap p \cdot I_{\mathbb{k}}(S)$ and use Proposition 3.6 to write $u = \sum_{i=1}^r \beta_i \Omega_{Q_i}$. Clearly, $u \in p \cdot A_{\mathbb{k}}(S)$ and by looking at the coefficients of S/Q_i , one easily deduces from Proposition 3.3 that $\beta_i = p \cdot \alpha_i$ for all i . It follows that $u = p \cdot (\sum_i \alpha_i \Omega_{Q_i})$. Now, $\epsilon(u) = 0$ and since \mathbb{k} is torsion free, we deduce that $\epsilon(\sum_i \alpha_i \Omega_{Q_i}) = 0$. Therefore $u \in p \cdot I_{\mathbb{k}}(S; \mathcal{F})$.

We now claim that

$$p^n \cdot I_{\mathbb{k}}(S; \mathcal{F}) \subseteq I_{\mathbb{k}}(S; \mathcal{F})^2.$$

To see this consider $\eta = (S/e - |S|)$ as an element in $I_{\mathbb{k}}(S; \mathcal{F})$. Observe that $\{(S/Q - |S/Q|)\}$ form a basis for $I_{\mathbb{k}}(S)$, where Q runs through the representatives for the S -conjugacy classes of the subgroups of S different from S . By inspection

$$\eta \cdot (S/Q - |S/Q|) = -|S| \cdot (S/Q - |S/Q|) = (-p^n) \cdot (S/Q - |S/Q|).$$

That is, multiplication by η results in multiplication by $(-p^n)$ in $I_{\mathbb{k}}(S)$. Therefore

$$I_{\mathbb{k}}(S; \mathcal{F})^2 \supseteq \eta \cdot I_{\mathbb{k}}(S; \mathcal{F}) = p^n \cdot I_{\mathbb{k}}(S; \mathcal{F}).$$

This completes the proof. \square

4.4. Corollary. *If \mathbb{k} is torsion-free then $I_{\mathbb{k}}(S; \mathcal{F})_{I_{\mathbb{k}}(S; \mathcal{F})}^{\wedge} = I_{\mathbb{k}}(S; \mathcal{F})_p^{\wedge} = I_{\mathbb{k}_p^{\wedge}}(S; \mathcal{F})$.*

Proof. The first equality follows from Proposition 4.3, the second follows since $I_{\mathbb{k}}(S; \mathcal{F})$ is a finitely generated free \mathbb{k} -module by Lemma 4.2 and therefore $I_{\mathbb{k}}(S; \mathcal{F})_p^{\wedge} = \mathbb{k}_p^{\wedge} \otimes_{\mathbb{k}} I_{\mathbb{k}}(S; \mathcal{F})$. \square

Any group homomorphism $\varphi: H \rightarrow G$ induces a ring homomorphism $\varphi^*: A_{\mathbb{k}}(G) \rightarrow A_{\mathbb{k}}(H)$. Note that φ^* carries $I_{\mathbb{k}}(G)$ into $I_{\mathbb{k}}(H)$ and hence there results a natural ring homomorphism $\varphi^*: A_{\mathbb{k}}(G)_{I_{\mathbb{k}}(G)}^{\wedge} \rightarrow A_{\mathbb{k}}(H)_{I_{\mathbb{k}}(H)}^{\wedge}$. Thus there is a contravariant functor $\widehat{A}_{\mathbb{k}}: \mathcal{F} \rightarrow \mathbf{Rings}$ which maps $P \leq S$ to $\widehat{A}_{\mathbb{k}}(P) = A_{\mathbb{k}}(P)_{I_{\mathbb{k}}(P)}^{\wedge}$.

4.5. Definition. Let \mathcal{F} be a saturated fusion system over the p -group S . We define the ring of \mathcal{F} -invariant completed S -sets as $\widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}} = \varprojlim_{\mathcal{F}} \widehat{A}_{\mathbb{k}}$.

Notice that $\widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}}$ is the subring of $\widehat{A}_{\mathbb{k}}(S)$ of the elements \hat{u} such that $\varphi^*(\hat{u}) = \text{res}_P^S(\hat{u})$ for all $\varphi \in \mathcal{F}(P, S)$ and any subgroup $P \leq S$.

4.6. Remark. Notice that for any pointed CW-complex X there is a natural equivalence $\Sigma^{\infty} X \vee \mathbb{S} \simeq \Sigma^{\infty}(X_+)$ in the homotopy category of spectra (see [1].)

Recall from §2 that the classifying spectrum of a saturated fusion system \mathcal{F} over S is equipped with a structure map $\sigma_{\mathcal{F}}: \mathbb{B}S \rightarrow \mathbb{B}\mathcal{F}$ where $\mathbb{B}S$ is the suspension spectrum of BS with some chosen basepoint. From the remark above we see that for any spectrum E there are isomorphisms $\{\mathbb{B}S \vee \mathbb{S}, E\} = \{\Sigma^{\infty} BS_+, E\} = E^0(BS_+)$. In particular, for $E = \mathbb{S}$ we have $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+)$.

4.7. Lemma. For any saturated fusion system \mathcal{F} over S the map $\sigma_{\mathcal{F}} \vee \mathbb{S}$ induces a split monomorphism

$$\pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \xrightarrow{(\sigma_{\mathcal{F}} \vee \mathbb{S})^*} \pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong \widehat{A}(S)$$

whose image is isomorphic to the subring $\widehat{A}(S)^{\mathcal{F}}$.

Proof. Consider the transfer map $t_{\mathcal{F}}: \mathbb{B}\mathcal{F} \rightarrow \mathbb{B}S$, see §2 and [9]. The composite $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}}$ is homotopic to $\text{id}_{\mathbb{B}\mathcal{F}}$ so $\pi^0(t_{\mathcal{F}} \vee \mathbb{S})$ is a left inverse for $\pi^0(\sigma_{\mathcal{F}} \vee \mathbb{S})$ which is therefore a split monomorphism. Set $\tilde{\omega}_{\mathcal{F}} = t_{\mathcal{F}} \circ \sigma_{\mathcal{F}}$, see [9].

For any spectrum E , the image of $E^0(\mathbb{B}\mathcal{F}) \xrightarrow{E^0(\sigma_{\mathcal{F}})} E^0(\mathbb{B}S)$ is equal to the image of $E^0(\mathbb{B}S) \xrightarrow{E^0(\tilde{\omega}_{\mathcal{F}})} E^0(\mathbb{B}S)$. By [9, Corollary 6.4], the image of $E^0(\tilde{\omega}_{\mathcal{F}})$ is equal to the set of homotopy classes $f \in \{\mathbb{B}S, E\}$ which are \mathcal{F} -invariant, that is

$$\{\mathbb{B}\mathcal{F}, E\} \xrightarrow[\cong]{E^0(\sigma_{\mathcal{F}})} \varinjlim_{P \in \mathcal{F}^{\text{op}}} \{\mathbb{B}P, E\} = E^0(\mathbb{B}S)^{\mathcal{F}}$$

where $\mathbb{B}: P \mapsto \mathbb{B}P$ is a functor from \mathcal{F} to the category of spectra. It follows that

$$(4.8) \quad E^0(\mathbb{B}\mathcal{F} \vee \mathbb{S}) \xrightarrow[\cong]{E^0(\sigma_{\mathcal{F}} \vee \mathbb{S})} \varinjlim_{P \in \mathcal{F}^{\text{op}}} E^0(\mathbb{B}P \vee \mathbb{S}) = E^0(\mathbb{B}S \vee \mathbb{S})^{\mathcal{F}}.$$

The assignments below induce isomorphic functors $\mathcal{F}^{\text{op}} \rightarrow \mathbf{Ab}$,

$$P \mapsto \{\mathbb{B}P \vee \mathbb{S}, E\} \quad \text{and} \quad P \mapsto \{\Sigma^{\infty} BP_+, E\} = E^0(BP_+).$$

This is because we have cofibre sequences $\mathbb{S} \rightarrow \Sigma^{\infty} BP_+ \rightarrow \mathbb{B}P$ and $\mathbb{S} \rightarrow \mathbb{B}P \vee \mathbb{S} \rightarrow \mathbb{B}P$ in which \mathbb{S} is a retract. Thus, $\{\mathbb{B}S \vee \mathbb{S}, E\}^{\mathcal{F}} \cong E^0(BS_+)^{\mathcal{F}}$. We now specialise to $E = \mathbb{S}$. Using the fact that the isomorphism $\pi^0(BG_+) \cong \widehat{A}(G)$ is natural in the group G , we obtain a natural isomorphism of rings $\pi^0(BS_+)^{\mathcal{F}} \cong \widehat{A}(S)^{\mathcal{F}}$ as a subring of $\pi^0(\mathbb{B}S \vee \mathbb{S}) \cong \pi^0(BS_+) \cong \widehat{A}(S)$. The result now follows from (4.8). \square

In order to simplify the notation, we will denote the $I_{\mathbb{k}}(S; \mathcal{F})$ -adic completion of $A_{\mathbb{k}}(S; \mathcal{F})$ by $\widehat{A}_{\mathbb{k}}(S; \mathcal{F})$. Similarly $\widehat{I}_{\mathbb{k}}(S; \mathcal{F})$ is the $I_{\mathbb{k}}(S; \mathcal{F})$ -adic completion of

$I_{\mathbb{k}}(S; \mathcal{F})$. When \mathcal{F} is the trivial fusion system over S this clearly becomes $\widehat{A}_{\mathbb{k}}(S)$ and $\widehat{I}_{\mathbb{k}}(S)$. Set $\widehat{I}_{\mathbb{k}}(S)^{\mathcal{F}} = \widehat{I}_{\mathbb{k}}(S) \cap \widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}}$.

4.9. Proposition. *Let \mathbb{k} be a torsion-free commutative ring such that \mathbb{k}_p^\wedge is also torsion-free. Assume that \mathcal{F} is a saturated fusion system over $S \neq 1$. Then*

- (1) $\widehat{I}_{\mathbb{k}}(S)^{\mathcal{F}} \cong \widehat{I}_{\mathbb{k}}(S; \mathcal{F}) \cong \mathbb{k}_p^\wedge \otimes_{\mathbb{k}} I_{\mathbb{k}}(S; \mathcal{F})$.
- (2) $\widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}} \cong \widehat{A}_{\mathbb{k}}(S; \mathcal{F})$.
- (3) $\widehat{A}_{\mathbb{k}}(S; \mathcal{F})$ contains $I_{\mathbb{k}_p^\wedge}(S; \mathcal{F})$ as an ideal with quotient \mathbb{k} . If \mathbb{k} is p -complete then $\widehat{A}_{\mathbb{k}}(S; \mathcal{F}) \cong A_{\mathbb{k}}(S; \mathcal{F})$.

Proof. We first observe that if M is an R -module and I is an ideal in R then the I -adic topology on M/IM is discrete and the short exact sequences $0 \rightarrow \frac{IM}{I^n M} \rightarrow \frac{M}{I^n M} \rightarrow \frac{M}{IM} \rightarrow 0$ for all $n \geq 1$ yield a short exact sequence

$$(4.10) \quad 0 \rightarrow (IM)_I^\wedge \rightarrow M_I^\wedge \rightarrow M/IM \rightarrow 0$$

because the tower $\{IM/I^n M\}_{n \geq 0}$ is Mittag-Leffler. In particular, the augmentation $A_{\mathbb{k}}(S) \xrightarrow{\epsilon} \mathbb{k}$ extends to a short exact sequence

$$(4.11) \quad 0 \rightarrow \widehat{I}_{\mathbb{k}}(S; \mathcal{F}) \rightarrow \widehat{A}_{\mathbb{k}}(S; \mathcal{F}) \xrightarrow{\epsilon} \mathbb{k} \rightarrow 0$$

which is split because \mathbb{k} is contained as a \mathbb{k} -subalgebra of $A_{\mathbb{k}}(S; \mathcal{F})$ generated by the trivial S -set $*$. Applying this to the trivial fusion system over S we obtain a short exact sequence

$$0 \rightarrow \widehat{I}_{\mathbb{k}}(S) \rightarrow \widehat{A}_{\mathbb{k}}(S) \xrightarrow{\epsilon} \mathbb{k} \rightarrow 0.$$

The inclusions $A_{\mathbb{k}}(S; \mathcal{F}) \subseteq A_{\mathbb{k}}(S)$ and $I_{\mathbb{k}}(S; \mathcal{F}) \subseteq I_{\mathbb{k}}(S)$ yield a ring homomorphism $\widehat{A}_{\mathbb{k}}(S; \mathcal{F}) \rightarrow \widehat{A}_{\mathbb{k}}(S)$ which factors through

$$\Psi: \widehat{A}_{\mathbb{k}}(S; \mathcal{F}) \rightarrow \widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}}$$

because $A_{\mathbb{k}}(S; \mathcal{F}) \subseteq A_{\mathbb{k}}(S)^{\mathcal{F}}$ by Proposition 3.6.

By definition $\widehat{I}_{\mathbb{k}}(S)^{\mathcal{F}} = \widehat{I}_{\mathbb{k}}(S) \cap \widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}}$ and therefore from (4.11) we now obtain the following morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{I}_{\mathbb{k}}(S; \mathcal{F}) & \longrightarrow & \widehat{A}_{\mathbb{k}}(S; \mathcal{F}) & \xrightarrow{\epsilon} & \mathbb{k} \longrightarrow 0 \\ & & \Psi' \downarrow & & \Psi \downarrow & & \parallel \\ 0 & \longrightarrow & \widehat{I}_{\mathbb{k}}(S)^{\mathcal{F}} & \longrightarrow & \widehat{A}_{\mathbb{k}}(S)^{\mathcal{F}} & \xrightarrow{\epsilon} & \mathbb{k} \longrightarrow 0 \end{array}$$

By Proposition 4.3 and from the fact that $I_{\mathbb{k}}(S; \mathcal{F})$ is a finitely generated free \mathbb{k} -module by Lemma 4.2, we see that

$$\begin{aligned} \widehat{I}_{\mathbb{k}}(S; \mathcal{F}) &\cong I_{\mathbb{k}}(S; \mathcal{F})_p^\wedge \cong I_{\mathbb{k}}(S; \mathcal{F}) \otimes_{\mathbb{k}} \mathbb{k}_p^\wedge = I_{\mathbb{k}_p^\wedge}(S; \mathcal{F}), \\ \widehat{I}_{\mathbb{k}}(S) &\cong I_{\mathbb{k}}(S)_p^\wedge \cong I_{\mathbb{k}}(S) \otimes_{\mathbb{k}} \mathbb{k}_p^\wedge = I_{\mathbb{k}_p^\wedge}(S). \end{aligned}$$

Thus, Ψ' is the map

$$I_{\mathbb{k}_p^\wedge}(S; \mathcal{F}) \subseteq I_{\mathbb{k}_p^\wedge}(S)^{\mathcal{F}}$$

induced by the inclusion $I(S; \mathcal{F}) \subseteq I(S)^{\mathcal{F}} = A(S)^{\mathcal{F}} \cap I(S)$. By Proposition 3.6 and Lemma 4.2, this inclusion is in fact an equality and therefore Ψ' is an isomorphism. This proves point (1) of this Proposition. The five-lemma now shows that Ψ is an isomorphism and this is point (2). The first assertion of point (3) follows from the

first row in the commutative ladder above. The second assertion follows because if \mathbb{k} is p -complete, namely $\mathbb{k}_p^\wedge = \mathbb{k}$, then $\widehat{I}_{\mathbb{k}}(S; \mathcal{F}) = I_{\mathbb{k}}(S; \mathcal{F})$. \square

Proof of Theorem B. This follows from Lemma 4.7 and Propositions 4.9 and 3.6. For part (2) of the theorem we also need Lemma 4.2. \square

Proof of Corollary C. In the presence of a p -local finite group, the canonical map $f: BS \rightarrow |\mathcal{L}|_p^\wedge$ induces $\omega_{\mathcal{F}}$. By Theorem B, it gives rise to a ring monomorphism $f^*: \pi^0(|\mathcal{L}|_p^\wedge \vee \mathbb{S}) \xrightarrow{\pi^0(f \vee \mathbb{S})} \pi^0(BS \vee \mathbb{S}) \cong \widehat{A}(S)$ whose image is $\widehat{A}(S)^{\mathcal{F}} \cong (A(S)^{\mathcal{F}})_I^\wedge$. Finally, we note that $\pi^0(f \vee \mathbb{S})$ can be replaced with $\pi^0(|\mathcal{L}|_{p^+}^\wedge) \xrightarrow{\pi^0(f_+)} \pi^0(BS_+)$, see Remark 4.6. \square

In the next result we obtain some information on $\mathbb{Z}_{(p)} \otimes \pi^0(\mathbb{B}\mathcal{F} \vee \mathbb{S})$.

4.12. Proposition. *Consider the rings $R_1 = \mathbb{Z}_{(p)} \otimes \widehat{A}(S; \mathcal{F})$ and $R_2 = A_{(p)}(S; \mathcal{F})$. Then*

- (1) *Both rings are local with residue field isomorphic to \mathbb{F}_p .*
- (2) *Their non-maximal prime ideals are in one-to-one correspondence with the \mathcal{F} -conjugacy classes $[H]$ of the subgroups of S and we denote them by $\widehat{\mathfrak{p}}_{[H],0} \triangleleft R_1$ and $\mathfrak{p}_{[H],0} \triangleleft R_2$. Each one of these ideals is contained only in the maximal ideal.*
- (3) *The quotient rings $R_2/\mathfrak{p}_{[H],0}$ are isomorphic to $\mathbb{Z}_{(p)}$. The quotient rings $R_1/\widehat{\mathfrak{p}}_{[H],0}$ are isomorphic to \mathbb{Z}_p^\wedge if $H \neq 1$ and to $\mathbb{Z}_{(p)}$ if $H = 1$.*

Proof. To avoid triviality we assume that $S \neq 1$. Set $R = A_{(p)}(S; \mathcal{F})$ and $I = I_{(p)}(S; \mathcal{F})$. By (4.11) the inclusion $A(S; \mathcal{F}) \subseteq A_{(p)}(S; \mathcal{F})$ induces a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{I}(S; \mathcal{F}) & \longrightarrow & \widehat{A}(S; \mathcal{F}) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{I}_{(p)}(S; \mathcal{F}) & \longrightarrow & \widehat{A}_{(p)}(S; \mathcal{F}) & \longrightarrow & \mathbb{Z}_{(p)} \longrightarrow 0 \end{array}$$

and by Proposition 4.9 the vertical arrow on the left is an isomorphism because

$$\widehat{I}(S; \mathcal{F}) \cong I_{\mathbb{Z}_p^\wedge}(S; \mathcal{F}) \cong I_{\mathbb{Z}_{(p)}^\wedge}(S; \mathcal{F}) \cong \widehat{I}_{(p)}(S; \mathcal{F}).$$

It follows that

$$R_1 = \mathbb{Z}_{(p)} \otimes \widehat{A}(S; \mathcal{F}) = \widehat{A}_{(p)}(S; \mathcal{F}) = R_I^\wedge.$$

Our goal now is to study the prime ideals in $\widehat{R} := R_I^\wedge$. We will write \widehat{I} for the I -completion of I and we note that this is an ideal of \widehat{R} . By Theorem 2.9 \widehat{R} is \widehat{I} -complete.

Observe that R is a subring of \widehat{R} because the I -adic topology on R is Hausdorff because the I -adic topology on I is equivalent to the p -adic topology and I is a free $\mathbb{Z}_{(p)}$ -module. Also note that R is a Noetherian ring because it is a finitely generated $\mathbb{Z}_{(p)}$ -module. As a consequence \widehat{R} is also Noetherian by Theorem 2.7.

Fix a prime ideal $\widehat{\mathfrak{p}} \triangleleft \widehat{R}$. By Theorem 2.8 and the fact that \widehat{R} is \widehat{I} -complete, we deduce that $\widehat{R}/\widehat{\mathfrak{p}}$ is \widehat{I} -complete as an \widehat{R} -module. However, the \widehat{I} -adic topology on $\widehat{R}/\widehat{\mathfrak{p}}$ is equivalent to the I -adic topology by Theorem 2.9. We now deduce that $\widehat{R}/\widehat{\mathfrak{p}}$ is I -complete as an R -module.

Recall from Proposition 3.6 that R has a $\mathbb{Z}_{(p)}$ -basis $\{\Omega_{Q_i}\}$ where Q_i are representatives for the \mathcal{F} -conjugacy classes of the subgroups of S . Let $\pi: \hat{R} \rightarrow \hat{R}/\hat{\mathfrak{p}}$ denote the quotient map and note that $R \subseteq \hat{R}$. Choose $H = Q_t$ of minimal order such that $\pi(\Omega_H) \neq 0$. For any $K = Q_j$ we know that all the isotropy groups of $\Omega_H \times \Omega_K$ are \mathcal{F} -conjugate to subgroups of H . Therefore, by Proposition 3.6

$$\Omega_H \times \Omega_K = |(\Omega_K)^H| \cdot \Omega_H + \sum_j \alpha_j \cdot \Omega_{L_j},$$

where L_j are \mathcal{F} -conjugate to proper subgroups of H , whence $\pi(\Omega_{L_j}) = 0$ by the minimality of $|H|$. It follows that $\pi(\Omega_H) \cdot \pi(\Omega_K) = |(\Omega_K)^H| \cdot \pi(\Omega_H)$. Since $\pi(\Omega_H) \neq 0$ and $\hat{R}/\hat{\mathfrak{p}}$ is an integral domain we deduce that

$$\pi(\Omega_K) = |(\Omega_K)^H| \cdot 1_{\hat{R}/\hat{\mathfrak{p}}} = \chi_H(\Omega_K) \cdot 1_{\hat{R}/\hat{\mathfrak{p}}}.$$

Thus, the following square in the category of R -modules is commutative

$$\begin{array}{ccc} R & \xrightarrow{\chi_H} & \mathbb{Z}_{(p)} \\ \text{incl} \downarrow & & \downarrow \iota_H: \xi \mapsto \xi \cdot 1 \\ \hat{R} & \xrightarrow{\pi} & \hat{R}/\hat{\mathfrak{p}}, \end{array}$$

where the R -module structure on $\mathbb{Z}_{(p)}$ is induced by χ_H . Since $\hat{R}/\hat{\mathfrak{p}}$ is I -complete, π is equal to the composite

$$\hat{R} \xrightarrow{\hat{\chi}_H} (\mathbb{Z}_{(p)})_I^\wedge \xrightarrow{(\iota_H)_I^\wedge} \hat{R}/\hat{\mathfrak{p}}.$$

Now, if $H = 1$ then the I -adic topology on $\mathbb{Z}_{(p)}$ is the discrete one because $\epsilon = \chi_{(1)}$ and if $H \neq 1$ then it is the p -adic topology because $\chi_H(S/e - |S|) \neq 0$. Since π is surjective, we deduce that $\hat{R}/\hat{\mathfrak{p}}$ is either $\mathbb{Z}_{(p)}$ or \mathbb{Z}_p^\wedge or it is \mathbb{F}_p .

Assume first that $\hat{R}/\hat{\mathfrak{p}} = \mathbb{F}_p$. Now, $\chi_H(\Omega_H) = 0 \pmod p$ if $H \leq S$ by Proposition 3.3, so from the way H was chosen we deduce that $H = S$. Therefore $\hat{\mathfrak{p}}$ is the kernel of

$$\hat{R} \xrightarrow{\hat{\chi}_S} \mathbb{Z}_p^\wedge \rightarrow \mathbb{F}_p$$

and it follows that \hat{R} is a local ring. Now assume that $\text{char}(\hat{R}/\hat{\mathfrak{p}}) = 0$. In this case, $\hat{R}/\hat{\mathfrak{p}} \cong (\mathbb{Z}_{(p)})_I^\wedge$ and so $\hat{\mathfrak{p}}$ is the kernel of

$$\begin{aligned} \hat{\chi}_H: \hat{R} &\rightarrow \mathbb{Z}_p^\wedge & \text{if } H \neq 1 \text{ or} \\ \hat{\chi}_1: \hat{R} &\rightarrow \mathbb{Z}_{(p)} & \text{if } H = 1. \end{aligned}$$

We denote these kernels by $\hat{\mathfrak{p}}_{[H],0}$ which are clearly prime ideals. It only remains to show that these ideals are distinct and that none is contained in the other.

Suppose that $\hat{\mathfrak{p}}_{[H],0} \subseteq \hat{\mathfrak{p}}_{[K],0}$. Then we get a surjection $\hat{R}/\hat{\mathfrak{p}}_{[H],0} \rightarrow \hat{R}/\hat{\mathfrak{p}}_{[K],0}$ where both rings are isomorphic to either $\mathbb{Z}_{(p)}$ or \mathbb{Z}_p^\wedge , whence this surjection must be an isomorphism and therefore $\hat{\mathfrak{p}}_{[H],0} = \hat{\mathfrak{p}}_{[K],0}$. Now suppose that $H = Q_i$ and $K = Q_j$ are not \mathcal{F} -conjugate. Without loss of generality we may assume that H is not \mathcal{F} -conjugate to a subgroup of K whence $(\Omega_K)^H = \emptyset$. This shows that $\Omega_K \in \ker(\hat{\chi}_H) = \hat{\mathfrak{p}}_{[H],0}$. However $\hat{\chi}_K(\Omega_K) = |\frac{N_S K}{K}| \neq 0$ so $\Omega_K \notin \hat{\mathfrak{p}}_{[K],0}$. Thus, $\hat{\mathfrak{p}}_{[H],0}$ and $\hat{\mathfrak{p}}_{[K],0}$ are distinct. This completes the analysis of the spectrum of the prime ideals in $R_1 = \hat{R}$.

The analysis of the spectrum of the prime ideals in $R_2 = R$ is similar. Let $\mathfrak{p}_{[H],0}$ be the kernel of $\chi_H: R \rightarrow \mathbb{Z}_{(p)}$ and let \mathfrak{m} denote the kernel of $\chi_S: R \rightarrow \mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$. Consider a prime ideal $\mathfrak{p} \triangleleft R$ and let $\pi: R \rightarrow R/\mathfrak{p}$ denote the quotient map. Set $H = Q_t$ where H has minimal order with the property that $\pi(\Omega_H) \neq 0$. One argues as above to prove that π may be identified with

$$R \xrightarrow{\chi_H} \mathbb{Z}_{(p)} \quad \text{or} \quad R \xrightarrow{\chi_H} \mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p.$$

If the second possibility happens then $H = S$ because $\pi(\Omega_H) = \chi_H(\Omega_H) = |\frac{N_S H}{H}| = 0 \pmod p$ if $H \lesssim S$. We therefore see that \mathfrak{m} is the unique maximal ideal in R . If $R/\mathfrak{p} \cong \mathbb{Z}_{(p)}$ then $\mathfrak{p} = \mathfrak{p}_{[H],0}$ and the argument above for $\hat{\mathfrak{p}}_{[H],0}$ shows that the ideals $\mathfrak{p}_{[K],0}$ are distinct for non- \mathcal{F} -conjugate K 's and that none is contained in the other. \square

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