RESEARCH ARTICLE

Homological stability for Iwahori–Hecke algebras

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Abstract

We show that the Iwahori–Hecke algebras $\mathcal{H}_n$ of type A_{n-1} satisfy homological stability, where homology is interpreted as an appropriate Tor group. Our result precisely recovers Nakaoka's homological stability result for the symmetric groups in the case that the defining parameter is equal to 1. We believe that this paper, and our joint work with Boyd on Temperley–Lieb algebras, are the first time that the techniques of homological stability have been applied to algebras that are not group algebras.

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1 | INTRODUCTION

1.1 | Homological stability

A family of discrete groups

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$$

satisfies *homological stability* if the maps

$$H_d(G_{n-1}) \longrightarrow H_d(G_n)$$

are isomorphisms when n is sufficiently large compared to d. Homological stability can similarly be formulated for sequences of topological groups, and for families of spaces that are not

necessarily classifying spaces of groups. Examples of families for which homological stability holds include symmetric groups [33], general linear groups [8, 34, 40], mapping class groups of surfaces and 3-manifolds [17, 23, 36, 41], diffeomorphism groups of highly connected manifolds [16], automorphism groups of free groups [21, 22], families of Coxeter groups [18] and Artin monoids [5], configuration spaces of manifolds [9, 35], and a great many others besides.

The homology $H_*(G; R)$ of a discrete group G with coefficients in a ring R can be written as the Tor group

$$\mathrm{Tor}^{RG}_*(\mathbb{1}, \mathbb{1})$$

over the group algebra RG, where $\mathbb{1}$ denotes the trivial representation. This formulation shows that the homology of a group depends only on the group algebra. We can therefore say that a family of algebras

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots$$

equipped with a consistent choice of 'trivial representation' $\mathbb{1}$ satisfies *homological stability* if the maps

$$\mathrm{Tor}^{\mathcal{A}_{n-1}}_d(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Tor}^{\mathcal{A}_n}_d(\mathbb{1}, \mathbb{1})$$

are isomorphisms when n is sufficiently large compared to d. Here the algebras need not be group algebras, and the only requirement on $\mathbb{1}$ is that it is a module for each $\mathcal{A}_n$ and that the module structures are compatible with the maps $\mathcal{A}_{n-1} \to \mathcal{A}_n$.

The purpose of this paper is to demonstrate that homological stability holds in this sense for Iwahori–Hecke algebras of type A_{n-1}, and moreover that it can be proved by adapting the suite of techniques used to study families of groups to the setting of algebras. In [2], Boyd and the author prove homological stability for the *Temperley–Lieb algebras*. There we again use the techniques of homological stability, but encounter — and resolve — novel obstructions that are not present in the setting of groups or of Iwahori–Hecke algebras.

To the best of our knowledge, the present paper and [2] are the first homological stability results of their kind for algebras that are not group algebras, and our hope is that they will serve as a proof of concept for the export of homological stability techniques into new algebraic contexts. Indeed, since the appearance of the present paper and [2], the author together with Boyd and Patzt have used the same set of techniques to study the homology of Brauer algebras [3]. On a related note, Sroka [38] has adapted techniques from the geometry and topology of Coxeter groups (specifically the Davis complex) to study the homology of odd Temperley–Lieb algebras.

1.2 | Iwahori–Hecke algebras

The symmetric group $\mathfrak{S}_n$ has presentation with generators

$$s_1, \dots, s_{n-1},$$

and with relations

$$s_i s_j = s_j s_i \qquad\qquad \text{for } |i - j| > 1,$$

$$s_i s_j s_i = s_j s_i s_j \qquad\qquad \text{for } |i - j| = 1,$$

$$s_i^2 = e \qquad\qquad \text{for all } i,$$

where s_i is the adjacent transposition $s_i = (i\ i+1)$. This is the presentation of $\mathfrak{S}_n$ as the *Coxeter group of type* A_{n-1}.

Now let R be a commutative ring and let $q \in R^\times$ be a unit. The *Iwahori–Hecke algebra of type* A_{n-1}, denoted $\mathcal{H}_n$, is the R-algebra with generators

$$T_1, \dots, T_{n-1}$$

and with relations

$$T_i T_j = T_j T_i \qquad\qquad \text{for } |i - j| > 1,$$

$$T_i T_j T_i = T_j T_i T_j \qquad\qquad \text{for } |i - j| = 1,$$

$$(T_i + 1)(T_i - q) = 0 \qquad\qquad \text{for all } i.$$

When $q = 1$, the final relation can be rewritten as $T_i^2 = 1$, so that $\mathcal{H}_n \cong R\mathfrak{S}_n$ by the isomorphism that sends T_i to s_i. Thus, $\mathcal{H}_n$ is a 'deformation' of $R\mathfrak{S}_n$ depending on the parameter q. Taking $R = \mathbb{C}$, then $\mathcal{H}_n \cong \mathbb{C}\mathfrak{S}_n$ unless q is a dth root of unity for $2 \leqslant d \leqslant n$ [43, Theorem 2.2], in which case no such isomorphism exists.

The algebras $\mathcal{H}_n$ are important from several points of view, and we mention just a couple. In knot theory, the $\mathcal{H}_n$ are a crucial ingredient in certain definitions of the HOMFLY-PT polynomial [13, 24], and their categorifications via Soergel bimodules are used to define categorifications of this polynomial [26]. In representation theory, if we take $R = \mathbb{C}$ and q a prime power, then $\mathcal{H}_n$ is isomorphic to the endomorphism algebra of a certain representation of $\mathrm{GL}_n(\mathbb{F}_q)$, and this allows the construction of an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ from each irreducible of $\mathfrak{S}_n$, see [32, pp. x–xi]. For general introductions to the $\mathcal{H}_n$, we suggest [32, chapter 1] and [29, chapters 4–5].

In general, there is an Iwahori–Hecke algebra associated to any Coxeter system, and these more general Iwahori–Hecke algebras are important in many parts of representation theory, see, for example, [15; 20, chapter 7; 27; 30]. We will often refer to the $\mathcal{H}_n$ as *Iwahori–Hecke algebras* without explicitly mentioning their type.

1.3 | Homological stability for Iwahori–Hecke algebras

The Iwahori–Hecke algebra $\mathcal{H}_n$ has two natural rank-1 modules, denoted $\mathbb{1}$ and ε, where each T_i acts on $\mathbb{1}$ as multiplication by q, and on ε as multiplication by (-1), see [32, Corollary 1.14]. When $q = 1$ the modules $\mathbb{1}$ and ε become the trivial representation and the sign representation, respectively. We may therefore consider

$$\mathrm{Tor}_*^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) \quad \text{and} \quad \mathrm{Ext}_{\mathcal{H}_n}^*(\mathbb{1}, \mathbb{1})$$

to be the *homology* and *cohomology* of $\mathcal{H}_n$, and indeed when $q = 1$ these become simply $H_*(\mathfrak{S}_n; R)$ and $H^*(\mathfrak{S}_n; R)$, respectively. We can now state our main result:

Theorem 1.1. *The maps*

$$\operatorname{Tor}_d^{\mathcal{H}_{n-1}}(\mathbb{1}, \mathbb{1}) \longrightarrow \operatorname{Tor}_d^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$$

and

$$\operatorname{Ext}_{\mathcal{H}_n}^d(\mathbb{1}, \mathbb{1}) \longrightarrow \operatorname{Ext}_{\mathcal{H}_{n-1}}^d(\mathbb{1}, \mathbb{1})$$

are isomorphisms for $d \leqslant \frac{n-1}{2}$.

When $q = 1$ then $\mathcal{H}_n \cong R\mathfrak{S}_n$, and Theorem 1.1 gives exactly Nakaoka's stability result for the homology and cohomology of symmetric groups. See [25, Theorem 2; 33, Corollary 6.7; 35, Theorem 5.1]. Nakaoka in fact gave a complete computation of $H_*(\mathfrak{S}_n; \mathbb{F}_p)$ for any prime p, and this can be used to show that for $k \geqslant 1$ the map $H_k(\mathfrak{S}_{2k-1}; \mathbb{F}_2) \to H_k(\mathfrak{S}_{2k}; \mathbb{F}_2)$ is not surjective. Thus, the range $d \leqslant \frac{n-1}{2}$ appearing in the theorem cannot be improved in general.

1.4 | Comparison with work of Benson–Erdmann–Mikaelian

The cohomology ring $\operatorname{Ext}_{\mathcal{H}_n}^*(\mathbb{1}, \mathbb{1})$ of $\mathcal{H}_n$ was explicitly computed by Benson, Erdmann and Mikaelian [1] in the case where $R = \mathbb{C}$ and q is a primitive ℓth root of unity with $\ell \geqslant 2$. In the case $\ell > n$, the result of Wenzl mentioned above shows that $\mathcal{H}_n \cong \mathbb{C}\mathfrak{S}_n$, so that $\operatorname{Ext}_{\mathcal{H}_n}^d(\mathbb{1}, \mathbb{1}) = H^*(\mathfrak{S}_n; \mathbb{C})$ is trivial, but when $2 \leqslant \ell \leqslant n$ then no such isomorphism holds, and indeed Benson–Erdmann–Mikaelian show that $\operatorname{Ext}_{\mathcal{H}_n}^d(\mathbb{1}, \mathbb{1})$ is nontrivial. Furthermore, one can use their results to observe that in this case the stabilisation maps $\operatorname{Ext}_{\mathcal{H}_n}^d(\mathbb{1}, \mathbb{1}) \to \operatorname{Ext}_{\mathcal{H}_{n-1}}^d(\mathbb{1}, \mathbb{1})$ are isomorphisms up to and including (at least) degree $(n - 2)$. So, [1] serves as an antecedent of the present work, but more interestingly, it demonstrates a much stronger stable range in this case, of slope 1 rather than slope $\frac{1}{2}$. This is reminiscent of the slope 1 rational homological stability results for configuration spaces of manifolds (see, for example, [9, Corollary 3; 35, Theorem B]). It suggests that there may be a slope 1 stability result for the $\mathcal{H}_n$ in characteristic 0.

1.5 | Discussion: Homological stability for Coxeter groups and Artin monoids

The present paper builds strongly on previous work of the author [18], which proved homological stability for families of Coxeter groups, and of Boyd [5], which proved homological stability for families of Artin monoids. These papers demonstrated that one can do all of the normal work of a homological stability proof purely in terms of a Coxeter or Artin-type presentation, rather than in terms of a concrete model of the group or monoid being studied. The defining presentation of the Iwahori–Hecke algebra $\mathcal{H}_n$ is of course very close to both of these, being a deformation of the Coxeter presentation of $\mathfrak{S}_n$, and a quotient of the Artin presentation of the braid group (or rather of their group rings).

In both [5] and [18], the results apply to families of groups or monoids obtained from sequences of Coxeter diagrams that 'grow a tail' of type A_{n-1} as n increases. These families are very general, but include as the basic case the families of type A, B and D. So, one may ask whether Theorem 1.1 can be extended to apply to any of these more general families. This seems likely, but we were not able to prove Theorem 1.1 by generalising the method of [18] from Coxeter groups to Iwahori–Hecke algebras; this is discussed further in Subsection 1.8.

1.6 | Discussion: Stable homology

Theorem 1.1 shows that, in a fixed degree d, then for n sufficiently large the groups $\mathrm{Tor}_d^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$ all agree and coincide with the *stable homology*

$$\mathrm{colim}_n \mathrm{Tor}_d^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) = \mathrm{Tor}_d^{\mathcal{H}_\infty}(\mathbb{1}, \mathbb{1}),$$

where $\mathcal{H}_\infty = \mathrm{colim}_n \mathcal{H}_n$ is the 'infinite' Iwahori–Hecke algebra. When $q = 1$, the stable homology $\mathrm{Tor}_*^{\mathcal{H}_\infty}(\mathbb{1}, \mathbb{1})$ coincides with the homology of the infinite symmetric group, $H_*(\Sigma_\infty; R)$, which is computed by the Barratt–Priddy–Quillen theorem [6, 12]:

$$H_*(\Sigma_\infty; R) \cong H_*(\Omega_0^\infty S^\infty; R).$$

Here $\Omega^\infty S^\infty = \mathrm{colim}_n \Omega^n S^n$ is the infinite loop space of the sphere spectrum, and $\Omega_0^\infty S^\infty$ is the path component of its basepoint. It is therefore natural to ask what is the stable homology $\mathrm{Tor}_*^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$ in general? To put it another way, what is the Iwahori–Hecke analogue of $H_*(\Omega_0^\infty S^\infty; R)$?

1.7 | Discussion: Homological stability for algebras

As we said earlier, we believe that the work of the present paper on Iwahori–Hecke algebras and of [2] on Temperley–Lieb algebras are the first time the techniques of homological stability have been applied to families of algebras that are not group algebras, and we hope that they will serve as a starting point for new work in this area. We refer the reader to the introduction of [2], where several possible directions are discussed in some detail.

1.8 | Method of proof

Proofs of homological stability for sequences of groups $(G_n)_{n \geqslant 0}$ can often be placed in the following broad framework.

- Find a complex (a simplicial complex, or semi-simplicial set, or chain complex) upon which the nth group G_n acts in such a way that the simplex stabilisers are of the form G_m for $m < n$. (Or at least, the simplex stabilisers must be associated to the previous groups in the sequence in some way).
- Prove that the complex is highly acyclic, that is, that its homology vanishes up to a certain point.

- Use an algebraic method (often but not always a spectral sequence argument) based on the complex in order to prove stability by induction.

While many different proofs fit this framework when viewed from a distance, there are many choices to be made and many variations are possible. It may be possible to prove stability for the same family of groups by choosing different complexes to begin with. It may be possible to prove high-acyclicity of the same complex in multiple ways. And it may be possible to use the same complex in different algebraic arguments to prove stability.

Our approach to proving Theorem 1.1 fits into the framework outlined above. There is a well-known complex, called the *complex of injective words*, that is used in many proofs of homological stability for the symmetric group. For our complex, we construct an Iwahori–Hecke analogue of the complex of injective words. While the complex of injective words has an action of $\mathfrak{S}_n$, our new complex is a chain complex of $\mathcal{H}_n$-modules; and while the generators of the complex of injective words have stabilisers given by smaller symmetric groups, our new complex is built out of tensor products like $\mathcal{H}_n \otimes_{\mathcal{H}_m} \mathbb{1}$ for $m < n$. The proof that our complex is highly acyclic is closely modelled on, but far more involved than, a proof that the complex of injective words is highly acyclic, and requires us to make careful use of the theory of distinguished coset representatives in Coxeter groups and the basis theorem for Iwahori–Hecke algebras. Furthermore, new difficulties arise because q is no longer equal to 1, so that one must now account for many hitherto-invisible powers of q. (Surprisingly, the formula $T_k^2 = (q-1)T_k + q$ explicitly surfaces in only one place, and it quickly disappears again.) The final step of our argument is a spectral sequence argument closely related to ones in the literature.

A lot of the difficulty in the present paper boils down to the fact that we are operating under two significant constraints. First, we are not working with the symmetric group, but with its Iwahori–Hecke algebra, and while $\mathcal{H}_n$ is closely related to $\mathfrak{S}_n$ *thought of as a Coxeter group*, it is not useful to think of $\mathcal{H}_n$ in terms of permutations of the set $\{1, \dots, n\}$. This means that we can approach the complex of injective words only in terms of the Coxeter presentation of $\mathfrak{S}_n$. Second, the *linear* nature of Iwahori–Hecke algebras heavily restricts the suite of topological tools that we can apply. For example, the approach of [18] to proving homological stability for Coxeter groups could not be adapted to this setting since it made use of simplicial complexes and barycentric subdivision, which do not seem to have analogues in the linear setting.

There are by now several systematic approaches to proving homological stability results, for example, Randal-Williams and Wahl's approach [37] via homogeneous categories, the author's approach via families of groups with multiplication [19], and Kupers, Galatius and Randal-Williams' approach via cellular E_k-algebras [14]. One may ask whether the present results could be proved using any of these frameworks. In the first two cases, the answer is no, since these are designed purely for the study of groups, though it is plausible that a 'linearised' version of [37] would produce the same complex that we use. In the final case, it seems that the methods of [14] could possibly be applied in the present situation, but we have taken a significantly more elementary approach.

1.9 | Outline of the paper

- We begin Section 2.3 with some detailed background on Iwahori–Hecke algebras.
- In Section 3, we give a short account of the complex of injective words and a proof that it is highly acyclic.

- In Section 4, we rephrase the complex of injective words entirely in terms of the group ring $R\mathfrak{S}_n$, where $\mathfrak{S}_n$ is regarded as a Coxeter group. This formulation places the existing work on the symmetric groups in a setting where it can be extended to Iwahori–Hecke algebras.
- In Section 5, we give an overview of the construction of our analogue of the complex of injective words, $\mathcal{D}(n)$, and the proof of its high-acyclicity. We set out how our approach builds on the rephrasing given in Section 4, we describe the difficulties that arise, and we preview the work that follows in Sections 6, 7 and 8.
- In Section 6, we define $\mathcal{D}(n)$.
- In Sections 7 and 8, we show that the homology of $\mathcal{D}(n)$ is zero up to and including degree $(n-2)$. Section 7 defines a filtration of $\mathcal{D}(n)$, while Section 8 identifies the filtration quotients in terms of the $\mathcal{D}(m)$ for $m < n$, allowing an inductive proof of high-acyclicity.
- In Section 9, we obtain a spectral sequence from $\mathcal{D}(n)$ and identify its E_1 and E_∞ terms.
- In Section 10, we use the spectral sequence to give an inductive proof of Theorem 1.1.

2 | BACKGROUND ON IWAHORI–HECKE ALGEBRAS

This section is a rapid run through the theory of Coxeter groups and Iwahori–Hecke algebras that is necessary for the applications in this paper. The intention is to give the reader a flavour of the extent and depth of the theory required. Everything that we recall here is basic in the theory of Coxeter groups and Iwahori–Hecke algebras, but it nevertheless amounts to a significant amount of nontrivial theory. However, none of this theory is strictly necessary until Subsection 8.3, and some readers may wish to skim or skip the section until then. For further reading, we recommend chapters 1, 2 and 4 of [15], chapters 3 and 4 of [10], or chapter 1 of [32] in the $\mathcal{H}_n$-case.

2.1 | Coxeter systems and Coxeter groups

A *Coxeter matrix* on a set S is a symmetric $S \times S$ matrix whose entries lie in $\{1, 2, 3, \ldots, \infty\}$ and satisfy $m_{ss} = 1$ for all $s \in S$, $m_{st} \geqslant 2$ if $s \neq t$. A Coxeter matrix determines a *Coxeter group*

$$W = \left\langle S \;\middle|\; s^2 = e \text{ for } s \in S, \;\; \underbrace{sts \cdots}_{m_{st} \text{ terms}} = \underbrace{tst \cdots}_{m_{st} \text{ terms}} \;\; \text{for } s, t \in S \right\rangle.$$

When $m_{st} = \infty$ no relation is applied. The relations

$$\underbrace{sts \cdots}_{m_{st} \text{ terms}} = \underbrace{tst \cdots}_{m_{st} \text{ terms}}$$

are called the *braid relations* or *braid moves*. The pair (W, S) is called a *Coxeter system*.

Example 2.1 (The Coxeter system of type A_{n-1}). Let $W = \mathfrak{S}_n$ be the symmetric group on n letters, and let $S_n = \{s_1, \ldots, s_{n-1}\}$ where s_i is the adjacent transposition $(i\ i+1)$. Then $(\mathfrak{S}_n, S_n)$ is a Coxeter system, called the *Coxeter system of type A_{n-1}*. Observe that

$$m_{s_i s_j} = \begin{cases} 2 & \text{if } |i - j| > 1 \\ 3 & \text{if } |i - j| = 1. \end{cases}$$

Indeed, if $|i - j| > 1$, then s_i and s_j are disjoint transpositions, so that $s_i s_j$ has order 2, while if $|i - j| = 1$, then $s_i s_j$ is a 3-cycle, and so has order 3. (The observation really just shows that if W is the Coxeter group of this type, then the relevant relations hold in $\mathfrak{S}_n$ so that there is a surjection $W \to \mathfrak{S}_n$. To show that this is an isomorphism, one must show that the relations of the Coxeter group are sufficient to relate any two words representing the same element of $\mathfrak{S}_n$. That is a simple exercise.)

Let (W, S) be a Coxeter system. A *word* in S is a tuple $\mathbf{s} = (s_1, \dots, s_l)$ of elements of S. We say that $w = w(\mathbf{s}) = s_1 \dots s_l$ is the element *represented* by $\mathbf{s}$, and we say equivalently that $\mathbf{s}$ is an *expression* for $w = w(\mathbf{s})$. The *length* of an element $w \in W$, denoted $\ell(w)$, is the minimum length of a word representing w. We say that $\mathbf{s}$ is a *reduced expression* for w if it is a word of minimum length representing w.

We will often blur the difference between words and their expressions, writing $w = s_1 \cdots s_l$ for an element of W, and referring to $s_1 \cdots s_l$ as a *word* or *expression* for w, hoping that it will be clear from what is written that the expression $(s_1, \dots, s_l)$ is to be understood.

Given an element $w \in W$, there are two possibilities for $\ell(sw)$.

- $\ell(sw) = \ell(w) + 1$. In this case one can obtain a reduced expression for sw by putting s in front of a reduced expression for w.
- $\ell(sw) = \ell(w) - 1$. In this case w has a reduced expression beginning with s.

(See [10, pp. 35–36].)

Here are two important results on reduced words in Coxeter groups.

Theorem 2.2 (Matsumoto's theorem [31], [15, section 1.2]). *Let (W, S) be a Coxeter system. Then any reduced expression for an element of W can be transformed into any other by repeatedly replacing subwords of the form $sts \cdots$ (with m_{st} terms) with $tst \cdots$ (again with m_{st} terms).*

Theorem 2.3 (The word problem, Tits [39], [10, 3.4.2]). *Let (W, S) be a Coxeter system. Then a word in S is a reduced expression if and only if it cannot be shortened by applying a sequence of the following* M-operations *or* M-moves.

- *Delete a subword of the form ss.*
- *Replace a subword of the form $\underbrace{sts \cdots}_{m_{st} \text{ terms}}$ with $\underbrace{tst \cdots}_{m_{st} \text{ terms}}$.*

Any two reduced expressions for the same element differ only by a sequence of moves of the second kind.

Let (W, S) be a Coxeter system. Let $T \subseteq S$. The associated *special subgroup* is the subgroup of W generated by T, and is denoted W_T. The pair (W_T, T) is then a Coxeter system, which is to say, W_T is precisely the Coxeter group with generators T and with Coxeter matrix obtained from the Coxeter matrix of (W, S) in the evident way [10, 4.1.6].

2.2 | Cosets in Coxeter groups

For the material in this subsection, we refer to [10, section 4.3; 15, section 2.1]. Let (W, S) be a Coxeter system, and let $J \subseteq S$. The cosets $W_J \backslash W$ are the subject of the following theory, which

will be extremely useful to us. Define

$$X_J = \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

Thus, X_J consists of all elements of W that have no reduced expressions beginning with an element of J. The elements of X_J are called $(J, \emptyset)$-*reduced*, and referred to as the *distinguished right coset representatives* for W_J, for reasons that the next theorem will make clear. If $J \subseteq K \subseteq S$, then we write X_J^K for the set of distinguished right-coset representatives for W_J in W_K.

Theorem 2.4.

(1) $x \in X_J$ if and only if $\ell(vx) = \ell(v) + \ell(x)$ for all $v \in W_J$.
(2) For each $w \in W$, there exist unique $x \in X_J$ and $v \in W_J$ such that $w = vx$.
(3) X_J forms a complete set of representatives for $W_J \backslash W$.
(4) If $x \in X_J$, then x is the unique shortest element in $W_J x$.

There is a similar theory for the cosets W/W_J, in which the role of X_J is now played by X_J^{-1}, elements of which are called $(\emptyset, J)$-*reduced*.

Moreover, if $J, K \subseteq S$, then there is also a theory for the double cosets $W_J \backslash W / W_K$. We define $X_{JK} = X_J \cap X_K^{-1}$. Thus, an element $x \in W$ lies in X_{JK} if and only if it has no reduced expressions beginning with a letter in J or ending with a letter in K. The elements of X_{JK} are called *distinguished double coset representatives of* W_J *and* W_K *in* W, and we also refer to them as (J, K)-*reduced*. They form a complete set of representatives for the double cosets $W_J \backslash W / W_K$, and each one is the unique shortest element in its double coset.

The *Mackey decomposition* states that for $J, K \subseteq S$,

$$X_J = \bigsqcup_{d \in X_{JK}} d \cdot X_{J^d \cap K}^K.$$

Inverting the Mackey decomposition gives us a version for the left cosets

$$X_J^{-1} = \bigsqcup_{d \in X_{KJ}} (X_{K \cap {}^d J}^K)^{-1} \cdot d.$$

Here, as is common in group theory, the notation J^d and ${}^d J$ denotes conjugation by d, with the positive power of d on the side indicated by the notation, so that

$$J^d = \{d^{-1} j d \mid j \in J\}, \qquad {}^d J = \{d j d^{-1} \mid j \in J\}.$$

Observe that in both Mackey decompositions the lengths add in products. For example, suppose that $d \in X_{JK}$ and $y \in X_{J^d \cap K}^K$, so that $d \in X_K^{-1}$ and $y \in W_K$, and hence $\ell(dy) = \ell(d) + \ell(y)$.

2.3 | Iwahori–Hecke algebras

Let (W, S) be a Coxeter system, let R be a commutative ring and let $q \in R^\times$ be a unit. The *Iwahori–Hecke algebra* associated to (W, S) is the algebra $\mathcal{H}_W$ with generators

$$T_s \text{ for } s \in S$$

and relations:

$$\underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}} \qquad \text{for } s, t \in S$$

$$(T_s + 1)(T_s - q) = 0 \qquad \text{for } s \in S.$$

Example 2.5 (The Iwahori–Hecke algebra of type A_{n-1}). Take $W = \mathfrak{S}_n$ and $S_n = \{s_1, \ldots, s_{n-1}\}$, as in Example 2.1, so that $(W, S) = (\mathfrak{S}_n, S_n)$ is the Coxeter system of type A_{n-1}. Then $m_{s_i s_j} = 2$ if $|s_i - s_j| > 1$, and and $m_{s_i s_j} = 3$ if $|s_i - s_j| = 1$, so that $\mathcal{H}_{\mathfrak{S}_n}$ has generators

$$T_{s_1}, \ldots, T_{s_{n-1}}$$

and relations

$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \qquad \text{for } |i - j| > 1,$$

$$T_{s_i} T_{s_j} T_{s_i} = T_{s_j} T_{s_i} T_{s_j} \qquad \text{for } |i - j| = 1,$$

$$(T_{s_i} + 1)(T_{s_i} - q) = 0 \qquad \text{for all } i.$$

Thus, if we write $T_i = T_{s_i}$, then $\mathcal{H}_{\mathfrak{S}_n}$ becomes exactly the algebra $\mathcal{H}_n$ defined in the introduction.

Let $w \in W$, and let $w = s_1 \cdots s_r$ be any reduced expression for w. Then by Matsumoto's Theorem 2.2, the quantity

$$T_w = T_{s_1} T_{s_2} \cdots T_{s_r}$$

depends only on w and not on the reduced expression. Suppose that $u, v \in W$ satisfy $\ell(uv) = \ell(u) + \ell(v)$. Then one can obtain a reduced expression for uv by combining reduced expressions for u and v. We therefore obtain

$$T_u T_v = T_{uv} \text{ for } u, v \in W \text{ such that } \ell(uv) = \ell(u) + \ell(v).$$

The significance of the elements T_w is the following central result, for which see [4, chapter IV, section 2, exercise 23], or [15, Theorem 4.4.6], or [32, Theorem 1.13] for the case $\mathcal{H}_W = \mathcal{H}_n$.

Theorem 2.6 (Basis theorem). *The elements T_w for $w \in W$ form a basis for $\mathcal{H}_W$ as an R-module, called the* standard basis.

And we have the following consequence, which is extremely important for the present paper.

Proposition 2.7. *Let (W, S) be a Coxeter system and let $J \subseteq S$. Then $\mathcal{H}_W$ is free as a left $\mathcal{H}_{W_J}$-module with basis $\{T_x \mid x \in X_J\}$. In particular, $\mathbb{1} \otimes_{\mathcal{H}_{W_J}} \mathcal{H}_W$ is free with basis $\{1 \otimes T_x \mid x \in X_J\}$.*

Similarly, $\mathcal{H}_W$ is free as a right $\mathcal{H}_{W_J}$-module with basis $\{T_x \mid x \in X_J^{-1}\}$, and $\mathcal{H}_W \otimes_{\mathcal{H}_{W_J}} \mathbb{1}$ is free with basis $\{T_x \otimes 1 \mid x \in X_J^{-1}\}$.

This follows by combining Theorems 2.4 and 2.6. The point is that there is a bijection $W_J \times X_J \to W$, $(v, x) \mapsto vx$ satisfying $\ell(vx) = \ell(v) + \ell(x)$ for every $(v, x) \in W_J \times X_J$, so that $T_{vx} = T_v T_x$. See [15, 4.4.7].

3 | SYMMETRIC GROUPS AND THE COMPLEX OF INJECTIVE WORDS

In this section, we will recall the definition of the complex of injective words, and we will give a proof that it is highly acyclic. This result is originally due to Farmer [11], and has since been proved in different ways by many authors, including Björner-Wachs [7], Kerz [25] and Randal-Williams [35]. The approach that we present here is closest to that of Kerz, but tailored to our later extension to Iwahori–Hecke algebras. Throughout the section, we fix a commutative ring R.

If A is a set, then an *injective word* on A is an ordered tuple $(a_0, \dots, a_r)$ of elements of A such that no element appears more than once. We allow the empty word ().

Definition 3.1 (The complex of injective words). Let $n \geqslant 0$. The *complex of injective words* $C(n)$ is the chain complex, concentrated in degrees $-1 \leqslant r \leqslant n - 1$, that in degree r is the R-module with basis consisting of the injective words $(a_0, \dots, a_r)$ of length $(r + 1)$ on the set $\{1, \dots, n\}$. The differential $\partial^r : C(n)_r \to C(n)_{r-1}$ is defined to be given by the alternating sum

$$\partial^r(a_0, \dots, a_r) = \sum_{j=0}^{r} (-1)^j (a_0, \dots, \widehat{a_j}, \dots, a_r).$$

We regard $C(n)$ as a chain complex of $\mathfrak{S}_n$-modules by allowing $\mathfrak{S}_n$ to act on the letters of a word in the evident way. Note that $C(n)_{-1}$ is a copy of R generated by the empty word ().

Remark 3.2. The complex of injective words appears in many forms, for example as the realisation of a poset in [7, 11], a chain complex in [25] and as a semi-simplicial set in [35]. We are working in the linear setting of $R\mathfrak{S}_n$-modules, and so our complex is a chain complex of $R\mathfrak{S}_n$-modules.

Note 3.3. Throughout the paper, we will use notation like ∂^r in Definition 3.1, where the superscript indicates the degree in which the differential originates. This causes visual clutter and is sometimes extraneous, but will be extremely helpful later on in keeping track of degrees.

Theorem 3.4 (Farmer [11]). $H_d(C(n)) = 0$ *for* $d \leqslant n - 2$.

To prove this theorem, we will define a filtration of $C(n)$ by looking at the position of the letter n. This is essentially the technique used by Kerz [25].

Definition 3.5 (The filtration of $C(n)$). Let $0 \leqslant p \leqslant n - 1$. Define $F_p \subseteq C(n)$ to be the subcomplex of $C(n)$ spanned by all words for which the letter n appears in the last $(p + 1)$ places, or not at all. Thus, we obtain a filtration

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_{n-1} = C(n).$$

Observe that the F_p are not submodules with respect to the $\mathfrak{S}_n$ action, since that can change the position of n, but that they are submodules with respect to the restricted action of $\mathfrak{S}_{n-1}$.

The following notation fixes our conventions for cones and suspensions of chain complexes. The conventions are chosen so as to make the subsequent parts of the proof as direct as possible.

Definition 3.6. Let X be a chain complex with differentials d_X^r. The *cone* on X, denoted CX, is the chain complex defined by

$$(CX)_r = X_r \oplus X_{r-1}$$

with

$$d_{CX}^r : (CX)_r \longrightarrow (CX)_{r-1}$$

defined by $d_{CX}^r(x, y) = (d_X^r(x) + (-1)^r y, d_X^{r-1}(y))$. The *suspension* ΣX is the chain complex defined by

$$(\Sigma X)_r = X_{r-1}$$

with

$$d_{\Sigma X}^r : (\Sigma X)_r \longrightarrow (\Sigma X)_{r-1}$$

defined by $d_{\Sigma X}^r = d_X^{r-1}$.

Lemma 3.7. *There is an isomorphism*

$$C(C(n-1)) \xrightarrow{\;\cong\;} F_0$$

of chain complexes of $R\mathfrak{S}_n$-modules.

Proof. F_0 is the span of all words in which either n does not appear, or appears in the final position. We define a map

$$C(C(n-1)) \longrightarrow F_0$$

in degree r by

$$((x_0, \dots, x_r), 0) \longmapsto (x_0, \dots, x_r), \qquad (0, (y_0, \dots, y_{r-1})) \longmapsto (y_0, \dots, y_{r-1}, n),$$

for $(x_0, \dots, x_r) \in C(n-1)_r$ and $(y_0, \dots, y_{r-1}) \in C(n-1)_{r-1}$. Since F_0 consists of words in which n appears in the final position or not at all, this is an isomorphism, and it is straightforward to check that it commutes with the differentials. $\square$

Lemma 3.8. *Let $1 \leqslant p \leqslant n-1$. There is an isomorphism*

$$R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1) \xrightarrow{\;\cong\;} F_p/F_{p-1}$$

of chain complexes of $R\mathfrak{S}_{n-1}$-modules. In particular, as a chain complex of R-modules, F_p/F_{p-1} is isomorphic to a direct sum of finitely many copies of $\Sigma^{p+1} C(n-p-1)$.

Proof. Let $-1 \leqslant r \leqslant n - p - 2$. Then in degree $(p+1) + r$, F_p is the span of all words with n in the last $(p+1)$ places or not at all, while F_{p-1} is the span of all words with n in the last p places or not at all, so that F_p/F_{p-1} has a basis consisting of all injective words of the form $(y_0, \ldots, y_r, n, z_1, \ldots, z_p)$, that is, with n in precisely the $(p+1)$-st place from the end. The effect of the boundary map $\partial^{(p+1)+r}$ on such a word is

$$(y_0, \ldots, y_r, n, z_1, \ldots, z_p) \longmapsto \sum_{j=0}^{r} (-1)^j (y_0, \ldots, \widehat{y_j}, \ldots, y_r, n, z_1, \ldots, z_p).$$

Here the last $(p+1)$ summands of $\partial^{(p+1)+r}$, in which one of the last $(p+1)$ letters is deleted, are not present because any resulting word would lie in F_{p-1}. So, the differential affects the segment $(y_0, \ldots, y_r)$ in the usual way, while leaving the remaining segment unchanged.

We now define our map in degree $(p+1) + r$, for $-1 \leqslant r \leqslant n - p - 2$, by

$$\sigma \otimes (x_0, \ldots, x_r) \longmapsto \sigma(x_0, \ldots, x_r, n, n-p, \ldots, n-1)$$

for $\sigma \in \mathfrak{S}_{n-1}$ and $(x_0, \ldots, x_r) \in \Sigma^{p+1} C(n-p-1)_{(p+1)+r} = C(n-p-1)_r$, where the final p terms increase from $n - p$ to $n - 1$. This is well-defined, and the fact that it is a chain map follows quickly from the computation above. To see that it is an isomorphism we choose, for each injective word $z = (z_1, \ldots, z_p)$ on $\{1, \ldots, n-1\}$, an element $\sigma_z \in \mathfrak{S}_{n-1}$ for which $z = \sigma_z(n-p, \ldots, n-1)$. Then the σ_z are a set of representatives for the cosets $\mathfrak{S}_{n-1}/\mathfrak{S}_{n-p-1}$, so that every $\sigma \otimes (x_0, \ldots, x_r)$ in $R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1)_r$ can be rewritten in the form $\sigma_z \otimes (x_0', \ldots, x_r')$ for a unique choice of z and $(x_0', \ldots, x_r')$. An inverse to our map can then be given by the rule

$$(x_0, \ldots, x_r, n, z_1, \ldots, z_p) \longmapsto \sigma_z \otimes \sigma_z^{-1}(x_0, \ldots, x_r). \qquad \square$$

Proof of Theorem 3.4. This is proved by induction on $n \geqslant 0$. In the case $n = 0$, $C(0)$ consists only of a copy of R in degree $-1 = n - 1$, and its homology has the same description. Now take $n > 0$ and suppose that the claim holds for all smaller values of n. Then $C(n)$ has filtration $F_0 \subseteq \cdots \subseteq F_{n-1}$. The subcomplex $F_0 \cong C(C(n-1))$ is chain-contractible by Lemma 3.7. And the induction hypothesis tells us that each $C(n-p-1)$ has zero homology in all degrees up to and including $(n-p-1) - 2$, so that $R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1))$ has zero homology in degrees up to and including $(n-p-1) - 2 + (p+1) = n - 2$, so that by Lemma 3.8 the same is true of F_p/F_{p-1}. Since F_0 and all F_p/F_{p-1} have vanishing homology in the stated range, the same follows for $C(n)$ itself. $\qquad \square$

4 | $C(n)$ VIA THE GROUP RING AND COXETER GENERATORS

In the last section, we recalled the complex of injective words, and we gave a proof that its homology vanishes up to and including degree $n - 2$. These are key ingredients in the proof of homological stability for the symmetric groups, and in order to prove stability for the Iwahori–Hecke algebras we must extend them to that setting. However, the Iwahori–Hecke algebra $\mathcal{H}_n$ is obtained by taking the presentation of the group ring $R\mathfrak{S}_n$, where $\mathfrak{S}_n$ is regarded as the group ring of a Coxeter group, and deforming one of the relations, so that we must begin by recasting the complex of injective words in the same terms. In this section, we will rephrase the chain

complex $C(n)$, the filtration $F_0 \subseteq \cdots \subseteq F_{n-1} = C(n)$, and the isomorphisms $F_0 \cong C(C(n-1))$ and $F_p/F_{p-1} \cong R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1)$ in terms of the group ring of $\mathfrak{S}_n$, regarded as a Coxeter group. The result will be a new chain complex $C'(n)$, with a filtration $F'_0 \subseteq \cdots \subseteq F'_{n-1} = C(n)$, and isomorphisms $F'_0 \cong C(C'(n-1))$ and $F'_p/F'_{p-1} \cong R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n-p-1)$, all identified with the originals under an appropriate isomorphism, and all described only in terms of group rings and Coxeter generators. This process of 'algebra-ising' the existing construction is a key step in the work of this paper, serving as motivation and explanation for everything that follows, and allowing us to generalise directly to Iwahori–Hecke algebras.

The account that follows will make repeated use of a particular family of elements of $R\mathfrak{S}_n$, which we define here.

Definition 4.1 (The elements s_{ba}). Given $n \geqslant b \geqslant a \geqslant 1$, we define

$$s_{ba} = s_{b-1} s_{b-2} \cdots s_a.$$

Note that the indices always decrease from left to right, so that if $b = a$, then the product is empty and s_{ba} is the unit element. When necessary for clarity, we will separate the indices with a comma, that is, $s_{b,a} = s_{ba}$. Note that $s_{a+1,a} = s_a$.

Remark 4.2. The element s_{ba} can be described in cycle notation as

$$\begin{aligned}
s_{ba} &= s_{b-1} \cdots s_a \\
&= (b\ b-1)(b-1\ b-2) \cdots (a+1\ a) \\
&= (b\ b-1\ \ldots\ a),
\end{aligned}$$

and in two-line notation as

$$s_{ba} = \begin{pmatrix} 1 & \cdots & a-1 & a & a+1 & \cdots & b & b+1 & \cdots & n \\ 1 & \cdots & a-1 & b & a & \cdots & b-1 & b+1 & \cdots & n \end{pmatrix}.$$

Thus, s_{ba} is the permutation that decreases each of $a+1, \ldots, b$ by 1, and that sends a to b. The elements s_{ba} will be used to encode deletion of letters, and to move the letter n into particular positions required by the filtration. See Remarks 4.7 and 4.10.

Before continuing, it will be useful to explain how the complex of injective words $C(n)$ fits into the theory of augmented semi-simplicial objects. An *augmented semi-simplicial object* $A_\bullet$ in some category is defined to be a sequence of objects $A_{-1}, A_0, A_1, \ldots$ in that category, together with *face maps* $\partial^r_j : A_r \to A_{r-1}$ for $r \geqslant 0$ and $j = 0, \ldots, r$, which satisfy the relations $\partial^{r-1}_i \partial^r_j = \partial^{r-1}_{j-1} \partial^r_i$ for $r \geqslant 1$ and $0 \leqslant i < j \leqslant r$. If $A_\bullet$ is an augmented semi-simplicial object $A_\bullet$ in an *abelian* category, then we can associate to it a chain complex A_* given in degree r by A_r, and with differential $\partial^r = \sum_{j=0}^r (-1)^j \partial^r_j$.

Recall that $C(n)$ in degree r is the free R-module with basis the injective words on the set $\{1, \ldots, n\}$, and that the differential $\partial^r : C(n)_r \to C(n)_{r-1}$ sends $(a_0, \ldots, a_r)$ to the alternating sum $\sum_{j=0}^r (-1)^j (a_0, \ldots, \hat{a}_j, \ldots, a_r)$. Thus, we may write $\partial^r = \sum_{j=0}^r (-1)^j \partial^r_j$, where ∂^r_j is defined by $\partial^r_j(a_0, \ldots, a_r) = (a_0, \ldots, \hat{a}_j, \ldots, a_r)$. This makes $C(n)$ into the chain complex associated to the augmented semi-simplicial $R\mathfrak{S}_n$-module $C(n)_\bullet$ with face maps ∂^r_j.

Augmented semi-simplicial objects give us an organised way of splitting differentials into their component summands, and this will be useful throughout the rest of the paper. We will point out such semi-simplicial structures where they occur.

We are now in a position to define the complex $C'(n)$, which is our rephrasing of $C(n)$ in terms of the group algebra of $\mathfrak{S}_n$, regarded as a Coxeter group.

Definition 4.3 (The chain complex $C'(n)$). Let $C'(n)$ denote the chain complex of $R\mathfrak{S}_n$-modules, concentrated in degrees $r = -1, \ldots, n-1$, which is given in degree r by

$$C'(n)_r = R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-r-1}} \mathbb{1},$$

and whose differential $\partial^r : C'(n)_r \to C'(n)_{r-1}$ is defined by

$$\partial^r = \sum_{j=0}^{r} (-1)^j \partial_j^r, \qquad \partial_j^r(\sigma \otimes 1) = \sigma s_{n-r+j,n-r} \otimes 1.$$

Thus, $C'(n)$ is the chain complex associated to an augmented semi-simplicial $R\mathfrak{S}_n$-module with face maps ∂_j^r.

Remark 4.4. Definition 4.3 is similar to several that already appear in the literature, for example, [5, Definition 7.1; 18, Definition 34; 28, Definition 7.5].

Definition 4.5 (The isomorphism Θ). Define $\Theta : C'(n) \to C(n)$ in degree r by

$$\Theta_r(\sigma \otimes 1) = \sigma(n-r, \ldots, n) = (\sigma(n-r), \ldots, \sigma(n)).$$

Proposition 4.6. Θ *is an isomorphism of chain complexes of $R\mathfrak{S}_n$-modules.*

Proof. Recall that $C(n)_r$ is the R-linear span of all injective words $(a_0, \ldots, a_r)$ on the set $\{1, \ldots, n\}$. The group $\mathfrak{S}_n$ acts transitively on the set of injective words, and the stabiliser of the word $(n-r, n-r+1, \ldots, n)$ is exactly $\mathfrak{S}_{n-r-1} \subseteq \mathfrak{S}_n$. It follows that Θ_r is a well-defined isomorphism in each degree.

Next we must check that Θ_r commutes with the differentials. It is enough to check that it commutes with the ∂_j^r, that is, that $\partial_j^r \Theta_r = \Theta_{r-1} \partial_j^r$, and so we compute

$$\partial_j^r \Theta_r(\sigma \otimes 1) = \partial_j^r(\sigma(n-r), \ldots, \sigma(n))$$

$$= (\sigma(n-r), \ldots, \sigma(\widehat{n-r+j}), \ldots, \sigma(n))$$

$$= (\sigma(s_{n-r+j,n-r}(n-r+1)), \ldots, \sigma(s_{n-r+j,n-r}(n)))$$

$$= \Theta_{r-1}(\sigma s_{n-r+j,n-r} \otimes 1)$$

as required. Here recall from Remark 4.2 that $s_{n-r+j,n-r}$ is the cycle $(n-r+j, \ldots, n-r)$, which decreases each of $n-r+1, \ldots, n-r+j$ by 1 while preserving each of $n-r+j+1, \ldots, n$. (Note that the proof in fact demonstrates that Θ is an isomorphism of augmented semi-simplicial $R\mathfrak{S}_n$-modules.) $\square$

Remark 4.7 (The element $s_{n-r+j,n-r}$ and deleting letters). In $C(n)$ the map ∂_j^r is given by erasing the jth letter, $\partial_j^r(a_0, \dots, a_r) = (a_0, \dots, \hat{a}_j, \dots, a_r)$, whereas in $C'(n)$ the map ∂_j^r is given by $\partial_j^r(\sigma \otimes 1) = \sigma s_{n-r+j,n-r} \otimes 1$. How do these two correspond, and in particular, why does $s_{n-r+j,n-r}$ appear? Well, in the last expression we had $\sigma \otimes 1 \in R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-r-1}} \mathbb{1}$ and $\sigma s_{n-r+j,n-r} \otimes 1 \in R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-r}} \mathbb{1}$. Increasing the algebra over which the tensor product is taken, from $R\mathfrak{S}_{n-r-1}$ to $R\mathfrak{S}_{n-r}$, corresponds under Θ to erasing the *first* letter of a word. The role played by $s_{n-r+j,n-r}$, then, is to take the jth letter $\sigma(n-r+j)$, which is to be deleted, and move it down to start of the word, where it is indeed deleted thanks to the change in the tensor product.

Having defined $C'(n)$ and the isomorphism $\Theta \colon C'(n) \xrightarrow{\cong} C(n)$, we will now transport the filtration of $C(n)$ to $C'(n)$.

Definition 4.8. Let $F'_0 \subseteq \cdots \subseteq F'_{n-1} = C'(n)$ be the filtration defined by letting F'_p in degree r be the $R\mathfrak{S}_{n-1}$-span of all elements of the following two kinds.

(1) In the cases $r = -1, \dots, n-2$, the element

$$s_{n,n-r-1} \otimes 1.$$

(2) In the cases $r = 0, \dots, n-1$, the elements

$$s_{n,n-t} \otimes 1$$

for $t = 0, \dots, \min(r, p)$,

It may seem unnecessary to distinguish between the two kinds of element above, especially when $p \geqslant r$, given the similarity between $s_{n,n-r-1} \otimes 1$ and the $s_{n,n-t} \otimes 1$. But in fact it will be important to maintain the distinction, see Remark 4.11.

Proposition 4.9. *Under the isomorphism Θ, the filtration $F'_0 \subseteq \cdots \subseteq F'_{n-1} = C'(n)$ corresponds precisely to the filtration $F_0 \subseteq \cdots \subseteq F_{n-1} = C(n)$.*

Proof. Recall that $F_p \subseteq C(n)$ is given in degree r by the span of all injective words $(x_0, \dots, x_r)$ in which the letter n appears in the final $p+1$ positions, or not at all. This means that F_p consists not of $\mathfrak{S}_n$-modules but of $\mathfrak{S}_{n-1}$-modules. The injective words in F_p consist of various $\mathfrak{S}_{n-1}$-orbits, determined by the position of the letter n. These orbits come to us in two distinct kinds, the first containing just a single orbit, and the second containing a family of orbits.

(1) The single orbit of words in which n does not appear at all. This orbit is only present in cases $-1 \leqslant r \leqslant n-2$, for if $r = n-1$, then our word has length n and all letters must appear.
(2) The orbits in which n appears in position $r - t$, for $t = 0, \dots, \min(r, p)$. Such orbits are only present when $0 \leqslant r \leqslant n-1$, for when $r = -1$ the only word is the empty word, and n does not appear there.

We now choose representatives of these orbits as follows.

(1) In the cases $r = -1, \dots, n-2$, we represent the $\mathfrak{S}_{n-1}$-orbit in which n does not appear by the word

$$(n-r-1, \dots, n-1).$$

(2) In the cases $r = 0, \dots, n-1$, for $t = 0, \dots, \min(r, p)$ we represent the orbit in which n appears in position $r - t$ by the word

$$(n-r, \dots, n-t-1, n, n-t, \dots, n-1).$$

This is the word in which $n-r, \dots, n-1$ appear in order, with n inserted in position $r - t$.

So, F_p is the $R\mathfrak{S}_{n-1}$-span of the words just listed.

To show that $\Theta F'_p = F_p$, it will suffice to check that the generators of F'_p listed in Definition 4.8 are mapped to the generators listed above. And indeed, $s_{n, n-r-1}$ is the cycle $(n\ n-1\ \cdots\ n-r-1)$, so that

$$\Theta_r(s_{n, n-r-1} \otimes 1) = s_{n, n-r-1}(n-r, \dots, n) = (n-r-1, \dots, n-1)$$

is our representative of the first kind of orbit. And $s_{n, n-t}$ is the cycle $(n\ n-1\ \cdots\ n-t)$, so that

$$\Theta_r(s_{n, n-t} \otimes 1) = s_{n, n-t}(n-r, \dots, n) = (n-r, \dots, n-t-1, n, n-t, \dots, n-1)$$

is our representative of the second variety of orbit.   □

Remark 4.10 (The role of $s_{n, n-r-1}$ and $s_{n, n-t}$). In this proof $s_{n, n-r-1}$ was used to produce a word in which n does not appear, and it did this by moving n into a position where it would be deleted under the relevant tensor product, much as in Remark 4.7. Somewhat differently, $s_{n, n-t}$ was used to move the letter n into position $r - t$ in our word while leaving the order of the remaining letters unaffected.

Remark 4.11 (A distinction). In the last definition and proposition, we see two instances of the same important distinction, namely the distinction between the two kinds of generator of F'_p in Definition 4.8, and the distinction between the two kinds of orbit in the proof of Proposition 4.9. Despite the similarity between the elements $s_{n, n-r-1}$ and $s_{n, n-t}$, it will be necessary to keep careful track of the distinction between the two. For example, under the identification $C(C(n-1)) \cong F_0$, the distinction between the two kinds becomes the distinction between the two parts of the cone, namely, the unshifted 'base' part and the shifted 'cone' part.

We will now transport the identifications of the filtration quotients of $C(n)$ to the setting of $C'(n)$. Recall from Definition 3.6 that $C(-)$ and $\Sigma(-)$ denote the cone and suspension of chain complexes.

Definition 4.12. Define maps

$$\Phi \colon C(C'(n-1)) \longrightarrow F'_0$$

$$\Psi \colon R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n-p-1) \longrightarrow F'_p / F'_{p-1}$$

by

$$\Phi_r((\sigma \otimes 1), 0) = (\sigma s_{n, n-r-1} \otimes 1)$$

$$\Phi_r(0, (\tau \otimes 1)) = (\tau \otimes 1)$$

$$\Psi_{(p+1)+r}(\sigma \otimes (1 \otimes 1)) = \sigma s_{n, n-p} \otimes 1$$

for $\sigma, \tau \in R\mathfrak{S}_{n-1}$.

Proposition 4.13. *Under the isomorphism* Θ*, the isomorphisms*

$$C(C(n-1)) \xrightarrow{\cong} F_0 \quad \text{and} \quad R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1) \xrightarrow{\cong} F_p/F_{p-1}$$

of Lemmas 3.7 and 3.8 correspond to the isomorphisms Φ *and* Ψ*, respectively.*

Proof. First we recall that the isomorphism $C(C(n-1)) \xrightarrow{\cong} F_0$ of Lemma 3.7 is given in degree r by

$$((x_0, \ldots, x_r), 0) \longmapsto (x_0, \ldots, x_r), \qquad (0, (y_0, \ldots, y_{r-1})) \longmapsto (y_0, \ldots, y_{r-1}, n).$$

We must show that under this map, we have

$$(\Theta_r(\sigma \otimes 1), 0) \longmapsto \Theta_r \Phi_r(\sigma \otimes 1, 0), \qquad (0, \Theta_r(\tau \otimes 1)) \longmapsto \Theta_r \Phi_r(0, \tau \otimes 1),$$

for $\sigma, \tau \in R\mathfrak{S}_{n-1}$. We check this by computing directly:

$$
\begin{aligned}
(\Theta_r(\sigma \otimes 1), 0) &= ((\sigma(n-r-1), \ldots, \sigma(n-1)), 0) \\
&\longmapsto (\sigma(n-r-1), \ldots, \sigma(n-1)) \\
&= (\sigma s_{n,n-r-1}(n-r), \ldots, \sigma s_{n,n-r-1}(n)) \\
&= \Theta_r(\sigma s_{n,n-r-1} \otimes 1) \\
&= \Theta_r(\Phi_r(\sigma \otimes 1, 0))
\end{aligned}
$$

and

$$
\begin{aligned}
(0, \Theta_{r-1}(\tau \otimes 1)) &= (0, (\tau(n-r), \ldots, \tau(n-1))) \\
&\longmapsto (\tau(n-r), \ldots, \tau(n-1), n) \\
&= \Theta_r(\tau \otimes 1) \\
&= \Theta_r(\Phi_r(0, \tau \otimes 1)).
\end{aligned}
$$

Next, the isomorphism $R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C(n-p-1) \xrightarrow{\cong} F_p/F_{p-1}$ of Lemma 3.8 is given in degree $(p+1)+r$, for $-1 \leqslant r \leqslant n-p-2$, by

$$\sigma \otimes (x_0, \ldots, x_r) \longmapsto \sigma(x_0, \ldots, x_r, n, n-p, \ldots, n-1).$$

Note that in this degree $R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n-p-1)$ is

$$R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} (R\mathfrak{S}_{n-p-1} \otimes_{R\mathfrak{S}_{n-p-1-r}} \mathbb{1}) \cong R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1-r}} \mathbb{1}$$

so that a typical generator may be written in the form $\sigma \otimes (1 \otimes 1)$ for $\sigma \in \mathfrak{S}_{n-1}$. We must show that, for such an element, we have

$$\sigma \otimes \Theta_r(1 \otimes 1) \longmapsto \Theta_{(p+1)+r}(\sigma \otimes 1).$$

Now we compute:

$$\sigma \otimes \Theta_r(1 \otimes 1) = \sigma \otimes (n - p - 1 - r, \dots, n - p - 1)$$

$$\longmapsto \sigma(n - p - 1 - r, \dots, n - p - 1, n, n - p, \dots, n - 1)$$

$$= \sigma s_{n,n-p}(n - p - 1 - r, \dots, n)$$

$$= \Theta_{(p+1)+r}(\sigma \otimes 1) \qquad \square$$

5 | AN OVERVIEW OF $\mathcal{D}(n)$

At the end of the last section, we summarised how to replace the complex of injective words, its filtration, and the identification of the filtration quotients with a version

$$C'(n)$$

$$F'_0 \subseteq \cdots \subseteq F'_{n-1} = C'(n)$$

$$\Phi : C(C'(n-1)) \xrightarrow{\cong} F'_0$$

$$\Psi : R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n - p - 1) \xrightarrow{\cong} F'_p / F'_{p-1}$$

that was phrased entirely in terms of the group ring of $\mathfrak{S}_n$, where $\mathfrak{S}_n$ is regarded as a Coxeter group. This functions as a framework for adapting the entire situation to the setting of Iwahori–Hecke algebras, by making the substitutions:

$$
\begin{array}{ccc}
R\mathfrak{S}_m & \rightsquigarrow & \mathcal{H}_m \\
s_i & \rightsquigarrow & T_i \\
s_i s_j s_i = s_j s_i s_j & \rightsquigarrow & T_i T_j T_i = T_j T_i T_j \\
s_i^2 = 1 & \rightsquigarrow & T_i^2 = (q - 1)T_i + q
\end{array}
$$

This is indeed the framework we will follow, and we shall follow it quite closely, though various obstacles will arise.

- We will need to find entirely new proofs, within the Iwahori–Hecke setting, of the following: That the ∂_j^r are well-defined, that the differentials compose to zero, that the differentials respect the filtration, that the filtration is exhaustive, that the maps Φ and Ψ are well-defined chain maps, and that they are isomorphisms.
- The group ring $R\mathfrak{S}_m$ is obtained from $\mathcal{H}_m$ in the case $q = 1$. This means that we must expect any and all quantities, once extended to the Iwahori–Hecke setting, to possibly contain hitherto-invisible powers of q.
- The relation $s_i^2 = 1$ is deformed to $T_i^2 = (q - 1)T_i + q$, and this can introduce new difficulties that are not present in the case $q = 1$.

The plan is as follows.

- In Section 6, we define our new chain complex $\mathcal{D}(n)$, and verify that its differentials are well-defined and compose to 0.

- In Section 7, we define the filtration $F_0 \subseteq \cdots \subseteq F_{n-1} \subseteq \mathcal{D}(n)$, and we prove that it consists of subcomplexes, and exhausts $\mathcal{D}(n)$.
- In Section 8, we study the filtration quotients, proving the following theorem.

Theorem 5.1. *There are isomorphisms*

$$\Phi \colon C(\mathcal{D}(n-1)) \xrightarrow{\cong} F_0, \qquad \Psi \colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} \mathcal{D}(n-p-1) \xrightarrow{\cong} F_p/F_{p-1}.$$

Here C denotes the cone and Σ denotes suspension, as in Definition 3.6, while the $\mathcal{D}(-)$ and F_p will be defined in 6.3 and 7.1.

At each step, it will be evident that the new constructions in the Iwahori–Hecke setting reduce to those in symmetric group setting in the case $q = 1$. Altogether, these results will prove the following theorem, which follows exactly as in the proof of Theorem 3.4.

Theorem 5.2. $H_d(\mathcal{D}(n)) = 0$ *for* $d \leqslant n - 2$.

6 | THE CHAIN COMPLEX $\mathcal{D}(n)$

In this section, we introduce our new chain complex $\mathcal{D}(n)$, which will be the Iwahori–Hecke analogue of the complex of injective words, or more accurately, of the replacement $C'(n)$ that we obtained in Section 4.

To begin with, we introduce elements $T_{ba} \in \mathcal{H}_n$ that are analogous to the elements $s_{ba} \in R\mathfrak{S}_n$ which arose in our discussion of $C'(n)$ and its filtration, and we establish the properties that we will need in what follows.

Definition 6.1 (The elements T_{ba}). Given $n \geqslant b \geqslant a \geqslant 1$ we define

$$T_{ba} = T_{b-1}T_{b-2} \cdots T_a.$$

Note that the indices always decrease from left to right, so that when $b = a$ the product is empty and $T_{aa} = 1$. Note also that $T_{a+1\,a} = T_a$. Note that $T_{ba} = T_{s_{ba}}$, and in particular that it specialises to the element s_{ba} of Definition 4.1 in the case $q = 1$. Where appropriate we will separate the indices of T_{ba} by a comma for clarity, for example, $T_{b,a} = T_{ba}$.

We now need to establish some properties of the T_{ba} for later use. In the following proposition we will denote closed intervals of natural numbers using the usual notation for subsets of $\mathbb{R}$, that is, $[a, b] = \{a, a + 1, \dots, b\} \subseteq \mathbb{N}$.

Proposition 6.2.

(1) *If* $[a, b] \cap [c, d] = \emptyset$, *then* $T_{dc}T_{ba} = T_{ba}T_{dc}$.
(2) *If* $b \in [a, c]$, *then* $T_{ca} = T_{cb}T_{ba}$. *In particular,* $T_{ba} = T_{b-1}T_{b-1,a} = T_{b,a+1}T_a$.
(3) *For* $[c, d], [c - 1, d - 1] \subseteq [a, b]$ *we have* $T_{ba}T_{dc} = T_{d-1,c-1}T_{ba}$. *In particular, if* $k \in (a, b)$, *then* $T_{ba}T_k = T_{k-1}T_{ba}$.
(4) *For* $a \leqslant b < c \leqslant d$, *we have* $T_{db}T_{ca} = (q - 1) \cdot T_{c-1,b}T_{da} + q \cdot T_{c-1,a}T_{d,b+1}$.

Proof. (1) holds because all letters of T_{dc} commute with all letters of T_{ba}. (2) is immediate from the definition. Next, the braid relation $T_k T_{k-1} T_k = T_{k-1} T_k T_{k-1}$ can be written as $T_{k+1,k-1} T_k = T_{k-1} T_{k+1,k-1}$. From this it follows that if $k \in (a, b)$, then:

$$\begin{aligned}
T_{ba} T_k &= (T_{b,k+1} T_{k+1,k-1} T_{k-1,a}) T_k \\
&= T_{b,k+1} (T_{k+1,k-1} T_k) T_{k-1,a} \\
&= T_{b,k+1} (T_{k-1} T_{k+1,k-1}) T_{k-1,a} \\
&= T_{k-1} (T_{b,k+1} T_{k+1,k-1}) T_{k-1,a} \\
&= T_{k-1} T_{ba}.
\end{aligned}$$

Now (3) follows by repeated applications of the last property:

$$T_{ba} T_{dc} = T_{ba} T_{d-1} \cdots T_c = T_{d-2} \cdots T_{c-1} T_{ba} = T_{d-1,c-1} T_{ba}.$$

We now prove (4), which boils down to the identity $T_b^2 = (q-1) T_b + q$.

$$\begin{aligned}
T_{db} T_{ca} &= T_{db} T_{c,b+1} T_{b+1,a} \\
&= T_{c-1,b} T_{db} T_{b+1,a} \\
&= T_{c-1,b} (T_{d,b+1} T_b)(T_b T_{ba}) \\
&= T_{c-1,b} T_{d,b+1} (T_b^2) T_{ba} \\
&= T_{c-1,b} T_{d,b+1} ((q-1) T_b + q) T_{ba} \\
&= (q-1) \cdot T_{c-1,b} T_{d,b+1} T_b T_{ba} + q \cdot T_{c-1,b} T_{d,b+1} T_{ba} \\
&= (q-1) \cdot T_{c-1,b} T_{da} + q \cdot T_{c-1,b} T_{ba} T_{d,b+1} \\
&= (q-1) \cdot T_{c-1,b} T_{da} + q \cdot T_{c-1,a} T_{d,b+1}.
\end{aligned}$$

Here, we used (2) multiple times, we used (3) on the second line, and on the seventh line we used (1). $\qquad\square$

Note that properties (1), (2) and (3) above were consequences of the braid relation $T_k T_{k-1} T_k = T_{k-1} T_k T_{k-1}$, and so reflect properties of the braid group itself.

We are now in a position to write down our complex $\mathcal{D}(n)$. It is a direct extension of the complex $\mathcal{C}'(n)$ of Definition 4.3, which was isomorphic to $\mathcal{C}(n)$ via the isomorphism Θ of Proposition 4.6. One can see that the complex $\mathcal{D}(n)$ below reduces to $\mathcal{C}'(n)$ in the case $q = 1$.

Definition 6.3 (The complex $\mathcal{D}(n)$). Let $n \geqslant 0$. The complex $\mathcal{D}(n)$ is defined to be the chain complex of left $\mathcal{H}_n$-modules with $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ for r in the range $-1 \leqslant r \leqslant n - 1$, and with

$D(n)_r = 0$ for r outside that range, so that

$$D(n)_{n-1} = \mathcal{H}_n \otimes_{\mathcal{H}_0} \mathbb{1}$$
$$\vdots$$
$$D(n)_r \;\; = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$$
$$\vdots$$
$$D(n)_{-1} = \mathcal{H}_n \otimes_{\mathcal{H}_n} \mathbb{1}.$$

Observe that $D(-1) \cong \mathbb{1}$ and $D(n-1) \cong \mathcal{H}_n$. The differential

$$\partial^r : D(n)_r \longrightarrow D(n)_{r-1}$$

of $D(n)$ is defined by

$$\partial^r = \sum_{j=0}^{r}(-1)^j q^{-j}\partial_j^r$$

where $\partial_j^r : D(n)_r \to D(n)_{r-1}$ is given by

$$\partial_j^r(x \otimes y) = (x \cdot T_{n-r+j,n-r}) \otimes y.$$

The following lemma explains that $D(n)$ really is a chain complex, that is, that $\partial^{r-1} \circ \partial^r = 0$, by explaining how to regard $D(n)$ as an augmented semi-simplicial $\mathcal{H}_n$-module. As we will see, the powers of q appearing in the definition of ∂^r compensate for the fact that $\partial_i^{r-1}\partial_j^r$ and $\partial_{j-1}^{r-1}\partial_i^r$ are only equal up to powers of q.

Lemma 6.4. *The maps $\partial_j^r : D(n)_r \to D(n)_{r-1}$ satisfy the relation*

$$\partial_i^{r-1}\partial_j^r = q \cdot \partial_{j-1}^{r-1}\partial_i^r$$

for $0 \leqslant i < j \leqslant r$. Consequently, the maps $q^{-j}\partial_j^r$ satisfy the relation

$$(q^{-i}\partial_i^{r-1}) \circ (q^{-j}\partial_j^r) = (q^{-(j-1)}\partial_{j-1}^{r-1}) \circ (q^{-i}\partial_i^r)$$

for $0 \leqslant i < j \leqslant r$, making the $D(n)_r$ into an augmented semi-simplicial $\mathcal{H}_n$-module with face maps $q^{-j}\partial_j^r$, and $D(n)$ into the associated chain complex. In particular, we have $\partial^{r-1} \circ \partial^r = 0$ for all $r \geqslant 1$.

Proof. The second relation is simply a rewriting of the first. The lemma's final claim then follows from standard semi-simplicial machinery. To prove the first relation, we compute

$$\partial_i^{r-1}\partial_j^r(x \otimes y) = \partial_i^{r-1}(xT_{n-r+j,n-r} \otimes y)$$

$$= xT_{n-r+j,n-r}T_{n-r+i+1,n-r+1} \otimes y$$

$$= xT_{n-r+i,n-r}T_{n-r+j,n-r} \otimes y$$

$$= xT_{n-r+i,n-r}T_{n-r+j,n-r+1}T_{n-r} \otimes y$$

$$= xT_{n-r+i,n-r}T_{n-r+j,n-r+1} \otimes T_{n-r}y$$

$$= q \cdot x T_{n-r+i,n-r} T_{n-r+j,n-r+1} \otimes y$$

$$= q \cdot \partial_{j-1}^{r-1}(x T_{n-r+i,n-r} \otimes y)$$

$$= q \cdot \partial_{j-1}^{r-1} \partial_i^r (x \otimes y).$$

Here the third and fourth equalities used, respectively, parts (3) and (2) of Proposition 6.2. The fifth equality comes from the fact that the elements lie in $\mathcal{D}(n)_{r-2} = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r+1}} \mathbb{1}$, and $\mathcal{H}_{n-r+1}$ contains the element T_{n-r}. □

The lemma above demonstrates that the powers of q appearing in the formula $\partial^r = \sum_{j=0}^r (-1)^j q^j \partial_j^r$ are there precisely to counter the q that appears in the formula $\partial_i^{r-1} \partial_j^r = q \cdot \partial_{j-1}^{r-1} \partial_i^r$. The lemma also explains that this single instance of q arises because the T_i act on $\mathbb{1}$ as multiplication by q.

Remark 6.5. In Remark 4.4, we explained that the definition of $\mathcal{D}(n)$ is similar to several that already appear in the literature. The same is true of Lemma 6.4, and the same properties of the braid group that are relevant there also underpin the fact that the objects constructed in [5, Definition 7.1; 18, Definition 34; 28, Definition 7.5] are semi-simplicial. See, in particular, [28, Remark 2.8] and the paragraph that precedes it, and [5, Lemma 7.3].

Lemma 6.6. *The maps ∂_i^r are well-defined.*

Proof. We must show that if $\lambda \in \mathcal{H}_{n-r-1}$, then $\partial_i^r(x\lambda \otimes y)$ and $\partial_i^r(x \otimes \lambda y)$ agree, or in other words that $(x\lambda T_{n-r+i,n-r}) \otimes y = (x T_{n-r+i,n-r}) \otimes (\lambda y)$. This amounts to showing that $T_{n-r+i,n-r}$ commutes with the generators of $\mathcal{H}_{n-r-1}$. To see this, observe that $T_{n-r+i,n-r}$ is a word in $T_{n-r}, \ldots, T_{n-r+i-1}$, while $\mathcal{H}_{n-r-1}$ is generated by $T_1, \ldots, T_{n-r-2}$, and each of the former commutes with each of the latter. □

We conclude the section with the following lemma, which examines the effect of the maps ∂_j^r on certain specific elements of $\mathcal{D}(n)$. It will be used in the sections that follow.

Lemma 6.7. *Suppose that $0 \leqslant r \leqslant n-1$, $0 \leqslant j \leqslant r$, and $0 \leqslant t \leqslant r+1$. Then we have:*

$$\partial_j^r(T_{n,n-t} \otimes 1) = \begin{cases} T_{n-r+j,n-r}(T_{n,n-t} \otimes 1) & j \leqslant r-t-1 \\[2ex] T_{n,n-r} \otimes 1 & j = r-t \\[2ex] (q-1)T_{n-r+j-1,n-t}(T_{n,n-(r-1)-1} \otimes 1) \\ \quad + q T_{n-r+j-1,n-r}(T_{n,n-(t-1)} \otimes 1) & j \geqslant r-t+1,\ r \geqslant t \\[2ex] q T_{n-r+j-1,n-r-1}(T_{n,n-r} \otimes 1) & t = r+1 \end{cases}$$

In the first case, the resulting quantity is equal to $q T_{n-r+j,n-r}(T_{n,n-(t-1)} \otimes 1)$. Note that the first three cases exclude the possibility that $t = r+1$, while the final case holds for $t = r+1$ regardless of the value of j in the range $0 \leqslant j \leqslant r$.

Proof. We consider the three cases in turn.

Case 1: $j \leqslant r - t - 1$. Here we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = T_{n,n-t}T_{n-r+j,n-r} \otimes 1$$

$$= T_{n-r+j,n-r}(T_{n,n-t} \otimes 1),$$

where in the second equality we used the upper bound on j to show that $n - r + j \leqslant n - t - 1$, so that $T_{n,n-t}$ and $T_{n-r+j,n-r}$ commute. For the final sentence of the lemma, we have

$$T_{n-r+j,n-r}(T_{n,n-t} \otimes 1) = T_{n-r+j,n-r}(T_{n,n-t+1}T_{n-t} \otimes 1)$$

$$= T_{n-r+j,n-r}(T_{n,n-t+1} \otimes T_{n-t} \cdot 1)$$

$$= qT_{n-r+j,n-r}(T_{n,n-(t-1)} \otimes 1),$$

where in the second equality we used the fact that $t \leqslant r$ to conclude that T_{n-t} lies in $\mathcal{H}_{n-r}$ and can therefore be moved past the tensor product (we are working in $\mathcal{D}(n)_{r-1} = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r}} \mathbb{1}$).

Case 2: $j = r - t$. Here we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = T_{n,n-t}T_{n-r+j,n-r} \otimes 1 = T_{n,n-t}T_{n-t,n-r} \otimes 1 = T_{n,n-r} \otimes 1$$

as claimed.

Case 3: $j \geqslant r - t + 1$, $r \geqslant t$. Here we have $n - r \leqslant n - t < n - r + j \leqslant n$ so that by part (4) of Proposition 6.2, we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = T_{n,n-t}T_{n-r+j,n-r} \otimes 1$$

$$= (q-1)T_{n-r+j-1,n-t}T_{n,n-r} \otimes 1$$

$$+ qT_{n-r+j-1,n-r}T_{n,n-t+1} \otimes 1$$

$$= (q-1)T_{n-r+j-1,n-t}(T_{n,n-(r-1)-1} \otimes 1)$$

$$+ qT_{n-r+j-1,n-r}(T_{n,n-(t-1)} \otimes 1)$$

as required.

Case 4: $t = r + 1$. Then by part (3) of Proposition 6.2, we have

$$\partial_j^r(T_{n,n-r-1} \otimes 1) = T_{n,n-r-1}T_{n-r+j,n-r} \otimes 1$$

$$= T_{n-r+j-1,n-r-1}T_{n,n-r-1} \otimes 1$$

$$= T_{n-r+j-1,n-r-1}T_{n,n-r}T_{n-r-1} \otimes 1$$

$$= T_{n-r+j-1,n-r-1}T_{n,n-r} \otimes T_{n-r-1} \cdot 1$$

$$= qT_{n-r+j-1,n-r-1}(T_{n,n-r} \otimes 1).$$

(Again, we are working in $\mathcal{D}(n)_{r-1} = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r}} \mathbb{1}$.) $\qquad\square$

7 | THE FILTRATION OF $\mathcal{D}(n)$

In the last section, we constructed our complex $\mathcal{D}(n)$, and now we will generalise the filtration of the complex of injective words to $\mathcal{D}(n)$. In Definition 4.8 and Proposition 4.9, we reformulated that filtration as $F'_0 \subseteq \cdots \subseteq F'_{n-1} = C'(n)$ where F'_p is defined, in degree r, to be the $R\mathfrak{S}_{n-1}$-span of the elements $s_{n,n-r-1} \otimes 1$ (only in the cases $r = -1, \dots, n-2$) and the elements $s_{n,n-t} \otimes 1$ for $t = 0, \dots, \min(r, p)$ (only in the cases $r = 0, \dots, n-1$). We generalise this in the next definition. Again, in the case $q = 1$ we will precisely recover the definition of the F'_p.

Definition 7.1 (The filtration F_p of $\mathcal{D}(n)$). Let $0 \leqslant p \leqslant (n-1)$. Recall that $\mathcal{D}(n)$ in degree r is given by the tensor product $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$. Define $F_p \subseteq \mathcal{D}(n)$ to be the subcomplex of $\mathcal{D}(n)$ that in degree r is generated as an $\mathcal{H}_{n-1}$-module by all elements of the following two kinds.

(1) In the cases $r = -1, \dots, n-2$, the element

$$T_{n,n-r-1} \otimes 1.$$

(2) In the cases $r = 0, \dots, n-1$, the elements

$$T_{n,n-t} \otimes 1$$

for $t = 0, \dots, \min(r, p)$.

Thus, we obtain a filtration

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq \mathcal{D}(n)$$

by $\mathcal{H}_{n-1}$-submodules. Note that F_p is *not* an $\mathcal{H}_n$-submodule of $\mathcal{D}(n)$.

Despite the similarity between $T_{n,n-r-1} \otimes 1$ and $T_{n,n-t} \otimes 1$, it will be important to maintain the distinction between the two kinds of generator of F_p, for example, to later construct the isomorphism $F_0 \cong C(\mathcal{D}(n-1))$. See Remark 4.10, which discussed the role of the words $s_{n,n-r-1}$ and $s_{n,n-t}$ in the definition of $C'(n)$, and Remark 4.11, which discussed the importance of maintaining the distinction between $s_{n,n-r-1}$ and $s_{n,n-t}$.

We must show that the filtration exhausts $\mathcal{D}(n)$, and that it consists of subcomplexes of $\mathcal{D}(n)$.

Lemma 7.2. $F_{n-1} = \mathcal{D}(n)$.

Proof. In degree r, F_{n-1} is the $\mathcal{H}_{n-1}$-span of the elements of the form $T_{n,n-t} \otimes 1$ for $t = 0, \dots, r+1$. (The case $t = r+1$ is $T_{n,n-r-1} \otimes 1$.) We must show that these elements span $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$.

To begin we claim that $\mathcal{H}_n$ is the $\mathcal{H}_{n-1}$-span of the elements T_{nb} for $b \in \{1, \dots, n\}$. To see this write $S_m = \{s_1, \dots, s_{m-1}\}$ and consider the Coxeter system $(\mathfrak{S}_n, S_n)$. The subgroup of $\mathfrak{S}_n$ generated by S_{n-1} is precisely $\mathfrak{S}_{n-1}$. In Lemma 8.7 we will see that an element of $\mathfrak{S}_n$ is $(\mathfrak{S}_{n-1}, \emptyset)$-reduced if and only if it has the form s_{nb} for some $b = 1, \dots, n$. Thus, any element of $\mathfrak{S}_n$ has a representation of the form $x \cdot s_{nb}$ for some $b \in \{1, \dots, n\}$, where $x \in \mathfrak{S}_{n-1}$, and $\ell(x \cdot s_{nb}) = \ell(x) + \ell(s_{nb})$. Recall that $\mathcal{H}_n$ has basis given by the elements T_σ for $\sigma \in \mathfrak{S}_n$. Consequently, $\mathcal{H}_n$ is spanned by elements

of the form $T_{x \cdot s_{nb}} = T_x \cdot T_{s_{nb}} = T_x \cdot T_{nb}$ where $x \in \mathfrak{S}_{n-1}$ and $1 \leqslant b \leqslant n$. But in any such product the factor T_x lies in $\mathcal{H}_{n-1}$, and this proves our claim.

Now, in any degree r we have $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$, and by the last paragraph this is the $\mathcal{H}_{n-1}$ span of the elements $T_{nb} \otimes 1$ for $b \in \{1, \dots, n\}$. However, note that if $b \leqslant n - r - 1$, then $T_{nb} \otimes 1 = T_{n,n-r-1} T_{n-r-1,b} \otimes 1 = T_{n,n-r-1} \otimes T_{n-r-1,b} \cdot 1 = q^{n-r-1-b}(T_{n,n-r-1} \otimes 1)$, because $T_{n-r-1,b} = T_{n-r-2} \cdots T_b$ lies in $\mathcal{H}_{n-r-1}$. It then follows that $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ is the $\mathcal{H}_{n-1}$-span of the elements $T_{n,n-t} \otimes 1$ for $t = 0, \dots, r+1$, as required. $\qquad\square$

Lemma 7.3. F_p is a subcomplex of $\mathcal{D}(n)$.

Proof. Fix $0 \leqslant p \leqslant (n-1)$ and $0 \leqslant r \leqslant (n-1)$. Definition 7.1 lists the generators of F_p, as an $\mathcal{H}_{n-1}$-module, in each degree r. So, it is enough for us to fix $0 \leqslant j \leqslant r$ and then check that if we apply ∂_j^r to one of the generators of F_p in degree r, then the result is an $\mathcal{H}_{n-1}$-linear combination of the generators of F_p in degree $r - 1$.

First let us take $0 \leqslant r \leqslant (n-2)$ and consider the generator $T_{n,n-r-1} \otimes 1$ of F_p in degree r. Then by the final case of Lemma 6.7 we have

$$\partial_j^r(T_{n,n-r-1} \otimes 1) = q T_{n-r+j-1,n-r-1}(T_{n,n-r} \otimes 1)$$

$$= q T_{(n-1)-r+j,(n-1)-r}(T_{n,n-(r-1)-1} \otimes 1).$$

Now observe that $T_{n,n-(r-1)-1} \otimes 1$ is one of the generators of F_p in degree $r - 1$, while $T_{n-r+j-1,n-r-1} \in \mathcal{H}_{n-1}$, so that the final result lies in F_p as required.

Next, let us take $0 \leqslant r \leqslant (n-1)$ and $0 \leqslant t \leqslant \min(r, p)$, and consider the generator $T_{n,n-t} \otimes 1$ of F_p in degree r. We must check that each $\partial_j^r(T_{n,n-t} \otimes 1)$ lies in F_p in degree $r - 1$. We split this into three cases depending on the value of j, and in each case we apply the relevant part of Lemma 6.7.

Case 1: $j \leqslant r - t - 1$. Here we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = q T_{n-r+j,n-r}(T_{n,n-(t-1)} \otimes 1).$$

The upper bound on j shows that $T_{n-r+j,n-r} \in \mathcal{H}_{n-1}$, while $T_{n,n-(t-1)}$ is a generator of F_p in degree $r - 1$, so that we can conclude that the above expression lies in F_p.

Case 2: $j = r - t$. Here we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = T_{n,n-r} \otimes 1 = T_{n,n-(r-1)-1} \otimes 1$$

which is one of the generators of F_p in degree $r - 1$.

Case 3: $j \geqslant r - t + 1$. Here we have

$$\partial_j^r(T_{n,n-t} \otimes 1) = (q-1) T_{n-r+j-1,n-t}(T_{n,n-(r-1)-1} \otimes 1)$$

$$+ q T_{n-r+j-1,n-r}(T_{n,n-(t-1)} \otimes 1)$$

Observe that $T_{n,n-(r-1)-1} \otimes 1$ and $T_{n,n-(t-1)} \otimes 1$ are generators of F_p in degree $r - 1$, while $T_{n-r+j-1,n-t}$ and $T_{n-r+j-1,n-r}$ both lie in $\mathcal{H}_{n-1}$ since $j \leqslant r$, so that the above expression lies in F_p, as required. $\qquad\square$

8 | IDENTIFYING THE FILTRATION QUOTIENTS

By now we have defined our Iwahori–Hecke analogue $\mathcal{D}(n)$ of the complex of injective words, together with the corresponding filtration $F_0 \subseteq \cdots \subseteq F_{n-1} \subseteq \mathcal{D}(n)$. The next step is to understand the filtration quotients $F_0 = F_0/F_{-1}$ and F_p/F_{p-1} for $p \geqslant 1$, and prove Theorem 5.1.

In Definition 4.12 and Proposition 4.13, we reformulated the identification of the filtration quotients of the complex of injective words as the isomorphisms

$$\Phi \colon C(C'(n-1)) \xrightarrow{\cong} F_0', \qquad \Psi \colon R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n-p-1) \xrightarrow{\cong} F_p'/F_{p-1}'$$

defined by

$$\Phi_r((\sigma \otimes 1), 0) = (\sigma s_{n,n-r-1} \otimes 1)$$

$$\Phi_r(0, (\tau \otimes 1)) = (\tau \otimes 1)$$

$$\Psi_{(p+1)+r}(\sigma \otimes (1 \otimes 1)) = \sigma s_{n,n-p} \otimes 1$$

for $\sigma, \tau \in R\mathfrak{S}_{n-1}$. In this section, we will extend these isomorphisms to the setting of Iwahori–Hecke algebras. In Subsections 8.1 and 8.2, we will define our extensions of Φ and Ψ, and prove that they are well-defined chain maps. Then in Subsection 8.3, we will prove that the extended maps are isomorphisms, completing the proof of Theorem 5.1.

8.1 | Defining Φ

Recall that the map $\Phi \colon C(C'(n-1)) \to F_0'$ is defined by $\Phi_r((\sigma \otimes 1), 0) = (\sigma s_{n,n-r-1} \otimes 1)$ and $\Phi_r(0, (\tau \otimes 1)) = (\tau \otimes 1)$ for $\sigma, \tau \in \mathfrak{S}_{n-1}$. In this subsection, we will extend this map to the Iwahori–Hecke setting and prove that it is a well-defined chain map.

Let us recall from Definition 3.6 that the cone $C(\mathcal{D}(n-1))$ is given in degree r by

$$C(\mathcal{D}(n-1))_r = \mathcal{D}(n-1)_r \oplus \mathcal{D}(n-1)_{r-1} = (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-2}} \mathbb{1}) \oplus (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1})$$

and therefore has one or two generators as an $\mathcal{H}_{n-1}$-module, namely $(1 \otimes 1, 0)$ (only when $-1 \leqslant r \leqslant n-2$) and $(0, 1 \otimes 1)$ (only when $0 \leqslant r \leqslant n-1$). Similarly, F_0 in degree r is the $\mathcal{H}_{n-1}$-submodule of $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ with generators $T_{n,n-r-1} \otimes 1$ (only when $-1 \leqslant r \leqslant n-2$) and $T_{n,n} \otimes 1 = 1 \otimes 1$ (only when $0 \leqslant r \leqslant n-1$). Observe that in most degrees, both $C(\mathcal{D}(n-1))$ and F_0 have two generators, but that in the extreme cases $r = -1, n-1$ they each have just one generator.

Definition 8.1. Define

$$\Phi \colon C(\mathcal{D}(n-1)) \longrightarrow F_0$$

to be the $\mathcal{H}_{n-1}$-linear map defined on generators in degree r by

$$\Phi_r(1 \otimes 1, 0) = q^{-r} \cdot T_{n,n-r-1} \otimes 1$$

and

$$\Phi_r(0, 1 \otimes 1) = q \cdot 1 \otimes 1.$$

In other words, given

$$(x \otimes 1, y \otimes 1) \in C(D(n-1))_r = (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-2}} \mathbb{1}) \oplus (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1})$$

we have

$$\Phi_r(x \otimes 1, y \otimes 1) = q^{-r} x T_{n,n-r-1} \otimes 1 + qy \otimes 1 \in D(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}.$$

One sees immediately that this restricts to the original definition in the case $q = 1$.

Lemma 8.2. Φ *is well-defined.*

Proof. For $(x \otimes 1, y \otimes 1)$ as in the definition, we must check that $\Phi_r(x \otimes 1, y \otimes 1)$ depends only on $x \otimes 1$ and $y \otimes 1$, and not on x and y themselves. This means that we must check that

$$\Phi_r(x T_k \otimes 1, y T_{k'} \otimes 1) = \Phi_r(x \otimes (T_k \cdot 1), y \otimes (T_{k'} \cdot 1))$$

for $1 \leqslant k \leqslant n - r - 3$ and $1 \leqslant k' \leqslant n - r - 2$. This amounts to showing that $x T_k T_{n,n-r-1} \otimes 1 = x T_{n,n-r-1} \otimes (T_k \cdot 1)$ and $y T_{k'} \otimes 1 = y \otimes (T_{k'} \cdot 1)$ in $\mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$. The first of these holds since T_k commutes with $T_{n,n-r-1}$ and lies in $\mathcal{H}_{n-r-1}$, and the second is immediate since $T_{k'}$ lies in $\mathcal{H}_{n-r-1}$. $\qquad\square$

Lemma 8.3. Φ *is a chain map.*

Proof. The map Φ is by definition $\mathcal{H}_{n-1}$-linear. In each degree r, $C(D(n-1))$ has one or two generators as an $\mathcal{H}_{n-1}$-module, namely $(1 \otimes 1, 0)$ (only when $-1 \leqslant r \leqslant n - 2$) and $(0, 1 \otimes 1)$ (only when $0 \leqslant r \leqslant n - 1$). So, it is sufficient for us to verify that Φ commutes with the differentials when applied to these two generators. Recall from Definition 3.6 that the differential of $C(D(n-1))$ is given by

$$\partial^r(x \otimes 1, y \otimes 1) = (\partial^r(x \otimes 1) + (-1)^r(y \otimes 1), \partial^{r-1}(y \otimes 1)).$$

First we consider $\partial^r(\Phi_r(1 \otimes 1, 0))$. The final case of Lemma 6.7 gives

$$\partial^r_j(T_{n,n-r-1} \otimes 1) = q T_{(n-1)-r+j,(n-1)-r}(T_{n,n-(r-1)-1} \otimes 1),$$

so that we have

$$\partial^r(\Phi_r(1 \otimes 1, 0)) = \sum_{j=0}^{r} (-1)^j q^{-j} \partial^r_j \Phi_r(1 \otimes 1, 0)$$

$$= \sum_{j=0}^{r} (-1)^j q^{-j} q^{-r} \partial^r_j (T_{n,n-r-1} \otimes 1)$$

$$= \sum_{j=0}^{r} (-1)^j q^{-j} q^{-(r-1)} T_{(n-1)-r+j,(n-1)-r}(T_{n,n-(r-1)-1} \otimes 1)$$

$$= \sum_{j=0}^{r} (-1)^j q^{-j} \Phi_{r-1}(T_{(n-1)-r+j,(n-1)-r} \otimes 1, 0)$$

$$= \sum_{j=0}^{r}(-1)^j q^{-j}\Phi_{r-1}(\partial_j^r(1\otimes 1), 0)$$

$$= \Phi_{r-1}(\partial^r(1\otimes 1), 0)$$

$$= \Phi_{r-1}(\partial^r(1\otimes 1, 0))$$

as required. Next we consider $\partial^r(\Phi_r(0, 1\otimes 1))$:

$$\partial^r(\Phi_r(0, 1\otimes 1)) = \sum_{j=0}^{r}(-1)^j q^{-j}\partial_j^r\Phi_r(0, 1\otimes 1)$$

$$= \sum_{j=0}^{r}(-1)^j q^{-j}q\partial_j^r(1\otimes 1)$$

$$= \sum_{j=0}^{r}(-1)^j q^{-j}q(T_{n-r+j,n-r}\otimes 1).$$

The sum of all but the final term can be simplified as:

$$\sum_{j=0}^{r-1}(-1)^j q^{-j}q(T_{n-r+j,n-r}\otimes 1) = \sum_{j=0}^{r-1}(-1)^j q^{-j}\Phi_{r-1}(0, T_{n-r+j,n-r}\otimes 1)$$

$$= \sum_{j=0}^{r-1}(-1)^j q^{-j}\Phi_{r-1}(0, T_{(n-1)-(r-1)+j,(n-1)-(r-1)}\otimes 1)$$

$$= \sum_{j=0}^{r-1}(-1)^j q^{-j}\Phi_{r-1}(0, \partial_j^{r-1}(1\otimes 1))$$

$$= \Phi_{r-1}(0, \partial^{r-1}(1\otimes 1)).$$

And the final term can be simplified as:

$$(-1)^r q^{-r+1}(T_{n,n-r}\otimes 1) = (-1)^r q^{-(r-1)}(T_{n,n-(r-1)-1}\otimes 1)$$

$$= (-1)^r\Phi_{r-1}(1\otimes 1, 0).$$

So, altogether we have

$$\partial^r(\Phi_r(0, 1\otimes 1)) = \Phi_{r-1}(0, \partial^{r-1}(1\otimes 1)) + (-1)^r\Phi_{r-1}(1\otimes 1, 0)$$

$$= \Phi_{r-1}((-1)^r(1\otimes 1), \partial^{r-1}(1\otimes 1))$$

$$= \Phi_{r-1}(\partial^r(0, 1\otimes 1)).$$

This completes the proof. $\square$

8.2 | Defining Ψ

Recall that the isomorphism

$$\Psi \colon R\mathfrak{S}_{n-1} \otimes_{R\mathfrak{S}_{n-p-1}} \Sigma^{p+1} C'(n-p-1) \xrightarrow{\cong} F'_p/F'_{p-1}$$

is defined by $\Psi_r(\sigma \otimes (1 \otimes 1)) = \sigma s_{n,n-p} \otimes 1$ for $\sigma \in R\mathfrak{S}_{n-1}$, or in other words, the map which sends the generator $1 \otimes (1 \otimes 1)$ to $s_{n,n-p} \otimes 1$. In the present section, we will extend this map to the Iwahori–Hecke setting, that is, a map

$$\Psi \colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} D(n-p-1) \to F_p/F_{p-1},$$

and we will prove that it is a well-defined chain map. Before defining our map, let us elaborate on its domain and codomain. We work in a fixed degree $(p+1)+r$ for $-1 \leqslant r \leqslant n-p-1$.

- In degree $(p+1)+r$, the domain is

$$\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} D(n-p-1)_{(p+1)+r}$$

$$= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} D(n-p-1)_r$$

$$= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \left(\mathcal{H}_{n-p-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1} \right)$$

$$\cong \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1}$$

  and so it is generated as an $\mathcal{H}_{n-1}$-module by the element $1 \otimes (1 \otimes 1)$.
- In degree $(p+1)+r$, the codomain F_p/F_{p-1} is a subquotient of

$$D(n)_{(p+1)+r} = \mathcal{H}_n \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1},$$

  and it is generated as an $\mathcal{H}_{n-1}$-module by the element $T_{n,n-p} \otimes 1$. (Recall that F_p in this degree is the $\mathcal{H}_{n-1}$ submodule generated by $T_{n,n-[(p+1)+r]-1} \otimes 1, T_{n,n} \otimes 1, \dots, T_{n,n-p} \otimes 1$, and that F_{p-1} is the $\mathcal{H}_{n-1}$-submodule generated by all these except the last.) For brevity, we will write elements of the quotient F_p/F_{p-1} as elements of F_p, that is, we will not indicate additive cosets or equivalence classes.

Definition 8.4. Let $p \geqslant 1$. We define

$$\Psi \colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} D(n-p-1) \longrightarrow F_p/F_{p-1}$$

to be the $\mathcal{H}_{n-1}$-linear map that in degree $(p+1)+r$ is defined on the generator by

$$\Psi_{(p+1)+r}(1 \otimes (1 \otimes 1)) = T_{n,n-p} \otimes 1.$$

One can see immediately that this specialises to the original definition in the case $q = 1$.

Lemma 8.5. Ψ is well-defined.

Proof. Taking $p \geqslant 1$ as in the definition of Ψ, and working in degree $(p+1)+r$ for $r \geqslant -1$, the domain of $\Psi_{(p+1)+r}$ is

$$\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \left(\mathcal{H}_{n-p-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1} \right) \cong \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1}$$

and so we must show that for $1 \leqslant k \leqslant n - r - p - 3$, $\Psi_{(p+1)+r}$ sends the elements $T_k \otimes (1 \otimes 1)$ and $1 \otimes (1 \otimes (T_k \cdot 1))$ to the same element. Since

$$\Psi_{(p+1)+r}(T_k \otimes (1 \otimes 1)) = T_k \cdot \Psi_{(p+1)+r}(1 \otimes (1 \otimes 1)) = T_k \cdot (T_{n,n-p} \otimes 1) = (T_k T_{n,n-p}) \otimes 1$$

and

$$\Psi_{(p+1)+r}(1 \otimes (1 \otimes (T_k \cdot 1))) = q \cdot \Psi_{(p+1)+r}(1 \otimes (1 \otimes 1)) = q(T_{n,n-p} \otimes 1) = T_{n,n-p} \otimes (T_k \cdot 1),$$

it is enough to show that T_k commutes with $T_{n,n-p}$. And indeed, since $k \leqslant n - r - p - 3 \leqslant n - p - 2$, this follows immediately. $\qquad \square$

Lemma 8.6. Ψ *is a chain map.*

Proof. We take $p \geqslant 1$ as in the definition of Ψ, and we take $r \geqslant -1$. Consider the differentials going from degree $r + p + 1$ to $r + p$. In the domain of Ψ the differential in this degree is induced by the boundary map

$$\partial^r = \sum_{j=0}^{r} (-1)^j q^{-j} \partial_j^r$$

of $\mathcal{D}(n - p - 1)$. And in the codomain of Ψ the differential in degree $(p+1) + r$ is

$$\partial^{(p+1)+r} = \sum_{j=0}^{(p+1)+r} (-1)^j q^{-j} \partial_j^{(p+1)+r}.$$

Now, the codomain is $(F_p/F_{p-1})_{(p+1)+r}$, which is generated as an $\mathcal{H}_{n-1}$-module by the single element $T_{n,n-p} \otimes 1$. The first three cases of Lemma 6.7 give us the following:

$$\partial_j^{r+p+1}(T_{n,n-p} \otimes 1) = \begin{cases} T_{n-r-p-1+j,n-r-p-1}(T_{n,n-p} \otimes 1) & j \leqslant r \\[2mm] T_{n,n-(p+r)-1} \otimes 1 & j = r + 1 \\[2mm] (q-1)T_{n-r-p+j-2,n-p}(T_{n,n-(r+p)-1} \otimes 1) \\ \quad + qT_{n-r-p+j-2,n-r-p-1}(T_{n,n-(p-1)} \otimes 1) & j \geqslant r + 2. \end{cases}$$

Both of $T_{n,n-(p+r)-1} \otimes 1$ and $T_{n,n-(p-1)} \otimes 1$ are generators of the $\mathcal{H}_{n-1}$-module F_{p-1} in degree $p + r$, and $T_{n-r-p+j-2,n-p}$ and $T_{n-r-p+j-2,n-r-p-1}$ lie in $\mathcal{H}_{n-1}$. So, in the second and third cases the results vanish in the quotient F_p/F_{p-1}, and we obtain

$$\partial_j^{r+p+1}(T_{n,n-p} \otimes 1) = \begin{cases} T_{n-r-p-1+j,n-r-p-1}(T_{n,n-p} \otimes 1) & j \leqslant r \\[2mm] 0 & j \geqslant r + 1. \end{cases}$$

This means that in order to verify that $\Psi_{(p+1)+(r-1)} \circ \partial^r = \partial^{(p+1)+r} \circ \Psi_{(p+1)+r}$, it will suffice to check that $\Psi_{(p+1)+(r-1)} \circ \partial_j^r = \partial_j^{(p+1)+r} \circ \Psi_{(p+1)+r}$ for $0 \leqslant j \leqslant r$. And since the domain is generated as an $\mathcal{H}_{n-1}$ module by the element $1 \otimes (1 \otimes 1)$, it is enough to verify that

$$\partial_j^{r+p+1} \Psi_{(p+1)+r}(1 \otimes (1 \otimes 1)) = \Psi_{(p+1)+(r-1)} \partial_j^r (1 \otimes (1 \otimes 1)).$$

And indeed,

$$
\begin{aligned}
\partial_j^{r+p+1} \Psi_{(p+1)+r}(1 \otimes (1 \otimes 1)) &= \partial_j^{r+p+1}(T_{n,n-p} \otimes 1) \\
&= T_{n-r-p-1+j, n-r-p-1}(T_{n,n-p} \otimes 1) \\
&= \Psi_{(p+1)+(r-1)}(1 \otimes (T_{n-r-p-1+j, n-r-p-1} \otimes 1)) \\
&= \Psi_{(p+1)+(r-1)}(1 \otimes (T_{(n-p-1)-r+j, (n-p-1)-r} \otimes 1)) \\
&= \Psi_{(p+1)+(r-1)}(1 \otimes (\partial_j^r(1 \otimes 1))) \\
&= \Psi_{(p+1)+(r-1)} \partial_j^r (1 \otimes (1 \otimes 1)).
\end{aligned}
$$

Here the third equality used the fact that $T_{n-r-p-1+j, n-r-p-1}$ lies in $\mathcal{H}_{n-1}$ and Ψ is $\mathcal{H}_{n-1}$-linear. $\qquad\square$

8.3 | Φ and Ψ are isomorphisms

So far in this section, we have defined our chain maps $\Phi \colon C(\mathcal{D}(n-1)) \to F_0$ and $\Psi \colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} D(n-p-1) \to F_p/F_{p-1}$. Now we will prove Theorem 5.1, which states that Φ and Ψ are isomorphisms, and this will conclude the programme set out in Section 5.

To prove that Φ and Ψ are isomorphisms, we will obtain bases for their domains and codomains, and prove that they induce bijections between these bases. Now, Φ and Ψ are maps of $\mathcal{H}_{n-1}$-modules, whose domains are built out of tensor products of the form $\mathcal{H}_{n-1} \otimes_{\mathcal{H}_k} \mathbb{1}$, and we understand from Proposition 2.7 how to give a basis for $\mathcal{H}_{n-1} \otimes_{\mathcal{H}_k} \mathbb{1}$ as an $\mathcal{H}_{n-1}$-module using the distinguished coset representatives for $\mathfrak{S}_{n-1}/\mathfrak{S}_k$. However, the codomains of Φ and Ψ are built from tensor products of the form $\mathcal{H}_n \otimes_{\mathcal{H}_k} \mathbb{1}$, and in order to obtain a basis of this *as an $\mathcal{H}_{n-1}$-module*, we will need to study the distinguished double coset representatives of $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n / \mathfrak{S}_k$. The relevant theory was described in Subsections 2.2 and 2.3.

In what follows, we will consider the Coxeter system $(\mathfrak{S}_n, S_n)$ where $S_n = \{s_1, \dots, s_{n-1}\}$. We will similarly write $S_k = \{s_1, \dots, s_{k-1}\}$, so that the parabolic subgroup of $\mathfrak{S}_n$ generated by S_k is precisely $\mathfrak{S}_k$. Note that $S_0 = S_1 = \emptyset$, corresponding to the fact that $\mathfrak{S}_0$ and $\mathfrak{S}_1$ are both trivial groups. We are interested in understanding generators of $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$, which is to say, the distinguished representatives $X_{S_{n-r-1}}^{-1}$ for the left cosets $\mathfrak{S}_n / \mathfrak{S}_{n-r-1}$. In particular, in order to study the filtration $\{F_p\}$ of $\mathcal{D}(n)$, we consider $\mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ as an $\mathcal{H}_{n-1}$-module, so that we will need to compute the distinguished representatives $X_{S_{n-1} S_{n-r-1}}$ of the double cosets $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n / \mathfrak{S}_{n-r-1}$.

We will need to once again make use of the elements

$$s_{ba} = s_{b-1} \cdots s_a, \qquad 1 \leqslant a \leqslant b \leqslant n$$

that we introduced in Definition 4.1, and which coincide with the T_{ba} in the case $q = 1$. Recall from Remark 4.2 that s_{ba} is the cycle $s_{ba} = (b \cdots a)$ which decreases each of $a + 1, \dots, b$ by 1, and which sends a to b. We will in particular need the elements

$$s_{nj} = (n \cdots j)$$

which move j into position n while preserving the order of the remaining letters.

We begin by identifying the distinguished representatives for $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n$.

Lemma 8.7. *An element of $\mathfrak{S}_n$ is $(S_{n-1}, \emptyset)$-reduced if and only if it has the form s_{nj} for some j in the range $1 \leqslant j \leqslant n$. In other words,*

$$X_{S_{n-1}} = \{s_{nn}, \dots, s_{n1}\}.$$

Proof. First we show that the given elements are all $(S_{n-1}, \emptyset)$-reduced. To do so, we need only show that they have no reduced expression beginning with an element of S_{n-1}. But the given expressions for the elements clearly admit no M-moves, and are therefore reduced, and since they do not begin with elements of S_{n-1}, this makes clear that the elements are $(S_{n-1}, \emptyset)$-reduced.

Now let w be $(\mathfrak{S}_{n-1}, \emptyset)$-reduced. Then either $w = e$ or the first letter of w must be s_{n-1}, so let $n \geqslant j \geqslant 1$ be the smallest element such that w has a reduced expression beginning $s_{n-1} \cdots s_j$. (Thus, the case $j = n$ corresponds to $w = e$.) We will show that $w = s_{n-1} \cdots s_j$. Suppose not: then w has a reduced expression beginning $s_{n-1} \cdots s_j s_i$ for some $i = 1, \dots, (n-1)$. We cannot have $i = j$ for then the expression is not reduced. We cannot have $i = j - 1$ by minimality of j. We cannot have $i < j - 1$ for then $s_{n-1} \cdots s_j s_i = s_i s_{n-1} \cdots s_j$ and w is not $(S_{n-1}, \emptyset)$-reduced. And finally we cannot have $i > j$ because then $s_{n-1} \cdots s_j s_i = s_{i-1} s_{n-1} \cdots s_j$ is again not $\mathfrak{S}_{n-1}$-reduced on the left. So, there is no such i. □

Now we wish to study the distinguished representatives of the double cosets $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n / \mathfrak{S}_{n-r-1}$.

Lemma 8.8. *For r in the range $-1 \leqslant r \leqslant (n-1)$, a complete set of distinguished double coset representatives of $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n / \mathfrak{S}_{n-r-1}$ is given by*

$$X_{S_{n-1} S_{n-r-1}} = \begin{cases} \{s_{nn}, \dots, s_{n,n-r-1}\}, & -1 \leqslant r \leqslant (n-2) \\ \{s_{nn}, \dots, s_{n1}\}, & r = (n-1). \end{cases}$$

Note that the cases $r = n - 2$ and $r = n - 1$ produce the same value for $X_{S_{n-1} S_{n-r-1}}$. This is due to the fact that S_1 and S_0 are both empty, corresponding to the fact that $\mathfrak{S}_1$ and $\mathfrak{S}_0$ are both the trivial group.

Proof. As in the proof of Lemma 8.7, the given expressions admit no M-moves. Since none of them begin with a generator of $\mathfrak{S}_{n-1}$ or end with a generator of $\mathfrak{S}_{n-r-1}$, they are (S_{n-1}, S_{n-r-1})-reduced. They are therefore minimal double coset representatives [15, Proposition 2.1.7].

It remains to show that they are a complete set of minimal double coset representatives. But a minimal double coset representative is $(S_{n-1}, \emptyset)$-reduced, so by Lemma 8.7 it has the form $s_{n,j}$ for some j. And for $s_{n,j}$ to be $(\emptyset, S_{n-r-1})$-reduced, we must have $j \geqslant n - r - 1$, so that s_{nj} is one of the given elements. □

Now we will apply the Mackey formula for left-cosets in order to understand distinguished representatives of $\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$ in terms of $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$:

Lemma 8.9. *Let* $-1 \leqslant r \leqslant (n-1)$. *Then,*

$$(X_{S_{n-r-1}})^{-1} = \underbrace{(X_{S_{n-r-1}}^{S_{n-1}})^{-1}s_{n,n} \sqcup \cdots \sqcup (X_{S_{n-r-1}}^{S_{n-1}})^{-1}s_{n,n-r}}_{r+1 \text{ terms}} \sqcup \underbrace{(X_{S_{n-r-2}}^{S_{n-1}})^{-1}s_{n,n-r-1}}.$$

Here the initial union of $r+1$ terms is empty in the case $r = -1$, while the final term is omitted in the case $r = (n-1)$. Note the difference between $X_{S_{n-r-1}}^{S_{n-1}}$ in the first $r+1$ terms and $X_{S_{n-r-2}}^{S_{n-1}}$ in the final term.

Proof. We will use the Mackey decomposition of Subsection 2.2, taking $S = S_n$, $J = S_{n-r-1}$ and $K = S_{n-1}$, so that we obtain

$$X_{S_{n-r-1}}^{-1} = \bigsqcup_{d \in X_{S_{n-1}S_{n-r-1}}} \left(X_{S_{n-1}\cap {}^dS_{n-r-1}}^{S_{n-1}} \right)^{-1} \cdot d. \tag{1}$$

Recall from Subsection 2.2 that in the expression $X_{S_{n-1}\cap {}^dS_{n-r-1}}^{S_{n-1}}$ the superscript d denotes conjugation, so that

$${}^dS_{n-r-1} = \{{}^dx \mid x \in S_{n-r-1}\} = \{dxd^{-1} \mid x \in S_{n-r-1}\}.$$

So, we must work out $S_{n-1} \cap {}^dS_{n-r-1}$ for $d \in X_{S_{n-1}S_{n-r-1}}$. For $0 \leqslant r \leqslant (n-1)$ and $0 \leqslant k \leqslant r$ we have

$$S_{n-1} \cap {}^{s_{n,n-k}}S_{n-r-1} = S_{n-1} \cap {}^{s_{n-1}\cdots s_{n-k}}\{s_1, \ldots, s_{n-r-2}\}$$
$$= S_{n-1} \cap \{s_1, \ldots, s_{n-r-2}\}$$
$$= S_{n-1} \cap S_{n-r-1}$$
$$= S_{n-r-1}$$

since $s_{n-1} \cdots s_{n-k}$ commutes with $s_1, \ldots, s_{n-r-2}$. And for $-1 \leqslant r \leqslant (n-2)$ we have

$$S_{n-1} \cap {}^{s_{n,n-r-1}}S_{n-r-1} = S_{n-1} \cap {}^{s_{n-1}\cdots s_{n-r-1}}\{s_1, \ldots, s_{n-r-2}\}$$
$$= S_{n-1} \cap \{s_1, \ldots, s_{n-r-3}, s_{n-1} \cdots s_{n-r-2} \cdots s_{n-1}\}$$
$$= \{s_1, \ldots, s_{n-r-3}\}$$
$$= S_{n-r-2}$$

since in this case $s_{n-1} \cdots s_{n-r-1}$ commutes with $s_1, \ldots, s_{n-r-3}$, but not with s_{n-r-2}. The Mackey decomposition (1) now gives us the required result. $\square$

Lemma 8.10. *Let* $-1 \leqslant r \leqslant (n-1)$. *Then* $(F_0)_r$ *has basis*

$$\{T_x T_{n,n} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\} \cup \{T_x T_{n,n-r-1} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}\}$$

with the first term omitted for $r = -1$ *and the last term omitted for* $r = n - 1$. *And for* $p \geqslant 1$, $(F_p/F_{p-1})_r$ *has basis given by*

$$\{T_x T_{n,n-p} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\}$$

so long as $r \geqslant p$, *with the basis being empty when* $r < p$.

Proof. $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ has basis $\{T_x \otimes 1 \mid x \in (X_{S_{n-r-1}})^{-1}\}$. By Lemma 8.9, this is equal to

$$\left\{ T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} s_{n,n} \right\} \cup \cdots \cup \left\{ T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} s_{n,n-r} \right\}$$

$$\cup \left\{ T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1} s_{n,n-r-1} \right\}.$$

Here the initial union of $r + 1$ terms is empty in the case $r = -1$, while the final term is omitted in the case $r = (n - 1)$. Observe that if $T_y \otimes 1$ is an element of the first set in the union above, then $y = x s_{n,n-r-1}$ for some $x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}$. Since $\ell(x s_{n,n-r-1}) = \ell(x) + \ell(s_{n,n-r-1})$ as in Theorem 2.4 we then have $T_y = T_x T_{s_{n,n-r-1}} = T_x T_{n,n-r-1}$. Similarly for the other sets in the union, so that the basis is given by

$$\left\{ T_x T_{n,n} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} \right\} \cup \cdots \cup \left\{ T_x T_{n,n-r} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} \right\}$$

$$\cup \left\{ T_x T_{n,n-r-1} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1} \right\}.$$

Here the initial union of $r + 1$ terms is empty in the case $r = -1$, while the final term is omitted in the case $r = (n - 1)$. Comparing with the definition of the filtration, we see that $(F_p)_r$ has basis given by the union of the first $\min(p, r) + 1$ of these sets, together with the last (in the cases that it is present). In particular, $(F_0)_r$ has basis

$$\left\{ T_x T_{n,n} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} \right\} \cup \left\{ T_x T_{n,n-r-1} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1} \right\}$$

with the first term omitted for $r = -1$ and the last term omitted for $r = n - 1$. And for $p \geqslant 1$, $(F_p/F_{p-1})_r$ has basis given by

$$\left\{ T_x T_{n,n-p} \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} \right\}$$

so long as $r \geqslant p$, with the basis being empty when $r < p$. This completes the proof. $\square$

Proof of Theorem 5.1. The last lemma established bases for F_0 and F_p/F_{p-1} in each degree. As we will see below, since $C(\mathcal{D}(n-1))$ and $\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} \mathcal{D}(n-p-1)$ are defined using tensor products $\mathcal{H}_n \otimes_{\mathcal{H}_k} \mathbb{1}$, one can write down bases for these in each degree directly using Propo-

sition 2.7. Then proof that Φ and Ψ are isomorphisms amounts to verifying that they induce bijections between these bases, at least up to multiplication by powers of q.

The domain of Φ in degree r is

$$C(\mathcal{D}(n-1))_r = \mathcal{D}(n-1)_r \oplus \mathcal{D}(n-1)_{r-1}$$

$$= (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-2}} \mathbb{1}) \oplus (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1})$$

and therefore has basis

$$\left\{ (0, T_x \otimes 1) \mid x \in (X_{S_{n-r-1}}^{S_{n-1}})^{-1} \right\} \cup \left\{ (T_x \otimes 1, 0) \mid x \in (X_{S_{n-r-2}}^{S_{n-1}})^{-1} \right\},$$

with the first term omitted when $r = -1$ and the second omitted when $r = (n-1)$. By the definition of Φ we have

$$\Phi_r(T_x \otimes 1, 0) = q^{-r} \cdot T_x T_{n,n-r-1} \otimes 1$$

and

$$\Phi_r(0, T_x \otimes 1) = q \cdot T_x \otimes 1 = q \cdot T_x T_{n,n} \otimes 1$$

so that, up to scaling by powers of q, Φ_r restricts to a bijection between the basis of $C(\mathcal{D}(n-1))_r$ and the basis of $(F_0)_r$ given in Lemma 8.10, so that Φ is an isomorphism.

Similarly, the domain of Ψ in degree $(p+1) + r$ is

$$\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1} \mathcal{D}(n-p-1)_{(p+1)+r} = \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \mathcal{D}(n-p-1)_r$$

$$= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} (\mathcal{H}_{n-p-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1})$$

$$\cong \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1}$$

and therefore has basis

$$\left\{ T_x \otimes (1 \otimes 1) \mid x \in (X_{S_{n-p-r-2}}^{S_{n-1}})^{-1} \right\}.$$

The definition of Ψ gives

$$\Psi_{(p+1)+r}(T_x \otimes (1 \otimes 1)) = T_x T_{n,n-p} \otimes 1$$

so that Ψ induces a bijection between this basis, and the basis of $(F_p/F_{p-1})_r$ given in Lemma 8.10, and therefore Ψ is an isomorphism. $\qquad\square$

9 | OBTAINING THE SPECTRAL SEQUENCE

The proof of Theorem 1.1 will consist of an intricate but well-known spectral sequence argument. Since this argument is not normally carried out in the context of algebras, we take some time here to explain how the relevant spectral sequence is constructed in this context. The results are summarised in the following proposition, which we will prove over the course of the section.

Proposition 9.1. *There is a homological spectral sequence $\{E^r\}_{r \geqslant 1}$ with the following properties.*

- $E^1_{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $E^1_{s,t} = \operatorname{Tor}^{\mathcal{H}_{n-s-1}}_t(\mathbb{1}, \mathbb{1})$.
- $d^1 : E^1_{s,t} \to E^1_{s-1,t}$ *is the stabilisation map when s is even, and vanishes when s is odd.*
- $E^\infty_{s,t} = 0$ *in total degrees $s + t \leqslant (n-2)$.*

Similarly, there is a cohomological spectral sequence $\{E^r\}_{r \geqslant 1}$ with the following properties.

- $E^{s,t}_1$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $E^{s,t}_1 = \operatorname{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1})$
- $d_1 : E^{s,t}_1 \to E^{s-1,t}_1$ *is the stabilisation map when s is even, and vanishes when s is odd.*
- $E^{s,t}_\infty = 0$ *in total degrees $s + t \leqslant (n-2)$.*

Throughout the section, we will need to fix a projective resolution P_* of $\mathbb{1}$ as a right $\mathcal{H}_n$-module, and an injective resolution I^* of $\mathbb{1}$ as a left $\mathcal{H}_n$-module. It will be important to note that by restricting the module structure to any $\mathcal{H}_{n-s-1}$, P_* can be regarded as a projective resolution of $\mathbb{1}$ as a right $\mathcal{H}_{n-s-1}$-module, and I^* can be regarded as an injective resolution I^* of $\mathbb{1}$ as a left $\mathcal{H}_{n-s-1}$-module. This is possible because $\mathcal{H}_n$ is free as both a right and left $\mathcal{H}_{n-s-1}$-module (see Proposition 2.7), so that restriction preserves injectivity and projectivity.

We now work towards a proof of the proposition.

Lemma 9.2. *There is a homological spectral sequence $\{^{II}E^r\}$ with the following properties.*

- $^{II}E^1_{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $^{II}E^1_{s,t} = \operatorname{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s)$.
- $d^1 : {}^{II}E^1_{s,t} \to {}^{II}E^1_{s-1,t}$ *is induced by $\partial^s : \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$.*
- $^{II}E^\infty_{s,t} = 0$ *in total degrees $s + t \leqslant (n-2)$.*

Similarly, there is a cohomological spectral sequence $\{^{II}E_r\}$ with the following properties.

- $^{II}E^{s,t}_1$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $^{II}E^{s,t}_1 = \operatorname{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1})$.
- $d^1 : {}^{II}E^{s-1,t}_1 \to {}^{II}E^{s,t}_1$ *is induced by $\partial^s : \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$.*
- $^{II}E^{s,t}_\infty = 0$ *in total degrees $s + t \leqslant (n-2)$.*

Proof. We prove the homological version first. Consider the (homological) double complex $P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*$. This double complex gives two spectral sequences, $\{^I E^r\}$ and $\{^{II}E^r\}$, obtained by filtering the totalisation by rows or columns. In our case, the first spectral sequence has E^1 term

$$^I E^1_{s,t} = H_t(P_s \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*)$$

with $d^1 : {}^I E^1_{s,t} \to {}^I E^1_{s-1,t}$ induced by the differential $P_s \to P_{s-1}$. The second spectral sequence has E^1 term

$$^{II}E^1_{s,t} = H_t(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_s) = \operatorname{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s)$$

and differential $d^1 : {}^{II}E^1_{s,t} \to {}^{II}E^1_{s-1,t}$ induced by $\partial^s : \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$. Both spectral sequences converge to the homology of the total complex $\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*)$. See [42, section 5.6] for details.

The E^1-term of $\{{}^{I}E^r\}$ can be identified using the fact that P_s is projective, so that the functor $(P_s \otimes_{\mathcal{H}_n} -)$ commutes with homology, giving us

$$
{}^{I}E^1_{s,t} = H_t(P_s \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*) \cong P_s \otimes_{\mathcal{H}_n} H_t(\mathcal{D}(n)_*).
$$

But by Theorem 5.2, the right-hand side vanishes for $t \leqslant (n-2)$. In particular, ${}^{I}E^1_{*,*}$ vanishes in total degrees $\leqslant (n-2)$. The same therefore holds for all subsequent pages of the spectral sequence, so that $H_*(\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*))$ vanishes in degrees $* \leqslant (n-2)$. Since $\{{}^{II}E^r_{s,t}\}$ also converges to $H_*(\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*))$, we obtain the conclusion.

For the second case, we consider instead the (cohomological) double complex $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{D}(n)_*, I^*)$. One obtains analogous spectral sequences $\{{}^{I}E_r\}$ and $\{{}^{II}E_r\}$, which are analysed in the same way as before. In the analysis of $\{{}^{I}E_r\}$ one uses the fact that I^* is injective and therefore $\mathrm{Hom}_{\mathcal{H}_n}(-, I^s)$ commutes with homology to show that

$$
{}^{I}E^{s,t}_1 = H^t(\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{D}(n)_*, I^s)) \cong \mathrm{Hom}_{\mathcal{H}_n}(H_t(\mathcal{D}(n)), I^s). \qquad \square
$$

Having obtained the spectral sequences $\{{}^{II}E^r\}$ and $\{{}^{II}E_r\}$, we now proceed to turn them into the ones required by Proposition 9.1. Recall that

$$
{}^{II}E^1_{s,t} = \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s) = \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}),
$$

$$
{}^{II}E^{s,t}_1 = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1}) = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, \mathbb{1}).
$$

Recall from Proposition 2.7 that $\mathcal{H}_n$ is free as a right $\mathcal{H}_{n-s-1}$-module, so that in particular $\mathcal{H}_n$ is flat as a right $\mathcal{H}_{n-s-1}$-module, and there is therefore a change-of-rings isomorphism

$$
\Xi_* : \mathrm{Tor}^{\mathcal{H}_{n-s-1}}_t(\mathbb{1}, \mathbb{1}) \xrightarrow{\cong} \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}) = \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s) = E^1_{s,t}
$$

given on the level of chain complexes by the isomorphism

$$
\Xi : P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \xrightarrow{\cong} P_* \otimes_{\mathcal{H}_n} (\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}), \qquad \Xi(p \otimes \mathbb{1}) = p \otimes (\mathbb{1} \otimes \mathbb{1}),
$$

with inverse $\Xi^{-1}(p \otimes (h \otimes \mathbb{1})) = ph \otimes \mathbb{1}$. And there is a change-of-rings isomorphism

$$
\Xi^* : \mathrm{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\cong} \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, \mathbb{1}) = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1}) = E^{s,t}_1.
$$

given on the level of chain complexes by the isomorphism

$$
\Xi : \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1}, I^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, I^*) \qquad \Xi(f)(h \otimes \mathbb{1}) = h \cdot f(\mathbb{1})
$$

with inverse $\Xi^{-1}(g)(\mathbb{1}) = g(\mathbb{1} \otimes \mathbb{1})$.

We now define $\{E^r\}$ to be simply the spectral sequence $\{{}^{II}E^r\}$, but with the E^1-term modified by replacing ${}^{II}E^r_{s,t} = \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s)$ with $\mathrm{Tor}^{\mathcal{H}_{n-s-1}}_t(\mathbb{1}, \mathbb{1})$ using the map Ξ_*, and then taking the

induced differentials. And we define $\{E_r\}$ to be $\{^{II}E_r\}$ but with E_1-term modified by replacing $^{II}E_r^{s,t} = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1})$ with $\mathrm{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1})$ using the map Ξ^*, and again taking the induced differentials. Then $\{E^r\}$ and $\{E_r\}$ have all the properties required by Proposition 9.1, except for the description of the differentials.

Lemma 9.3. *The composites*

$$\Xi_*^{-1} \circ d^1 \circ \Xi_* : \ \mathrm{Tor}_*^{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Tor}_*^{\mathcal{H}_{n-s}}(\mathbb{1}, \mathbb{1}),$$

$$\Xi^{*-1} \circ d_1 \circ \Xi^* : \ \mathrm{Ext}^*_{\mathcal{H}_{n-s}}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Ext}^*_{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1})$$

vanish when s is odd, and are given by the relevant stabilisation map when s is even.

Proof. Recall that d^1 is induced by the differential of $\mathcal{D}(n)$, so that it is given on the level of chains by the map

$$\mathrm{id} \otimes \partial^r : \ P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_s \longrightarrow P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_{s-1},$$

$$p \otimes (h \otimes 1) \longmapsto \sum_{j=0}^{s} (-1)^j q^{-j} (p \otimes (h T_{n-s+j,n-s} \otimes 1)).$$

Thus, $\Xi_*^{-1} \circ d^1 \circ \Xi_*$ is given on the level of chains by the composite

$$P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \xrightarrow{\Xi} P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_s \xrightarrow{\mathrm{id} \otimes \partial_s} P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_{s-1} \xrightarrow{\Xi^{-1}} P_* \otimes_{\mathcal{H}_{n-s}} \mathbb{1},$$

whose effect on the element $p \otimes 1$ is

$$p \otimes 1 \mapsto p \otimes (1 \otimes 1)$$

$$\mapsto \sum_{j=0}^{s} (-1)^j q^{-j} (p \otimes (T_{n-s+j,n-s} \otimes 1))$$

$$\mapsto \sum_{j=0}^{s} (-1)^j q^{-j} (p T_{n-s+j,n-s} \otimes 1).$$

By Lemma 9.4, this composite is chain homotopic to the map

$$P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \longrightarrow P_* \otimes_{\mathcal{H}_{n-s}} \mathbb{1}$$

$$p \otimes 1 \mapsto \sum_{j=0}^{s} (-1)^j q^{-j} q^j (p \otimes 1) = \sum_{j=0}^{s} (-1)^j (p \otimes 1) = \begin{cases} p \otimes 1 & s \text{ even} \\ 0 & s \text{ odd} \end{cases}$$

and the result follows in the homological case. In the cohomological case the proof is similar, and we leave the details to the reader. $\qquad\square$

Lemma 9.4. *The map $P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \to P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}$, $p \otimes 1 \mapsto p T_{n-s+j,n-s} \otimes 1$ is chain homotopic to the map given by multiplication by q^j. Consequently, the map $P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \to P_* \otimes_{\mathcal{H}_{n-s}} \mathbb{1}$,*

$p \otimes 1 \mapsto p T_{n-s+j,n-s} \otimes 1$ *is chain homotopic to the reduction map* $P_* \otimes_{\mathcal{H}_{n-s-1}} 1 \to P_* \otimes_{\mathcal{H}_{n-s}} 1$
multiplied by q^j.

Analogously, the map $\mathrm{Hom}_{\mathcal{H}_{n-s-1}}(1, I^*) \to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(1, I^*)$ *defined by* $f \mapsto (1 \mapsto T_{n-s+j,n-s} \cdot f(1))$ *is chain homotopic to the map given by multiplication by* q^j. *Consequently, the map* $\mathrm{Hom}_{\mathcal{H}_{n-s}}(1, I^*) \to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(1, I^*)$, $f \mapsto (1 \mapsto T_{n-s+j,n-s} \cdot f(1))$ *is chain homotopic to the restriction map* $\mathrm{Hom}_{\mathcal{H}_{n-s}}(1, I^*) \to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(1, I^*)$ *multiplied by* q^j.

Proof. Let us begin with the homological case. The next paragraph will show that right-multiplication by $T_{n-s+j,n-s}$ on P_* is a map of $\mathcal{H}_{n-s-1}$-modules, and that its effect on homology is multiplication by q^j. Another chain map with the same properties is multiplication by q^j. But since P_* is a projective resolution by $\mathcal{H}_{n-s-1}$-modules, these two maps are chain homotopic.

To see that right-multiplication on P_* by $T_{n-s+j,n-s}$ is a map of $\mathcal{H}_{n-s-1}$-modules, observe that $T_{n-s+j,n-s} = T_{n-s+j-1} \cdots T_{n-s}$ commutes with $\mathcal{H}_{n-s-1}$. The effect of the map on homology is the map $1 \to 1$ that is again given by right multiplication by $T_{n-s+j,n-s}$, and since $T_{n-s+j,n-s}$ is a product of j factors T_k, this is multiplication by q^j.

The proof in the cohomological case is similar, left-multiplication by $T_{n-s+j,n-s}$ on I^* is a map of $\mathcal{H}_{n-s-1}$-modules given on cohomology by multiplication by q^j, and since I^* an injective resolution by $\mathcal{H}_{n-s-1}$-modules, this map is chain homotopic to multiplication by q^j. $\square$

10 | PROOF OF THEOREM 1.1

We are now able to prove Theorem 1.1, in the homological case. The following argument is essentially what appears in [35, section 5.2], or in the proof of [25, Theorem 2], except for changes in indexing and notation.

We prove that $\mathrm{Tor}_d^{\mathcal{H}_{n-1}}(1,1) \to \mathrm{Tor}_d^{\mathcal{H}_n}(1,1)$ is an isomorphism in degrees d satisfying $2d \leqslant n - 1$. We do this by induction on n. The cases $n = 1$ and $n = 2$ only make a statement about degree $d = 0$ and therefore hold trivially.

Suppose now that $n \geqslant 3$ and that the induction hypothesis holds for all smaller values of n. In the spectral sequence $\{E^r\}_{r \geqslant 1}$ of Proposition 9.1, the differential

$$d^1 : E^1_{s,t} \to E^1_{s-1,t}$$

is the stabilisation map

$$\mathrm{Tor}_t^{\mathcal{H}_{n-s-1}}(1,1) \to \mathrm{Tor}_t^{\mathcal{H}_{n-s}}(1,1)$$

when s is even, and vanishes when s is odd. In particular, our aim is to show that the maps $d^1 : E^1_{0,t} \to E^1_{-1,t}$ are isomorphisms for $2t \leqslant n - 1$, or in other words that $E^2_{0,t} = 0$ and $E^2_{-1,t} = 0$ for $2t \leqslant n - 1$.

Now let $u \geqslant 1$ and consider the differential

$$d^1 : E^1_{2u,t} \to E^1_{2u-1,t}.$$

Since this is the stabilisation map, our induction hypothesis states that it is an isomorphism for $2t \leqslant n - 2u - 1$. This gives the first property below. The second property follows easily from it.

(1) For $r \geqslant 2$, $E^r_{*,*}$ vanishes in bidegrees $(2u, t)$ and $(2u - 1, t)$ for $u \geqslant 1$, $2t \leqslant n - 2u - 1$.

(2) For $r \geqslant 2$, $E^r_{*,*}$ vanishes in bidegrees (s, t) satisfying $2t \leqslant n - s - 2$ and $s \geqslant 1$.

We now claim that for $r \geqslant 2$ there are no differentials d^r affecting terms in bidegrees $(-1, t)$ and $(0, t)$ for $2t \leqslant n - 1$. In the case of bidegrees $(-1, t)$, observe that a d^r landing there must originate in bidegree $(-1 + r, t - r + 1)$, but that $E^r_{-1+r, t-r+1} = 0$ by property (2) above. In the case of bidegrees $(0, t)$ and $r \geqslant 3$, the same reasoning applies. In the case of bidegrees $(0, t)$ and $r = 2$, the differential d^2 landing there must originate in $(2, t - 1)$, which is $(2u, t - 1)$ for $u = 1$, and $E^2_{2u, t-1} = 0$ by property (1) above.

It follows that if $2t \leqslant n - 1$, then $E^\infty_{-1, t} = E^2_{-1, t}$ and $E^\infty_{0, t} = E^2_{0, t}$. These terms lie in total degrees d satisfying $d \leqslant (n - 2)$ (this requires our assumption that $n \geqslant 3$). But by Proposition 9.1, we know that E^∞ vanishes in these total degrees, so that these terms vanish, and this completes the proof in the homological case.

The proof in the cohomological case is entirely similar.

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