# UNIFORM BOUNDEDNESS FOR ALGEBRAIC GROUPS AND LIE GROUPS 

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#### Abstract

Let $G$ be a semisimple linear algebraic group over a field $k$ and let $G^{+}(k)$ be the subgroup generated by the subgroups $R_{u}(Q)(k)$, where $Q$ ranges over all the minimal $k$-parabolic subgroups $Q$ of $G$. We prove that if $G^{+}(k)$ is bounded then it is uniformly bounded. Under extra assumptions we get explicit bounds for $\Delta\left(G^{+}(k)\right)$ : we prove that if $k$ is algebraically closed then $\Delta\left(G^{+}(k)\right) \leq 4 \operatorname{rank}(G)$, and if $G$ is split over $k$ then $\Delta\left(G^{+}(k)\right) \leq 28 \operatorname{rank}(G)$. We deduce some analogous results for real and complex semisimple Lie groups.


## 1. Introduction

In this paper we investigate the boundedness behaviour of a semisimple linear algebraic group $G$ over an infinite field $k$. (For definitions of boundedness and related notions, see Section 2.) If $k=\mathbb{R}$ then $G$ is a semisimple Lie group, and it is well known that $G$ is compact in the real topology if and only if it is anisotropic. The authors showed in [5, Thm. 1.2] that if $G$ is compact then $G$ is bounded but is not uniformly bounded; on the other hand, if $G$ has no simple compact factors then $G$ is uniformly bounded. Motivated by this, we make the following conjecture.

Conjecture 1.1. Let $G$ be a semisimple linear algebraic group over an infinite field $k$. Then $G^{+}(k)$ is uniformly bounded.

Here $G^{+}(k)$ denotes the subgroup of $G(k)$ generated by the subgroups $R_{u}(Q)(k)$, where $Q$ ranges over the minimal $k$-parabolic subgroups of $G$. If $k=\bar{k}$ then $G^{+}(k)=G(k)$, while if $G$ is anisotropic over $k$ then $G^{+}(k)=1$. If $G$ has no anisotropic $k$-simple factors then $G^{+}(k)$ is dense in $G$. Note that a finite group is clearly uniformly bounded,

[^0]so Conjecture 1.1 and the other results below all hold trivially for a semisimple linear algebraic group over a finite field $k$.

We make some steps towards proving the conjecture.
Theorem 1.2. Let $G$ be a semisimple linear algebraic group over an infinite field $k$, and suppose $G(k)=G^{+}(k)$. Then $G(k)$ is finitely normally generated. Moreover, if $G(k)$ is bounded then $G(k)$ is uniformly bounded.

We want to give explicit bounds for $\Delta(G)$ in terms of Lie-theoretic quantities such as $\operatorname{rank} G$ and $\operatorname{dim} G$. We can do this in some special cases. The first improves the bound $4 \operatorname{dim} G$ from [5, Thm. 4.3].

Theorem 1.3. Let $G$ be a semisimple linear algebraic group over an algebraically closed field $k$. Then $\Delta(G(k)) \leq 4 \operatorname{rank} G$.

Theorem 1.4. Let $G$ be a split semisimple linear algebraic group over an infinite field $k$. Then $\Delta\left(G^{+}(k)\right) \leq 28 \operatorname{rank} G$.

When $k=\mathbb{R}$, we get the following result.
Theorem 1.5. Let $H$ be a real semisimple linear algebraic group with no compact simple factors. Then $H$ is uniformly bounded. Moreover, if $H$ is split then $\Delta(H) \leq 28 \operatorname{rank} G$.

When $k=\mathbb{C}$, we get the following result.
Theorem 1.6. Let $H$ be a complex semisimple linear algebraic group. Then $H$ is uniformly bounded and $\Delta(H) \leq 4 \operatorname{rank} G$.

The idea of the proofs is as follows. First we prove Theorem 1.3 (Section 4); the new ingredient is that we work in the quotient variety $G / \operatorname{Inn}(G)$ rather than in $G$, which allows us to improve on the bound in [5, Thm. 4.3]. A key result underpinning our theorems for nonalgebraically closed $k$ is Proposition 5.5. We prove this in Section 5 and deduce Theorem 1.2. When $G$ is split we obtain Theorem 1.4 from Proposition 5.5 and the Bruhat decomposition; see Section 6. In Section 7 we prove Theorems 1.5 and 1.6.

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## 2. Boundedness And uniform boundedness

A conjugation-invariant norm on a group $H$ is a non-negative function $\|\|: H \rightarrow \mathbb{R}$ such that $\| \|$ is constant on conjugacy classes, $\|g\|=0$ if and only if $g=1$ and $\|g h\| \leq\|g\|+\|h\|$ for all $g, h \in H$. The diameter of $H$, denoted $\|H\|$, is $\sup _{g \in H}\|g\|$. A group $H$ is called bounded
if every conjugation-invariant norm has finite diameter. In [5] we introduced two stronger notions of boundedness. We briefly recall them now.

A subset $S \subseteq H$ is said to normally generate $H$ if the union of the conjugacy classes of its elements generates $H$. Thus, every element of $H$ can be written as a word in the conjugates of the elements of $S$ and their inverses. Given $g \in H$, the length of the shortest such word that is needed to express $g$ is the word norm of $g$ denoted $\|g\|_{S}$. It is a conjugation-invariant norm on $H$. The diameter of $H$ with respect to this word norm is denoted $\|H\|_{S}$. For every $n \geq 0$ we define

$$
B_{S}^{H}(n)=\left\{g \in H \mid\|g\|_{S} \leq n\right\}
$$

the ball of radius $n$ (of all elements that can be written as a product of $n$ or fewer conjugates of the elements of $S$ and their inverses). When there is no danger of confusion we simply write $B_{S}(n)$ (cf. Notation 3.1).

We will use the following result [5, Lem. 2.3] repeatedly: if $X, Y \subseteq H$ and $Y \subseteq B_{X}(m)$ then $B_{Y}(n) \subseteq B_{X}(m n)$.

We say that $H$ is finitely normally generated if it admits a finite normally generating set. In this case we define

$$
\begin{aligned}
& \Delta_{k}(H)=\sup \left\{\|H\|_{S}: S \text { normally generates } H \text { and }|S| \leq k\right\} \\
& \Delta(H)=\sup \left\{\|H\|_{S}: S \text { normally generates } H \text { and }|S|<\infty\right\}
\end{aligned}
$$

A finitely normally generated group $H$ is called strongly bounded if $\Delta_{k}(H)<\infty$ for all $k$. It is called uniformly bounded if $\Delta(H)<\infty$. Notice that $\Delta_{k}(H) \leq \Delta(H)$ for all $k \in \mathbb{N}$, so uniform boundedness implies strong boundedness. It follows from [5, Corollary 2.9] that strong boundedness implies boundedness.

## 3. LINEAR ALGEBRAIC GROUPS

We recall some material on linear algebraic groups; see [2] and [9] for further details. Below $k$ denotes an infinite field and $G$ denotes a semisimple linear algebraic $k$-group; we write $r$ for $\operatorname{rank} G$. We adopt the notation of [2]: we regard $G$ as a linear algebraic group over the algebraic closure $\bar{k}$ together with a choice of $k$-structure. We identify $G$ with its group of $\bar{k}$-points $G(\bar{k})$. If $H$ is any $k$-subgroup of $G$ then we denote by $H(k)$ the group of $k$-points of $H$. More generally, if $C$ is any subset of $G$-not necessarily closed or $k$-defined-then we set $C(k)=C \cap G(k)$. By [2, V.18.3 Cor.], $G(k)$ is dense in $G$.

Fix a maximal split $k$-torus $S$ of $G$. Let $L=C_{G}(S)$ and fix a $k$-parabolic subgroup $P$ such that $L$ is a Levi subgroup of $P$. Set $U=R_{u}(P)$. Then $P$ is a minimal $k$-parabolic subgroup of $G, L$ and $S$ are $k$-defined and $P, S$ are unique up to $G^{+}(k)$-conjugacy [9, 15.4.7].

Fix a maximal $k$-torus $T$ of $G$ such that $S \subseteq T$ and a (not necessarily $k$-defined) Borel subgroup $B$ of $G$ such that $T \subseteq B \subseteq P$.

Notation 3.1. If $X \subseteq G^{+}(k)$ then we write $B_{X}(n)$ for $B_{X}^{G^{+}(k)}(n)$.
Lemma 3.2. Let $O, O^{\prime}$ be nonempty open subsets of $G$. For any $g \in$ $G(k)$, there exist $h \in O(k)$ and $h^{\prime} \in O^{\prime}(k)$ such that $g=h h^{\prime}$.

Proof. Since $G$ is irreducible as a variety, $O^{-1} g \cap O^{\prime}$ is an open dense subset of $G$. Since $G(k)$ is dense in $G$, we can choose $h^{\prime} \in\left(O^{-1} g\right)(k) \cap$ $O^{\prime}(k)$. We can write $h^{\prime}=h^{-1} g$ for some $h \in O(k)$. This yields $g=h h^{\prime}$, as required.

For the rest of the section we assume that $G$ is split over $k$; then $S=T$ and $P=B$. Let $\Psi_{T}$ denote the set of roots of $G$ with respect to $T$. For $\alpha \in \Psi_{T}$, we denote by $U_{\alpha}$ the corresponding root group. Let $\alpha_{1}, \ldots, \alpha_{r}$ be the base for the set of positive roots associated to $B$. Note that $U_{\alpha_{i}}$ commutes with $U_{-\alpha_{j}}$ if $i \neq j$ because $\alpha_{i}-\alpha_{j}$ is not a root. Let $U^{-}$be the opposite unipotent subgroup to $U$ with respect to $T$. Let $G_{\alpha}=\left\langle U_{\alpha} \cup U_{-\alpha}\right\rangle$ for $\alpha \in \Psi_{T}$; then $G_{\alpha}$ is $k$-isomorphic to either $\mathrm{SL}_{2}$ or $\mathrm{PGL}_{2}$. Let $\alpha^{\vee}: \mathbb{G}_{m} \rightarrow G_{\alpha}$ be the coroot associated to $\alpha$. The image $T_{\alpha}$ of $\alpha^{\vee}$ is $G_{\alpha} \cap T$, and this is a maximal torus of $G_{\alpha}$.

We use the Bruhat decomposition for $G(k)$. We recall the necessary facts [2, Sec. V.14, Sec. V.21]. Fix a set $\widetilde{W} \subseteq N_{G}(T)(k)$ of representatives for the Weyl group; we denote by $n_{0} \in \widetilde{W}$ the representative corresponding to the longest element of $W$ (note that $n_{0}^{2} \in T(k)$ and $n_{0} U n_{0}^{-1}=U^{-}$). The Bruhat decomposition $G=\bigsqcup_{n \in \widetilde{W}} B n B$ for $G$ yields a decomposition $G(k)=\bigsqcup_{n \in \widetilde{W}} B(k) n B(k)$ for $G(k)$ [2, Thm. V.21.15]. The double coset $B n_{0} B$ is open and $k$-defined. The map $U \times B \rightarrow B n_{0} B,(u, b) \mapsto u n_{0} b$ is an isomorphism of varieties. Hence if $g \in B n_{0} B(k)$ then $g=u n_{0} b$ for unique $u \in U$ and $b \in B$, and it follows that $u \in U(k)$ and $b \in B(k)$. Likewise, multiplication gives $k$-isomorphisms of varieties

$$
U^{-} \times T \times U \rightarrow U^{-} \times B \rightarrow U^{-} B=n_{0}\left(B n_{0} B\right)
$$

so $U^{-} B$ is open and $\left(U^{-} B\right)(k)=U^{-}(k) B(k)=U^{-}(k) T(k) U(k)$.

## 4. The ALGEBRAICALLY CLOSED CASE

Throughout this section $k$ is algebraically closed. We need to recall some results from geometric invariant theory [7, Ch. 3]. Let $H$ be a reductive group acting on an affine variety $X$ over $\bar{k}$. We denote the orbit of $x \in X$ by $H \cdot x$ and the stabiliser of $x$ by $H_{x}$. One may form the affine quotient variety $X / H$. The points of $X / H$ correspond to the
closed $H$-orbits. We have a canonical projection $\pi_{X}: X \rightarrow X / H$. The closure $\overline{H \cdot x}$ of any orbit $H \cdot x$ contains a unique closed orbit $H \cdot y$, and we have $\pi_{X}(x)=\pi_{X}(y)$. If $C \subseteq X$ is closed and $H$-stable then $\pi_{X}(C)$ is closed.

In particular, $H$ acts on itself by inner automorphisms-that is, by conjugation - and the orbit $H \cdot h$ is the conjugacy class of $h$. We denote the quotient variety by $H / \operatorname{Inn}(H)$ and the canonical projection by $\pi_{H}: H \rightarrow H / \operatorname{Inn}(H)$. If $h=h_{s} h_{u}$ is the Jordan decomposition of $h$ then $H \cdot h_{s}$ is the unique closed orbit contained in $\overline{H \cdot h}$; so $H \cdot h$ is closed if and only if $h$ is semisimple, and $\pi_{H}(h)=\pi_{H}(1)$ if and only if $h$ is unipotent. Fix a maximal torus $T$ of $H$. The Weyl group $W$ acts on $T$ by conjugation. The inclusion of $T$ in $G$ gives rise to a map $\psi_{T}: T / W \rightarrow H / \operatorname{Inn}(H)$; it is well known that $\psi_{T}$ is an isomorphism of varieties.

Now assume $G$ is simply connected. We can write $G \cong G_{1} \times \cdots \times G_{m}$, where the $G_{i}$ are simple. Let $\nu_{i}: G \rightarrow G_{i}$ be the canonical projection. Set $r_{i}=\operatorname{rank}\left(G_{i}\right)$ for $1 \leq i \leq m$.

Lemma 4.1. Let $C$ be a closed $G$-stable subset of $G$ such that $C \nsubseteq$ $Z(G)$. Then there exist $g \in C$ and $x \in G$ such that $[g, x]$ is not unipotent.

Proof. Let $g \in C$ such that $g \notin Z(G)$. Note that $g_{s} \in C$ as $C$ is closed and conjugation-invariant. If $g_{s}$ is not central in $G$ then we can choose a maximal torus $T^{\prime}$ of $G$ such that $g_{s} \in T^{\prime}$; then $\left[g_{s}, x\right]$ is a nontrivial element of $T$ for some $x \in N_{G}(T)$, and we are done. So we can assume $g_{s}$ is central in $G$. Then $g_{u}$ is a nontrivial unipotent element of $G$. By [3, Lem. 3.2], $\overline{G \cdot g}$ contains an element of the form $g_{s} u$, where $1 \neq u$ belongs to some root group $U_{\alpha}$. Let $n \in N_{G_{\alpha}}\left(T_{\alpha}\right)$ represent the nontrivial element of the Weyl group $N_{G_{\alpha}}\left(T_{\alpha}\right) / T_{\alpha}$. Recall that $G_{\alpha}$ is isomorphic to $\mathrm{SL}_{2}$ or $\mathrm{PGL}_{2}$. Explicit calculations with $2 \times 2$ matrices (cf. the proof of Lemma 6.1 below) show that $[u, n]=\left[g_{s} u, n\right]$ is not unipotent. This completes the proof.

Suppose we are given $G$-conjugacy classes $C_{1}, \ldots, C_{m}$ of $G$ such that for each $i, \nu_{i}\left(C_{i}\right)$ is noncentral in $G_{i}$ (we do not insist that the $C_{i}$ are all distinct). Set $D_{i}=\left[C_{i}, G_{i}\right]$ and $\left.E_{i}=\overline{D_{i}}=\overline{\left[\overline{C_{i}}, G_{i}\right.}\right]$. Note that for each $i, D_{i}$ is conjugation-invariant and constructible, and $D_{i}^{-1}=D_{i}$; likewise, $E_{i}$ is conjugation-invariant and irreducible, and $E_{i}^{-1}=E_{i}$.

Proposition 4.2. Let $G$, etc., be as above, and set $X=D_{1} \cup \cdots \cup D_{m}$. Then $B_{X}(r)$ contains a constructible dense subset of $G$.

Proof. It suffices to prove that the constructible set $D_{i_{1}} \cdots D_{i_{r}}$ is dense in $G$ for some $i_{1}, \ldots, i_{r}$. It is enough to show that the constructible set $E_{i_{1}} \cdots E_{i_{r}}$ is a dense subset of $G$ for some $i_{1}, \ldots, i_{r}$.

Fix a maximal torus $T$ of $G$ and set $T_{i}=T \cap G_{i}$ for each $i$. Clearly it is enough to prove that $\left(E_{i}\right)^{r_{i}}$ is a dense subset of $G_{i}$ for each $i$. For notational convenience, we assume therefore that $m=1$ and $G=G_{1}$ is simple; then $T=T_{1}$. Set $C=C_{1}=\nu_{1}\left(C_{1}\right)$ and $E=E_{1}$; we prove that $E^{r}$ is a dense subset of $G$. By hypothesis, $E=\overline{[\bar{C}, G]}$ is an irreducible positive-dimensional subvariety of $G$. Set $A=E \cap T$. We claim that $A$ has an irreducible component $A^{\prime}$ such that $\operatorname{dim}\left(A^{\prime}\right)>0$.

Set $F=\pi_{G}(E)$; note that $F$ is closed and irreducible because $E$ is closed, conjugation-invariant and irreducible. Suppose $\operatorname{dim}(F)=0$. Since $1 \in E$, we have $F=\left\{\pi_{G}(1)\right\}$, which forces $E$ to consist of unipotent elements. But this is impossible by Lemma 4.1. We deduce that $\operatorname{dim}(F)>0$. Clearly $\pi_{G}(A) \subseteq F$. Conversely, given $g \in E$, write $g=g_{s} g_{u}$ (Jordan decomposition). Since $E$ is conjugation-invariant, we can, by conjugating $g$, assume without loss that $g_{s} \in T$. We have $g_{s} \in \overline{G \cdot g} \cap T \subseteq A$ and $\pi_{G}\left(g_{s}\right)=\pi_{G}(g)$. This shows that $F \subseteq \pi_{G}(A)$. Hence $F=\pi_{G}(A)$. Let $\pi_{W}: T \rightarrow T / W$ be the canonical projection. Now $F^{\prime}:=\psi_{T}^{-1}(F)$ is an irreducible closed positive-dimensional subset of $T / W$, with $A=\pi_{W}^{-1}\left(F^{\prime}\right)$. Since $W$ is finite, $\pi_{W}$ is a finite map and the fibres of $\pi_{W}$ are precisely the $W$-orbits. Hence the irreducible components of $A$ are permuted transitively by $W$, and each surjects onto $F^{\prime}$. Thus any irreducible component $A^{\prime}$ of $A$ has the desired properties.

Let $A_{1}, \ldots, A_{t}$ be the $W$-conjugates of $A^{\prime}$. The $A_{i}$ generate a nontrivial $W$-stable subtorus $S$ of $T$. Hence the subset $V$ of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\{\chi \in X(T) \mid \chi(S)=1\}$ is proper and $W$-stable. But $W$ acts absolutely irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, so $V=0$. This forces $S$ to be the whole of $T$. So the $A_{i}$ generate $T$. By the argument of [5, Sec. 5] or [4, 7.5 Prop.], there exist $i_{1}, \ldots, i_{r} \in\{1, \ldots, t\}$ and $\epsilon_{1}, \ldots, \epsilon_{r} \in\{ \pm 1\}$ such that $A_{i_{1}}^{\epsilon_{1}} \cdots A_{i_{r}}^{\epsilon_{r}}$ is a constructible dense subset of $T$. Hence $E^{r}$ contains a constructible dense subset of $T$, and we deduce that $E^{r}$ is a constructible dense subset of $G$. This completes the proof.

Proof of Theorem 1.3. We have $\Delta(\widetilde{G}) \leq \Delta(G)$ by [5, Lem. 2.16], where $\widetilde{G}$ is the simply connected cover of $G$. Hence there is no harm in assuming $G$ is simply connected. Let $X$ be a finite normal generating set for $G$. We can choose $x_{1}, \ldots, x_{m} \in X$ such that $\nu_{i}\left(x_{i}\right)$ is noncentral in $G_{i}$ for $1 \leq i \leq m$. Let $C_{i}=G \cdot x_{i} \subseteq X$, let $D_{i}=\left[C_{i}, G\right]$ and let $X^{\prime}=$ $D_{1} \cup \cdots \cup D_{m}$. By Proposition 4.2, $B_{X^{\prime}}(r)$ contains a dense constructible subset of $G$. Since $D_{i} \subseteq B_{C_{i}}(2)$ for each $i, B_{X}(2 r)$ contains a nonempty
open subset $U$ of $G$. Now $U^{2}=G$ by [2, I.1.3 Prop.], so $B_{X}(4 r) \supseteq$ $B_{X}(2 r) B_{X}(2 r) \supseteq U^{2}=G$. It follows that $\Delta(G) \leq 4 r$, as required.

## 5. The isotropic case

Now we consider the case of arbitrary semisimple $G$. There is no harm in replacing $G$ with the Zariski closure of $G^{+}(k)$, which is the product of the isotropic $k$-simple factors of $G$. Hence we assume in this section that $G^{+}(k)$ is dense in $G$.

We start by noting a corollary of Proposition 4.2. Let $X \subseteq G^{+}(k)$ such that $X$ is a finite normal generating set for $G$. By Proposition 4.2, there exist $i_{1}, \ldots, i_{r} \in\{1, \ldots, m\}$ such that the image of the map $f: G^{2 r} \rightarrow G$ defined by

$$
f\left(h_{1}, \ldots, h_{r}, g_{1}, \ldots, g_{r}\right)=\left(h_{1} x_{1} h_{1}^{-1} g_{1} x_{1}^{-1} g_{1}^{-1}\right) \cdots\left(h_{r} x_{r} h_{r}^{-1} g_{r} x_{r}^{-1} g_{r}^{-1}\right)
$$

contains a nonempty open subset $G^{\prime}$ of $G$. Now let $O$ be a nonempty open subset of $G$. Then $f^{-1}\left(G^{\prime} \cap O\right)$ is a nonempty open subset of $G^{2 r}$. But $G^{+}(k)$ is dense in $G$, so $G^{+}(k)^{2 r}$ is dense in $G^{2 r}$. It follows that $f\left(h_{1}, \ldots, h_{r}, g_{1}, \ldots, g_{r}\right) \in O$ for some $h_{1}, \ldots, h_{r}, g_{1}, \ldots, g_{r} \in G^{+}(k)^{2 r}$. We deduce that for any nonempty open subset $O$ of $G$,

$$
\begin{equation*}
B_{X}(2 r) \cap O \neq \emptyset . \tag{5.1}
\end{equation*}
$$

Remark 5.1. Let $C=\operatorname{im}(f)$, where $f$ is as above. It follows from Eqn. (5.1) and Lemma 3.2 that $C(k)^{2}=G(k)$. We cannot, however, conclude directly from this that $B_{X}(2 r)^{2}=G(k)$ : the problem is that although the map $f: G^{2 r} \rightarrow C$ is surjective on $\bar{k}$-points, it need not be surjective on $k$-points.
Lemma 5.2. There exists $t \in P(k)$ such that $t$ is regular semisimple.
Proof. Define $f: G \times P \rightarrow G$ by $f(g, h)=g h g^{-1}$. Then $f$ is surjective since every element of $G$ belongs to a Borel subgroup of $G$. Let $O$ be the set of regular semisimple elements of $G$, a nonempty open subset of $G$. By [2, Thm. 21.20(ii)], $P(k)$ is dense in $P$, and we know that $G(k)$ is dense in $G$, so $G(k) \times P(k)$ is dense in $G \times P$. It follows that there is a point $(g, t) \in(G(k) \times P(k)) \cap f^{-1}(O)$. Then $g t g^{-1}$ is regular semisimple, so $t \in P(k)$ is regular semisimple also.

Lemma 5.3. Let $t \in P(k)$ be regular semisimple. Then $U(k) \subseteq B_{t}(2)$.
Proof. Define $f: U \rightarrow U$ by $f(u)=u t u^{-1} t^{-1}$. The conjugacy class $U \cdot t$ is closed because orbits of unipotent groups are closed, so $\operatorname{im}(f)$ is a closed subvariety of $U$. Since $t$ is regular, it is easily checked that $f$ is injective and the derivative $d f_{u}$ is an isomorphism for each $u \in U$. It follows from Zariski's Main Theorem that $f$ is an isomorphism of
varieties. As $f$ is defined over $k, f$ gives a bijection from $U(k)$ to $U(k)$, and the result follows.

Lemma 5.4. Let $X$ be a finite normal generating subset for $G^{+}(k)$. Then $X$ normally generates $G$.

Proof. There exists $d \in \mathbb{N}$ such that $(G(k) \cdot X)^{d}=G^{+}(k)$. So the constructible set $(G \cdot X)^{d}$ contains $G^{+}(k)$ and is therefore dense in $G$. This implies that $(G \cdot X)^{d}$ contains a nonempty open subset of $G$, so $(G \cdot X)^{d}(G \cdot X)^{d}=G$. Hence $X$ is a finite normal generating set for $G$.

Proposition 5.5. Let $X$ be a finite subset of $G^{+}(k)$ such that $X$ normally generates $G$. Then $U(k) \subseteq B_{X}(8 r)$.

Proof. The big cell $P n_{0} P$ is open, so by Eqn. (5.1), we can choose $g \in B_{X}(2 r) \cap P n_{0} P$. We can write $g=x n_{0} x^{\prime}$ for some $x, x^{\prime} \in P(k)$. Since $B_{X}(2 r)$ is conjugation-invariant, there is no harm in replacing $g$ with $\left(x^{\prime}\right)^{-1} g x^{\prime}$, so we can assume that $x^{\prime}=1$ and $g=x n_{0}$. Let $C_{1}=$ $\left\{n_{0} x_{1} \mid x_{1} \in P, x n_{0}^{2} x_{1}\right.$ is regular semisimple $\}$. Let $O_{1}=P \cdot C_{1}=U \cdot C_{1}$; then $O_{1}$ is a constructible dense subset of $G$. By Eqn. (5.1), there exists $g \in B_{X}(2 r) \cap O_{1}$. We can write $g=u n_{0} x_{1} u^{-1}$ where $x n_{0}^{2} x_{1}$ is regular semisimple and $u \in U$. Since $g \in G(k)$, both $u$ and $x_{1} u^{-1}$ belong to $G(k)$. Hence $n_{0} x_{1} \in B_{X}(2 r) \cap C_{1}$. It follows that $t:=x n_{0}^{2} x_{1}$ is regular semisimple and belongs to $B_{X}(4 r)$. We have $t \in B_{X}(4 r) \cap P(k)$, so $U(k) \subseteq B_{t}(2) \subseteq B_{X}(8 r)$ by Lemma 5.3. This completes the proof.

Proof of Theorem 1.2. Suppose $G(k)=G^{+}(k)$. By Lemma 5.2, there exists $t \in P(k)$ such that $t$ is regular semisimple. By Lemma 5.3, $B_{t}(2)$ contains $U(k)$. Since $G(k)$ is generated by the $G(k)$-conjugates of $U(k)$, we deduce that $\{t\}$ normally generates $G(k)$. Hence $G(k)$ is finitely normally generated.

Now suppose further that $G(k)$ is bounded. Fix a finite normal generating set $Y$ for $G(k)$. Then $G(k)=B_{Y}(s)$ for some $s \in \mathbb{N}$ and $Y \subseteq B_{U(k)}(d)$ for some $d \in \mathbb{N}$. Let $X$ be any finite normal generating set for $G(k)$. Then $X$ is normally generates $G$ by Lemma 5.4. By Proposition 5.5, $U(k) \subseteq B_{X}(8 r)$. So

$$
G(k)=B_{Y}(s) \subseteq B_{U(k)}(s d) \subseteq B_{X}(8 r s d)
$$

This shows that $G(k)$ is uniformly bounded, as required.
Remark 5.6. The hypothesis that $G^{+}(k)=G(k)$ holds in many cases if $G$ is $k$-simple and simply connected-this is the content of the KneserTits conjecture, which holds, for example, when $k$ is a local field.

Example 5.7. It is well known that the abelianisation of $\mathrm{SO}_{3}(\mathbb{Q})$ is $\mathbb{Q}^{*} /\left(\mathbb{Q}^{*}\right)^{2}$, which is an infinitely generated abelian group. It follows that $\mathrm{SO}_{3}(\mathbb{Q})$ is not finitely normally generated. Note that $\mathrm{SO}_{3}^{+}(\mathbb{Q})=1$ since $\mathrm{SO}_{3}$ is anisotropic over $\mathbb{Q}$.

## 6. The split case

In this section we assume $G$ is split over $k$. If $G$ is simply connected then the Kneser-Tits Conjecture holds for $G$, so $G^{+}(k)=G(k)$ in this case.

Lemma 6.1. Suppose ( $*$ ) each $G_{\alpha}$ is isomorphic to $\mathrm{SL}_{2}$. Let $t_{i} \in T_{\alpha_{i}}(k)$ for $1 \leq i \leq r$ and set $t=t_{1} \cdots t_{r}$. There exist $u_{i}, w_{i} \in U_{\alpha_{i}}(k)$ and $v_{i}, x_{i} \in U_{-\alpha_{i}}(k)$ for $1 \leq i \leq r$ such that $t=x_{r} \cdots x_{1} u_{r} \cdots u_{1} v_{1} \cdots v_{r} w_{1} \cdots w_{r}$.

Proof. We use induction on $r$. The case $r=0$ is vacuous. Now consider the case $r=1$. Then $G \cong \mathrm{SL}_{2}$. For any $a, b, c, d \in k$ we have

$$
\left(\begin{array}{ll}
1 & a \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
b & 1
\end{array}\right)\left(\begin{array}{ll}
1 & c \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
d & 1
\end{array}\right)=\left(\begin{array}{cc}
1+a b & a \\
b & 1
\end{array}\right)\left(\begin{array}{cc}
1+c d & c \\
d & 1
\end{array}\right)=\binom{1+a b+c d+a b c d+a d}{b+b c d+d}
$$

Let $x \in k^{*}$. Set $a=-x, b=x^{-1}-1, c=1$ and $d=x-1$; then the matrix above becomes $\left(\begin{array}{cc}x & 0 \\ 0 & x^{-1}\end{array}\right)$. Hence the result holds when $r=1$.

Now suppose $r>1$. Let $H$ be the semisimple group with root system spanned by $\pm \alpha_{1}, \ldots, \pm \alpha_{r-1}$. Clearly condition $(*)$ holds for $H$. Let $s=t_{1} \cdots t_{r-1}$. By our induction hypothesis, there exist $u_{i}, w_{i} \in U_{\alpha_{i}}(k)$ and $v_{i}, x_{i} \in U_{-\alpha_{i}}(k)$ for $1 \leq i \leq r-1$ such that

$$
s=x_{r-1} \cdots x_{1} u_{r-1} \cdots u_{1} v_{1} \cdots v_{r-1} w_{1} \cdots w_{r-1}
$$

By the $\mathrm{SL}_{2}$ case considered above, $t_{r}=x_{r}^{\prime} u_{r}^{\prime} v_{r}^{\prime} w_{r}^{\prime}$ for some $u_{r}, w_{r} \in U_{\alpha_{r}}$ and some $v_{r}, x_{r} \in U_{-\alpha_{r}}$. Set $x_{r}=s x_{r}^{\prime} s^{-1}, u_{r}=s u_{r}^{\prime} s^{-1}, v_{r}=v_{r}^{\prime}$ and $w_{r}=w_{r}^{\prime}$. We have

$$
\begin{aligned}
& x_{r} x_{r-1} \cdots x_{1} u_{r} u_{r-1} \cdots u_{1} v_{1} \cdots v_{r-1} v_{r} w_{1} \cdots w_{r-1} w_{r} \\
= & x_{r} u_{r} x_{r-1} \cdots x_{1} u_{r-1} \cdots u_{1} v_{1} \cdots v_{r-1} w_{1} \cdots w_{r-1} v_{r} w_{r} \\
= & x_{r} u_{r} s v_{r} w_{r} \\
= & s x_{r}^{\prime} u_{r}^{\prime} v_{r}^{\prime} w_{r}^{\prime} \\
= & s t_{r}=t .
\end{aligned}
$$

The result follows by induction.
Proposition 6.2. Suppose $G$ is simply connected. Let $X \subseteq G^{+}(k)$ such that $U(k) \subseteq X$. Then $B_{X}(7)=G^{+}(k)$.

Proof. Since $G$ is simply connected, $(*)$ holds for $G$ and the map $\psi: \mathbb{G}_{m}^{r} \rightarrow T$ given by $\psi\left(a_{1}, \ldots, a_{r}\right)=\alpha_{1}^{\vee}\left(a_{1}\right) \cdots \alpha_{r}^{\vee}\left(a_{r}\right)$ is a $k$-isomorphism. It follows that $T(k)=T_{\alpha_{1}}(k) \cdots T_{\alpha_{r}}(k)$, so $T(k) \subseteq B_{X}(4)$ by Lemma 6.1.

Hence $U^{-}(k) B(k)=U^{-}(k) T(k) U(k) \subseteq B_{X}(1) B_{X}(4) B_{X}(1) \subseteq B_{X}(6)$. Now $G(k)=\left(U^{-} B\right)^{-1}(k)\left(U^{-} B\right)(k)$ by Lemma 3.2. But

$$
\begin{aligned}
& \left(U^{-} B\right)^{-1}(k)\left(U^{-} B\right)(k)=B(k) U^{-}(k) U^{-}(k) B(k)=U(k) T(k) U^{-}(k) T(k) U(k) \\
& \quad=U(k) U^{-}(k) T(k) U(k)=U(k) U^{-}(k) B(k) \subseteq B_{X}(1) B_{X}(6) \subseteq B_{X}(7)
\end{aligned}
$$

so we are done.
Proof of Theorem 1.4. Let $\widetilde{G}$ be the split form of the simply connected cover of $G$ and let $\psi: \widetilde{G} \rightarrow G$ be the canonical projection. Then $\psi$ is a $k$-defined central isogeny, so by [2, V.22.6 Thm.], the map $\widetilde{B} \mapsto \psi(\widetilde{B})$ gives a bijection between the set of $k$-Borel subgroups of $\widetilde{G}$ and the set of $k$-Borel subgroups of $G$; moreover, for each $\widetilde{B}, \psi$ gives rise to a $k$-isomorphism from $R_{u}(\widetilde{B})$ to $R_{u}(B)$ [2, Prop. V.22.4]. It follows that $\psi\left(\widetilde{G}^{+}(k)\right)=G^{+}(k)$. By [5, Lem. 2.16] we have $\Delta\left(G^{+}(k)\right) \leq$ $\Delta\left(\widetilde{G}^{+}(k)\right)$, so we can assume without loss that $G$ is simply connected. In particular, $G^{+}(k)=G(k)$.

Let $X$ be a finite normal generating set for $G(k)$. Then $X$ is a finite normal generating set for $G$ (Lemma 5.4), so by Eqn. (5.1) there exists $t \in B_{X}(2 r)$ such that $t$ is regular semisimple. We have $U(k) \subseteq B_{t}(2)$ by Lemma 5.3 and $G(k) \subseteq B_{U(k)}(7)$ by Proposition 6.2. So

$$
G(k) \subseteq B_{U(k)}(7) \subseteq B_{t}(14) \subseteq B_{X}(28 r)
$$

This shows that $\Delta(G(k)) \leq 28 r$, as required.
Example 6.3. (a) Let $G=\mathrm{SL}_{n}(k)$ where $n \geq 3$, let $g$ be the elementary matrix $E_{1 n}(1)$ and let $X=G(k) \cdot g$. By [5, Prop. 6.23], $X$ generates $G(k)$. One sees easily by direct computation that the centraliser $C_{G}(g)$ has dimension $n^{2}-2 n+1$, so $\operatorname{dim}(G \cdot g)=2 n-2$. A simple dimension-counting argument shows that if $t<\frac{1}{2} \operatorname{rank}(G)$ then $\overline{X^{t}}$ is a proper closed subvariety of $G$. Since $G(k)$ is dense in $G$, it follows that $\overline{X^{t}}$ does not contain $G(k)$, so $X(k)^{t}$ does not contain $G(k)$. We deduce that $\Delta(G(k)) \geq \frac{1}{2} \operatorname{rank}(G)$.
(b) The bounds in Theorems 1.3 and 1.4 are far from sharp. Aseeri has shown by direct calculation that $3 \leq \Delta\left(\mathrm{SL}_{3}(\mathbb{C})\right) \leq 6$ and that $\Delta\left(\mathrm{SL}_{2}(\mathbb{C})^{m}\right)=3 m$ and $\Delta\left(\mathrm{PGL}_{2}(\mathbb{C})^{m}\right)=2 m$ for every $k \in \mathbb{N}[1$, Thm. 8.0.2, Thm. 7.2.10, Thm. 7.2.6], whereas Theorem 1.3 yields the bounds $\Delta\left(\mathrm{SL}_{3}(\mathbb{C})\right) \leq 8$ and $\Delta\left(\mathrm{SL}_{2}(\mathbb{C})^{m}\right), \Delta\left(\mathrm{PGL}_{2}(\mathbb{C})^{m}\right) \leq 4 m$. Aseeri also showed that $3 \leq \Delta\left(\mathrm{SL}_{3}(\mathbb{R})\right) \leq 4$, whereas Theorem 1.4 gives $\Delta\left(\mathrm{SL}_{3}(\mathbb{R})\right) \leq 56$.

## 7. Semisimple Lie groups

Proof of Theorems 1.5 and 1.6. Let $H$ be a linear semisimple Lie group such that $H$ has no compact simple factors. By [6, Thm. III.2.13], there is a complex semisimple algebraic group $G$ defined over $\mathbb{R}$ such that $G^{+}(\mathbb{R})=H$. Now $Z(H)$ is finite, so $H$ is finitely normally generated and bounded by [5, Thm. 1.2]. It follows from Theorem 1.2 that $H$ is uniformly bounded. If $H$ is split then $G$ is split over $\mathbb{R}$, so $\Delta(H) \leq$ $28 \operatorname{rank}(H)$ by Theorem 1.4.

The argument for the complex case is similar: if $H$ is a semisimple linear complex Lie group then there is a semisimple complex algebraic group $G$ such that the complex Lie group associated to $G$ is $H$ (cf. [8, Ch. 4, Sec. 2, Problem 12], and $G$ is isomorphic to $H$. The result now follows from Theorem 1.3.

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