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UNIFORM BOUNDEDNESS FOR ALGEBRAIC GROUPS AND LIE GROUPS

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ABSTRACT. Let G be a semisimple linear algebraic group over a field k and let $G^+(k)$ be the subgroup generated by the subgroups $R_u(Q)(k)$, where Q ranges over all the minimal k-parabolic subgroups Q of G. We prove that if $G^+(k)$ is bounded then it is uniformly bounded. Under extra assumptions we get explicit bounds for $\Delta(G^+(k))$: we prove that if k is algebraically closed then $\Delta(G^+(k)) \leq 4 \operatorname{rank}(G)$, and if G is split over k then $\Delta(G^+(k)) \leq 28 \operatorname{rank}(G)$. We deduce some analogous results for real and complex semisimple Lie groups.

1. INTRODUCTION

In this paper we investigate the boundedness behaviour of a semisimple linear algebraic group G over an infinite field k. (For definitions of boundedness and related notions, see Section 2.) If $k = \mathbb{R}$ then G is a semisimple Lie group, and it is well known that G is compact in the real topology if and only if it is anisotropic. The authors showed in [5, Thm. 1.2] that if G is compact then G is bounded but is not uniformly bounded; on the other hand, if G has no simple compact factors then G is uniformly bounded. Motivated by this, we make the following conjecture.

Conjecture 1.1. Let G be a semisimple linear algebraic group over an infinite field k. Then $G^+(k)$ is uniformly bounded.

Here $G^+(k)$ denotes the subgroup of G(k) generated by the subgroups $R_u(Q)(k)$, where Q ranges over the minimal k-parabolic subgroups of G. If $k = \overline{k}$ then $G^+(k) = G(k)$, while if G is anisotropic over k then $G^+(k) = 1$. If G has no anisotropic k-simple factors then $G^+(k)$ is dense in G. Note that a finite group is clearly uniformly bounded,

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so Conjecture 1.1 and the other results below all hold trivially for a semisimple linear algebraic group over a finite field k.

We make some steps towards proving the conjecture.

Theorem 1.2. Let G be a semisimple linear algebraic group over an infinite field k, and suppose $G(k) = G^+(k)$. Then G(k) is finitely normally generated. Moreover, if G(k) is bounded then G(k) is uniformly bounded.

We want to give explicit bounds for $\Delta(G)$ in terms of Lie-theoretic quantities such as rank G and dim G. We can do this in some special cases. The first improves the bound $4 \dim G$ from [5, Thm. 4.3].

Theorem 1.3. Let G be a semisimple linear algebraic group over an algebraically closed field k. Then $\Delta(G(k)) \leq 4 \operatorname{rank} G$.

Theorem 1.4. Let G be a split semisimple linear algebraic group over an infinite field k. Then $\Delta(G^+(k)) \leq 28 \operatorname{rank} G$.

When $k = \mathbb{R}$, we get the following result.

Theorem 1.5. Let H be a real semisimple linear algebraic group with no compact simple factors. Then H is uniformly bounded. Moreover, if H is split then $\Delta(H) \leq 28 \operatorname{rank} G$.

When $k = \mathbb{C}$, we get the following result.

Theorem 1.6. Let H be a complex semisimple linear algebraic group. Then H is uniformly bounded and $\Delta(H) \leq 4 \operatorname{rank} G$.

The idea of the proofs is as follows. First we prove Theorem 1.3 (Section 4); the new ingredient is that we work in the quotient variety G/Inn(G) rather than in G, which allows us to improve on the bound in [5, Thm. 4.3]. A key result underpinning our theorems for non-algebraically closed k is Proposition 5.5. We prove this in Section 5 and deduce Theorem 1.2. When G is split we obtain Theorem 1.4 from Proposition 5.5 and the Bruhat decomposition; see Section 6. In Section 7 we prove Theorems 1.5 and 1.6.

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2. Boundedness and uniform boundedness

A conjugation-invariant norm on a group H is a non-negative function $|| ||: H \to \mathbb{R}$ such that || || is constant on conjugacy classes, ||g|| = 0if and only if g = 1 and $||gh|| \le ||g|| + ||h||$ for all $g, h \in H$. The diameter of H, denoted ||H||, is $\sup_{g \in H} ||g||$. A group H is called *bounded*

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if every conjugation-invariant norm has finite diameter. In [5] we introduced two stronger notions of boundedness. We briefly recall them now.

A subset $S \subseteq H$ is said to normally generate H if the union of the conjugacy classes of its elements generates H. Thus, every element of H can be written as a word in the conjugates of the elements of S and their inverses. Given $g \in H$, the length of the shortest such word that is needed to express g is the word norm of g denoted $||g||_S$. It is a conjugation-invariant norm on H. The diameter of H with respect to this word norm is denoted $||H||_S$. For every $n \ge 0$ we define

$$B_{S}^{H}(n) = \{ g \in H \mid ||g||_{S} \le n \},\$$

the ball of radius n (of all elements that can be written as a product of n or fewer conjugates of the elements of S and their inverses). When there is no danger of confusion we simply write $B_S(n)$ (cf. Notation 3.1).

We will use the following result [5, Lem. 2.3] repeatedly: if $X, Y \subseteq H$ and $Y \subseteq B_X(m)$ then $B_Y(n) \subseteq B_X(mn)$.

We say that H is *finitely normally generated* if it admits a finite normally generating set. In this case we define

$$\Delta_k(H) = \sup\{\|H\|_S : S \text{ normally generates } H \text{ and } |S| \le k\}$$

$$\Delta(H) = \sup\{\|H\|_S : S \text{ normally generates } H \text{ and } |S| < \infty\}.$$

A finitely normally generated group H is called *strongly bounded* if $\Delta_k(H) < \infty$ for all k. It is called *uniformly bounded* if $\Delta(H) < \infty$.

Notice that $\Delta_k(H) \leq \Delta(H)$ for all $k \in \mathbb{N}$, so uniform boundedness implies strong boundedness. It follows from [5, Corollary 2.9] that strong boundedness implies boundedness.

3. LINEAR ALGEBRAIC GROUPS

We recall some material on linear algebraic groups; see [2] and [9] for further details. Below k denotes an infinite field and G denotes a semisimple linear algebraic k-group; we write r for rank G. We adopt the notation of [2]: we regard G as a linear algebraic group over the algebraic closure \overline{k} together with a choice of k-structure. We identify G with its group of \overline{k} -points $G(\overline{k})$. If H is any k-subgroup of G then we denote by H(k) the group of k-points of H. More generally, if C is any subset of G—not necessarily closed or k-defined—then we set $C(k) = C \cap G(k)$. By [2, V.18.3 Cor.], G(k) is dense in G.

Fix a maximal split k-torus S of G. Let $L = C_G(S)$ and fix a k-parabolic subgroup P such that L is a Levi subgroup of P. Set $U = R_u(P)$. Then P is a minimal k-parabolic subgroup of G, L and S are k-defined and P, S are unique up to $G^+(k)$ -conjugacy [9, 15.4.7].

Fix a maximal k-torus T of G such that $S \subseteq T$ and a (not necessarily k-defined) Borel subgroup B of G such that $T \subseteq B \subseteq P$.

Notation 3.1. If $X \subseteq G^+(k)$ then we write $B_X(n)$ for $B_X^{G^+(k)}(n)$.

Lemma 3.2. Let O, O' be nonempty open subsets of G. For any $g \in G(k)$, there exist $h \in O(k)$ and $h' \in O'(k)$ such that g = hh'.

Proof. Since G is irreducible as a variety, $O^{-1}g \cap O'$ is an open dense subset of G. Since G(k) is dense in G, we can choose $h' \in (O^{-1}g)(k) \cap$ O'(k). We can write $h' = h^{-1}g$ for some $h \in O(k)$. This yields g = hh', as required.

For the rest of the section we assume that G is split over k; then S = T and P = B. Let Ψ_T denote the set of roots of G with respect to T. For $\alpha \in \Psi_T$, we denote by U_{α} the corresponding root group. Let $\alpha_1, \ldots, \alpha_r$ be the base for the set of positive roots associated to B. Note that U_{α_i} commutes with $U_{-\alpha_j}$ if $i \neq j$ because $\alpha_i - \alpha_j$ is not a root. Let U^- be the opposite unipotent subgroup to U with respect to T. Let $G_{\alpha} = \langle U_{\alpha} \cup U_{-\alpha} \rangle$ for $\alpha \in \Psi_T$; then G_{α} is k-isomorphic to either SL₂ or PGL₂. Let $\alpha^{\vee} \colon \mathbb{G}_m \to G_{\alpha}$ be the coroot associated to α . The image T_{α} of α^{\vee} is $G_{\alpha} \cap T$, and this is a maximal torus of G_{α} .

We use the Bruhat decomposition for G(k). We recall the necessary facts [2, Sec. V.14, Sec. V.21]. Fix a set $\widetilde{W} \subseteq N_G(T)(k)$ of representatives for the Weyl group; we denote by $n_0 \in \widetilde{W}$ the representative corresponding to the longest element of W (note that $n_0^2 \in T(k)$ and $n_0Un_0^{-1} = U^-$). The Bruhat decomposition $G = \bigsqcup_{n \in \widetilde{W}} BnB$ for Gyields a decomposition $G(k) = \bigsqcup_{n \in \widetilde{W}} B(k)nB(k)$ for G(k) [2, Thm. V.21.15]. The double coset Bn_0B is open and k-defined. The map $U \times B \to Bn_0B$, $(u, b) \mapsto un_0b$ is an isomorphism of varieties. Hence if $g \in Bn_0B(k)$ then $g = un_0b$ for unique $u \in U$ and $b \in B$, and it follows that $u \in U(k)$ and $b \in B(k)$. Likewise, multiplication gives k-isomorphisms of varieties

$$U^- \times T \times U \to U^- \times B \to U^- B = n_0(Bn_0B),$$

-B is open and $(U^-B)(k) = U^-(k)B(k) = U^-(k)T(k)U(k)$

4. The Algebraically closed case

Throughout this section k is algebraically closed. We need to recall some results from geometric invariant theory [7, Ch. 3]. Let H be a reductive group acting on an affine variety X over \overline{k} . We denote the orbit of $x \in X$ by $H \cdot x$ and the stabiliser of x by H_x . One may form the affine quotient variety X/H. The points of X/H correspond to the

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so U^{-}

closed *H*-orbits. We have a canonical projection $\pi_X \colon X \to X/H$. The closure $\overline{H \cdot x}$ of any orbit $H \cdot x$ contains a unique closed orbit $H \cdot y$, and we have $\pi_X(x) = \pi_X(y)$. If $C \subseteq X$ is closed and *H*-stable then $\pi_X(C)$ is closed.

In particular, H acts on itself by inner automorphisms—that is, by conjugation—and the orbit $H \cdot h$ is the conjugacy class of h. We denote the quotient variety by H/Inn(H) and the canonical projection by $\pi_H \colon H \to H/\text{Inn}(H)$. If $h = h_s h_u$ is the Jordan decomposition of h then $H \cdot h_s$ is the unique closed orbit contained in $\overline{H \cdot h}$; so $H \cdot h$ is closed if and only if h is semisimple, and $\pi_H(h) = \pi_H(1)$ if and only if h is unipotent. Fix a maximal torus T of H. The Weyl group Wacts on T by conjugation. The inclusion of T in G gives rise to a map $\psi_T \colon T/W \to H/\text{Inn}(H)$; it is well known that ψ_T is an isomorphism of varieties.

Now assume G is simply connected. We can write $G \cong G_1 \times \cdots \times G_m$, where the G_i are simple. Let $\nu_i \colon G \to G_i$ be the canonical projection. Set $r_i = \operatorname{rank}(G_i)$ for $1 \leq i \leq m$.

Lemma 4.1. Let C be a closed G-stable subset of G such that $C \not\subseteq Z(G)$. Then there exist $g \in C$ and $x \in G$ such that [g, x] is not unipotent.

Proof. Let $g \in C$ such that $g \notin Z(G)$. Note that $g_s \in C$ as C is closed and conjugation-invariant. If g_s is not central in G then we can choose a maximal torus T' of G such that $g_s \in T'$; then $[g_s, x]$ is a nontrivial element of T for some $x \in N_G(T)$, and we are done. So we can assume g_s is central in G. Then g_u is a nontrivial unipotent element of G. By [3, Lem. 3.2], $\overline{G \cdot g}$ contains an element of the form $g_s u$, where $1 \neq u$ belongs to some root group U_{α} . Let $n \in N_{G_{\alpha}}(T_{\alpha})$ represent the nontrivial element of the Weyl group $N_{G_{\alpha}}(T_{\alpha})/T_{\alpha}$. Recall that G_{α} is isomorphic to SL₂ or PGL₂. Explicit calculations with 2×2 matrices (cf. the proof of Lemma 6.1 below) show that $[u, n] = [g_s u, n]$ is not unipotent. This completes the proof.

Suppose we are given G-conjugacy classes C_1, \ldots, C_m of G such that for each $i, \nu_i(C_i)$ is noncentral in G_i (we do not insist that the C_i are all distinct). Set $D_i = [C_i, G_i]$ and $E_i = \overline{D_i} = [\overline{C_i}, G_i]$. Note that for each i, D_i is conjugation-invariant and constructible, and $D_i^{-1} = D_i$; likewise, E_i is conjugation-invariant and irreducible, and $E_i^{-1} = E_i$.

Proposition 4.2. Let G, etc., be as above, and set $X = D_1 \cup \cdots \cup D_m$. Then $B_X(r)$ contains a constructible dense subset of G. *Proof.* It suffices to prove that the constructible set $D_{i_1} \cdots D_{i_r}$ is dense in G for some i_1, \ldots, i_r . It is enough to show that the constructible set $E_{i_1} \cdots E_{i_r}$ is a dense subset of G for some i_1, \ldots, i_r .

Fix a maximal torus T of G and set $T_i = T \cap G_i$ for each i. Clearly it is enough to prove that $(E_i)^{r_i}$ is a dense subset of G_i for each i. For notational convenience, we assume therefore that m = 1 and $G = G_1$ is simple; then $T = T_1$. Set $C = C_1 = \nu_1(C_1)$ and $E = E_1$; we prove that E^r is a dense subset of G. By hypothesis, $E = [\overline{C}, \overline{G}]$ is an irreducible positive-dimensional subvariety of G. Set $A = E \cap T$. We claim that A has an irreducible component A' such that $\dim(A') > 0$.

Set $F = \pi_G(E)$; note that F is closed and irreducible because Eis closed, conjugation-invariant and irreducible. Suppose dim(F) = 0. Since $1 \in E$, we have $F = \{\pi_G(1)\}$, which forces E to consist of unipotent elements. But this is impossible by Lemma 4.1. We deduce that dim(F) > 0. Clearly $\pi_G(A) \subseteq F$. Conversely, given $g \in E$, write $g = g_s g_u$ (Jordan decomposition). Since E is conjugation-invariant, we can, by conjugating g, assume without loss that $g_s \in T$. We have $g_s \in \overline{G \cdot g} \cap T \subseteq A$ and $\pi_G(g_s) = \pi_G(g)$. This shows that $F \subseteq \pi_G(A)$. Hence $F = \pi_G(A)$. Let $\pi_W \colon T \to T/W$ be the canonical projection. Now $F' := \psi_T^{-1}(F)$ is an irreducible closed positive-dimensional subset of T/W, with $A = \pi_W^{-1}(F')$. Since W is finite, π_W is a finite map and the fibres of π_W are precisely the W-orbits. Hence the irreducible components of A are permuted transitively by W, and each surjects onto F'. Thus any irreducible component A' of A has the desired properties.

Let A_1, \ldots, A_t be the *W*-conjugates of *A'*. The A_i generate a nontrivial *W*-stable subtorus *S* of *T*. Hence the subset *V* of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\{\chi \in X(T) \mid \chi(S) = 1\}$ is proper and *W*-stable. But *W* acts absolutely irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, so V = 0. This forces *S* to be the whole of *T*. So the A_i generate *T*. By the argument of [5, Sec. 5] or [4, 7.5 Prop.], there exist $i_1, \ldots, i_r \in \{1, \ldots, t\}$ and $\epsilon_1, \ldots, \epsilon_r \in \{\pm 1\}$ such that $A_{i_1}^{\epsilon_1} \cdots A_{i_r}^{\epsilon_r}$ is a constructible dense subset of *T*. Hence E^r contains a constructible dense subset of *T*, and we deduce that E^r is a constructible dense subset of *G*. This completes the proof. \Box

Proof of Theorem 1.3. We have $\Delta(\tilde{G}) \leq \Delta(G)$ by [5, Lem. 2.16], where \tilde{G} is the simply connected cover of G. Hence there is no harm in assuming G is simply connected. Let X be a finite normal generating set for G. We can choose $x_1, \ldots, x_m \in X$ such that $\nu_i(x_i)$ is noncentral in G_i for $1 \leq i \leq m$. Let $C_i = G \cdot x_i \subseteq X$, let $D_i = [C_i, G]$ and let $X' = D_1 \cup \cdots \cup D_m$. By Proposition 4.2, $B_{X'}(r)$ contains a dense constructible subset of G. Since $D_i \subseteq B_{C_i}(2)$ for each $i, B_X(2r)$ contains a nonempty

open subset U of G. Now $U^2 = G$ by [2, I.1.3 Prop.], so $B_X(4r) \supseteq B_X(2r)B_X(2r) \supseteq U^2 = G$. It follows that $\Delta(G) \leq 4r$, as required. \Box

5. The isotropic case

Now we consider the case of arbitrary semisimple G. There is no harm in replacing G with the Zariski closure of $G^+(k)$, which is the product of the isotropic k-simple factors of G. Hence we assume in this section that $G^+(k)$ is dense in G.

We start by noting a corollary of Proposition 4.2. Let $X \subseteq G^+(k)$ such that X is a finite normal generating set for G. By Proposition 4.2, there exist $i_1, \ldots, i_r \in \{1, \ldots, m\}$ such that the image of the map $f: G^{2r} \to G$ defined by

$$f(h_1,\ldots,h_r,g_1,\ldots,g_r) = (h_1 x_1 h_1^{-1} g_1 x_1^{-1} g_1^{-1}) \cdots (h_r x_r h_r^{-1} g_r x_r^{-1} g_r^{-1})$$

contains a nonempty open subset G' of G. Now let O be a nonempty open subset of G. Then $f^{-1}(G' \cap O)$ is a nonempty open subset of G^{2r} . But $G^+(k)$ is dense in G, so $G^+(k)^{2r}$ is dense in G^{2r} . It follows that $f(h_1, \ldots, h_r, g_1, \ldots, g_r) \in O$ for some $h_1, \ldots, h_r, g_1, \ldots, g_r \in G^+(k)^{2r}$. We deduce that for any nonempty open subset O of G,

$$(5.1) B_X(2r) \cap O \neq \emptyset.$$

Remark 5.1. Let $C = \operatorname{im}(f)$, where f is as above. It follows from Eqn. (5.1) and Lemma 3.2 that $C(k)^2 = G(k)$. We cannot, however, conclude directly from this that $B_X(2r)^2 = G(k)$: the problem is that although the map $f: G^{2r} \to C$ is surjective on \overline{k} -points, it need not be surjective on k-points.

Lemma 5.2. There exists $t \in P(k)$ such that t is regular semisimple.

Proof. Define $f: G \times P \to G$ by $f(g, h) = ghg^{-1}$. Then f is surjective since every element of G belongs to a Borel subgroup of G. Let O be the set of regular semisimple elements of G, a nonempty open subset of G. By [2, Thm. 21.20(ii)], P(k) is dense in P, and we know that G(k) is dense in G, so $G(k) \times P(k)$ is dense in $G \times P$. It follows that there is a point $(g, t) \in (G(k) \times P(k)) \cap f^{-1}(O)$. Then gtg^{-1} is regular semisimple, so $t \in P(k)$ is regular semisimple also. \Box

Lemma 5.3. Let $t \in P(k)$ be regular semisimple. Then $U(k) \subseteq B_t(2)$.

Proof. Define $f: U \to U$ by $f(u) = utu^{-1}t^{-1}$. The conjugacy class $U \cdot t$ is closed because orbits of unipotent groups are closed, so im(f) is a closed subvariety of U. Since t is regular, it is easily checked that f is injective and the derivative df_u is an isomorphism for each $u \in U$. It follows from Zariski's Main Theorem that f is an isomorphism of

varieties. As f is defined over k, f gives a bijection from U(k) to U(k), and the result follows.

Lemma 5.4. Let X be a finite normal generating subset for $G^+(k)$. Then X normally generates G.

Proof. There exists $d \in \mathbb{N}$ such that $(G(k) \cdot X)^d = G^+(k)$. So the constructible set $(G \cdot X)^d$ contains $G^+(k)$ and is therefore dense in G. This implies that $(G \cdot X)^d$ contains a nonempty open subset of G, so $(G \cdot X)^d (G \cdot X)^d = G$. Hence X is a finite normal generating set for G.

Proposition 5.5. Let X be a finite subset of $G^+(k)$ such that X normally generates G. Then $U(k) \subseteq B_X(8r)$.

Proof. The big cell Pn_0P is open, so by Eqn. (5.1), we can choose $g \in B_X(2r) \cap Pn_0P$. We can write $g = xn_0x'$ for some $x, x' \in P(k)$. Since $B_X(2r)$ is conjugation-invariant, there is no harm in replacing g with $(x')^{-1}gx'$, so we can assume that x' = 1 and $g = xn_0$. Let $C_1 = \{n_0x_1 \mid x_1 \in P, xn_0^2x_1 \text{ is regular semisimple}\}$. Let $O_1 = P \cdot C_1 = U \cdot C_1$; then O_1 is a constructible dense subset of G. By Eqn. (5.1), there exists $g \in B_X(2r) \cap O_1$. We can write $g = un_0x_1u^{-1}$ where $xn_0^2x_1$ is regular semisimple and $u \in U$. Since $g \in G(k)$, both u and x_1u^{-1} belong to G(k). Hence $n_0x_1 \in B_X(2r) \cap C_1$. It follows that $t := xn_0^2x_1$ is regular semisimple and belongs to $B_X(4r)$. We have $t \in B_X(4r) \cap P(k)$, so $U(k) \subseteq B_t(2) \subseteq B_X(8r)$ by Lemma 5.3. This completes the proof. \Box

Proof of Theorem 1.2. Suppose $G(k) = G^+(k)$. By Lemma 5.2, there exists $t \in P(k)$ such that t is regular semisimple. By Lemma 5.3, $B_t(2)$ contains U(k). Since G(k) is generated by the G(k)-conjugates of U(k), we deduce that $\{t\}$ normally generates G(k). Hence G(k) is finitely normally generated.

Now suppose further that G(k) is bounded. Fix a finite normal generating set Y for G(k). Then $G(k) = B_Y(s)$ for some $s \in \mathbb{N}$ and $Y \subseteq B_{U(k)}(d)$ for some $d \in \mathbb{N}$. Let X be any finite normal generating set for G(k). Then X is normally generates G by Lemma 5.4. By Proposition 5.5, $U(k) \subseteq B_X(8r)$. So

$$G(k) = B_Y(s) \subseteq B_{U(k)}(sd) \subseteq B_X(8rsd).$$

This shows that G(k) is uniformly bounded, as required.

Remark 5.6. The hypothesis that $G^+(k) = G(k)$ holds in many cases if G is k-simple and simply connected—this is the content of the Kneser-Tits conjecture, which holds, for example, when k is a local field.

Example 5.7. It is well known that the abelianisation of $SO_3(\mathbb{Q})$ is $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, which is an infinitely generated abelian group. It follows that $SO_3(\mathbb{Q})$ is not finitely normally generated. Note that $SO_3^+(\mathbb{Q}) = 1$ since SO_3 is anisotropic over \mathbb{Q} .

6. The split case

In this section we assume G is split over k. If G is simply connected then the Kneser-Tits Conjecture holds for G, so $G^+(k) = G(k)$ in this case.

Lemma 6.1. Suppose (*) each G_{α} is isomorphic to SL_2 . Let $t_i \in T_{\alpha_i}(k)$ for $1 \leq i \leq r$ and set $t = t_1 \cdots t_r$. There exist $u_i, w_i \in U_{\alpha_i}(k)$ and $v_i, x_i \in U_{-\alpha_i}(k)$ for $1 \leq i \leq r$ such that $t = x_r \cdots x_1 u_r \cdots u_1 v_1 \cdots v_r w_1 \cdots w_r$.

Proof. We use induction on r. The case r = 0 is vacuous. Now consider the case r = 1. Then $G \cong SL_2$. For any $a, b, c, d \in k$ we have

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix} \begin{pmatrix} 1+cd & c \\ d & 1 \end{pmatrix} = \begin{pmatrix} 1+ab+cd+abcd+ad & c+abc+a \\ b+bcd+d & bc+1 \end{pmatrix}.$$

Let $x \in k^*$. Set a = -x, $b = x^{-1} - 1$, c = 1 and d = x - 1; then the matrix above becomes $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. Hence the result holds when r = 1.

Now suppose r > 1. Let H be the semisimple group with root system spanned by $\pm \alpha_1, \ldots, \pm \alpha_{r-1}$. Clearly condition (*) holds for H. Let $s = t_1 \cdots t_{r-1}$. By our induction hypothesis, there exist $u_i, w_i \in U_{\alpha_i}(k)$ and $v_i, x_i \in U_{-\alpha_i}(k)$ for $1 \le i \le r-1$ such that

$$s = x_{r-1} \cdots x_1 u_{r-1} \cdots u_1 v_1 \cdots v_{r-1} w_1 \cdots w_{r-1}.$$

By the SL₂ case considered above, $t_r = x'_r u'_r v'_r w'_r$ for some $u_r, w_r \in U_{\alpha_r}$ and some $v_r, x_r \in U_{-\alpha_r}$. Set $x_r = sx'_r s^{-1}$, $u_r = su'_r s^{-1}$, $v_r = v'_r$ and $w_r = w'_r$. We have

$$x_r x_{r-1} \cdots x_1 u_r u_{r-1} \cdots u_1 v_1 \cdots v_{r-1} v_r w_1 \cdots w_{r-1} w_r$$

$$= x_r u_r x_{r-1} \cdots x_1 u_{r-1} \cdots u_1 v_1 \cdots v_{r-1} w_1 \cdots w_{r-1} v_r w_r$$

$$= x_r u_r s v_r w_r$$

$$= s x'_r u'_r v'_r w'_r$$

$$= s t_r = t.$$

The result follows by induction.

Proposition 6.2. Suppose G is simply connected. Let $X \subseteq G^+(k)$ such that $U(k) \subseteq X$. Then $B_X(7) = G^+(k)$.

Proof. Since G is simply connected, (*) holds for G and the map $\psi \colon \mathbb{G}_m^r \to T$ given by $\psi(a_1, \ldots, a_r) = \alpha_1^{\vee}(a_1) \cdots \alpha_r^{\vee}(a_r)$ is a k-isomorphism. It follows that $T(k) = T_{\alpha_1}(k) \cdots T_{\alpha_r}(k)$, so $T(k) \subseteq B_X(4)$ by Lemma 6.1.

Hence $U^{-}(k)B(k) = U^{-}(k)T(k)U(k) \subseteq B_{X}(1)B_{X}(4)B_{X}(1) \subseteq B_{X}(6).$ Now $G(k) = (U^{-}B)^{-1}(k)(U^{-}B)(k)$ by Lemma 3.2. But $(U^{-}B)^{-1}(k)(U^{-}B)(k) = B(k)U^{-}(k)U^{-}(k)B(k) = U(k)T(k)U^{-}(k)T(k)U(k)$ $= U(k)U^{-}(k)T(k)U(k) = U(k)U^{-}(k)B(k) \subseteq B_{X}(1)B_{X}(6) \subseteq B_{X}(7),$

so we are done.

Proof of Theorem 1.4. Let \widetilde{G} be the split form of the simply connected cover of G and let $\psi: \widetilde{G} \to G$ be the canonical projection. Then ψ is a k-defined central isogeny, so by [2, V.22.6 Thm.], the map $\widetilde{B} \mapsto \psi(\widetilde{B})$ gives a bijection between the set of k-Borel subgroups of \widetilde{G} and the set of k-Borel subgroups of G; moreover, for each \widetilde{B} , ψ gives rise to a k-isomorphism from $R_u(\widetilde{B})$ to $R_u(B)$ [2, Prop. V.22.4]. It follows that $\psi(\widetilde{G}^+(k)) = G^+(k)$. By [5, Lem. 2.16] we have $\Delta(G^+(k)) \leq \Delta(\widetilde{G}^+(k))$, so we can assume without loss that G is simply connected. In particular, $G^+(k) = G(k)$.

Let X be a finite normal generating set for G(k). Then X is a finite normal generating set for G (Lemma 5.4), so by Eqn. (5.1) there exists $t \in B_X(2r)$ such that t is regular semisimple. We have $U(k) \subseteq B_t(2)$ by Lemma 5.3 and $G(k) \subseteq B_{U(k)}(7)$ by Proposition 6.2. So

$$G(k) \subseteq B_{U(k)}(7) \subseteq B_t(14) \subseteq B_X(28r).$$

This shows that $\Delta(G(k)) \leq 28r$, as required.

Example 6.3. (a) Let $G = \operatorname{SL}_n(k)$ where $n \geq 3$, let g be the elementary matrix $E_{1n}(1)$ and let $X = G(k) \cdot g$. By [5, Prop. 6.23], X generates G(k). One sees easily by direct computation that the centraliser $C_G(g)$ has dimension $n^2 - 2n + 1$, so $\dim(G \cdot g) = 2n - 2$. A simple dimension-counting argument shows that if $t < \frac{1}{2} \operatorname{rank}(G)$ then $\overline{X^t}$ is a proper closed subvariety of G. Since G(k) is dense in G, it follows that $\overline{X^t}$ does not contain G(k), so $X(k)^t$ does not contain G(k). We deduce that $\Delta(G(k)) \geq \frac{1}{2} \operatorname{rank}(G)$.

(b) The bounds in Theorems 1.3 and 1.4 are far from sharp. Aseeri has shown by direct calculation that $3 \leq \Delta(\mathrm{SL}_3(\mathbb{C})) \leq 6$ and that $\Delta(\mathrm{SL}_2(\mathbb{C})^m) = 3m$ and $\Delta(\mathrm{PGL}_2(\mathbb{C})^m) = 2m$ for every $k \in \mathbb{N}$ [1, Thm. 8.0.2, Thm. 7.2.10, Thm. 7.2.6], whereas Theorem 1.3 yields the bounds $\Delta(\mathrm{SL}_3(\mathbb{C})) \leq 8$ and $\Delta(\mathrm{SL}_2(\mathbb{C})^m), \Delta(\mathrm{PGL}_2(\mathbb{C})^m) \leq 4m$. Aseeri also showed that $3 \leq \Delta(\mathrm{SL}_3(\mathbb{R})) \leq 4$, whereas Theorem 1.4 gives $\Delta(\mathrm{SL}_3(\mathbb{R})) \leq 56$.

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7. Semisimple Lie groups

Proof of Theorems 1.5 and 1.6. Let H be a linear semisimple Lie group such that H has no compact simple factors. By [6, Thm. III.2.13], there is a complex semisimple algebraic group G defined over \mathbb{R} such that $G^+(\mathbb{R}) = H$. Now Z(H) is finite, so H is finitely normally generated and bounded by [5, Thm. 1.2]. It follows from Theorem 1.2 that H is uniformly bounded. If H is split then G is split over \mathbb{R} , so $\Delta(H) \leq$ $28 \operatorname{rank}(H)$ by Theorem 1.4.

The argument for the complex case is similar: if H is a semisimple linear complex Lie group then there is a semisimple complex algebraic group G such that the complex Lie group associated to G is H (cf. [8, Ch. 4, Sec. 2, Problem 12], and G is isomorphic to H. The result now follows from Theorem 1.3.

References

- Fawaz Aseeri. Uniform boundedness of groups. PhD thesis, University of Aberdeen, 2022.
- [2] Armand Borel. Linear algebraic groups, volume 126 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.
- [3] Robert M. Guralnick and Gunter Malle. Classification of 2F-modules. II. In Finite groups 2003, pages 117–183. Walter de Gruyter, Berlin, 2004.
- [4] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.
- [5] Jarek Kędra, Assaf Libman, and Ben Martin. On boundedness properties of groups. J. Topol. Anal., 2021. DOI: 10.1142/S1793525321500497.
- [6] James S. Milne. Lie algebras, algebraic groups, and Lie groups. Version 2.00, https://www.jmilne.org/math/CourseNotes/LAG.pdf, 2013.
- P. E. Newstead. Introduction to moduli problems and orbit spaces, volume 51 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.
- [8] A. L. Onishchik and E. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.
- [9] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

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