HEDGING OF EUROPEAN TYPE CONTINGENT CLAIMS IN DISCRETE TIME BINOMIAL MARKET MODELS

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Abstract. We consider a discrete-time binomial model of a market consisting of $m \geq 1$ risky securities and one bond. For a European type contingent claim we give an explicit formula for the minimum-cost maximal hedging strategy.

1. The main result

In this note we consider a discrete-time binomail model for a market with m risky securities $S_1, \dots, S_m$ and one bond S_0 with return $R > 0$. Time has values $k = 0, \dots, n$, and we write $S_i(k)$ for the price of the i-th security at time k. The model comes with a choice of numbers $0 < D_i < R < U_i$ for each $1 \leq i \leq m$. To describe the random process of the values of S_i, suppose that the prices of $S_0, \dots, S_m$ are known at time $k < n$. Their values at time $k+1$ is determined as follows.

(a) For the bond process,

$$S_0(k+1) = S_0(k) \cdot R.$$

(b) For the remaining securities, flip m coins and according to the results set

$$S_i(k+1) = S_i(k) \cdot U_i \qquad \text{or} \qquad S_i(k+1) = S_i(k) \cdot D_i$$

The coins are not assumed to be independent, nor do the flips at time k and time $k' \neq k$.

We consider a European contingent claim X with pay-off at time n (of maturity) given by

$$F = \left(\sum_{i=0}^{m} \gamma_i S_i(n) - K \right)^+, \tag{1.1}$$

where $\gamma_1, \dots, \gamma_n \geq 0$, $K \geq 0$ and $x^+ \stackrel{\text{def}}{=} \max\{x, 0\}$ for any real number x.

It is known that the set of rational values of X at time k, (i.e its no-arbitrage price range at time k, forms an open interval whose upper bound we denote by $C_{\max}(X, k)$. In [1, Section 6A eqns. (6.1) and (6.2)] we have shown that $C_{\max}(X, k)$ can be expressed solely by means of the prices of $S_0, \dots, S_m$ at time k and the parameters of the model (see Proposition 2.6 below).

A ***minimum cost maximal hedging strategy*** for X consist of a choice, at each time $k = 0, \dots, n-1$, of numbers $\alpha_0(k), \dots, \alpha_m(k)$ which minimize (the cost of the hedging portfolio)

$$V_\alpha(k) = \sum_{i=0}^{m} \alpha_i(k) S_i(k) \tag{1.2}$$

subject to the (maximal-hedging) condition that at time $k+1$ the value of this portfolio satisfies

$$\sum_{i=0}^{m} \alpha_i(k) \cdot S_i(k+1) \geq C_{\max}(X, k+1). \tag{1.3}$$

In particular the value of the portfolio $V_\alpha(k)$ acquired at time k is guaranteed to exceed the value of the option X at time $k+1$. Notice that the values chosen for $\alpha_i(k)$ depend on the "state of the world" at time k, and in particular on the prices of $S_0, \dots, S_m$ at time k. In [1, Proposition 4.2] we showed that a minimum cost maximal hedging strategy exists and that its set up cost at each time k is exactly $C_{\max}(X, k)$, namely the maximal rational value of X at time k.

The purpose of this note is to give an explicit formula for the values of $\alpha_0(k), \dots, \alpha_m(k)$. In the remainder of this section we describe this formula.

For every $1 \leq i \leq m$ set

$$b_i = \frac{R - D_i}{U_i - D_i}. \tag{1.4}$$

If necessary, reorder the securities $S_1, \dots, S_m$ so that $b_1, \dots, b_m$ is non-increasing, namely

$$b_1 \geq b_2 \geq \cdots \geq b_m. \tag{1.5}$$

Notice that $0 < b_i < 1$ for all i. Define for any $0 \leq j \leq m$

$$q_j = \begin{cases} 1 - b_1 & j = 0 \\ b_j - b_{j+1} & 1 \leq j \leq m-1 \\ b_m & j = m. \end{cases} \tag{1.6}$$

Define for $0 \leq i \leq m$ and for $0 \leq j \leq m$ numbers $\chi_i(j)$ as follows

$$\chi_0(j) = R \qquad \text{and} \qquad \chi_i(j) = \begin{cases} U_i & i \leq j \\ D_i & i > j. \end{cases} \tag{1.7}$$

We denote

$$\mathcal{P}_k(m) = \{0, \dots, m\}^k = \{(j_1, \dots, j_k) : 0 \leq j_i \leq m\}.$$

For any $J \in \mathcal{P}_k(m)$ set

$$q_J = \prod_{j \in J} q_j \qquad \text{and} \qquad \chi_i(J) = \prod_{j \in J} \chi_i(j).$$

Assume that at time $0 \leq k \leq n-1$, the prices $S_0(k), \dots, S_i(m)$ are known. Define for any $0 \leq j \leq m$

$$Y_j(k) = R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_J \left(\sum_{i=0}^{m} \gamma_i \chi_i(J) \chi_i(j) S_i(k) - K \right)^+. \tag{1.8}$$

Finally, define the following $(m+1)\times(m+1)$ matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots\cdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots\cdots & -1 & 1 \end{bmatrix}$$

$$N = \left[\begin{array}{c|cccc} 1 & D_1 & D_2 & \cdots & D_m \\ \hline 0 & U_1 - D_1 & 0 & \dots & 0 \\ 0 & 0 & U_2 - D_2 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & & U_m - D_m \end{array}\right]$$

$$T = \begin{bmatrix} RS_0(k) & & & \\ & S_1(k) & & \\ & & \ddots & \\ & & & S_m(k) \end{bmatrix}$$

Observe that the matrix N only depend on the parameters of the model, and is easily seen to be invertible. Only the matrix T depends on the state of the world at time k, and since $S_i(k) > 0$ it is clearly invertible. Also, Q is invertible.

Theorem 1.1. *With the set up and notation above, at time $0 \le k \le n-1$ the portfolio $V_\alpha(k) = \sum_{i=0}^m \alpha_i(k)\cdot S_i(k)$ of a minimum cost maximal hedge (1.2), (1.3) for a European contingent claim X in (1.1) is given by*

$$\begin{bmatrix} \alpha_0(k) \\ \alpha_1(k) \\ \vdots \\ \alpha_m(k) \end{bmatrix} = T^{-1}\cdot N^{-1}\cdot Q \cdot \begin{bmatrix} Y_0(k) \\ Y_1(k) \\ \vdots \\ Y_m(k) \end{bmatrix}$$

Moreover, $V_\alpha(k) = C_{\max}(F,k)$.

REMARK: The formula given for $Y_j(k)$ has computational complexity $O((m+1)^{n-k})$ (the number of terms in the sum). This is exponential in n. As discussed in [1], the action of the symmetric group S_{n-k-1} on $\mathcal{P}_{n-k-1}(m)$ gives a formula for $Y_j(k)$ whose complexity is only $O((n-k)^{m+1})$, polynomial in n.

REMARK: As is the case in [1], the function $h(x) = x^+$ can be replaced with any convex function. Thus, our results apply to several contingent claims other than European basket call options. The reader is referred to [1] for details.

2. Formalisation of the model and proof of the main result

2.1. **Single time-step.** Each step of the model consists of flipping m coins. A natural sample space for this experiment is the set

$$\mathcal{L} = \{0,1\}^m$$

of all the sequences of length m consisting of 0's and 1's. We denote its elements by $\lambda = (\lambda(1), \dots, \lambda(m))$ and view $\mathcal{L}$ as a subset of $\mathbb{R}^m$. Let ℓ_i denote the (random variable of the) result of the i-th coin, namely $\ell_i \colon \mathcal{L} \to \mathbb{R}$ is the projection to the i-th factor:

$$\ell_i \colon \lambda \mapsto \lambda(i).$$

Observe that ℓ_i is the restriction to $\mathcal{L}$ of the linear projection function $\pi_i \colon \mathbb{R}^m \to \mathbb{R}$.

Let ψ_i be the "price jump" of the i-th security. One checks that $\psi_0 = R$ and that $\psi_i(\lambda) = D_i + (U_i - D_i)\lambda(i)$ for $1 \le i \le m$, namely

$$\psi_0 = R \qquad \text{and} \qquad \psi_i = D_i + (U_i - D_i)\ell_i. \tag{2.1}$$

Thus, for every $1 \le i \le m$, ψ_i is the restriction to $\mathcal{L}$ of an affine function $f_i \colon \mathbb{R}^m \to \mathbb{R}$ where $f_i(x_1, \dots, x_m) = D_i + (U_i - D_i)x_i$.

Let us consider probability measures on $\mathcal{L}$ on the σ-algebra $\wp(\mathcal{L})$ of all the subsets of $\mathcal{L}$. These are equivalent to probability density functions $p \colon \mathcal{L} \to \mathbb{R}$ and we will abuse notation and write p for both the density function and the probability measure it induces. The requirement that p is risk-neural in a single-step model is the condition

$$E_p(\psi_i) = R, \qquad (1 \le i \le m).$$

By the linearity of the expectation and the definition (1.4), this is equivalent to

$$E_p(\ell_i) = b_i.$$

Throughout we assume that $b_1, \dots, b_m$ is non-increasing (1.5).

Definition 2.1. Let $P(\mathcal{L}, b)$ denote the set of all probability density functions $p \colon \mathcal{L} \to \mathbb{R}$ such that $E_p(\ell_i) = b_i$ for all $1 \le i \le m$.

Define $\rho_0, \dots \rho_m \in \mathcal{L}$ as follows

$$\rho_j = (\underbrace{1, \dots, 1}_{j \text{ times}}, 0, \dots, 0). \tag{2.2}$$

Thus, ρ_j describes the event of a run of j heads followed by a run of $n - j$ tails. Use (1.6) to define a probability density function $q \colon \mathcal{L} \to \mathbb{R}$ by

$$q(\lambda) = \begin{cases} q_j & \text{if } \lambda = \rho_j \text{ for some } 0 \le j \le m \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

We call q the ***upper supermodular vertex*** of $P(\mathcal{L}, b)$. Compare with [1, Appendix A equation (A.6)] where we denoted ρ_j by μ_j. One checks, see [1, Appendix A], that indeed $q \in P(\mathcal{L}, b)$. In particular

$$E_q(\psi_i) = D_i + (U_i - D_i)E_q(\ell_i) = D_i + (U_i - D_i)b_i = R. \tag{2.4}$$

There is a canonical bijection of $\mathcal{L}$ with the set $\wp(\{1, \dots, m\})$. This gives rise to a partial order $\preceq$ on $\mathcal{L}$ induced by the partial order $\subseteq$ on $\wp(\{1, \dots, m\})$. Thus,

$$\lambda \preceq \lambda' \iff \operatorname{supp}(\lambda) \subseteq \operatorname{supp}(\lambda'). \tag{2.5}$$

Union and intersection of sets render $\mathcal{L}$ a lattice with join $\vee$ and meet $\wedge$,

$$\begin{aligned}\lambda \vee \lambda' &= (\max\{\lambda(1), \lambda'(1)\}, \dots, \max\{\lambda(m), \lambda'(m)\}) \\ \lambda \wedge \lambda' &= (\min\{\lambda(1), \lambda'(1)\}, \dots, \min\{\lambda(m), \lambda'(m)\})\end{aligned} \tag{2.6}$$

The concept of submodular functions is originally due to Lovász in [2]. In this note, its variant, supermodular functions [3], is used.

Definition 2.2. A function $f: \mathcal{L} \to \mathbb{R}$ is called ***supermodular*** if for any $\lambda, \lambda' \in \mathcal{L}$

$$f(\lambda \vee \lambda') + f(\lambda \wedge \lambda') \geq f(\lambda) + f(\lambda').$$

It is called ***modular*** if equality holds.

Proposition 2.3. *(i) A linear combination with non-negative coefficients of supermodular functions is supermodular.*

(ii) Let $g: \mathbb{R}^m \to \mathbb{R}$ be an affine function. Then $g|_{\mathcal{L}}$ is modular.

(iii) Let $g: \mathbb{R}^m \to \mathbb{R}$ be an affine function of the form $g(x_1, \dots, x_m) = \sum_{i=1}^m a_i x_i + b$ where $a_1, \dots, a_m \geq 0$. If $h: \mathbb{R} \to \mathbb{R}$ is convex then the restriction of $h \circ g$ to $\mathcal{L}$ is supermodular.

Proof. Part (i) is [3, Proposition 2.2.5(a)]. Part (ii) follows from [3, Theorem 2.2.3] (and is straightforward). Part (iii) follows from [3, Theorem 2.2.6(a)]. $\square$

The crucial property of the upper supermodlar vertex q (2.3) is given by the following result.

Proposition 2.4 ([1, Theorem A.5(i)])**.** *Let $f: \mathcal{L} \to \mathbb{R}$ be supermodular. Let q be the upper supermodular vertex in $P(\mathcal{L}, b)$. Then*

$$\sup_{p \in P(\mathcal{L},b)} E_p(f) = \max_{p \in P(\mathcal{L},b)} E_p(f) = E_q(f).$$

2.2. **Multi time-step model.** For the n-step model one performs n iterations (not necessarily independent) of the experiment of flipping m coins. Thus, the natural sample space for an n-step discrete time binomial market model is $\mathcal{L}^n$. We equip it with the σ-algebra $\mathcal{F} = \wp(\mathcal{L}^n)$ of all subsets of $\mathcal{L}^n$.

The "state of the world" at time $0 \leq k \leq n$ is described by a k-tuple $(\lambda^1, \dots, \lambda^k) \in \mathcal{L}^k$. Thus, the set of the states of the world at time k is naturally identified with $\mathcal{L}^k$. We obtain a partition $\{\omega \times \mathcal{L}^{n-k} : \omega \in \mathcal{L}^k\}$ of $\mathcal{L}^n$ which generates a sub-σ-algebra $\mathcal{F}_k$.

The price jump of the i-th security at time $1 \leq k \leq n$, where $1 \leq i \leq m$, is the random variable $\Psi_i(k): \mathcal{L}^n \to \mathbb{R}$

$$\Psi_i(k): (\lambda^1, \dots, \lambda^n) \mapsto \psi_i(\lambda^k).$$

Of course, $\Psi_0(k): \mathcal{L}^n \to \mathbb{R}$ is the constant function (random variable) with value R.

The random process $S_i(0), \dots, S_i(n)$ of the prices of the i-th security at time $0 \leq k \leq n$ are random variables $S_i(k): \mathcal{L}^n \to \mathbb{R}$. Clearly, they are given by

$$S_i(k) = S_i(0) \cdot \Psi_i(1) \cdots \Psi_i(k) \tag{2.7}$$

Where $S_i(0) > 0$ are constant (the initial prices of the securities at time 0). Clearly, the value of $S_i(k)$ at $\omega \in \mathcal{L}^n$ depend only on the first k entries of ω. Hence, $S_i(k)$ are $\mathcal{F}_k$-measurable random variables, namely their values only depend on the state of the world at time k. We will therefore abuse notation and regard $S_i(k)$ as functions with domain $\mathcal{L}^k$.

We now fix $\gamma_0, \dots, \gamma_m$ where $\gamma_i \geq 0$ for all $1 \leq i \leq m$ and fix some K and set

$$F = \left(\sum_{i=0}^{m} \gamma_i S_i(n) - K \right)^+. \tag{2.8}$$

This random variable is the pay-off of the European contingent claim which is the subject of study of this note.

Recall the elements $\rho_j \in \mathcal{L}$ from (2.2). Observe that by definition of ψ_i (2.1) and of χ_i (1.7) we have

$$\psi_i(\rho_j) = \chi_i(j), \qquad (0 \leq i, j \leq m). \tag{2.9}$$

Proposition 2.5. *Consider some $\omega = (\lambda^1, \dots, \lambda^k) \in \mathcal{L}^k$, a state of the world at time $0 \leq k \leq n$, and some $J = (j_1, \dots, j_{n-k}) \in \mathcal{P}_{n-k}(m)$. Set $\tau = (\rho_{j_1}, \dots, \rho_{j_{n-k}}) \in \mathcal{L}^{n-k}$. Then*

$$F(\omega\tau) = \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \chi_i(J) - K \right)^+.$$

Proof. By the definition of $S_i(n)$ (2.7)

$$F(\omega\tau) = \left(\sum_{i=0}^{m} \gamma_i S_i(0) \cdot \prod_{p=1}^{n} \Psi_i(p)(\omega\tau) - K \right)^+ =$$

$$\left(\sum_{i=0}^{m} \gamma_i S_i(0) \cdot \prod_{p=1}^{k} \psi_i(\lambda^p) \cdot \prod_{p=k+1}^{n} \psi_i(\rho_{j_p}) - K \right)^+ = \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \cdot \chi_i(J) - K \right)^+.$$

□

For every $0 \leq k \leq n$ we will denote by

$$C_{\max}(F, k) \qquad \text{and} \qquad C_{\min}(F, k)$$

the upper and lower bounds of the rational values of F at time k. Of course, these numbers depend only on the state of the world at time k, and so $C_{\max/\min}(F, k)$ are $\mathcal{F}_k$-measurable random variables (on $\mathcal{L}^n$).

Proposition 2.6. *With the pay-off F (2.8) of European basket call, at time $0 \leq k \leq n$*

$$C_{\max}(F, k) = R^{k-n} \sum_{J \in \mathcal{P}_{n-k}(m)} q_J \left(\sum_{i=0}^{m} \gamma_i S_i(k) \chi_i(J) - K \right)^+.$$

Proof. This is an immediate consequence of [1, Example 7.3 and Section 6A eqns. (6.1) and (6.2)] and Proposition 2.5. We note that in [1] the elements $\rho_j \in \mathcal{L}$ are denoted μ_j and $\mathcal{P}_{n-k}(m)$ is denoted I^{n-k}. $\square$

Notice that $C_{\max}(F,k)$ is an $\mathcal{F}_k$ measurable random variable on $\mathcal{L}^n$. Hence, it will be convenient to think of it as a function with domain $\mathcal{L}^k$.

Proposition 2.7. *Consider some $0 \le k \le n-1$ and some $\omega \in \mathcal{L}^k$ representing the state of the world at time k. Then the function $f \colon \mathcal{L} \to \mathbb{R}$ defined by*

$$f(\lambda) = C_{\max}(F, k+1)(\omega\lambda)$$

is supermodular. Moreover, with respect to the upper supermodular vertex (2.3)

$$E_q(f) = R \cdot C_{\max}(F,k)(\omega).$$

Proof. It follows from Proposition 2.6 and since $S_i(k+1) = S_i(k) \cdot \Psi_i(k+1)$ that

$$f(\lambda) = R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_J \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \cdot \psi_i(\lambda) \cdot \chi_i(J) - K \right)^+ .$$

Proposition 2.3(i),(iii) implies that f is supermodular since $h(x) = x^+$ is convex and since $\psi_i = D_i + (U_i - D_i)\ell_i$ is an affine function with non-negative coefficients and since $R > 0$ and $\gamma_i, S_i(k), \chi_i(J) \ge 0$ for all $1 \le i \le m$. Moreover, by (2.9)

$$\begin{aligned}
E_q(f) &= \sum_{\lambda \in \mathcal{L}} q(\lambda) f(\lambda) \\
&= \sum_{j=0}^{m} q(\rho_j) f(\rho_j) \\
&= \sum_{j=0}^{m} q_j \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_J R^{k+1-n} \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \cdot \psi_i(\rho_j) \cdot \chi_i(J) - K \right)^+ \\
&= \sum_{j=0}^{m} q_j \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_J R^{k+1-n} \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \cdot \chi_i(J)\chi_i(j) - K \right)^+ \\
&= R \cdot R^{k-n} \sum_{J \in \mathcal{P}_{n-k}(m)} q_J \left(\sum_{i=0}^{m} \gamma_i S_i(k)(\omega) \cdot \chi_i(J) - K \right)^+ \\
&= R \cdot C_{\max}(F,k)(\omega).
\end{aligned}$$

$\square$

Consider some $0 \le k \le n-1$ and recall $Y_j(k)$ from (1.8) where $0 \le j \le m$. Observe that $Y_j(k)$ is a function of the random variables $S_i(k)$, so $Y_j(k)$ is a random variable (on $\mathcal{L}^n$) whose values depend only on the state of the world at time k, namely $Y_j(k)$ is $\mathcal{F}_k$-measurable.

Proposition 2.8. *For any $0 \le j \le m$ and any $\omega = (\lambda^1, \dots, \lambda^k) \in \mathcal{L}^k$*

$$C_{\max}(F, k+1)(\omega\rho_j) = Y_j(\omega).$$

Proof. This follows immediately from Propositions 2.6 and equation (2.9). ☐

Proof of Theorem 1.1. Fix some $0 \le k \le n-1$ and some state of the world $\omega = (\lambda^1, \dots, \lambda^k) \in \mathcal{L}^k$ at time k. Any subsequent state of the world at time $k+1$ has the form $\omega\lambda$ for $\lambda \in \mathcal{L}$. Our goal is to find numbers $\alpha_0(k)(\omega), \dots, \alpha_m(k)(\omega)$, which for simplicity we denote by $\alpha_0, \dots, \alpha_m$, which fulfil the inequality (1.3), namely for every $\lambda \in \mathcal{L}$

$$\sum_{i=0}^{m} \alpha_i S_i(k+1)(\omega\lambda) \ge C_{\max}(F, k+1)(\omega\lambda)$$

and which minimize

$$V_\alpha(k)(\omega) = \sum_{i=0}^{m} \alpha_i S_i(k)(\omega). \tag{2.10}$$

We rewrite the first inequality as a set of inequalities (indexed by $\lambda \in \mathcal{L}$)

$$\underbrace{\sum_{i=0}^{m} \alpha_i S_i(k)(\omega) \cdot \psi_i(\lambda)}_{\Phi_\alpha(\lambda)} \ge \underbrace{C_{\max}(F, k+1)(\omega\lambda)}_{\Xi(\lambda)} \qquad (\lambda \in \mathcal{L}). \tag{2.11}$$

We obtain two functions $\Phi_\alpha \colon \mathcal{L} \to \mathbb{R}$ and $\Xi \colon \mathcal{L} \to \mathbb{R}$, and (2.11) is the inequality

$$\Phi_\alpha \ge \Xi.$$

The first step of the proof is to show that the following system of $m+1$ linear equations with the $m+1$ unknowns $\alpha_0, \dots, \alpha_m$ has a unique solution

$$\sum_{i=0}^{m} \alpha_i S_i(k)(\omega) \cdot \psi_i(\rho_j) = C_{\max}(F, k+1)(\omega\rho_j), \qquad (0 \le j \le m).$$

Notice that these equations are obtained by imposing equalities in the inequalities (2.11) for $\lambda = \rho_0, \dots, \rho_m$. By (2.9) and by Proposition 2.8, this is the system of equations

$$\sum_{i=0}^{m} \alpha_i S_i(k)(\omega) \cdot \chi_i(j) = Y_j(\omega), \qquad (0 \le j \le m). \tag{2.12}$$

Write $\alpha_\bullet$ for the column vector $(\alpha_0, \dots, \alpha_m)$ and $Y_\bullet$ for the column vector $(Y_0(\omega), \dots, Y_m(\omega))$. Then this system of $m+1$ linear equations is $M\alpha_\bullet = Y_\bullet$ where M is the $(m+1) \times (m+1)$ matrix

$$(M_{j,i}) = (S_i(k)(\omega) \cdot \chi_i(j)) =$$

$$\underbrace{\begin{bmatrix} 1 & \chi_1(0) & \cdots & \chi_m(0) \\ 1 & \chi_1(1) & \cdots & \chi_m(1) \\ \vdots & \vdots & & \vdots \\ 1 & \chi_1(m) & \cdots & \chi_m(m) \end{bmatrix}}_{M'} \cdot \underbrace{\begin{bmatrix} RS_0(k)(\omega) & & & \\ & S_1(k)(\omega) & & \\ & & \ddots & \\ & & & S_m(k)(\omega) \end{bmatrix}}_{T}.$$

Clearly, T is invertible since $R, S_i(k)(\omega) > 0$. Observe that for any $1 \leq j \leq m$ and any $0 \leq i \leq m-1$

$$\chi_j(i) - \chi_j(i+1) = \begin{cases} 0 & i \neq j \\ U_i - D_i & i = j \end{cases}$$

By inspection we get

$$Q \cdot M' = \underbrace{\left[\begin{array}{c|cccc} 1 & D_1 & D_2 & \cdots & D_m \\ \hline 0 & U_1 - D_1 & 0 & \cdots & 0 \\ 0 & 0 & U_2 - D_2 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & U_m - D_m \end{array}\right]}_{N}.$$

Clearly, Q and N are invertible, hence so is M'. It follows that $M = M'T$ is invertible, so the system (2.12) has a unique solution given by

$$\alpha_\bullet = T^{-1} N^{-1} Q \cdot Y_\bullet.$$

This gives the values of $\alpha_i(k)(\omega)$ stated in the theorem. It remains to show that these α_i solve all the inequalities in (2.11) (one for each $\lambda \in \mathcal{L}$) and minimize (2.10) and $V_\alpha(k)(\omega) = C_{\max}(F, k)(\omega)$.

Claim 1: $\alpha_0, \ldots, \alpha_m$ solve the inequalities (2.11).

Proof: Suppose that not all these inequalities are solved, namely $\Phi_\alpha(\lambda) > \Xi(\lambda)$ for some $\lambda \in \mathcal{L}$. Since $\alpha_0, \ldots, \alpha_m$ solve the equations (2.12), we certainly get $\Phi_\alpha(\rho_i) = \Xi(\rho_i)$ for all $0 \leq i \leq m$.

Among all $\lambda \in \mathcal{L}$ for which $\Phi(\lambda) > \Xi(\lambda)$ choose one with maximal possible j such that $\rho_j \preceq \lambda$ (2.5). Observe that $j < m$ because if $j = m$ then $\rho_m \preceq \lambda$ implies that $\lambda = \rho_m$ which we have seen is impossible. Set $\lambda' = \lambda \vee \rho_{j+1}$ (2.6). By the maximality of j we get that $\lambda \wedge \rho_{j+1} = \rho_j$. Observe that Φ_α is an affine function, hence it is modular by Proposition 2.3(ii), so

$$\Phi(\lambda') + \Phi(\rho_j) = \Phi(\lambda) + \Phi(\rho_{j+1}).$$

By Proposition 2.7 Ξ is supermodular, so

$$\Xi(\lambda') + \Xi(\rho_j) \geq \Xi(\lambda) + \Xi(\rho_{j+1}).$$

Subtracting the first equality from the second inequality, and recalling that $\Phi(\rho_i) = \Xi(\rho_i)$ for all i, we get

$$\Xi(\lambda') - \Phi(\lambda') \geq \Xi(\lambda) - \Psi(\lambda) > 0.$$

So $\Phi(\lambda') > \Xi(\lambda')$ and $\rho_{j+1} \preceq \lambda'$ which contradicts the maximality of j. q.e.d

Recall that any $\beta_0, \ldots, \beta_m$ define $V_\beta(k)(\omega)$ in (2.10).

Claim 2: For any $\beta_0, \ldots, \beta_m$ for which the inequalities (2.11) hold,

$$V_\beta(k)(\omega) \geq C_{\max}(F, k)(\omega).$$

In addition, $\alpha_0, \ldots, \alpha_m$ attain this lower bound, namely

$$V_\alpha(k)(\omega) = C_{\max}(F, k)(\omega).$$

Proof: Suppose β_i solve the inequalities (2.11). It follows from (2.4) that

$$E_q(\Phi_\beta) = \sum_{i=0}^{m} \beta_i S_i(k)(\omega) \cdot E_q(\psi_i) = R \cdot \sum_{i=0}^{m} \beta_i S_i(k)(\omega) = RV_\beta(k)(\omega). \tag{2.13}$$

Proposition 2.7 implies that $E_q(\Xi) = RC_{\max}(F,k)(\omega)$. Since β_i solve the inequalities (2.11), this means $\Phi_\beta \geq \Xi$. By the monotonicity of the expectation, $E_q(\Phi_\beta) \geq E_q(\Xi)$, and since $R > 0$, it follows that $V_\beta(k)(\omega) \geq C_{\max}(F,k)(\omega)$ as needed.

By construction $\Phi_\alpha(\rho_j) = \Xi(\rho_j)$ for all $j = 0, \ldots, m$. Since q is supported on $\rho_0, \ldots, \rho_m$, we deduce from (2.13) that

$$RV_\alpha(k)(\omega) = E_q(\Phi_\alpha) = E_q(\Xi) = R \cdot C_{\max}(F,k)(\omega).$$

Hence $V_\alpha(k)(\omega) = C_{\max}(F,k)(\omega)$. q.e.d

The theorem follows from Claims 1 and 2. □

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