Received: 29 April 2022 | Revised: 17 March 2023 | Accepted: 13 April 2023

DOI: 10.1112/tlm3.12054

RESEARCH ARTICLE

Transactions of the London Mathematical Society

Punctured groups for exotic fusion systems

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Abstract

The transporter systems of Oliver and Ventura and the localities of Chermak are classes of algebraic structures that model the p-local structures of finite groups. Other than the transporter categories and localities of finite groups, important examples include centric, quasicentric, and subcentric linking systems for saturated fusion systems. These examples are, however, not defined in general on the full collection of subgroups of the Sylow group. We study here *punctured groups*, a short name for transporter systems or localities on the collection of nonidentity subgroups of a finite p-group. As an application of the existence of a punctured group, we show that the subgroup homology decomposition on the centric collection is sharp for the fusion system. We also prove a Signalizer Functor Theorem for punctured groups and use it to show that the smallest Benson–Solomon exotic fusion system at the prime 2 has a punctured group, while the others do not. As for exotic fusion systems at odd primes p, we survey several classes and find that in almost all cases, either the subcentric linking system is a punctured group for the system, or the system has no punctured group because the normalizer of some subgroup of order p is exotic. Finally, we classify punctured groups restricting to the centric linking system for certain fusion systems on extraspecial p-groups of order p^3.

MSC 2020
20D05, 20D20, 20J06, 55R35 (primary)

1 INTRODUCTION

Let $\mathcal{F}$ be a fusion system over the finite p-group S. Thus, $\mathcal{F}$ is a category with objects the subgroups of S, and with morphisms injective group homomorphisms that contain among them the conjugation homomorphisms induced by elements of S plus one more weak axiom. A fusion system is said to be saturated if it satisfies two stronger "saturation" axioms that were originally formulated by Puig [47] and reformulated by Broto, Levi, and Oliver [8]. Those axioms hold whenever G is a finite group, S is a Sylow p-subgroup of G, and $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_G(P, Q)$ is the set of conjugation maps c_g from P to Q that are induced by elements $g \in G$. The fusion system of a finite group is denoted $\mathcal{F}_S(G)$.

A saturated fusion system $\mathcal{F}$ is said to be *exotic* if it is not of the form $\mathcal{F}_S(G)$ for any finite group G with Sylow p-subgroup S. The Benson–Solomon fusion systems at $p = 2$ form one family of examples of exotic fusion systems [1, 35]. They are essentially the only known examples at the prime 2, and they are in some sense the oldest known examples, having been studied in the early 1970s by Solomon in the course of the classification of finite simple groups (although not with the more recent categorical framework in mind) [49]. In contrast with the case $p = 2$, a fast-growing literature describes many exotic fusion systems on finite p-groups when p is odd.

In replacing a group by its fusion system at a prime, one retains information about conjugation homomorphisms between p-subgroups, but otherwise loses information about the group elements themselves. It is therefore natural that a recurring theme throughout the study of saturated fusion systems is the question of how to "enhance" or "rigidify" a saturated fusion system to make it again more group-like, and also to study which fusion systems have such rigidifications.

The study of the existence and uniqueness of centric linking systems was a first instantiation of this theme of rigidifying saturated fusion systems. A centric linking system is an important extension category of a fusion system $\mathcal{F}$ that provides just enough algebraic information to recover a p-complete classifying space. For example, it recovers the homotopy type of the p-completion of BG in the case where $\mathcal{F} = \mathcal{F}_S(G)$. Centric linking systems of finite groups are easily defined, and Oliver proved that the centric linking systems of finite groups are unique [38, 39]. Then, Chermak proved that each saturated fusion system, possibly exotic, has a unique associated centric linking system [14]. A proof that does not rely on the classification of finite simple groups can be obtained through [24, 41].

More generally, there are at least two frameworks for considering extensions, or rigidifications, of saturated fusion systems: the *transporter systems* of Oliver and Ventura [45] and the *localities* of Chermak [14]. In particular, one can consider centric linking systems in either setting. While centric linking systems in either setting have a specific set of objects, the object sets in transporter systems and localities can be any conjugation-invariant collection of subgroups that is closed under passing to overgroups. The categories of transporter systems and isomorphisms and of localities and isomorphisms are equivalent [14, appendix] and [25, Theorem 2.11]. However, depending on the intended application, it is sometimes advantageous to work in the setting of transporter systems, and sometimes in localities. The reader is referred to Section 2 for an introduction to localities and transporter systems.

In this paper, we study *punctured groups*. These are transporter systems, or localities, with objects the *nonidentity* subgroups of a finite p-group S. To motivate the terminology, recall that every finite group G with Sylow p-subgroup S admits a transporter system $\mathcal{T}_S(G)$ whose objects are *all* subgroups of S and $\mathrm{Mor}_{\mathcal{T}}(P,Q) = N_G(P,Q)$, the transporter set consisting of all $g \in G$ that conjugate P into Q. Conversely, [45, Proposition 3.11] shows that a transporter system $\mathcal{T}$ whose set of objects consists of all the subgroups of S is necessarily the transporter system $\mathcal{T}_S(G)$ where $G = \mathrm{Aut}_{\mathcal{T}}(1)$, and the fusion system $\mathcal{F}$ with which $\mathcal{T}$ is associated is $\mathcal{F}_S(G)$. Thus, a punctured group $\mathcal{T}$ is a transporter system whose object set is missing the trivial subgroup, an object whose inclusion forces $\mathcal{T}$ to be the transporter system of a finite group.

If we consider localities rather than transporter systems, then the punctured group of G is the *locality* $\mathcal{L}_{\mathscr{S}^*(S)}(G) \subseteq G$ consisting of those elements $g \in G$ that conjugate a nonidentity subgroup of S back into S. This is equipped with the multivariable partial product $w := (g_1, \dots, g_n) \mapsto g_1 \cdots g_n$, defined only when each initial subword of the word w conjugates some fixed nonidentity subgroup of S back into S. Thus, the product is defined on words that correspond to sequences of composable morphisms in the transporter category $\mathcal{T}_S^*(G)$. See Definition 2.6 for more details.

By contrast with the existence and uniqueness of linking systems, we will see that punctured groups for exotic fusion systems do not necessarily exist. The existence of a punctured group for an exotic fusion system seems to indicate that the fusion system is “close to being realizable” in some sense. Therefore, considering punctured groups might provide some insight into how exotic systems arise.

It is also not reasonable to expect that a punctured group is unique when it does exist. To give one example, the fusion systems $PSL_2(q)$ with $q \equiv 9 \pmod{16}$ all have a single class of involutions and equivalent fusion systems at the prime 2. On the other hand, the centralizer of an involution is dihedral of order $2(q-1)$, so the associated punctured groups are distinct for distinct q. Examples like this one occur systematically in groups of Lie type in nondefining characteristic. Later we will give examples of realizable fusion systems with punctured groups that do not occur as a full subcategory of the punctured group of a finite group.

We will now describe our results in detail. To start, we present a result that gives some motivation for studying punctured groups.

1.1 | Sharpness of the subgroup homology decomposition

As an application of the existence of the structure of a punctured group for a saturated fusion system $\mathcal{F}$, we prove that it implies the sharpness of the subgroup homology decomposition for that system. Recall from [8, Definition 1.8] that given a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ its classifying space is the Bousfield–Kan p-completion of the geometric realization of the category $\mathcal{L}$. This space is denoted by $|\mathcal{L}|_p^\wedge$.

The orbit category of $\mathcal{F}$, see [8, Definition 2.1], is the category $\mathcal{O}(\mathcal{F})$ with the same objects as $\mathcal{F}$ and whose morphism sets $\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q)$ is the set of orbits of $\mathcal{F}(P,Q)$ under the action of $\mathrm{Inn}(Q)$. The full subcategory of the $\mathcal{F}$-centric subgroups is denoted $\mathcal{O}(\mathcal{F}^c)$. For every $j \geqslant 0$ there is a functor

$\mathcal{H}^j : \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$:

$$\mathcal{H}^j : P \mapsto H^j(P; \mathbb{F}_p), \qquad (P \in \mathcal{O}(\mathcal{F}^c)).$$

The stable element theorem for p-local finite groups [8, Theorem B, see also Theorem 5.8] asserts that for every $j \geqslant 0$,

$$H^j(|\mathcal{L}|^\wedge_p; \mathbb{F}_p) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} \mathcal{H}^j = \varprojlim_{P \in \mathcal{O}(\mathcal{F}^c)} H^j(P; \mathbb{F}_p).$$

The proof of this theorem in [8] is indirect and requires heavy machinery such as Lannes's T-functor theory. From the conceptual point of view, the stable element theorem is only a shadow of a more general phenomenon. By [8, Proposition 2.2], there is a functor

$$\tilde{B} : \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$$

with the property that $\tilde{B}(P)$ is homotopy equivalent to the classifying space of P (denoted BP) and moreover there is a natural homotopy equivalence

$$|\mathcal{L}| \simeq \operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B}.$$

The Bousfield–Kan spectral sequence for this homotopy colimit [6, chapter XII, section 4.5] takes the form

$$E_2^{i,j} = \varprojlim_{\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}}{}^i \mathcal{H}^j \Rightarrow H^{i+j}(|\mathcal{L}|^\wedge_p; \mathbb{F}_p)$$

and is called the *subgroup decomposition* of $(S, \mathcal{F}, \mathcal{L})$. We call the subgroup decomposition *sharp*, see [20], if the spectral sequence collapses to the vertical axis, namely $E_2^{i,j} = 0$ for all $i > 0$. When this is the case, the stable element theorem is a direct consequence. Indeed, whenever $\mathcal{F}$ is induced from a finite group G with a Sylow p-subgroup S, the subgroup decomposition is sharp (and the stable element theorem goes back to Cartan–Eilenberg [13, Theorem XII.10.1]). This follows immediately from Dwyer's work [20, section 1.11] and [7, Lemma 1.3], see, for example, [18, Theorem B].

It is still an open question as to whether the subgroup decomposition is sharp for every saturated fusion system. We will prove the following theorem.

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system that affords the structure of a punctured group. Then the subgroup decomposition on the $\mathcal{F}$-centric subgroups is sharp. In other words,*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}}{}^i \mathcal{H}^j = 0$$

for every $i \geqslant 1$ and $j \geqslant 0$.

We will prove this theorem in Section 3. We remark that our methods apply to any functor $\mathcal{H}$ which in the language of [18] is the pullback of a Mackey functor on the orbit category of $\mathcal{F}$ denoted

$\mathcal{O}(\mathcal{F})$ such that $\mathcal{H}(e) = 0$ where $e \leqslant S$ is the trivial subgroup. In the absence of applications in sight for this level of generality, we have confined ourselves to the functors $\mathcal{H} = \mathcal{H}^j$.

1.2 | Signalizer functor theorem for punctured groups

It is natural to ask for which exotic fusion systems punctured groups exist. We will answer this question for specific families of exotic fusion systems. As a tool for proving the nonexistence of punctured groups we define and study signalizer functors for punctured groups thus mirroring a concept from finite group theory.

Definition 1.2. Let $(\mathcal{L}, \Delta, S)$ be a punctured group. If P is a subgroup of S, write $\mathcal{I}_p(P)$ for the set of elements of P of order p. A *signalizer functor of* $(\mathcal{L}, \Delta, S)$ on elements of order p is a map θ from $\mathcal{I}_p(S)$ to the set of subgroups of $\mathcal{L}$, which associates to each element $a \in \mathcal{I}_p(S)$ a normal p'-subgroup $\theta(a)$ of $C_{\mathcal{L}}(a)$ such that the following two conditions hold.

- (Conjugacy condition) $\theta(a^g) = \theta(a)^g$ for any $g \in \mathcal{L}$ and $a \in \mathcal{I}_p(S)$ such that a^g is defined and an element of S.
- (Balance condition) $\theta(a) \cap C_{\mathcal{L}}(b) \leqslant \theta(b)$ for all $a, b \in \mathcal{I}_p(S)$ with $[a, b] = 1$.

Notice in the above definition that, as $(\mathcal{L}, \Delta, S)$ is a punctured group, for any $a \in S$, the normalizer $N_{\mathcal{L}}(\langle a \rangle)$ and thus also the centralizer $C_{\mathcal{L}}(a)$ is a subgroup.

Theorem 1.3 (Signalizer functor theorem for punctured groups). *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and suppose θ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order p. Then*

$$\widehat{\Theta} := \bigcup_{x \in \mathcal{I}_p(S)} \theta(x)$$

is a partial normal subgroup of $\mathcal{L}$ with $\widehat{\Theta} \cap S = 1$. In particular, the canonical projection $\rho : \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^\rho$. Upon identifying S with S^ρ, the following properties hold.

(a) *$(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality and $\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.*
(b) *For each $P \in \Delta$, the projection ρ restricts to an epimorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ with kernel $\Theta(P)$ and thus induces an isomorphism $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

1.3 | Punctured groups for families of exotic fusion systems

Let $\mathcal{F}$ be a saturated fusion system on the p-group S. If $\mathcal{L}$ is a locality or transporter system associated with $\mathcal{F}$, then for each fully $\mathcal{F}$-normalized object P of $\mathcal{L}$, the normalizer fusion system $N_{\mathcal{F}}(P)$ is the fusion system of the group $N_{\mathcal{L}}(P)$ if $\mathcal{L}$ is a locality, and of the group $\mathrm{Aut}_{\mathcal{L}}(P)$ if $\mathcal{L}$ is a transporter system. This gives an easy necessary condition for the existence of a punctured group: for each fully $\mathcal{F}$-normalized nonidentity subgroup $P \leqslant S$, the normalizer $N_{\mathcal{F}}(P)$ is realizable.

Conversely, there is a sufficient condition for the existence of a punctured group: $\mathcal{F}$ is of characteristic p-type, that is, for each fully $\mathcal{F}$-normalized nonidentity subgroup $P \leqslant S$, the normalizer $N_{\mathcal{F}}(P)$ is constrained. This follows from the existence of linking systems (or similarly linking localities) of a very general kind, a result that was shown in [28, Theorem A] building on the existence and uniqueness of centric linking systems.

The Benson–Solomon fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ at the prime 2 have the property that the normalizer fusion system of each nonidentity subgroup P is realizable, and moreover, $C_{\mathcal{F}}(Z(S))$ is the fusion system at $p = 2$ of $\mathrm{Spin}_7(q)$, and hence not constrained. So, $\mathcal{F}_{\mathrm{Sol}}(q)$ satisfies the obvious necessary condition for the existence of a punctured group, and does not satisfy the sufficient one.

Based on results of Solomon [49], Levi and Oliver showed that $\mathcal{F}_{\mathrm{Sol}}(q)$ is exotic [35, Theorem 3.4], that is, it has no locality with objects *all* subgroups of a Sylow 2-group. In Section 4, we show the following theorem.

Theorem 1.4. *For any odd prime power q, the Benson–Solomon fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ has a punctured group if and only if $q \equiv \pm 3 \pmod 8$.*

If l is the nonnegative integer with the property that 2^{l+3} is the 2-part of $q^2 - 1$, then $\mathcal{F}_{\mathrm{Sol}}(q) \cong \mathcal{F}_{\mathrm{Sol}}(3^{2^l})$. So, the theorem says that only the smallest Benson–Solomon system, $\mathcal{F}_{\mathrm{Sol}}(3)$, has a punctured group, and the larger ones do not. Further details and a uniqueness statement are given in Theorem 4.1.

When showing the nonexistence of a punctured group in the case $q \equiv \pm 1 \pmod 8$, the Signalizer Functor Theorem 1.3 plays an important role in showing that a putative minimal punctured group has no nontrivial partial normal p'-subgroups. This is similar to the way signalizer functor theory was used by Solomon in [49, section 3]. To construct a punctured group in the case $q \equiv \pm 3 \pmod 8$, we turn to a procedure we call Chermak descent. It is an important tool in Chermak's proof of the existence and uniqueness of centric linking systems [14, section 5] and allows us (under some assumptions) to "expand" a given locality to produce a new locality with a larger object set. Starting with a linking locality, we use Chermak descent to construct a punctured group $\mathcal{L}$ for $\mathcal{F}_{\mathrm{Sol}}(q)$ in which the centralizer of an involution is $C_{\mathcal{L}}(Z(S)) \cong \mathrm{Spin}_7(3)$.

It is possible that there could be other examples of punctured groups for $\mathcal{F}_{\mathrm{Sol}}(3)$ in which the centralizer of an involution is $\mathrm{Spin}_7(q)$ for certain $q = 3^{1+6a}$; a necessary condition for existence is that each prime divisor of $q^2 - 1$ is a square modulo 7. However, given this condition, we can neither prove or disprove the existence of an example with the prescribed involution centralizer.

In Section 5, we survey a few families of known exotic fusion systems at odd primes to determine whether or not they have a punctured group. A summary of the findings is contained in Theorem 5.2. For nearly all the exotic systems we consider, either the system is of characteristic p-type, or the normalizer of some p-subgroup is exotic and therefore a punctured group does not exist. Indeed, it might be that a similar result can be shown for all known exotic fusion systems at odd primes. At least we are not aware of any counterexample.

In particular, when considering the family of Clelland–Parker systems [17] in which each essential subgroup is special, we find that $O^{p'}(C_{\mathcal{F}}(Z(S))/Z(S))$ is simple, exotic, and had not appeared elsewhere in the literature as of the time of our writing. We dedicate part of Subsection 5.3 to describing these systems and to proving that they are exotic.

Applying Theorem 1.1 to the results of Sections 4 and 5 establish the sharpness of the subgroup decomposition for new families of exotic fusion systems, notably

- Benson–Solomon's system $\mathcal{F}_{\mathrm{Sol}}(3)$ [35],
- all Parker–Stroth systems [46],
- all Clelland–Parker systems [17] in which each essential subgroup is abelian.

It also recovers the sharpness for certain fusion systems on p-groups with an abelian subgroup of index p, a result that was originally established in full generality by Dıaz and Park [18].

1.4 | Classification of punctured groups over p_+^{1+2}

In general, it seems difficult to classify all the punctured groups associated with a given saturated fusion system. However, for fusion systems over an extraspecial p-group of exponent p, which by [48] are known to contain among them three exotic fusion systems at the prime 7, we are able to work out such an example. There is always a punctured group $\mathcal{L}$ associated to such a fusion system, and when $\mathcal{F}$ has one class of subgroups of order p and the full subcategory of $\mathcal{L}$ with objects the $\mathcal{F}$-centric subgroups is the centric linking system, a classification is obtained in Theorem 6.4. Conversely, the cases we list in that theorem all occur in an example for a punctured group. This demonstrates on the one hand that there can be more than one punctured group associated to the same fusion system and indicates on the other hand that examples for punctured groups are still somewhat limited.

Outline of the paper and notation

The paper proceeds as follows. In Section 2, we recall the definitions and basic properties of transporter systems and localities, and we prove the Signalizer Functor Theorem in Subsection 2.8. In Section 3, we prove sharpness of the subgroup decomposition for fusion systems with associated punctured groups. Section 4 examines punctured groups for the Benson–Solomon fusion systems, while Section 5 looks at several families of exotic fusion systems at odd primes. Finally, in Section 6 classifies certain punctured groups over an extraspecial p-group of order p^3 and exponent p. The Appendix sets notation and provides certain general results on finite groups of Lie type that are needed in Section 4.

The first four sections of the paper do not use the classification of the finite simple groups (CFSG). The CFSG is always used indirectly in Sections 5 and 6 whenever we need to apply known results that certain exotic fusion systems at odd primes are indeed exotic. Each time this occurs (e.g., Proposition 5.7), the results could be stated so as to avoid indirect use of the CFSG. Aside from this, there are two direct applications of the CFSG. The first occurs in the proof of Lemma 5.10(c) when showing that fusion systems related to the Clelland–Parker systems are exotic. The second occurs in the proof of Lemma 6.5(b).

Throughout most of the paper, we write conjugation like maps on the right side of the argument and compose from left to right. There are two exceptions: when working with transporter systems, such as in Section 3, we compose morphisms from right to left. Also, we apply certain maps in Section 4 on the left of their arguments (e.g., roots, when viewed as characters of a torus). The notation for Section 4 is outlined in more detail in the Appendix.

2 | LOCALITIES AND TRANSPORTER SYSTEMS

As already mentioned in the introduction, transporter systems as defined by Oliver and Ventura [45] and localities in the sense of Chermak [14] are algebraic structures that carry essentially the same information. In this section, we will give an introduction to both subjects and outline briefly the connection between localities and transporter systems. At the end we present some signalizer functor theorems for localities.

2.1 Partial groups

We refer the reader to Chermak's papers [14] or [15] for a detailed introduction to partial groups and localities. However, we will briefly summarize the most important definitions and results here. Following Chermak's notation, we write $\mathbf{W}(\mathcal{L})$ for the set of words in a set $\mathcal{L}$, and $\emptyset$ for the empty word. The concatenation of words $u_1, \dots, u_k \in \mathbf{W}(\mathcal{L})$ is denoted by $u_1 \circ u_2 \circ \cdots \circ u_k$.

Definition 2.1 (Partial group). Let $\mathcal{L}$ be a nonempty set, let $\mathbf{D}$ be a subset of $\mathbf{W}(\mathcal{L})$, let $\Pi : \mathbf{D} \to \mathcal{L}$ be a map and let $(-)^{-1} : \mathcal{L} \to \mathcal{L}$ be an involutory bijection, which we extend to a map

$$(-)^{-1} : \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}), w = (g_1, \dots, g_k) \mapsto w^{-1} = (g_k^{-1}, \dots, g_1^{-1}).$$

We say that $\mathcal{L}$ is a *partial group* with product Π and inversion $(-)^{-1}$ if the following hold.

- $\mathcal{L} \subseteq \mathbf{D}$ (i.e., $\mathbf{D}$ contains all words of length 1), and

$$u \circ v \in \mathbf{D} \Longrightarrow u, v \in \mathbf{D}.$$

  (So in particular, $\emptyset \in \mathbf{D}$.)
- Π restricts to the identity map on $\mathcal{L}$.
- $u \circ v \circ w \in \mathbf{D} \Longrightarrow u \circ (\Pi(v)) \circ w \in \mathbf{D}$, and $\Pi(u \circ v \circ w) = \Pi(u \circ (\Pi(v)) \circ w)$.
- $w \in \mathbf{D} \Longrightarrow w^{-1} \circ w \in \mathbf{D}$ and $\Pi(w^{-1} \circ w) = \mathbf{1}$ where $\mathbf{1} := \Pi(\emptyset)$.

Note that any group G can be regarded as a partial group with product defined in $\mathbf{D} = \mathbf{W}(G)$ by extending the "binary" product to a map $\mathbf{W}(G) \to G, (g_1, g_2, \dots, g_n) \mapsto g_1 g_2 \cdots g_n$.

For the remainder of this section, let $\mathcal{L}$ be a partial group with product $\Pi : \mathbf{D} \to \mathcal{L}$ defined on the domain $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$.

Because of the group-like structure of partial groups, the product $\mathcal{X}\mathcal{Y}$ of two subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{L}$ is naturally defined by

$$\mathcal{X}\mathcal{Y} := \{\Pi(x, y) : x \in \mathcal{X}, \ y \in \mathcal{Y} \text{ such that } (x, y) \in \mathbf{D}\}.$$

Similarly, there is a natural notion of conjugation, which we consider next.

Definition 2.2. For every $g \in \mathcal{L}$, we define

$$\mathbf{D}(g) = \{x \in \mathcal{L} \mid (g^{-1}, x, g) \in \mathbf{D}\}.$$

The map $c_g : \mathbf{D}(g) \to \mathcal{L}, x \mapsto x^g = \Pi(g^{-1}, x, g)$ is the *conjugation map* by g. If $\mathcal{H}$ is a subset of $\mathcal{L}$ and $\mathcal{H} \subseteq \mathbf{D}(g)$, then we set

$$\mathcal{H}^g = \{h^g \mid h \in \mathcal{H}\}.$$

Whenever we write x^g (or $\mathcal{H}^g$), we mean implicitly that $x \in \mathbf{D}(g)$ (or $\mathcal{H} \subseteq \mathbf{D}(g)$, respectively). Moreover, if $\mathcal{M}$ and $\mathcal{H}$ are subsets of $\mathcal{L}$, we write $N_{\mathcal{M}}(\mathcal{H})$ for the set of all $g \in \mathcal{M}$ such that $\mathcal{H} \subseteq \mathbf{D}(g)$ and $\mathcal{H}^g = \mathcal{H}$. Similarly, we write $C_{\mathcal{M}}(\mathcal{H})$ for the set of all $g \in \mathcal{M}$ such that $\mathcal{H} \subseteq \mathbf{D}(g)$ and $h^g = h$ for all $h \in \mathcal{H}$. If $\mathcal{M} \subseteq \mathcal{L}$ and $h \in \mathcal{L}$, set $C_{\mathcal{M}}(h) := C_{\mathcal{M}}(\{h\})$.

Definition 2.3. Let $\mathcal{H}$ be a nonempty subset of $\mathcal{L}$. The subset $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$ if

- $g \in \mathcal{H} \Longrightarrow g^{-1} \in \mathcal{H}$; and
- $w \in \mathbf{D} \cap \mathbf{W}(\mathcal{H}) \Longrightarrow \Pi(w) \in \mathcal{H}$.

If $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$ with $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$, then $\mathcal{H}$ is called a *subgroup* of $\mathcal{L}$.

A partial subgroup $\mathcal{N}$ of $\mathcal{L}$ is called a *partial normal subgroup* of $\mathcal{L}$ (denoted $\mathcal{N} \trianglelefteq \mathcal{L}$) if for all $g \in \mathcal{L}$ and $n \in \mathcal{N}$,

$$n \in \mathbf{D}(g) \Longrightarrow n^g \in \mathcal{N}.$$

We remark that a subgroup $\mathcal{H}$ of $\mathcal{L}$ is always a group in the usual sense with the group multiplication defined by $hg = \Pi(h, g)$ for all $h, g \in \mathcal{H}$.

2.2 | Localities

Roughly speaking, localities are partial groups with some some extra structure, in particular with a “Sylow p-subgroup” and a set Δ of “objects” which in a certain sense determines the domain of the product. This is made more precise in Definition 2.5. We continue to assume that $\mathcal{L}$ is a partial group with product $\Pi : \mathbf{D} \to \mathcal{L}$. We will use the following notation.

Notation 2.4. If S is a subset of $\mathcal{L}$ and $g \in \mathcal{L}$, set

$$S_g := \{s \in S \cap \mathbf{D}(g) : s^g \in S\}.$$

More generally, if $w = (g_1, \ldots, g_n) \in \mathbf{W}(\mathcal{L})$ with $n \geqslant 1$, define S_w to be the set of elements $s \in S$ for which there exists a sequence of elements $s = s_0, s_1, \ldots, s_n \in S$ with $s_{i-1} \in \mathbf{D}(g_i)$ and $s_{i-1}^{g_i} = s_i$ for all $i = 1, \ldots, n$.

Definition 2.5. We say that $(\mathcal{L}, \Delta, S)$ is a *locality* if the partial group $\mathcal{L}$ is finite as a set, S is a p-subgroup of $\mathcal{L}$, Δ is a nonempty set of subgroups of S, and the following conditions hold.

(L1) S is maximal with respect to inclusion among the p-subgroups of $\mathcal{L}$.

(L2) For any word $w = (f_1, \ldots, f_n) \in \mathbf{W}(\mathcal{L})$, we have $w \in \mathbf{D}$ if and only if there exist $P_0, \ldots, P_n \in \Delta$ with
 (∗) $P_{i-1} \subseteq \mathbf{D}(f_i)$ and $P_{i-1}^{f_i} = P_i$ for all $i = 1, \ldots, n$.

(L3) The set Δ is closed under passing to $\mathcal{L}$-conjugates and overgroups in S, that is, Δ is overgroup-closed in S and, for every $P \in \Delta$ and $g \in \mathcal{L}$ such that $P \subseteq S_g$, we have $P^g \in \Delta$.

If $(\mathcal{L}, \Delta, S)$ is a locality, $w = (f_1, \ldots, f_n) \in \mathbf{W}(\mathcal{L})$, and $P_0, \ldots, P_n$ are elements of Δ such that (∗) holds, then we say that $w \in \mathbf{D}$ via $P_0, \ldots, P_n$ (or $w \in \mathbf{D}$ via P_0).

It is argued in [28, Remark 5.2] that Definition 2.5 is equivalent to the definition of a locality given by Chermak [15, Definition 2.7] (which is essentially the same as the one given in [14, Definition 2.9]).

Example 2.6. Let M be a finite group and $S \in \mathrm{Syl}_p(M)$. Set $\mathcal{F} = \mathcal{F}_S(M)$ and let Δ be a nonempty collection of subgroups of S, which is closed under $\mathcal{F}$-conjugacy and overgroup-closed in S. Set

$$\mathcal{L}_\Delta(M) := \{g \in G : S \cap S^g \in \Delta\} = \{g \in G : \exists P \in \Delta \text{ with } P^g \leqslant S\}$$

and let $\mathbf{D}$ be the set of tuples $(g_1, \dots, g_n) \in \mathbf{W}(M)$ such that there exist $P_0, P_1, \dots, P_n \in \Delta$ with $P_{i-1}^{g_i} = P_i$. Then $\mathcal{L}_\Delta(M)$ forms a partial group whose product is the restriction of the multivariable product on M to $\mathbf{D}$, and whose inversion map is the restriction of the inversion map on the group M to $\mathcal{L}_\Delta(M)$. Moreover, $(\mathcal{L}_\Delta(M), \Delta, S)$ forms a locality.

In the next lemma, we summarize the most important properties of localities that we will use throughout, most of the time without reference.

Lemma 2.7 (Important properties of localities). *Let $(\mathcal{L}, \Delta, S)$ be a locality. Then the following hold.*

(a) *$N_{\mathcal{L}}(P)$ is a subgroup of $\mathcal{L}$ for each $P \in \Delta$.*
(b) *Let $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq S_g$. Then $Q := P^g \in \Delta$, $N_{\mathcal{L}}(P) \subseteq \mathbf{D}(g)$ and*

$$c_g : N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q), x \mapsto x^g$$

is an isomorphism of groups.
(c) *Let $w = (g_1, \dots, g_n) \in \mathbf{D}$ via $(X_0, \dots, X_n)$. Then*

$$c_{g_1} \circ \dots \circ c_{g_n} = c_{\Pi(w)}$$

is a group isomorphism $N_{\mathcal{L}}(X_0) \to N_{\mathcal{L}}(X_n)$.
(d) *For every $g \in \mathcal{L}$, we have $S_g \in \Delta$. In particular, S_g is a subgroup of S. Moreover, $S_g^g = S_{g^{-1}}$ and $c_g : S_g \to S, x \mapsto x^g$ is an injective group homomorphism.*
(e) *For every $g \in \mathcal{L}$, $c_g : \mathbf{D}(g) \to \mathbf{D}(g^{-1}), x \mapsto x^g$ is a bijection with inverse map $c_{g^{-1}}$.*
(f) *For any $w \in \mathbf{W}(\mathcal{L})$, S_w is a subgroup of S with $S_w \in \Delta$ if and only if $w \in \mathbf{D}$. Moreover, $w \in \mathbf{D}$ implies $S_w \leqslant S_{\Pi(w)}$.*

Proof. Properties (a), (b), and (c) correspond to the statements (a), (b), and (c) in [15, Lemma 2.3] except for the fact stated in (b) that $Q \in \Delta$, which is, however, clearly true if one uses the definition of a locality given above. Property (d) holds by [15, Proposition 2.5(a),(b)] and property (e) is stated in [14, Lemma 2.5(c)]. Property (f) corresponds to [15, Corollary 2.6]. □

Let $(\mathcal{L}, \Delta, S)$ be a locality. Then it follows from Lemma 2.7(d) that, for every $P \in \Delta$ and every $g \in \mathcal{L}$ with $P \subseteq S_g$, the map $c_g : P \to P^g, x \mapsto x^g$ is an injective group homomorphism. The fusion system $\mathcal{F}_S(\mathcal{L})$ is the fusion system over S generated by such conjugation maps. Equivalently, $\mathcal{F}_S(\mathcal{L})$ is generated by the conjugation maps between subgroups of S, or by the conjugation maps of the form $c_g : S_g \to S, x \mapsto x^g$ with $g \in \mathcal{L}$.

Definition 2.8. If $\mathcal{F}$ is a fusion system, then we say that the locality $(\mathcal{L}, \Delta, S)$ is a locality over $\mathcal{F}$ if $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$.

If $(\mathcal{L}, \Delta, S)$ is a locality over $\mathcal{F}$, then notice that the set Δ is always overgroup-closed in S and closed under $\mathcal{F}$-conjugacy. Definition 2.8 says precisely that every morphism in $\mathcal{F}$ is a composite of conjugation maps. It is, however, not true in general that every $\mathcal{F}$-morphism is itself a conjugation map. In the following lemma, the assumption that P is an object in Δ (rather than just a subgroup of S) is therefore important.

Lemma 2.9. *Let $(\mathcal{L}, \Delta, S)$ be a locality over a fusion system $\mathcal{F}$ and $P \in \Delta$. Then the following hold.*

(a) *For every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, there exists $g \in \mathcal{L}$ such that $P \leqslant S_g$ and $\varphi(x) = x^g$ for all $x \in P$.*
(b) $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$.

Proof. For (a), see [28, Lemma 5.6]. As $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, one sees that $\mathcal{F}_{N_S(P)}(N_{\mathcal{L}}(P))$ is a subsystem of $N_{\mathcal{F}}(P)$. Conversely, by definition of the normalizer system, each morphism in $N_{\mathcal{F}}(P)$ extends to a morphism with source a subgroup of $N_S(P)$ containing P, so $N_{\mathcal{F}}(P)$ is generated as a fusion system by morphisms in $\mathcal{F}$ between objects of $\mathcal{L}$. Part (a) then gives equality. □

Suppose $(\mathcal{L}^+, \Delta^+, S)$ is a locality with partial product $\Pi^+ : \mathbf{D}^+ \to \mathcal{L}^+$. If Δ is a nonempty subset of Δ^+ which is closed under taking $\mathcal{L}^+$-conjugates and overgroups in S, we set

$$\mathcal{L}^+|_\Delta := \{f \in \mathcal{L}^+ : \exists P \in \Delta \text{ such that } P \subseteq \mathbf{D}^+(f) \text{ and } P^f \leqslant S\}$$

and write $\mathbf{D}$ for the set of words $w = (f_1, \dots, f_n)$ such that $w \in \mathbf{D}^+$ via $P_0, \dots, P_n$ for some $P_0, \dots, P_n \in \Delta$. Note that $\mathbf{D}$ is a set of words in $\mathcal{L}^+|_\Delta$ that is contained in $\mathbf{D}^+$. It is easy to check that $\mathcal{L}^+|_\Delta$ forms a partial group with partial product $\Pi^+|_{\mathbf{D}} : \mathbf{D} \to \mathcal{L}^+|_\Delta$, and that $(\mathcal{L}^+|_\Delta, \Delta, S)$ forms a locality; see [14, Lemma 2.21] for details. We call $\mathcal{L}^+|_\Delta$ the *restriction* of $\mathcal{L}^+$ to Δ.

2.3 | Projections of localities

Throughout this subsection, let $\mathcal{L}$ and $\mathcal{L}'$ be partial groups with products $\Pi : \mathbf{D} \to \mathcal{L}$ and $\Pi' : \mathbf{D}' \to \mathcal{L}'$, respectively.

Definition 2.10. Let $\beta : \mathcal{L} \to \mathcal{L}', g \mapsto g^\beta$ be a map. By abuse of notation, we denote by β also the induced map on words

$$\mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}'), \quad w = (f_1, \dots, f_n) \mapsto w^\beta = (f_1^\beta, \dots, f_n^\beta)$$

and set $\mathbf{D}^\beta = \{w^\beta : w \in \mathbf{D}\}$. We say that β is a homomorphism of partial groups if

(1) $\mathbf{D}^\beta \subseteq \mathbf{D}'$; and
(2) $\Pi(w)^\beta = \Pi'(w^\beta)$ for every $w \in \mathbf{D}$.

If moreover $\mathbf{D}^\beta = \mathbf{D}'$ (and thus β is in particular surjective), then we say that β is a *projection* of partial groups. If β is a bijective projection of partial groups, then β is called an *isomorphism*.

There is no accepted notion of a morphism of localities (at the same prime p) in the literature. One could, however, form a category of localities with morphisms the partial group homomorphisms – or alternatively there are several full subcategories of this which one might want to consider. For our purposes, it will be enough to consider the category of localities with projections of localities as defined next.

Definition 2.11. Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities and let $\beta : \mathcal{L} \to \mathcal{L}'$ be a projection of partial groups. We say that β is a *projection of localities* from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ if, setting $\Delta^\beta = \{P^\beta \mid P \in \Delta\}$, we have $\Delta^\beta = \Delta'$ (and thus $S^\beta = S'$).

If β is in addition bijective, then β is a called an *isomorphism* of localities. If $S = S'$, then an isomorphism of localities from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S)$ is called a *rigid isomorphism* if it restricts to the identity on S.

The notion of a rigid isomorphism will be important later on when talking about the uniqueness of certain localities attached to a given fusion system.

We will now describe some naturally occurring projections of localities. Suppose $(\mathcal{L}, \Delta, S)$ is a locality and $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$. A *coset* of $\mathcal{N}$ in $\mathcal{L}$ is a subset of the form

$$\mathcal{N}f := \{\Pi(n, f) : n \in \mathcal{N} \text{ such that } (n, f) \in \mathbf{D}\}$$

for some $f \in \mathcal{L}$. Unlike in groups, the set of cosets does not form a partition of $\mathcal{L}$ in general. Instead, one needs to focus on the *maximal cosets*, that is, the elements of the set of cosets of $\mathcal{N}$ in $\mathcal{L}$ that are maximal with respect to inclusion. By [15, Lemma 3.15], the set $\mathcal{L}/\mathcal{N}$ of maximal cosets of $\mathcal{N}$ in $\mathcal{L}$ forms a partition of $\mathcal{L}$. Thus, there is a natural map

$$\beta : \mathcal{L} \to \mathcal{L}/\mathcal{N}$$

sending each element $g \in \mathcal{L}$ to the unique maximal coset of $\mathcal{N}$ in $\mathcal{L}$ containing g. Set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$ and $\overline{\mathbf{D}} := \mathbf{D}^\beta := \{w^\beta : w \in \mathbf{D}\}$. By [15, Lemma 3.16], there is a unique map $\overline{\Pi} : \overline{\mathbf{D}} \to \overline{\mathcal{L}}$ and a unique involutory bijection $\overline{\mathcal{L}} \to \overline{\mathcal{L}}, \overline{f} \mapsto \overline{f}^{-1}$ such that $\overline{\mathcal{L}}$ with these structures is a partial group, and such that β is a projection of partial groups. Moreover, setting $\overline{S} := S^\beta$ and $\overline{\Delta} := \{P^\beta : P \in \Delta\}$, the triple $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is by [15, Corollary 4.5] a locality, and β is a projection from $(\mathcal{L}, \Delta, S)$ to $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$. The map β is called the *natural projection* from $\mathcal{L} \to \overline{\mathcal{L}}$.

The notation used above suggests already that we will use a "bar notation" similar to the one commonly used in finite groups. Namely, if we set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$, then for every subset or element P of $\mathcal{L}$, we will denote by $\overline{P}$ the image of P under the natural projection $\beta : \mathcal{L} \to \overline{\mathcal{L}}$. We conclude this section with a little lemma needed later on.

Lemma 2.12. *Let $(\mathcal{L}, \Delta, S)$ be a locality with partial normal subgroup $\mathcal{N}$. Setting $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$, the preimage of $\overline{S}$ under the natural projection equals $\mathcal{N}S$.*

Proof. For every $s \in S$, the coset $\mathcal{N}s$ is maximal by [15, Lemma 3.7(a) and Proposition 3.14(c)]. Thus, for every $s \in S$, we have $\overline{s} = \mathcal{N}s$. Hence, the preimage of $\overline{S} = \{\overline{s} : s \in S\}$ equals $\bigcup_{s \in S} \mathcal{N}s = \mathcal{N}S$. □

2.4 | Transporter systems

Throughout this section, fix a finite p-group S, a fusion system $\mathcal{F}$ over S, and a collection Δ of nonidentity subgroups of S that is overgroup-closed in S and closed under $\mathcal{F}$-conjugacy. As the literature about transporter systems is written in left-hand notation, in this section, we will also *write our maps on the left-hand side of the argument*. Accordingly, we will conjugate from the left.

The *transporter category* $\mathcal{T}_S(G)$ (at the prime p) of a finite group G with Sylow p-subgroup S is the category with objects the nonidentity subgroups of S and with morphisms given by the transporter sets $N_G(P,Q) = \{g \in G \mid {}^gP \leqslant Q\}$. More precisely, the morphisms in $\mathcal{T}_S(G)$ between P and Q are the triples (g, P, Q) with $g \in N_G(P,Q)$. We also write $\mathcal{T}_\Delta(G)$ for the full subcategory of $\mathcal{T}_S(G)$ with objects in Δ.

As we conjugate in this section from the left, for $P, Q \leqslant S$ and $g \in N_G(P,Q)$, we write c_g for the conjugation map from P to Q given by *left conjugation*, that is,

$$c_g : P \to Q,\ x \mapsto {}^gx.$$

Definition 2.13 [45, Definition 3.1]. A *transporter system* associated to $\mathcal{F}$ is a nonempty finite category $\mathcal{T}$ having object set $\Delta \subseteq \mathrm{Ob}(\mathcal{F})$, together with functors

$$\mathcal{T}_\Delta(S) \xrightarrow{\ \epsilon\ } \mathcal{T} \xrightarrow{\ \rho\ } \mathcal{F}$$

that satisfy the following axioms.

(A1) Δ is closed under $\mathcal{F}$-conjugacy and passing to overgroups, ϵ is the identity on objects, and ρ is the inclusion on objects.

(A2) For each $P, Q \in \Delta$, the kernel

$$E(P) := \ker(\rho_{P,P} : \mathrm{Aut}_{\mathcal{T}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P))$$

acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by right composition, and $\rho_{P,Q}$ is the orbit map for this action. Also, $E(Q)$ acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by left composition.

(B) For each $P, Q \in \Delta$, $\epsilon_{P,Q} : N_S(P,Q) \to \mathrm{Mor}_{\mathcal{T}}(P,Q)$ is injective, and the composite $\rho_{P,Q} \circ \epsilon_{P,Q}$ sends $s \in N_S(P,Q)$ to $c_s \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.

(C) For all $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ and all $g \in P$, the diagram

$$\begin{tikzcd}
P \arrow[r, "\phi"] \arrow[d, "\epsilon_{P,P}(g)"'] & Q \arrow[d, "\epsilon_{Q,Q}(\rho(\phi)(g))"] \\
P \arrow[r, "\phi"'] & Q
\end{tikzcd}$$

commutes in $\mathcal{T}$.

(I) $\epsilon_{S,S}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{T}}(S)$.

(II) Let $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$, let $P \trianglelefteq \bar{P} \leqslant S$, and let $Q \trianglelefteq \bar{Q} \leqslant S$ be such that $\varphi \circ \epsilon_{P,P}(\bar{P}) \circ \varphi^{-1} \leqslant \epsilon_{Q,Q}(\bar{Q})$. Then there exists $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(\bar{P}, \bar{Q})$ such that $\bar{\varphi} \circ \epsilon_{P,\bar{P}}(1) = \epsilon_{Q,\bar{Q}}(1) \circ \varphi$.

If we want to be more precise, we say that $(\mathcal{T}, \epsilon, \rho)$ is a transporter system.

Note that, by [45, Lemmas 3.2(b) and 3.8], every morphism in a transporter system is both a monomorphism and an epimorphism.

A centric linking system in the sense of [8] is a transporter system in which Δ is the set of $\mathcal{F}$-centric subgroups and $E(P)$ is precisely the center $Z(P)$ viewed as a subgroup of $N_S(P)$ via the map $\epsilon_{P,P}$. A more general notion of linking system will be introduced in Subsection 2.6.

We next want to state the definition of an isomorphism of transporter systems as used in [25]. First we prove a lemma that helps to explain that a property implicitly assumed there, and which is needed for the definition to make sense, does in fact hold.

Given a fusion system $\mathcal{F}$ on a finite p-group S and a set Δ of subgroups of S, we write $\mathcal{F}|_\Delta$ for the full subcategory of $\mathcal{F}$ with object set Δ.

Lemma 2.14. *Let $(\mathcal{T}, \epsilon, \rho)$ and $(\mathcal{T}', \epsilon', \rho')$ be two transporter systems having object sets Δ and Δ', associated with the fusion systems $\mathcal{F}$ and $\mathcal{F}'$ over the p-groups S and S', respectively. Let $\alpha : \mathcal{T} \to \mathcal{T}'$ be any equivalence of categories. Then the following hold.*

(a) *$\alpha(S) = S'$, and $\alpha(P) \leqslant \alpha(S)$ for each $P \in \Delta$.*

(b) *Suppose α has the following two additional properties:*
 (typ) *$\alpha_{P,P}(\epsilon_{P,P}(P)) = \epsilon'_{\alpha(P),\alpha(P)}(\alpha(P))$ for each $P \in \Delta$, and*
 (inc) *$\alpha_{P,S}(\epsilon_{P,S}(1)) = \epsilon'_{\alpha(P),\alpha(S)}(1)$ for each $P \in \Delta$,*
 and set

$$\beta = (\epsilon'_{S',S'})^{-1} \circ \alpha_{S,S} \circ \epsilon_{S,S}.$$

 Then
 (i) *β is an isomorphism of groups from S to S',*
 (ii) *$\alpha(P) = \beta(P)$ for all $P \in \Delta$, and for each $P, Q \in \Delta$ with $P \leqslant Q$, we have $\alpha(P) \leqslant \alpha(Q)$ and $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$,*
 (iii) *the functor $c_\beta : \mathcal{F}|_\Delta \to \mathcal{F}'|_{\alpha(\Delta)}$ defined by $c_\beta(P) = \beta(P)$ for each $P \in \Delta$, and by $c_\beta(\varphi) = \beta \circ \varphi \circ \beta^{-1}$ for each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ with $P, Q \in \Delta$, is well-defined and an isomorphism of categories.*

(c) *If α satisfies (typ) and (inc), then α is a bijection on objects, and hence an isomorphism of categories.*

Proof.

(a) The subgroup S (resp., S') is characterized as the unique object of $\mathcal{T}$ (resp., $\mathcal{T}'$) that receives a morphism from every object. As α is an equivalence, this implies $\alpha(S) = S'$, and hence also $\alpha(P) \leqslant S' = \alpha(S)$ for all $P \in \Delta$.

(b) By (a), $\epsilon'_{\alpha(P),\alpha(S)}(1)$ is defined, so the right-hand side of the equality in (inc) makes sense. Assume (typ) and (inc) and set $\beta = (\epsilon'_{S',S'})^{-1} \circ \alpha_{S,S} \circ \epsilon_{S,S}$ as above.
 (i) As α is an equivalence, it is a bijection on morphism sets. As $\alpha_{S,S}(\epsilon_{S,S}(S)) = \epsilon'_{S',S'}(S')$ by (a) and (typ), and as ϵ is injective on morphism sets (Axiom (B)), β is thus a well-defined isomorphism of groups from $\mathrm{Aut}_{\mathcal{T}_\Delta(S)}(S) = N_S(S) = S$ to $\mathrm{Aut}_{\mathcal{T}_{\Delta'}(S')}(S') = N_{S'}(S') = S'$.
 (ii) Proving $\alpha(P) = \beta(P)$ for all $P \in \Delta$ means showing $\alpha_{S,S}(\epsilon_{S,S}(P)) = \epsilon'_{S',S'}(\alpha(P))$ for all such P. For the proof fix an object P and let $x \in P$. By (typ), there exist $y \in \alpha(S) = S'$ and

$y' \in \alpha(P)$ such that $\alpha_{S,S}(\epsilon_{S,S}(x)) = \epsilon'_{S',S'}(y)$ and $\alpha_{P,P}(\epsilon_{P,P}(x)) = \epsilon'_{\alpha(P),\alpha(P)}(y')$. Note that

$$\epsilon_{P,S}(1) \circ \epsilon_{P,P}(x) = \epsilon_{P,S}(x) = \epsilon_{S,S}(x) \circ \epsilon_{P,S}(1).$$

Applying α we obtain from this and our assumption that

$$\epsilon'_{\alpha(P),S'}(1) \circ \epsilon'_{\alpha(P),\alpha(P)}(y') = \epsilon'_{S',S'}(y) \circ \epsilon'_{\alpha(P),S'}(1)$$

and thus $\epsilon'_{\alpha(P),S'}(y') = \epsilon'_{\alpha(P),S'}(y)$. By axiom (B), $\epsilon'_{\alpha(P),S'}$ is injective. Thus, $y = y' \in \alpha(P)$ and thus $\alpha_{S,S}(\epsilon_{S,S}(x)) = \epsilon'_{S',S'}(y) \in \epsilon'_{S',S'}(\alpha(P))$. This proves $\alpha_{S,S}(\epsilon_{S,S}(P)) \leqslant \epsilon'_{S',S'}(\alpha(P))$.

To prove the opposite inclusion, let $a \in \alpha(P)$. By the first assumption, there exist $x \in S$ and $x' \in P$ such that $\epsilon'_{S',S'}(a) = \alpha_{S,S}(\epsilon_{S,S}(x))$ and $\epsilon'_{\alpha(P),\alpha(P)}(a) = \alpha_{P,P}(\epsilon_{P,P}(x'))$. Then

$$\begin{aligned}
\alpha_{P,S}(\epsilon_{P,S}(x')) &= \alpha_{P,S}(\epsilon_{P,S}(1)) \circ \alpha_{P,P}(\epsilon_{P,P}(x')) \\
&= \epsilon'_{\alpha(P),S'}(1) \circ \epsilon'_{\alpha(P),\alpha(P)}(a) \\
&= \epsilon'_{\alpha(P),S'}(a) \\
&= \epsilon'_{S',S'}(a) \circ \epsilon'_{\alpha(P),S'}(1) \\
&= \alpha_{S,S}(\epsilon_{S,S}(x)) \circ \alpha_{P,S}(\epsilon_{P,S}(1)) \\
&= \alpha_{P,S}(\epsilon_{P,S}(x)).
\end{aligned}$$

As $\alpha_{P,S}$ and $\epsilon_{P,S}$ are injective (Axiom B), it follows $x = x' \in P$ and thus $\epsilon'_{S',S'}(a) \in \alpha_{S,S}(\epsilon_{S,S}(P))$. This shows $\alpha_{S,S}(\epsilon_{S,S}(P)) = \epsilon'_{S',S'}(\alpha(P))$ and completes the proof that α and β coincide on objects.

Now let $P \leqslant Q$ be an inclusion of subgroups in Δ. Then $\alpha(P) = \beta(P) \leqslant \beta(Q) = \alpha(Q)$. The equality $\epsilon_{Q,S}(1) \circ \epsilon_{P,Q}(1) = \epsilon_{P,S}(1)$ holds in $\mathcal{T}$. Applying α to this and using (inc) we have

$$\epsilon'_{\alpha(Q),\alpha(S)}(1) \circ \alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(S)}(1).$$

On the other hand, $\epsilon'_{\alpha(P),\alpha(Q)}(1)$ is a morphism in $\mathcal{T}'$ because $\alpha(P) \leqslant \alpha(Q)$, and we have a similar equality

$$\epsilon'_{\alpha(Q),\alpha(S)}(1) \circ \epsilon'_{\alpha(P),\alpha(Q)}(1) = \epsilon'_{\alpha(P),\alpha(S)}(1).$$

As $\epsilon'_{\alpha(Q),\alpha(S)}(1)$ is a monomorphism in $\mathcal{T}'$, we have $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$, completing the proof of (ii).

(iii) This is more or less shown in the proof of [25, Proposition 2.5], but not all details are given there. In any case, our stated hypotheses here are weaker (we do not assume Δ, Δ' contain $\mathcal{F}^{cr}$, $\mathcal{F}'^{cr}$) and our conclusion is weaker (we only are claiming an isomorphism of full subcategories of the fusion systems, not of the fusion systems themselves). So, we repeat the proof and add the details.

Step 1: Let $\beta_* : \mathcal{T}_\Delta(S) \to \mathcal{T}_{\Delta'}(S')$ be the functor defined by β on objects and also β on morphism sets. We first show $\epsilon' \circ \beta_* = \alpha \circ \epsilon$.

The functors $\epsilon' \circ \beta_*$ and $\alpha \circ \epsilon$ agree on objects by (ii) and (A1). They also agree on morphism sets as we now show. Let $P, Q \in \Delta$ and $s \in N_S(P, Q)$. In $\mathcal{T}$ we have

$$\epsilon_{S,S}(s) \circ \epsilon_{P,S}(1) = \epsilon_{Q,S}(1) \circ \epsilon_{P,Q}(s).$$

Applying α, and using (inc) and the definition of β, we have

$$\epsilon'_{S',S'}(\beta(s)) \circ \epsilon'_{\alpha(P),S'}(1) = \epsilon'_{\alpha(Q),S'}(1) \circ \alpha_{P,Q}(\epsilon_{P,Q}(s)).$$

We also have the same equality when $\alpha_{P,Q}(\epsilon_{P,Q}(s))$ is replaced by $\epsilon'_{\alpha(P),\alpha(Q)}(\beta(s))$. As every morphism of $\mathcal{T}'$ is a monomorphism, this forces $\alpha_{P,Q}(\epsilon_{P,Q}(s)) = \epsilon'_{\alpha(P),\alpha(Q)}(\beta(s))$. Thus,

$$\epsilon' \circ \beta_* = \alpha \circ \epsilon. \tag{2.1}$$

Step 2: We next show that for each $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ with $P, Q \in \Delta$,

$$c_\beta(\rho_{P,Q}(\varphi)) = \rho'_{\alpha(P),\alpha(Q)}(\alpha_{P,Q}(\varphi)),$$

but let us omit subscripts on ρ, ρ', and α in the proof to lighten the notation, after which the equality read reads

$$c_\beta(\rho(\varphi)) = \rho'(\alpha(\varphi)).$$

This then implies $\beta \circ \rho(\varphi) \circ \beta^{-1} \in \mathrm{Hom}(\beta(P), \beta(Q))$ is a morphism in $\mathcal{F}'$. Using that ρ is surjective on morphisms by (A2), c_β as defined in (iii) is thus a well-defined functor.

Let $x \in P$. By Axiom (C) for $\mathcal{T}$, we have

$$\varphi \circ \epsilon_{P,P}(x) = \epsilon_{Q,Q}(\rho(\varphi)(x)) \circ \varphi.$$

Applying α and using (2.1), this gives

$$\alpha(\varphi) \circ \epsilon'_{\alpha(P),\alpha(P)}(\beta(x))) = \epsilon'_{\alpha(Q),\alpha(Q)}(\beta(\rho(\varphi)(x))) \circ \alpha(\varphi).$$

On the other hand, Axiom (C) for $\mathcal{T}'$ says we have the same equality if we replace the instance of $\epsilon'_{\alpha(Q),\alpha(Q)}(\beta(\rho(\varphi)(x)))$ by $\epsilon'_{\alpha(Q),\alpha(Q)}(\rho'(\alpha(\varphi))(\beta(x)))$. As $\alpha(\varphi)$ is an epimorphism in $\mathcal{T}'$ and $\epsilon'_{\alpha(Q),\alpha(Q)}$ is injective, we have shown

$$c_\beta(\rho(\varphi))(\beta(x)) = \beta(\rho(\varphi)(x)) = \rho'(\alpha(\varphi))(\beta(x))$$

for all $x \in P$, and hence $c_\beta(\rho(\varphi)) = \rho'(\alpha(\varphi))$ as claimed.

Step 3. We finish the proof of (iii). By (i) and (ii), c_β is a bijection on objects, and it is also an injection on morphisms. By assumption on α and axiom (A2) for $\mathcal{T}'$, the composite functor $\rho' \circ \alpha$ is surjective on morphisms, so c_β is surjective on morphisms by Step 2.

(c) By assumption on α and (i)–(ii), α is injective on objects, and it remains to prove that α is surjective on objects. Assume this is not the case, and choose $Q' \in \Delta'$ of minimal index in S' subject to $Q' \notin \alpha(\Delta)$. As α is essentially surjective, there is $P \in \Delta$ and an isomorphism $\varphi' : \alpha(P) \to Q'$ in $\mathcal{T}'$. By Alperin's Fusion Theorem for transporter systems [45, Proposition 3.9], there are subgroups $\alpha(P) = Q'_0, \dots, Q'_n = Q'$ in Δ', subgroups $T'_i \geqslant \langle Q'_{i-1}, Q'_i \rangle$, automorphisms $\tau'_i \in \mathrm{Aut}_{\mathcal{T}'}(T'_i)$, and isomorphisms $\varphi'_i \in \mathrm{Iso}_{\mathcal{T}'}(Q'_{i-1}, Q'_i)$ for $i = 1, \dots, n$, such that $\varphi'_i = \tau'_i|_{Q'_{i-1}, Q'_i}$ for all i, and $\varphi' = \varphi'_n \circ \cdots \circ \varphi'_1$.

Choose the least index $m \in \{0, 1, \dots, n\}$ such that $Q'_m \notin \alpha(\Delta)$. Then $m > 0$ and $Q'_{m-1} \in \alpha(\Delta)$, so that $Q'_{m-1} \neq Q'_m$. The subgroup T'_m therefore has smaller index in S' than Q', and hence $T'_m \in \alpha(\Delta)$. Fix $T_m, Q_{m-1} \in \Delta$ and $\tau_m \in \mathrm{Aut}_{\mathcal{T}}(T_m)$ such that $\beta(Q_{m-1}) = \alpha(Q_{m-1}) = Q'_{m-1}$, $\beta(T_m) = \alpha(T_m) = T'_m$ and $\alpha_{T_m,T_m}(\tau_m) = \tau'_m$.

Now $\rho'(\tau'_m)$ is an $\mathcal{F}'$-automorphism of T'_m sending Q'_{m-1} onto Q'_m. Thus, by (iii), $c_\beta^{-1}(\rho'(\tau'_m)) = \beta^{-1} \circ \rho'(\tau'_m) \circ \beta$ is an $\mathcal{F}$-automorphism of T_m sending $Q_{m-1} = \beta^{-1}(Q'_{m-1})$ onto $\beta^{-1}(Q'_m)$. As $Q_{m-1} \in \Delta$ and Δ is closed under $\mathcal{F}$-conjugacy by (A1), we have $\beta^{-1}(Q'_m) \in \Delta$. Hence, $Q'_m = \alpha(\beta^{-1}(Q'_m)) \in \alpha(\Delta)$ by (ii), and this contradicts the choice of m. □

An equivalence satisfying (typ) is said to be *isotypical*. One satisfying (inc) is said to *send inclusions to inclusions*. Given Lemma 2.14, it is now sensible to define an isomorphism of transporter systems to be an isotypical equivalence sending inclusions to inclusions.

Definition 2.15. Let $\mathcal{F}$ and $\mathcal{F}'$ be fusion systems over the finite p-groups S and S', and let $(\mathcal{T}, \epsilon, \rho)$ and $(\mathcal{T}', \epsilon', \rho')$ be transporter systems over $\mathcal{F}$ and $\mathcal{F}'$, respectively. An equivalence of categories $\alpha : \mathcal{T} \to \mathcal{T}'$ is called an *isomorphism* of transporter systems if for all objects P of $\mathcal{T}$,

(typ) $\alpha_{P,P}(\epsilon_{P,P}(P)) = \epsilon'_{\alpha(P),\alpha(P)}(\alpha(P))$ and
(inc) $\alpha_{P,S}(\epsilon_{P,S}(1)) = \epsilon'_{\alpha(P),\alpha(S)}(1)$.

An isomorphism $\alpha : \mathcal{T} \to \mathcal{T}'$ of transporter systems is said to be *rigid* if $S = S'$ and $\alpha_{S,S} \circ \epsilon_{S,S} = \epsilon'_{S,S}$ as homomorphisms $S \to \mathrm{Aut}_{\mathcal{T}'}(S)$.

Thus, an isomorphism of transporter systems is in particular an isomorphism of categories by Lemma 2.14(c), and a rigid isomorphism is one for which the isomorphism β of Lemma 2.14(b) is the identity map on $S = S'$.

This version of the definition of isomorphism of transporter systems is equivalent to the one given in [25, Definition 2.3]. The definition of an isotypical equivalence is the same here as there. The authors formulated the condition that α sends inclusions to inclusions by requiring that $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$ for each pair of objects $P \leqslant Q$ in [25]. But as was pointed out to the third author by Julian Kaspczyk, this assumes implicitly that $P \leqslant Q$ implies $\alpha(P) \leqslant \alpha(Q)$, which presumably need not hold for an arbitrary equivalence. Lemma 2.14(a), however, shows that (inc) makes sense when α is isotypical, and then Lemma 2.14(ii) gives indeed that $\alpha(P) \leqslant \alpha(Q)$ and $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$ whenever $P \leqslant Q$ are objects of $\mathcal{T}$. Thus, (typ) and the seemingly stronger condition

(inc') $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$ whenever $P \leqslant Q$ are objects of $\mathcal{T}$ with $\alpha(P) \leqslant \alpha(Q)$

are together equivalent to conditions (typ) and (inc). We will refer to property (inc') also by saying that "α sends inclusions to inclusions".

2.5 | The correspondence between transporter systems and localities

Throughout this subsection, let $\mathcal{F}$ be a fusion system over S.

Every locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ leads to a transporter system associated to $\mathcal{F}$. To see that we need to consider conjugation from the *left*. If $f, x \in \mathcal{L}$ such that $(f, x, f^{-1}) \in \mathbf{D}$ (or equivalently $x \in \mathbf{D}(f^{-1})$), then we set ${}^f x := \Pi(f, x, f^{-1}) = x^{f^{-1}}$. Similarly, if $f \in \mathcal{L}$ and $\mathcal{H} \subseteq \mathbf{D}(f^{-1})$, then set

$$^{f}\mathcal{H} := \mathcal{H}^{f^{-1}} := \{{}^f x : x \in \mathcal{H}\}.$$

Define $\mathcal{T}_\Delta(\mathcal{L})$ to be the category whose object set is Δ with the morphism set $\mathrm{Mor}_{\mathcal{T}_\Delta(\mathcal{L})}(P, Q)$ between two objects $P, Q \in \Delta$ given as the set of triples (f, P, Q) with $f \in \mathcal{L}$ such that $P \subseteq \mathbf{D}(f^{-1})$ and ${}^f P \leqslant Q$. This leads to a transporter system $(\mathcal{T}_\Delta(\mathcal{L}), \epsilon, \rho)$, where for all $P, Q \in \Delta$, $\epsilon_{P,Q}$ is the inclusion map and $\rho_{P,Q}$ sends (f, P, Q) to the conjugation map $P \to Q,\ x \mapsto {}^f x$.

Conversely, Chermak showed in [14, appendix] essentially that every transporter system leads to a locality. More precisely, it is proved in [25, Theorem 2.11] that there is an equivalence of categories between the category of transporter systems with morphisms the isomorphisms and the category of localities with morphisms the isomorphisms, and such an equivalence can be chosen to preserve the rigid isomorphisms. The definition of a locality in [25] differs slightly from the one given in this paper, but the two definitions can be seen to be equivalent if one uses first that conjugation by $f \in \mathcal{L}$ from the left corresponds to conjugation by f^{-1} from the right, and second that for every partial group $\mathcal{L}$ with product $\Pi : \mathbf{D} \to \mathcal{L}$ the axioms of a partial group yield $\mathbf{D} = \{w \in \mathbf{W}(\mathcal{L}) : w^{-1} \in \mathbf{D}\}$.

We will consider punctured groups in either setting thus using the term "punctured group" slightly abusively.

Definition 2.16. We call a transporter system $\mathcal{T}$ over $\mathcal{F}$ a *punctured group* if the object set of $\mathcal{T}$ equals the set of all nonidentity subgroups. Similarly, a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ is said to be a *punctured group* if Δ is the set of all nonidentity subgroups of S.

Observe that a transporter system over $\mathcal{F}$ that is a punctured group exists if and only if a locality over $\mathcal{F}$ that is a punctured group exists. If it matters it will always be clear from the context whether we mean by a punctured group a transporter system or a locality.

2.6 | Linking localities and linking systems

As we have seen in the previous subsection, localities correspond to transporter systems. Of fundamental importance in the theory of fusion systems are (centric) linking systems, which are special cases of transporter systems. It is therefore natural to look at localities corresponding to linking systems. Thus, we will introduce special kinds of localities called linking localities. We will moreover introduce a (slightly nonstandard) definition of linking systems and summarize some of the most important results about the existence and uniqueness of linking systems and linking localities. Throughout this subsection let $\mathcal{F}$ be a saturated fusion system over S.

We refer the reader to [2] for the definitions of $\mathcal{F}$-centric and $\mathcal{F}$-centric radical subgroups denoted by $\mathcal{F}^c$ and $\mathcal{F}^{cr}$ respectively. Moreover, we will use the following definition that was introduced in [28].

Definition 2.17. A subgroup $P \leqslant S$ is called $\mathcal{F}$-*subcentric* if $O_p(N_{\mathcal{F}}(Q))$ is centric in $\mathcal{F}$ for every fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate Q of P. The set of subcentric subgroups is denoted by $\mathcal{F}^s$.

Recall that $\mathcal{F}$ is called constrained if there is an $\mathcal{F}$-centric normal subgroup of $\mathcal{F}$. It is shown in [28, Lemma 3.1] that a subgroup $P \leqslant S$ is $\mathcal{F}$-subcentric if and only if for some (and thus for every) fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate Q of P, the normalizer $N_{\mathcal{F}}(Q)$ is constrained.

Definition 2.18.

- A finite group G is said to be of *characteristic* p if $C_G(O_p(G)) \leqslant O_p(G)$.
- Define a locality $(\mathcal{L}, \Delta, S)$ to be of *objective characteristic* p if, for any $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic p.
- A locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ is called a *linking locality*, if $\mathcal{F}^{cr} \subseteq \Delta$ and $(\mathcal{L}, \Delta, S)$ is of objective characteristic p.
- A *subcentric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^s, S)$ over $\mathcal{F}$. Similarly, a *centric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^c, S)$ over $\mathcal{F}$.

If $(\mathcal{L}, \Delta, S)$ is a centric linking locality, then it is shown in [28, Proposition 1] that the corresponding transporter system $\mathcal{T}_\Delta(\mathcal{L})$ is a centric linking system. Also, if $(\mathcal{L}, \Delta, S)$ is a centric linking locality, then it is a centric linking system in the sense of Chermak [14], that is, we have the property that $C_{\mathcal{L}}(P) \leqslant P$ for every $P \in \Delta$.

The term linking system is used in [28] for all transporter systems coming from linking localities, as such transporter systems have properties similar to the ones of linking systems in Oliver's definition [40] and can be seen as a generalization of such linking systems. We adapt this slightly nonstandard definition here.

Definition 2.19. A *linking system* over $\mathcal{F}$ is a transporter system $\mathcal{T}$ over $\mathcal{F}$ such that $\mathcal{F}^{cr} \subseteq \mathrm{obj}(\mathcal{T})$ and $\mathrm{Aut}_{\mathcal{T}}(P)$ is of characteristic p for every $P \in \mathrm{obj}(\mathcal{T})$. A *subcentric linking system* over $\mathcal{F}$ is a linking system $\mathcal{T}$ whose set of objects is the set $\mathcal{F}^s$ of subcentric subgroups.

Proving the existence and uniqueness of centric linking systems was a long-standing open problem, which was solved by Chermak [14]. Building on a basic idea in Chermak's proof, Oliver [41] gave a new one via an earlier developed cohomological obstruction theory. Both proofs depend *a priori* on the classification of finite simple groups, but work of Glauberman and the third author of this paper [24] removes the dependence of Oliver's proof on the classification. The precise theorem proved is the following.

Theorem 2.20 (Chermak [14], Oliver [41], Glauberman–Lynd [24]). *There exists a centric linking system associated to $\mathcal{F}$ that is unique up to an isomorphism of transporter systems. Similarly, there exists a centric linking locality over $\mathcal{F}$ that is unique up to a rigid isomorphism.*

Using the existence and uniqueness of centric linking systems one can relatively easily prove the following theorem.

Theorem 2.21 (Henke [28]). *The following hold.*

(a) *If $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^s$ such that Δ is overgroup-closed in S and closed under $\mathcal{F}$-conjugacy, then there exists a linking locality over $\mathcal{F}$ with object set Δ, and such a linking locality is unique up to a rigid isomorphism. Similarly, there exists a linking system $\mathcal{T}$ associated to $\mathcal{F}$ whose set of objects is Δ, and such a linking system is unique up to an isomorphism of transporter systems. Moreover, the nerve $|\mathcal{T}|$ is homotopy equivalent to the nerve of a centric linking system associated to $\mathcal{F}$.*

(b) *The set $\mathcal{F}^s$ is overgroup-closed in S and closed under $\mathcal{F}$-conjugacy. In particular, there exists a subcentric linking locality over $\mathcal{F}$ that is unique up to a rigid isomorphism, and there exists a subcentric linking system associated to $\mathcal{F}$ that is unique up to an isomorphism of transporter systems.*

The existence of subcentric linking systems stated in part (b) of the above theorem gives often a way of proving the existence of a punctured group, and indeed yields a punctured group directly when the fusion system is of characteristic p-type.

Definition 2.22. The saturated fusion system $\mathcal{F}$ is of *characteristic p-type* if $N_{\mathcal{F}}(Q)$ is constrained for every nontrivial fully $\mathcal{F}$-normalized subgroup Q of S.

Note that a fusion system $\mathcal{F}$ is constrained if and only if the trivial subgroup of S is $\mathcal{F}$-subcentric, and thus if and only if every subgroup of S is $\mathcal{F}$-subcentric. The next lemma gives an analogous characterization for fusion systems of characteristic p-type. Properties (c) and (c') of it will be the usual way we use the characteristic p-type condition in Section 5, for example.

Lemma 2.23. *The following are equivalent.*

(a) *$\mathcal{F}$ is of characteristic p-type.*
(b) *Every nonidentity subgroup of S is $\mathcal{F}$-subcentric.*
(c) *$N_{\mathcal{F}}(Q)$ is constrained for each fully normalized $Q \leqslant S$ of order p.*
(c') *$C_{\mathcal{F}}(Q)$ is constrained for each fully normalized $Q \leqslant S$ of order p.*

Proof. Points (a) and (b) are equivalent by the definition of subcentric subgroup. For the equivalence of (c) and (c'), we refer to [28, Lemma 2.13]. Finally, (b) and (c) are equivalent by Theorem 2.21(b) (i.e., $\mathcal{F}^s$ is overgroup-closed and closed under $\mathcal{F}$-conjugacy). □

Thus, if $\mathcal{F}$ is of characteristic p-type but not constrained, then the set $\mathcal{F}^s$ equals the set of all nonidentity subgroups. In any case, by Theorem 2.21(b) and Lemma 2.23, there exists a canonical punctured group associated to each $\mathcal{F}$ of characteristic p-type, namely the subcentric linking locality (or the subcentric linking system if one uses the language of transporter systems).

2.7 | Partial normal p'-subgroups

Normal p'-subgroups are often considered in finite group theory. We will now introduce a corresponding notion in localities and prove some basic properties. Throughout this subsection, let $(\mathcal{L}, \Delta, S)$ be a locality.

Definition 2.24. A *partial normal p'-subgroup* of $\mathcal{L}$ is a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $\mathcal{N} \cap S = 1$. The locality $(\mathcal{L}, \Delta, S)$ is said to be *p'-reduced* if there is no nontrivial partial normal p'-subgroup of $\mathcal{L}$.

Remark 2.25. If $(\mathcal{L}, \Delta, S)$ is a locality over a fusion system $\mathcal{F}$, then for any p'-group N, the direct product $(\mathcal{L} \times N, \Delta, S)$ is a locality over $\mathcal{F}$ such that N is a partial normal p'-subgroup of $\mathcal{L} \times N$ and $(\mathcal{L} \times N)/N \cong \mathcal{L}$; see [27] for details of the construction of direct products of localities. Thus, if we want to prove classification theorems for localities, it is actually reasonable to restrict attention to p'-reduced localities.

Recall that, for a finite group G, the largest normal p'-subgroup is denoted by $O_{p'}(G)$. Indeed, a similar notion can be defined for localities. Namely, it is a special case of [15, Theorem 5.1] that the product of two partial normal p'-subgroups is again a partial normal p'-subgroup. Thus, the following definition makes sense.

Definition 2.26. The largest normal p'-subgroup of $\mathcal{L}$ is denoted by $O_{p'}(\mathcal{L})$.

We will now prove some properties of partial normal p'-subgroups. To start, we show two lemmas that generalize corresponding statements for groups. The first of these lemmas gives a way of passing from an arbitrary locality to a p'-reduced locality.

Lemma 2.27. *Set $\overline{\mathcal{L}} := \mathcal{L}/O_{p'}(\mathcal{L})$. Then $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is p'-reduced.*

Proof. Let $\mathcal{N}$ be the preimage of $O_{p'}(\overline{\mathcal{L}})$ under the natural projection $\mathcal{L} \to \overline{\mathcal{L}}$. Then by [15, Proposition 4.7], $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$ containing $O_{p'}(\mathcal{L})$. Moreover, $\overline{\mathcal{N} \cap S} \subseteq \overline{\mathcal{N}} \cap \overline{S} = 1$, which implies $\mathcal{N} \cap S \subseteq O_{p'}(\mathcal{L})$ and thus $\mathcal{N} \cap S \subseteq O_{p'}(\mathcal{L}) \cap S = 1$. Thus, $\mathcal{N}$ is a partial normal p' subgroup of $\mathcal{L}$ and so by definition contained in $O_{p'}(\mathcal{L})$. This implies $O_{p'}(\overline{\mathcal{L}}) = \overline{\mathcal{N}} = 1$. □

Lemma 2.28. *Given a partial normal p'-subgroup $\mathcal{N}$ of $\mathcal{L}$, the image of $O_{p'}(\mathcal{L})$ in $\mathcal{L}/\mathcal{N}$ under the canonical projection is a partial normal p'-subgroup of $\mathcal{L}/\mathcal{N}$. In particular, if $\mathcal{L}/\mathcal{N}$ is p'-reduced, then $\mathcal{N} = O_{p'}(\mathcal{L})$.*

Proof. Set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$. Then by [15, Proposition 4.7], $\overline{O_{p'}(\mathcal{L})}$ is a partial normal subgroup of $\overline{\mathcal{L}}$. By Lemma 2.12, the preimage of $\overline{S}$ equals $\mathcal{N}S$. As $\mathcal{N} \subseteq O_{p'}(\mathcal{L})$, the preimage of $\overline{O_{p'}(\mathcal{L})} \cap \overline{S}$ is thus contained in $O_{p'}(\mathcal{L}) \cap (\mathcal{N}S)$. By the Dedekind lemma [15, Lemma 1.10], we have $O_{p'}(\mathcal{L}) \cap (\mathcal{N}S) = \mathcal{N}(O_{p'}(\mathcal{L}) \cap S) = \mathcal{N}$. Hence, $\overline{O_{p'}(\mathcal{L})} \cap \overline{S} = \mathbf{1}$ and $\overline{O_{p'}(\mathcal{L})}$ is a normal p'-subgroup of $\overline{\mathcal{L}}$. If $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ is p'-reduced, it follows that $\overline{O_{p'}(\mathcal{L})} = \mathbf{1}$ and so $O_{p'}(\mathcal{L}) = \mathcal{N}$. □

We now proceed to prove some technical results that are needed in the next subsection.

Lemma 2.29. *If $\mathcal{N}$ is a partial normal p'-subgroup of $\mathcal{L}$, then $f \in C_{\mathcal{N}}(S_f)$ for every $f \in \mathcal{N}$.*

Proof. Let $f \in \mathcal{N}$, set $P := S_f$ and let $s \in P$. Then $P^f \leqslant S$ and thus $P^{fs} \leqslant S$. Moreover, $P^s = P$. Thus, $w = (s^{-1}, f^{-1}, s, f) \in \mathbf{D}$ via P^{fs}. Now $\Pi(w) = (f^{-1})^s f = s^{-1}s^f \in \mathcal{N} \cap S = \mathbf{1}$ and hence $s^f = s$. As $s \in P$ was arbitrary, this proves $f \in C_{\mathcal{N}}(P)$. □

Lemma 2.30. *If $\mathcal{N}$ is a nontrivial partial normal p' subgroup of $\mathcal{L}$, then there exists $P \in \Delta$ such that $N_{\mathcal{N}}(P) = C_{\mathcal{N}}(P) \neq 1$. In particular, if $O_{p'}(N_{\mathcal{L}}(P)) = 1$ for all $P \in \Delta$, then $O_{p'}(\mathcal{L}) = 1$.*

Proof. Let $\mathcal{N}$ be a nontrivial partial normal p'-subgroup and pick $1 \neq f \in \mathcal{N}$. Then $P := S_f \in \Delta$ by Lemma 2.7(d), and it follows from Lemma 2.29 that $1 \neq f \in C_{\mathcal{N}}(P)$. As $N_{\mathcal{N}}(P)$ is a normal p'-subgroup of $N_{\mathcal{L}}(P)$ and P is a normal p-subgroup of $N_{\mathcal{L}}(P)$, we have $C_{\mathcal{N}}(P) = N_{\mathcal{N}}(P)$. Hence, $C_{\mathcal{N}}(P) = N_{\mathcal{N}}(P) \neq 1$ is a normal p'-subgroup of $N_{\mathcal{L}}(P)$ and the assertion follows. □

Corollary 2.31. *If $(\mathcal{L}, \Delta, S)$ is a linking locality or, more generally, a locality of objective characteristic p, then $O_{p'}(\mathcal{L}) = 1$.*

Proof. If, for every $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic p, then it is in particular p'-reduced. Thus, the assertion follows from Corollary 2.30. □

2.8 | A signalizer functor theorem for punctured groups

In this section, we provide some tools for showing that a locality has a nontrivial partial normal p'-subgroup. Corresponding problems for groups are typically treated using signalizer functor theory. A similar language will be used here for localities. We will start by investigating how a nontrivial partial normal p'-subgroup can be produced if some information is known on the level of normalizers of objects. We will then use this to show a theorem for punctured groups that looks similar to the signalizer functor theorem for finite groups, but is much more elementary to prove. Throughout this subsection, let $(\mathcal{L}, \Delta, S)$ be a locality.

Definition 2.32. A *signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects* is a map from Δ to the set of subgroups of $\mathcal{L}$, which associates to $P \in \Delta$ a normal p'-subgroup $\Theta(P)$ of $N_{\mathcal{L}}(P)$ such that the following conditions hold.

- (Conjugacy condition) $\Theta(P)^g = \Theta(P^g)$ for all $P \in \Delta$ and all $g \in \mathcal{L}$ with $P \leqslant S_g$.
- (Balance condition) $\Theta(P) \cap C_{\mathcal{L}}(Q) = \Theta(Q)$ for all $P, Q \in \Delta$ with $P \leqslant Q$.

As seen in Lemma 2.30, given a locality $(\mathcal{L}, \Delta, S)$ with $O_{p'}(\mathcal{L}) \neq 1$, there exists $P \in \Delta$ with $O_{p'}(N_{\mathcal{L}}(P)) \neq 1$. The next theorem says basically that, under suitable extra conditions, the converse holds.

Proposition 2.33. *If Θ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects, then*

$$\widehat{\Theta} := \bigcup_{P \in \Delta} \Theta(P)$$

is a partial normal p'-subgroup of $\mathcal{L}$. In particular, the canonical projection $\rho : \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^\rho$. Upon identifying S with S^ρ, the following properties hold.

(a) *$(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality and $\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.*
(b) *For each $P \in \Delta$, the restriction $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ of ρ is an epimorphism with kernel $\Theta(P)$. In particular, $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

Proof. We proceed in three steps, where in the first step, we prove a technical property, which allows us in the second step to show that $\widehat{\Theta}$ is a partial normal p'-subgroup, and in the third step to conclude that the remaining properties hold.

Step 1: We show $x \in \Theta(S_x)$ for any $x \in \widehat{\Theta}$. Let $x \in \widehat{\Theta}$. Then by definition of $\widehat{\Theta}$, the element x lies in $\Theta(P)$ for some $P \in \Delta$. Choose such P maximal with respect to inclusion. Notice that $[P, x] = 1$. In particular, $P \leqslant S_x$ and $[N_{S_x}(P), x] \leqslant \Theta(P) \cap N_S(P) = \mathbf{1}$. Hence, using the balance condition, we conclude $x \in \Theta(P) \cap C_{\mathcal{L}}(N_{S_x}(P)) = \Theta(N_{S_x}(P))$. So, the maximality of P yields $P = N_{S_x}(P)$ and thus $P = S_x$. Hence, $x \in \Theta(S_x)$ as required.

Step 2: We show that $\widehat{\Theta}$ is a partial normal p'-subgroup of $\mathcal{L}$. Clearly, $\widehat{\Theta}$ is closed under inversion, as $\Theta(P)$ is a group for every $P \in \Delta$. Note also that $\Pi(\emptyset) = \mathbf{1} \in \widehat{\Theta}$ as $\mathbf{1} \in \Theta(P)$ for any $P \in \Delta$. Let now $(x_1, \dots, x_n) \in \mathbf{D} \cap \mathbf{W}(\widehat{\Theta})$ with $n \geqslant 1$. Then $R := S_{(x_1,\dots,x_n)} \in \Delta$ by Lemma 2.7(f). Induction on i together with the balance condition and Step 1 show $R \leqslant S_{x_i}$ and $x_i \in \Theta(S_{x_i}) \leqslant \Theta(R) \leqslant C_{\mathcal{L}}(R)$ for each $i = 1, \dots, n$. Hence, $\Pi(x_1, x_2, \dots, x_n) \in \Theta(R) \subseteq \widehat{\Theta}$. Thus, $\widehat{\Theta}$ is a partial subgroup of $\mathcal{L}$.

Let $x \in \widehat{\Theta}$ and $f \in \mathcal{L}$ with $(f^{-1}, x, f) \in \mathbf{D}$. Then $X := S_{(f^{-1},x,f)} \in \Delta$ by Lemma 2.7(f). Moreover, $X^{f^{-1}} \leqslant S_x$. By Step 1, we have $x \in \Theta(S_x)$, and then by the balance condition, $x \in \Theta(X^{f^{-1}})$. It follows now from the conjugacy condition that $x^f \in \Theta(X^{f^{-1}})^f = \Theta(X) \subseteq \widehat{\Theta}$. Hence, $\widehat{\Theta}$ is a partial normal subgroup of $\mathcal{L}$. Notice that $\widehat{\Theta} \cap S = \mathbf{1}$, as $\Theta(P) \cap S = \Theta(P) \cap N_S(P) = \mathbf{1}$ for each $P \in \Delta$. Thus, $\widehat{\Theta}$ is a partial normal p'-subgroup of $\mathcal{L}$.

Step 3: We are now in a position to complete the proof. By [15, Corollary 4.5], the quotient map $\rho : \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ is a projection of partial groups with $\ker(\rho) = \widehat{\Theta}$. Moreover, by the same result, setting $\Delta^\rho := \{P^\rho : P \in \Delta\}$, the triple $(\mathcal{L}/\widehat{\Theta}, \Delta^\rho, S^\rho)$ is a locality. Notice that $\rho|_S : S \to S^\rho$ is a homomorphism of groups with kernel $S \cap \widehat{\Theta} = 1$ and thus an isomorphism of groups. Upon identifying S with S^ρ, it follows now that $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality. Moreover, by [28, Theorem 5.7(b)], we have $\mathcal{F}_S(\mathcal{L}) = \mathcal{F}_S(\mathcal{L}/\widehat{\Theta})$. So, (a) holds.

Let $P \in \Delta$. By [15, Theorem 4.3(c)], the restriction of ρ to a map $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ is an epimorphism with kernel $N_{\mathcal{L}}(P) \cap \widehat{\Theta}$. For any $x \in N_{\mathcal{L}}(P) \cap \widehat{\Theta}$, we have $P \leqslant S_x$ and then $x \in \Theta(S_x) \leqslant \Theta(P)$ by the balance condition and Step 1. This shows $N_{\mathcal{L}}(P) \cap \widehat{\Theta} = \Theta(P)$ and so (b) holds. □

The property stated in Proposition 2.33(a) holds indeed for every partial normal p'-subgroup $\widehat{\Theta}$ of $\mathcal{L}$. More generally, for any partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$, setting $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$ and $\overline{\Delta} := \{\overline{P} : P \in \Delta\}$, the triple $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality with $\mathcal{F}_{\overline{S}}(\overline{\mathcal{L}}) \cong \mathcal{F}_S(\mathcal{L})/(S \cap \mathcal{N})$ (see [15, Corollary 4.5] and [28, Theorem 5.7]).

Remark 2.34. If $P, Q, R \in \Delta$ such that $P \leqslant Q \leqslant R$ and the balance condition in Definition 2.32 holds for the pair $P \leqslant Q$, then

$$\Theta(Q) \cap C_{\mathcal{L}}(R) = \Theta(P) \cap C_{\mathcal{L}}(Q) \cap C_{\mathcal{L}}(R) = \Theta(P) \cap C_{\mathcal{L}}(R)$$

so $\Theta(R) = \Theta(Q) \cap C_{\mathcal{L}}(R)$ if and only if $\Theta(R) = \Theta(P) \cap C_{\mathcal{L}}(R)$. Hence, in this situation balance holds for the pair $Q \leqslant R$ if and only if balance holds for the pair $P \leqslant R$.

Definition 2.35. Let G be a finite group. Then G is said to be *p-constrained* if $G/O_{p'}(G)$ is of characteristic p. The group G is called *Sylow p-constrained*, if $C_T(O_p(G)) \leqslant O_p(G)$ for some (and thus for every) Sylow p-subgroup T of G.

The following proposition is essentially a restatement of [28, Proposition 6.4], but we will give an independent proof building on the previous proposition.

Proposition 2.36. *Let $(\mathcal{L}, \Delta, S)$ be a locality such that $N_{\mathcal{L}}(P)$ is p-constrained for all $P \in \Delta$. For each $P \in \Delta$, set*

$$\Theta(P) := O_{p'}(N_{\mathcal{L}}(P)).$$

Then the assignment Θ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects and $O_{p'}(\mathcal{L})$ equals $\widehat{\Theta} := \bigcup\{\Theta(P) : P \in \Delta\}$. In particular, the canonical projection $\rho : \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^{\rho}$. Upon identifying S with S^{ρ}, the following properties hold.

(a) *$(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality of objective characteristic p.*
(b) *$\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.*
(c) *For every $P \in \Delta$, the restriction $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ of ρ is an epimorphism with kernel $\Theta(P)$. In particular, $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

Proof. We remark first that, as any normal p'-subgroup of $N_{\mathcal{L}}(P)$ centralizes P and $O_{p'}(C_{\mathcal{L}}(P))$ is characteristic in $C_{\mathcal{L}}(P) \trianglelefteq N_{\mathcal{L}}(P)$, we have $\Theta(P) = O_{p'}(C_{\mathcal{L}}(P))$ for every $P \in \Delta$.

We show now that the assignment Θ is a signalizer functor of $\mathcal{L}$ on objects. It follows from Lemma 2.7(b) that the conjugacy condition holds. Thus, it remains to show the balance condition, that is, that $\Theta(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$ for any $P, Q \in \Delta$ with $P \leqslant Q$. For the proof note that P is subnormal in Q. So, by induction on the subnormal length and by Remark 2.34, we may assume that $P \trianglelefteq Q$. Set $G := N_{\mathcal{L}}(P)$. Then $Q \leqslant G$ and $C_{\mathcal{L}}(Q) = C_G(Q)$. As G is p-constrained, it follows from [34, 8.2.12] that $O_{p'}(N_G(Q)) = O_{p'}(G) \cap N_G(Q) = O_{p'}(G) \cap C_G(Q)$. Hence, $\Theta(Q) = O_{p'}(C_{\mathcal{L}}(Q)) = O_{p'}(C_G(Q)) = O_{p'}(N_G(Q)) = O_{p'}(G) \cap C_G(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$. This proves that the assignment Θ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects. In particular, by Proposition 2.33, the subset

$$\widehat{\Theta} := \bigcup_{P \in \Delta} \Theta(P)$$

is a partial normal p'-subgroup of $\mathcal{L}$. Moreover, upon identifying S with its image in $\mathcal{L}/\widehat{\Theta}$, the triple $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality and properties (b) and (c) hold. Part (c) and our assumption yield (a). Hence, by Corollary 2.31, we have $O_{p'}(\mathcal{L}/\widehat{\Theta}) = \mathbf{1}$. So, by Lemma 2.28, we have $\widehat{\Theta} = O_{p'}(\mathcal{L})$ and the proof is complete. □

Lemma 2.37. *If G is a Sylow p-constrained finite group, then G is p-constrained.*

Proof. Write T for a Sylow p-subgroup of G, and set $P := O_p(G)$. Then the centralizer $C_T(P)$ equals $Z(P)$ and is thus a central Sylow p-subgroup of $C_G(P)$. So, for example, by the Schur–Zassenhaus theorem [34, 6.2.1], we have $C_G(P) = Z(P) \times O_{p'}(C_G(P)) \leqslant Z(P) \times O_{p'}(G)$. Set $\overline{G} = G/O_{p'}(G)$, write C for the preimage of $C_{\overline{G}}(\overline{P})$ in G. As P is normal in G, it follows that $[P, C] \leqslant P \cap O_{p'}(G) = 1$. So, $C = C_G(P)$ and thus $\overline{C} = \overline{C_G(P)} \leqslant \overline{P}$. Thus, $\overline{G}$ has characteristic p and G is p-constrained. □

We now turn attention to the case that $(\mathcal{L}, \Delta, S)$ is a punctured group and we are given a signalizer functor *on elements of order p* in the sense of Definition 1.2 in the introduction. We show first that, if θ is such a signalizer functor on elements of order p and $a \in \mathcal{I}_p(S)$, the subgroup $\theta(a)$ depends only on $\langle a \rangle$.

Lemma 2.38. *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and let θ be a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order p. Then $\theta(a) = \theta(b)$ for all $a, b \in \mathcal{I}_p(S)$ with $\langle a \rangle = \langle b \rangle$.*

Proof. If $\langle a \rangle = \langle b \rangle$, then $[a, b] = 1$ and $\theta(a) \subseteq C_{\mathcal{L}}(a) = C_{\mathcal{L}}(b)$. So, the balance condition implies $\theta(a) = \theta(a) \cap C_{\mathcal{L}}(b) \subseteq \theta(b)$. A symmetric argument gives the opposite inclusion $\theta(b) \subseteq \theta(a)$, so the assertion holds. $\square$

Theorem 1.3 in the introduction follows directly from the following theorem, which explains at the same time how a signalizer functor on objects can be constructed from a signalizer functor on elements of order p.

Theorem 2.39 (Signalizer functor theorem for punctured groups). *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and suppose θ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order p. Then a signalizer functor Θ of $(\mathcal{L}, \Delta, S)$ on objects is defined by*

$$\Theta(P) := \left(\bigcap_{x \in \mathcal{I}_p(P)} \theta(x) \right) \cap C_{\mathcal{L}}(P) \text{ for all } P \in \Delta.$$

In particular,

$$\widehat{\Theta} := \bigcup_{P \in \Delta} \Theta(P) = \bigcup_{x \in \mathcal{I}_p(S)} \theta(x)$$

is a partial normal p' subgroup of $\mathcal{L}$ and the other conclusions in Proposition 2.33 hold.

Proof. As $\theta(x)$ is a p'-subgroup for each $x \in \mathcal{I}_p(S)$, the subgroup $\Theta(P)$ is a p'-subgroup for each object $P \in \Delta$. Moreover, it follows from the conjugacy condition for θ (as stated in Definition 1.2) that $\Theta(P)$ is a normal subgroup of $N_{\mathcal{L}}(P)$, and that the conjugacy condition stated in Definition 2.32 holds for Θ; to obtain the latter conclusion notice that Lemma 2.7(b) implies $C_{\mathcal{L}}(P)^g = C_{\mathcal{L}}(P^g)$ for every $P \in \Theta$ and every $g \in \mathcal{L}$ with $P \leqslant S_g$.

To prove that Θ is a signalizer functor on objects, it remains to show that the balance condition $\Theta(P) \cap C_{\mathcal{L}}(Q) = \Theta(Q)$ holds for every pair $P \leqslant Q$ with $P \in \Delta$. Notice that P is subnormal in Q whenever $P \leqslant Q$. Hence, if the balance condition for Θ fails for some pair $P \leqslant Q$ with $P \in \Delta$, then by Remark 2.34, it fails for some pair $P \trianglelefteq Q$ with $P \in \Delta$. Suppose this is the case. Among all pairs $P \trianglelefteq Q$ such that $P \in \Delta$ and the balance condition fails, choose one such that Q is of minimal order.

Notice that $P < Q$, as the balance condition would otherwise trivially hold. So as $1 \neq P_0 := C_P(Q) \in \Delta$ and $P_0 \trianglelefteq P$, the minimality of $|Q|$ yields that the balance condition holds for the pair $P_0 \leqslant P$. If the balance condition holds also for the pair $P_0 \trianglelefteq Q$, then the balance condition holds for the pair $P \leqslant Q$ by Remark 2.34, contradicting our assumption. So, the balance condition does not hold for $P_0 \leqslant Q$. Therefore, replacing P by P_0, we can and will assume from now on that $P \leqslant Z(Q)$.

It is clear from the definition that $\Theta(Q) \leqslant \Theta(P) \cap C_{\mathcal{L}}(Q)$. Hence, it remains to prove the opposite inclusion. By definition of $\Theta(Q)$, this means that we need to show $\Theta(P) \cap C_{\mathcal{L}}(Q) \leqslant \theta(b)$ for all $b \in \mathcal{I}_p(Q)$. To show this fix $b \in \mathcal{I}_p(Q)$. As $P \in \Delta$, we have $P \neq 1$ and so we can pick $a \in \mathcal{I}_p(P)$. As $P \leqslant Z(Q)$, the elements a and b commute. Hence, the balance condition for θ yields

$$\Theta(P) \cap C_{\mathcal{L}}(Q) \leqslant \theta(a) \cap C_{\mathcal{L}}(b) \leqslant \theta(b).$$

This completes the proof that Θ is a signalizer functor on objects.

Given $P \in \Delta$, we can pick any $x \in \mathcal{I}_p(P)$ and have $\Theta(P) \subseteq \theta(x)$. Hence, $\widehat{\Theta} := \bigcup_{P\in\Delta} \Theta(P)$ is contained in $\bigcup_{x\in\mathcal{I}_p(S)} \theta(x)$. The opposite inclusion holds as well, as Lemma 2.38 implies $\theta(x) = \Theta(\langle x\rangle)$ for every $x \in \mathcal{I}_p(S)$. The assertion follows now from Proposition 2.33. □

3 | SHARPNESS OF THE SUBGROUP DECOMPOSITION

3.1 | Additive extensions of categories

Let $\mathcal{C}$ be a (small) category. Define a category $\mathcal{C}_{\amalg}$ as follows, see [32, section 4]. The objects of $\mathcal{C}_{\amalg}$ are pairs $(I, \mathbf{X})$ where I is a finite set and $\mathbf{X} : I \to \mathrm{obj}(\mathcal{C})$ is a function. A morphisms $(I, \mathbf{X}) \to (J, \mathbf{Y})$ is a pair $(\sigma, \mathbf{f})$ where $\sigma : I \to J$ is a function and $\mathbf{f} : I \to \mathrm{mor}(\mathcal{C})$ is a function such that $\mathbf{f}(i) \in \mathcal{C}(\mathbf{X}(i), \mathbf{Y}(\sigma(i)))$. We leave it to the reader to check that this defines a category.

There is a fully faithful inclusion $\mathcal{C} \subseteq \mathcal{C}_{\amalg}$ by sending $X \in \mathcal{C}$ to the function $\mathbf{X} : \{\varnothing\} \to \mathrm{obj}(\mathcal{C})$ with $\mathbf{X}(\varnothing) = X$. We will write X (not boldface) to denote these objects in $\mathcal{C}_{\amalg}$.

The category $\mathcal{C}_{\amalg}$ has a monoidal structure $\coprod$ where $(I, \mathbf{X}) \coprod (J, \mathbf{Y}) \overset{\text{def}}{=} (I \coprod J, \mathbf{X} \coprod \mathbf{Y})$. One checks that this is the categorical coproduct in $\mathcal{C}_{\amalg}$. For this reason, we will often write objects of $\mathcal{C}_{\amalg}$ in the form $\coprod_{i\in I} X_i$ where $X_i \in \mathcal{C}$. Also, when the indexing set I is understood we will simply write $\mathbf{X}$ instead of $(I, \mathbf{X})$.

When $(I, \mathbf{X})$ is an object and $J \subseteq I$ we will refer to $(J, \mathbf{X}|_J)$ as a "subobject" of $(I, \mathbf{X})$ and we leave it to the reader to check that the inclusion is a monomorphism, namely for any two morphisms $\mathbf{f}, \mathbf{g} : \mathbf{Y} \to \mathbf{X}|_J$, if $\mathrm{incl}^{\mathbf{X}}_{\mathbf{X}|_J} \circ \mathbf{f} = \mathrm{incl}^{\mathbf{X}}_{\mathbf{X}|_J} \circ \mathbf{g}$ then $\mathbf{f} = \mathbf{g}$. One also checks that

$$\mathcal{C}_{\amalg}\Big(\coprod_{i\in I} X_i, Y\Big) = \prod_{i\in I} \mathcal{C}(X_i, Y), \tag{3.1}$$

$$\mathcal{C}_{\amalg}\Big(X, \coprod_{i\in I} Y_i\Big) = \coprod_{i\in I} \mathcal{C}(X, Y_i).$$

Definition 3.1 (Compare [32, p. 123]). We say that $\mathcal{C}$ satisfies (PB $\times_{\amalg}$) if the product of each pair of objects in $\mathcal{C}$ exists in $\mathcal{C}_{\amalg}$ and if the pullback of each diagram $c \to e \leftarrow d$ of objects in $\mathcal{C}$ exists in $\mathcal{C}_{\amalg}$.

Definition 3.2 (Compare [32, p. 124 and Lemma 5.13]). Assume that $\mathcal{C}$ is a small category satisfying (PB $\times_{\amalg}$). A functor $M : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is called a *proto-Mackey functor* if there is a functor $M_* : \mathcal{C} \to \mathbf{Ab}$ such that the following hold.

(a) $M(C) = M_*(C)$ for any $C \in \mathrm{obj}(\mathcal{C})$.
(b) For any isomorphism $\varphi \in \mathcal{C}$, $M_*(\varphi) = M(\varphi^{-1})$.

(c) By applying M and M_* to a pullback diagram in $\mathcal{C}_{\amalg}$ of the form

$$\begin{tikzcd} \coprod_{i\in I} B_i \arrow[r, "\sum_i \varphi_i"] \arrow[d, "\sum_i \psi_i"'] & D \arrow[d, "\beta"] \\ C \arrow[r, "\alpha"] & E \end{tikzcd}$$

where $B_i, C, E \in \mathcal{C}$, there results the following commutative square in **Ab**

$$\begin{tikzcd} \bigoplus_{i\in I} M(B_i) \arrow[r, "\sum_i M_*(\varphi_i)"] & M(D) \\ M(C) \arrow[u, "\oplus_i M(\psi_i)"] \arrow[r, "M_*(\alpha)"] & M(E) \arrow[u, "M(\beta)"'] \end{tikzcd}$$

We remark that every pullback diagram in $\mathcal{C}_{\amalg}$ defined by objects in $\mathcal{C}$ is isomorphic in $\mathcal{C}_{\amalg}$ to a commutative square as in (c) in this definition.

Given a small category $\mathbf{D}$ and a functor $M : \mathbf{D} \to \mathbf{Ab}$, we write

$$H^*(\mathbf{D}; M) \overset{\text{def}}{=} \varprojlim_{\mathbf{D}}{}^* M$$

for the derived functors of M. We say that M is *acyclic* if $H^i(\mathbf{D}; M) = 0$ for all $i > 0$.

Proposition 3.3 (See [32, Corollary 5.16]). *Fix a prime p. Let $\mathcal{C}$ be a small category that satisfies (PB $\times_{\amalg}$) and in addition*

(B1) *$\mathcal{C}$ has finitely many isomorphism classes of objects, all morphism sets are finite and all self maps in $\mathcal{C}$ are isomorphisms;*
(B2) *for every object $C \in \mathcal{C}$ there exists an object D such that $|\mathcal{C}(C, D)| \neq 0 \mod p$.*

Then any proto-Mackey functor $M : \mathcal{C}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-$\mathfrak{mod}$ is acyclic, namely $H^i(\mathcal{C}^{\mathrm{op}}, M) = 0$ for all $i > 0$.

3.2 | Transporter categories

Let $\mathcal{F}$ be a saturated fusion system over S and let $\mathcal{T}$ be a transporter system associated with $\mathcal{F}$ (Definition 2.13). By [45, Lemmas 3.2(b) and 3.8] every morphism in $\mathcal{T}$ is both a monomorphism and an epimorphism. For any $P, Q \in \mathrm{obj}(\mathcal{T})$ such that $P \leqslant Q$ denote $\iota_P^Q = \epsilon_{P,Q}(e) \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$. We think of these as "inclusion" morphisms in $\mathcal{T}$. We obtain a notion of "extension" and "restriction" of morphisms in $\mathcal{T}$ as follows. Suppose $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ and $P' \leqslant P$ and $Q' \leqslant Q$ and $\psi \in \mathrm{Mor}_{\mathcal{T}}(P', Q')$ are such that $\varphi \circ \iota_{P'}^P = \iota_{Q'}^Q \circ \psi$. Then we say that ψ is a restriction of φ and that φ is an extension of ψ. Notice that as $\iota_{Q'}^Q$ is a monomorphism, given φ then its restriction ψ if it exists, is unique and we will write $\psi = \varphi|_{P'}^{Q'}$. Similarly, as $\iota_{P'}^P$ is an epimorphism, given ψ, if an extension φ exists then it is unique. By [45, Lemma 3.2(c)], given $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ and subgroups $P' \leqslant P$ and $Q' \leqslant Q$ such that $\rho(\varphi)(P') \leqslant Q'$, then φ restricts to a (unique morphism) $\varphi' \in \mathrm{Mor}_{\mathcal{T}}(P', Q')$.

We will use this fact repeatedly. In particular, every morphism $\varphi : P \to Q$ in $\mathcal{T}$ factors uniquely $P \xrightarrow{\bar{\varphi}} \bar{P} \xrightarrow{\iota_{\bar{P}}^{Q}} Q$ where $\bar{\varphi}$ is an isomorphism in $\mathcal{T}$ and $\bar{P} = \rho(\varphi)(P)$.

For any $P, Q \in \mathrm{obj}(\mathcal{T})$, set

$$K_{P,Q} = \{(A, \alpha) : \ A \leqslant P,\ A \in \mathrm{obj}(\mathcal{T}),\ \alpha \in \mathrm{Mor}_{\mathcal{T}}(A, Q)\}.$$

This set is partially ordered where $(A, \alpha) \preceq (B, \beta)$ if $A \leqslant B$ and $\alpha = \beta|_A$. As $K_{P,Q}$ is finite we may consider the set $K_{P,Q}^{\max}$ of the maximal elements.

For any $x \in N_S(P, Q)$ we write $\widehat{x}$ instead of $\epsilon_{P,Q}(x)$. There is an action of $Q \times P$ on $K_{P,Q}$ defined by

$$(y, x) \cdot (A, \alpha) = (xAx^{-1}, \widehat{y} \circ \alpha \circ \widehat{x}^{-1}), \qquad (x \in P,\ y \in Q).$$

This action is order preserving and therefore it leaves $K_{P,Q}^{\max}$ invariant. We will write $\overline{K_{P,Q}^{\max}}$ for the set of orbits. For any $P, Q \in \mathcal{T}$ we will choose *once and for all* a subset

$$\mathcal{K}_{P,Q}^{\max} \subseteq K_{P,Q}^{\max}$$

of representatives for the orbits of $Q \times P$ on $K_{P,Q}^{\max}$.

Lemma 3.4. *For any $(A, \alpha) \in K_{P,Q}$ there exists a unique $(B, \beta) \in K_{P,Q}^{\max}$ such that $(A, \alpha) \preceq (B, \beta)$.*

Proof. We use induction on $[S : A]$. Fix $(A, \alpha) \in K_{P,Q}$ and (B_1, β_1) and (B_2, β_2) in $K_{P,Q}^{\max}$ such that $(A, \alpha) \preceq (B_i, \beta_i)$. Thus, $\beta_1|_A = \alpha = \beta_2|_A$. We may assume that $A < B_i$ because if say $A = B_1$ then $(B_1, \beta_1) \preceq (B_2, \beta_2)$ and maximality implies $(B_1, \beta_1) = (B_2, \beta_2)$.

For $i = 1, 2$ set $N_i = N_{B_i}(A)$. Then N_i contain A properly and we set $D = \langle N_1, N_2 \rangle$. Then $A \trianglelefteq D$. Set $T = \alpha(A)$ and $\overline{T} = N_Q(T)$. For $i = 1, 2$, if $x \in N_i$ then Axiom (C) of Definition 2.13 applied to β_i yields

$$\alpha \circ \widehat{x}|_A^A = ((\beta_i|_{N_i}) \circ \widehat{x}|_{N_i}^{N_i})|_A = \widehat{\beta_i(x)}|_Q^Q \circ \beta_i|_A = \widehat{\beta_i(x)}|_Q^Q \circ \alpha.$$

Notice that $\beta_i(x) \in N_Q(T)$, so Axiom (II) of Definition 2.13 implies that α extends to $\delta \in \mathrm{Mor}_{\mathcal{T}}(D, Q)$. As for $i = 1, 2$ the morphisms $\iota_A^{N_i} : A \to N_i$ are epimorphisms in $\mathcal{T}$, the equality $\beta_i|_{N_i} \circ \iota_A^{N_i} = \alpha = \delta|_A = (\delta|_{N_i}) \circ \iota_A^{N_i}$ shows that $\delta|_{N_i} = \beta_i|_{N_i}$. Now we have $(N_i, \beta_i|_{N_i}) \preceq (D, \delta)$ and $(N_i, \beta_i|_{N_i}) \preceq (B_i, \beta_i)$ in $K_{P,Q}$. As $|A| < |N_i|$ the induction hypothesis implies that (B_i, β_i) is the unique maximal extension of $(N_i, \beta_i|_{N_i})$ for each $i = 1, 2$, and both must coincide with the unique maximal extension of (D, δ). It follows that $(B_1, \beta_1) = (B_2, \beta_2)$. □

The *orbit category* of $\mathcal{T}$ is a category $\mathcal{OT}$ with the same set of objects as $\mathcal{T}$. For any $P, Q \in \mathcal{OT}$ the morphism set $\mathrm{Mor}_{\mathcal{OT}}(P, Q)$ is the set of orbits of $\mathrm{Mor}_{\mathcal{T}}(P, Q)$ under the action of $\widehat{Q} = \epsilon_{Q,Q}(Q) \subseteq \mathrm{Mor}_{\mathcal{T}}(Q, Q)$ via postcomposition. See [45, section 4, p. 1010]. Axiom (C) guarantees that composition in $\mathcal{OT}$ is well-defined. Given $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ we will denote its image in $\mathrm{Mor}_{\mathcal{OT}}(P, Q)$ by $[\varphi]$.

We notice that every morphism in $\mathcal{OT}$ is an epimorphism, namely for every $[\alpha] \in \mathrm{Mor}_{\mathcal{OT}}(P,Q)$ and $[\beta],[\gamma] \in \mathrm{Mor}_{\mathcal{OT}}(Q,R)$, if $[\beta]\circ[\alpha] = [\gamma]\circ[\alpha]$ then $[\beta] = [\gamma]$. This follows from the fact that every morphism in $\mathcal{T}$ is an epimorphism.

Consider $P, Q \in \mathrm{obj}(\mathcal{OT})$ such that $P \trianglelefteq Q$. Precomposition with $[\iota_P^Q]$ gives a "restriction" map

$$\mathrm{Mor}_{\mathcal{OT}}(Q,S) \xrightarrow{\mathrm{res}} \mathrm{Mor}_{\mathcal{OT}}(P,S).$$

Observe that Q acts on $\mathrm{Mor}_{\mathcal{OT}}(P,S)$ by precomposing morphisms with $[\widehat{x}|_P^P]$ for any $x \in Q$. This action has P in its kernel by Axiom (C) of transporter systems.

Lemma 3.5. *Let $\mathcal{T}$ be a transporter system and $\mathcal{OT}$ its orbit category.*

(a) *For any $P, Q \in \mathrm{obj}(\mathcal{OT})$ such that $P \trianglelefteq Q$ the map $\mathrm{Mor}_{\mathcal{OT}}(Q,S) \to \mathrm{Mor}_{\mathcal{OT}}(P,S)$ induced by the restriction $[\varphi] \mapsto [\varphi|_P]$, gives rise to a bijection*

$$\mathrm{res}: \mathrm{Mor}_{\mathcal{OT}}(Q,S) \to \mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}. \tag{3.2}$$

(b) *For any $P \in \mathcal{OT}$, we have $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| \neq 0 \mod p$.*

Proof.

(a) First, observe that if $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(Q,S)$ then $[\varphi|_P]$ is fixed by Q/P by Axiom (C), hence the image of res is contained in $\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$. Now suppose that $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$ and set $\bar{P} = \rho(\varphi)(P)$. As $[\varphi]$ is fixed by Q/P this exactly means that for every $x \in Q$ there exists $y \in N_S(\bar{P})$ such that $\varphi \circ \widehat{x}|_P^P = \widehat{y} \circ \varphi$ and Axiom (II) implies that φ extends to a morphism $\psi \in \mathrm{Mor}_{\mathcal{T}}(Q,S)$. This shows that the map res in (3.2) is onto $\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$. It is injective because $[\iota_P^Q]$ is an epimorphism in $\mathcal{OT}$.

(b) Use induction on $[S : P]$. If $P = S$ then $\epsilon_{S,S}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{T}}(S) = \mathrm{Mor}_{\mathcal{T}}(S,S)$ and therefore $|\mathrm{Mor}_{\mathcal{OT}}(S,S)| \neq 0 \mod p$. Suppose $P < S$ and set $Q = N_S(P)$. Then $Q > P$ and because Q/P is a finite p-group, $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| = |\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}| \mod p$. It follows from part (a) and the induction hypothesis on $[S : Q]$ that $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| \neq 0 \mod p$. □

In the remainder of this subsection, we will prove that $\mathcal{OT}$ satisfies (PB $\times_{\amalg}$), keeping the notation from above.

Definition 3.6. Let $\mathcal{T}$ be a transporter system with orbit category $\mathcal{OT}$. For $P, Q \in \mathcal{OT}$, consider the object $P \boxtimes Q$ of $\mathcal{OT}_{\amalg}$ given by

$$P \boxtimes Q = \coprod_{(L,\lambda)\in\mathcal{K}_{P,Q}^{\max}} L.$$

That is, $P \boxtimes Q : \mathcal{K}_{P,Q}^{\max} \to \mathrm{obj}(\mathcal{OT})$ is the function $(L,\lambda) \mapsto L$. Let $\pi_1 : P \boxtimes Q \to P$ and $\pi_2 : P \boxtimes Q \to Q$ be the morphisms in $\mathcal{OT}_{\amalg}$ defined by $\pi_1 = \sum_{(L,\lambda)}[\iota_L^P]$ and $\pi_2 = \sum_{(L,\lambda)}[\lambda]$.

Proposition 3.7. *Let $\mathcal{T}$ be a transporter system with orbit category $\mathcal{OT}$. Then $P \boxtimes Q$ is the product in $\mathcal{OT}_{\amalg}$ of $P, Q \in \mathrm{obj}(\mathcal{OT})$.*

Proof. It follows from (3.1) that it suffices to show that

$$\mathcal{OT}_{\amalg}(R, P \boxtimes Q) \xrightarrow{(\pi_{1*},\pi_{2*})} \mathrm{Mor}_{\mathcal{OT}}(R,P) \times \mathrm{Mor}_{\mathcal{OT}}(R,Q)$$

is a bijection for any $R \in \mathrm{obj}(\mathcal{OT})$. Write $\pi = (\pi_{1*}, \pi_{2*})$.

Surjectivity of π: Consider $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(R,P)$ and $[\psi] \in \mathrm{Mor}_{\mathcal{OT}}(R,Q)$. Set $A = \rho(\varphi)(R)$. Then $A \leqslant P$ and there exists an isomorphism $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(R,A)$ such that $\varphi = \iota_A^P \circ \bar{\varphi}$.

Set $\alpha = \psi \circ (\bar{\varphi})^{-1} \in \mathrm{Mor}_{\mathcal{T}}(A,Q)$. Then $(A,\alpha) \in K_{P,Q}$. Choose $(B,\beta) \in K_{P,Q}^{\max}$ such that $(A,\alpha) \preceq (B,\beta)$. There exists a unique $(L,\lambda) \in \mathcal{K}_{P,Q}^{\max}$ and some $x \in P$ and $y \in Q$ such that

$$(L,\lambda) = (y,x)\cdot(B,\beta) = (xBx^{-1}, \hat{y} \circ \beta \circ (\hat{x}|_B^L)^{-1}).$$

Set $\mu = (\hat{x}|_B^L) \circ \iota_A^B \circ \bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(R,L)$. It defines a morphism $[\mu] : R \to P \boxtimes Q$ in $\mathcal{OT}_{\amalg}$ via the inclusion $(L,\lambda) \subseteq P \boxtimes Q$. We claim that $\pi([\mu]) = ([\varphi],[\psi])$, completing the proof of the surjectivity of π. By definition of $\pi_1 : P \boxtimes Q \to P$ and $\pi_2 : P \boxtimes Q \to Q$,

$$\begin{aligned}
\pi_{1*}([\mu]) &= [\iota_L^P] \circ [\mu] = [\iota_L^P \circ (\hat{x}|_B^L) \circ \iota_A^B \circ \bar{\varphi}] = [(\hat{x}|_A^P) \circ \bar{\varphi}] = [(\hat{x}|_P^P) \circ \varphi] = [\varphi] \\
\pi_{2*}([\mu]) &= [\lambda] \circ [\mu] = [\hat{y}^{-1} \circ \lambda \circ \mu] = [\hat{y}^{-1} \circ \lambda \circ (\hat{x}|_B^L) \circ \iota_A^B \circ \bar{\varphi}] \\
&= [\beta \circ \iota_A^B \circ \bar{\varphi}] = [\alpha \circ \bar{\varphi}] = [\psi].
\end{aligned}$$

Injectivity of π: Suppose that $h, h' \in \mathcal{OT}_{\amalg}(R, P \boxtimes Q)$ are such that $\pi(h) = \pi(h')$. From (3.1) there are $(L,\lambda), (L',\lambda') \in \mathcal{K}_{P,Q}^{\max}$ and $\varphi \in \mathrm{Mor}_{\mathcal{T}}(R,L)$ and $\varphi' \in \mathrm{Mor}_{\mathcal{T}}(R,L')$ such that $h = [\varphi]$ and $h' = [\varphi']$ via the inclusions $L, L' \subseteq P \boxtimes Q$. The hypothesis $\pi(h) = \pi(h')$ then becomes $[\iota_L^P \circ \varphi] = [\iota_{L'}^P \circ \varphi']$ and $[\lambda \circ \varphi] = [\lambda' \circ \varphi']$. Thus,

$$\begin{aligned}
\iota_{L'}^P \circ \varphi' &= \hat{x} \circ \iota_L^P \circ \varphi \qquad \text{for some } x \in P \\
\lambda' \circ \varphi' &= \hat{y} \circ \lambda \circ \varphi \qquad \text{for some } y \in Q.
\end{aligned} \tag{3.3}$$

Set $A = \rho(\varphi)(R)$ and $A' = \rho(\varphi')(R)$. There are factorizations $\varphi = \iota_A^L \circ \bar{\varphi}$ and $\varphi' = \iota_{A'}^{L'} \circ \bar{\varphi'}$ for isomorphisms $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(R,A)$ and $\bar{\varphi'} \in \mathrm{Mor}_{\mathcal{T}}(R,A')$ in $\mathcal{T}$. We get from (3.3) that $\iota_{A'}^P \circ \bar{\varphi'} = \hat{x}|_A^P \circ \bar{\varphi}$. From this we deduce that $A' = xAx^{-1}$ and that $\bar{\varphi'} = \hat{x}|_A^{A'} \circ \bar{\varphi}$. The second equation in (3.3) gives

$$\lambda' \circ \iota_{A'}^{L'} = \hat{y} \circ \lambda \circ \iota_A^L \circ \widehat{x^{-1}}|_{A'}^A.$$

We deduce that $(A', \lambda'|_{A'}) = (y,x) \cdot (A, \lambda|_A)$. Clearly $(A', \lambda'|_{A'}) \preceq (L', \lambda')$ and $(A, \lambda|_A) \preceq (L, \lambda)$ so Lemma 3.4 implies that $(L', \lambda') = (y,x) \cdot (L,\lambda)$. As (L,λ) and (L',λ') are elements of $\mathcal{K}_{P,Q}^{\max}$ and are in the same orbit of $Q \times P$ it follows that $(L,\lambda) = (L',\lambda')$. In particular $x \in N_P(L)$, and it follows from (3.3) that $\varphi' = \hat{x} \circ \varphi$ and that $\lambda = \hat{y} \circ \lambda \circ \hat{x}^{-1}$ (as φ is an epimorphism in $\mathcal{T}$). By Axiom (II) of Definition 2.13, there exists an extension of λ to a morphism $\tilde{\lambda} : \langle L, x \rangle \to Q$ in $\mathcal{T}$. Notice that $\langle L, x \rangle \subseteq P$ so the maximality of (L,λ) implies that $x \in L$. As $\varphi' = \hat{x} \circ \varphi$ we deduce $[\varphi'] = [\varphi]$ namely $h = h'$ as needed. □

Definition 3.8. Let $P \xrightarrow{f} R \xleftarrow{g} Q$ be morphisms in $\mathcal{OT}$. Let $U(f,g)$ be the subobject of $P \boxtimes Q$ obtained by restriction of $P \boxtimes Q : \mathcal{K}^{\max}_{P,Q} \xrightarrow{(L,\lambda)\mapsto L} \mathrm{obj}(\mathcal{OT})$ to the set I of those $(L,\lambda) \in \mathcal{K}^{\max}_{P,Q}$ such that $f \circ [\iota_L^P] = g \circ [\lambda]$.

Proposition 3.9. *Let $\mathcal{T}$ be a transporter system with orbit category $\mathcal{OT}$, and let $P \xrightarrow{f} R \xleftarrow{g} Q$ be morphisms in $\mathcal{OT}$. Then $(U(f,g), \pi_1|_{U(f,g)}, \pi_2|_{U(f,g)})$ is the pullback of P and Q along f and g in $\mathcal{OT}_{\amalg}$. Moreover, the pullback of $P \xrightarrow{\iota_P^R} R \xleftarrow{\iota_Q^R} Q$ is*

$$\coprod_{x \in (Q\backslash R/P)_{\mathcal{T}}} Q^x \cap P$$

where x runs through representatives of the double cosets such that $Q^x \cap P = x^{-1}Qx \cap P$ is an object of $\mathcal{T}$.

Proof. It follows from (3.1) that in order to check the universal property of $U = U(f,g)$ it suffices to test objects $T \in \mathcal{OT}$. Suppose that we are given morphisms $T \xrightarrow{[\varphi]} P$ and $T \xrightarrow{[\psi]} Q$ that satisfy $f \circ [\varphi] = g \circ [\psi]$. We obtain $T \xrightarrow{([\varphi],[\psi])} P \boxtimes Q$ which factors $T \xrightarrow{\bar{h}} L \subseteq P \boxtimes Q$ for some $(L,\lambda) \in \mathcal{K}^{\max}_{P,Q}$. Then

$$f \circ \pi_1|_L \circ \bar{h} = f \circ \pi_1 \circ ([\varphi],[\psi]) = f \circ [\varphi]$$
$$g \circ \pi_2|_L \circ \bar{h} = g \circ \pi_2 \circ ([\varphi],[\psi]) = g \circ [\psi].$$

As $\bar{h}$ is an epimorphism in $\mathcal{OT}$ and because $f \circ [\varphi] = g \circ [\psi]$ by assumption, it follows that $f \circ \pi_1|_L = g \circ \pi_2|_L$ which is the statement $f \circ [\iota_L^P] = g \circ [\lambda]$. This precisely means that $(L,\lambda) \in I$ where I is as in Definition 3.8, hence $h = ([\varphi],[\psi])$ factors through U and clearly $\pi_1 \circ h = [\varphi]$ and $\pi_2 \circ h = [\psi]$. As the inclusion $U \subseteq P \boxtimes Q$ is a monomorphism in $\mathcal{OT}_{\amalg}$, there can be only one morphism $h : T \to U$ such that $\pi_1 \circ h = [\varphi]$ and $\pi_2 \circ h = [\psi]$. This shows that $U = U(f,g)$ is the pullback.

Now assume we are given $P \xrightarrow{\iota} R \xleftarrow{\iota} Q$. The indexing set of the object $U(f,g)$ consists of $(L,\lambda) \in \mathcal{K}^{\max}_{P,Q}$ such that $[\iota_L^R] = [\iota_Q^R \circ \lambda]$, namely $\iota_Q^R \circ \lambda = \hat{x}|_L^R$ for some $x \in N_R(L,Q)$, which is furthermore unique. As ι_Q^R is a monomorphism, this implies that $\lambda = \hat{x}|_L^Q$. As (L,λ) is maximal, $L = Q^x \cap P$. We obtain a map $U(\iota_P^R, \iota_Q^R) \to (Q\backslash R/P)_{\mathcal{T}}$ that sends (L,λ) to PxQ with $x \in N_R(L,Q)$ described above. This map is injective because if $QxP = Qx'P$ are the images of (L,λ) and (L',λ') then $x' = qxp$ for some $p \in P$ and $q \in Q$ and it follows that $L' = p^{-1}Lp$ and that $\lambda = \hat{x}|_L^Q$ and $\lambda' = \hat{x}'|_{L'}^Q$ and therefore $\lambda = \hat{q} \circ \lambda' \circ \hat{p}|_L^{L'}$, so (L,λ) and (L',λ') are in the same orbit of $Q \times P$, hence they must be equal. It is surjective because for any $PxQ \in (Q\backslash R/P)_{\mathcal{T}}$ we obtain a summand in $U(\iota_P^R, \iota_Q^R)$ that is equivalent in $K_{P,Q}$ to (L,λ) with $L = Q^x \cap P$ and $\lambda = \hat{x}|_L^Q$. □

3.3 | The Λ-functors

Let Γ be a finite group and M a (right) Γ-module. Let p be a fixed prime and let $\mathcal{O}_p(\Gamma)$ be the full subcategory of the category of Γ-sets whose objects are the transitive Γ-sets whose isotropy groups are p-groups. Let $F_M : \mathcal{O}_p(\Gamma)^{\mathrm{op}} \to \mathbf{Ab}$ be the functor that assigns M to the free orbit $\Gamma/1$ and 0 to all orbits with nontrivial isotropy. Define [33, Definition 5.3]

$$\Lambda^*(\Gamma, M) \overset{\text{def}}{=} \varprojlim_{\mathcal{O}_p(\Gamma)^{\mathrm{op}}}{}^{*} F_M \ \big(= H^*(\mathcal{O}_p(\Gamma)^{\mathrm{op}}; F_M)\big).$$

These functors have the following important properties.

Lemma 3.10. *Suppose that M is a $\mathbb{Z}_{(p)}[\Gamma]$-module.*

(a) *If $C_\Gamma(M)$ contains an element of order p, then $\Lambda^*(\Gamma; M) = 0$.*
(b) *If $\Gamma/C_\Gamma(M)$ has order prime to p, then $\Lambda^*(\Gamma; M) = 0$ for all $* \geqslant 1$.*

Proof. Point (a) is [33, Proposition 6.1(ii)]. Point (b) follows from [33, Proposition 6.1(ii)] when p divides $|C_\Gamma(M)|$ and from [33, Proposition 6.1(i),(iii)] when p does not divide $|C_\Gamma(M)|$. □

Observe that $\rho : \mathcal{T} \to \mathcal{F}$ reflects isomorphisms. Hence, the isomorphism classes of objects of $\mathcal{T}$ and of $\mathcal{O}\mathcal{T}$ are $\mathcal{F}$-conjugacy classes.

A functor $\Phi : \mathcal{O}\mathcal{T}^{\mathrm{op}} \to \mathbf{Ab}$ is called *atomic* if there exists $Q \in \mathrm{obj}(\mathcal{T})$ such that Φ vanishes outside the $\mathcal{F}$-conjugacy class of Q. The fundamental property of Λ-functors is that they calculate the higher limits of atomic functors:

Lemma 3.11 [45, Lemma 4.3]. *Let $\mathcal{T}$ be a transporter system associated with a fusion system $\mathcal{F}$ over S. Let $\Phi : \mathcal{O}\mathcal{T}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-$\mathfrak{mod}$ be an atomic functor concentrated on the $\mathcal{F}$-conjugacy class of Q. Then there is a natural isomorphism*

$$H^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}; \Phi) \cong \Lambda^*(\mathrm{Aut}_{\mathcal{O}\mathcal{T}}(Q); \Phi(Q)).$$

We remark that the result holds, in fact, for any functor Φ into the category of abelian groups (indeed, the proof given by Oliver and Ventura only uses [45, Proposition A.2]).

Notice that $\rho : \mathcal{T} \to \mathcal{F}$ induces a functor $\bar{\rho} : \mathcal{O}\mathcal{T} \to \mathcal{O}(\mathcal{F})$. We will write $\mathcal{O}\mathcal{T}^c$ for the full subcategory of $\mathcal{T}$ spanned by $P \in \mathcal{T}$ that are $\mathcal{F}$-centric.

Corollary 3.12. *Let $\mathcal{T}$ be a transporter category for $\mathcal{F}$. Let $\bar{\Phi} : \mathcal{O}(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-$\mathfrak{mod}$ be a functor and set $\Phi = \bar{\Phi} \circ \bar{\rho}$. Then Φ is a functor $\mathcal{O}\mathcal{T}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-$\mathfrak{mod}$ and let Ψ be the restriction of Φ to $\mathcal{O}\mathcal{T}^c$. Then the restriction induces an isomorphism*

$$H^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}, \Phi) \xrightarrow{\cong} H^*((\mathcal{O}\mathcal{T}^c)^{\mathrm{op}}; \Psi).$$

Proof. Let $\Phi' : \mathcal{O}\mathcal{T}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-$\mathfrak{mod}$ be the functor obtained from Φ by setting $\Phi'(Q) = 0$ for all $Q \in \mathrm{obj}(\mathcal{T} \setminus \mathcal{T}^c)$ and $\Phi'(Q) = \Phi(Q)$ otherwise. This is a well-defined functor because the $\mathcal{F}$-centric subgroups are closed to overgroups. As there is no morphism in $\mathcal{O}\mathcal{T}$ from a centric object to a noncentric one, and as $\Psi = \Phi'|_{(\mathcal{O}\mathcal{T}^c)^{\mathrm{op}}}$ there is an isomorphism of cochain complexes $C^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}, \Phi') \cong C^*((\mathcal{O}\mathcal{T}^c)^{\mathrm{op}}, \Psi)$ (cf. the description of the bar resolution in [2, section III.5.1]), and hence an isomorphism

$$H^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}, \Phi') \cong H^*((\mathcal{O}\mathcal{T}^c)^{\mathrm{op}}, \Psi).$$

It remains to show that $H^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{O}\mathcal{T}^{\mathrm{op}}, \Phi')$.

Suppose that $Q \in \mathrm{obj}(\mathcal{T} \setminus \mathcal{T}^c)$ has minimal order. Set $M = \Phi(Q)$ and let $F_M : \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the induced atomic functor. The minimality of Q implies that there is an injective natural transformation $F_M \to \Phi$. By possibly replacing it with an $\mathcal{F}$-conjugate, we may assume that Q is fully centralized in $\mathcal{F}$. As Q is not $\mathcal{F}$-centric, choose some $x \in C_S(Q) \setminus Q$. Its image in $\Gamma = \mathrm{Aut}_{\mathcal{OT}}(Q)$ is a nontrivial element (as $x \notin Q$) of order p-power. It acts trivially on $\Phi(Q)$ because its image in $\mathrm{Out}_{\mathcal{F}}(Q)$ is trivial (as the image of $C_S(Q)$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$ is trivial) and because $\Phi = \bar{\Phi} \circ \bar{\rho}$. Lemma 3.10(a) implies that $\Lambda^*(\mathrm{Aut}_{\mathcal{OT}}(Q), \Phi(Q)) = 0$. It follows from Lemma 3.11 and the long exact sequence in derived limits associated with the short exact sequence $0 \to F_M \to \Phi \to \Phi/F_M \to 0$ that $H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{OT}^{\mathrm{op}}, \Phi/F_M)$. But Φ/F_M is obtained from Φ by annihilating the groups $\Phi(Q')$ for all Q' in the $\mathcal{F}$-conjugacy class of Q. We may now apply the same process to Φ/F_M and continue inductively (on the number of $\mathcal{F}$-conjugacy classes in $\mathcal{T} \setminus \mathcal{T}^c$) to show that $H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{OT}^{\mathrm{op}}, \Phi')$ as needed. □

Proof of Theorem 1.1. Let $\mathcal{F}$ be a saturated fusion system over S that affords a punctured group $\mathcal{T}$. That is, $\mathcal{T}$ is a transporter system associated to $\mathcal{F}$ with object set Δ containing all the nontrivial subgroups of S.

Let $\mathcal{H}^j : \mathcal{O}(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the functor

$$\mathcal{H}^j : P \mapsto H^j(P; \mathbb{F}_p)$$

and let $M^j : \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the composite $\mathcal{OT}^{\mathrm{op}} \xrightarrow{\bar{\rho}^{\mathrm{op}}} \mathcal{O}(\mathcal{F})^{\mathrm{op}} \xrightarrow{\mathcal{H}^j} \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$. Our goal is to show that for every $j \geqslant 0$,

$$H^i(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}; \mathcal{H}^j) = 0 \qquad \text{for all } i \geqslant 1.$$

Choose a fully normalized $P \in \mathrm{obj}(\mathcal{T})$. As $N_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{T}}(P)$, see [45, Proposition 3.4(a)], it follows that $C_S(P)$ is a Sylow p-subgroup of the kernel of $\mathrm{Aut}_{\mathcal{T}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$ and hence $C_S(P)/Z(P)$ is a Sylow p-subgroup of the kernel of $\mathrm{Aut}_{\mathcal{OT}}(P) \to \mathrm{Out}_{\mathcal{F}}(P)$. Thus, if P is $\mathcal{F}$-centric, then this kernel has order prime to p, and so [7, Lemma 1.3] implies the first isomorphism in

$$H^*(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}; \mathcal{H}^j) \cong H^*((\mathcal{OT}^c)^{\mathrm{op}}; M^j) \cong H^*(\mathcal{OT}^{\mathrm{op}}; M^j),$$

while Corollary 3.12 gives the second. It remains to show that $H^*(\mathcal{OT}^{\mathrm{op}}; M^j) = 0$ for all $j \geqslant 0$ and all $* \geqslant 1$.

Assume first that $j \geqslant 1$. We will show that M^j is a proto-Mackey functor for $\mathcal{OT}$ in this case. The transfer homomorphisms give rise to a (covariant) functor $\mathcal{H}^j_* : \mathcal{O}(\mathcal{F}) \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ where $P \mapsto H^j(P; \mathbb{F}_p)$ and to any $\varphi \in \mathcal{F}(P, Q)$ we assign $\mathrm{tr}(\varphi) : H^j(P; \mathbb{F}_p) \to H^j(Q; \mathbb{F}_p)$. The composition $M^j_* := \mathcal{H}^j_* \circ \bar{\rho}$ is a covariant functor $\mathcal{OT} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$.

Now, $\mathcal{OT}$ satisfies (PB $\times_{\amalg}$) by Propositions 3.7 and 3.9. Clearly, M^j and M^j_* have the same values on objects; this is the first condition in Definition 3.2. The transfer homomorphisms have the property that if $\varphi : P \to Q$ is an isomorphism then $\mathrm{tr}_P^Q(\varphi) = H^j(\varphi^{-1}; \mathbb{F}_p)$. This is the second condition in Definition 3.2. The factorization of morphisms in $\mathcal{T}$ as isomorphisms followed by inclusions imply that any pullback diagram $P' \xrightarrow{f} R \xleftarrow{g} Q'$ in $\mathcal{OT}$ is isomorphic to one of the form $P \xrightarrow{[\iota_P^R]} R \xleftarrow{[\iota_Q^R]} Q$. If $U = U([\iota_P^R], [\iota_Q^R])$ is the pullback (Definition 3.8), then by Proposition 3.9, $U = \coprod_{x \in X} Q^x \cap P$ where x runs through a set $X = (Q\backslash R/P)_{\mathcal{T}}$ of representatives of those double cosets QxP with $x \in R$ and $Q^x \cap P \in \mathcal{T}$, namely $Q^x \cap P \neq 1$ (because obj($\mathcal{T}$) is the set of all nontrivial subgroups of S). As $j \geqslant 1$ we have that $H^j(1; \mathbb{F}_p) = 0$ so $\bigoplus_{x \in X} H^j(Q^x \cap P; \mathbb{F}_p) = \bigoplus_{x \in Q\backslash R/P} H^j(Q^x \cap P; \mathbb{F}_p)$, where here $Q\backslash R/P$ is a full set of double coset representatives. Mackey's formula [10, Proposition 9.5(iii)] then gives the commutativity of the diagram

$$\begin{tikzcd}
\bigoplus_{x \in X} H^j(Q^x \cap P; \mathbb{F}_p) \arrow[r, "\sum_x \mathrm{tr}^P_{Q^x \cap P} \circ c_x"] & H^j(Q; \mathbb{F}_p) \\
H^j(P; \mathbb{F}_p) \arrow[u, "(\mathrm{res}^P_{Q^x \cap P})_{x \in X}"] \arrow[r, "\mathrm{tr}^R_P"'] & H^j(R; \mathbb{F}_p) \arrow[u, "\mathrm{res}^R_Q"']
\end{tikzcd}$$

This shows that the third condition in Definition 3.2 also holds and that M^j is a proto-Mackey functor. Now, Condition (B1) in Proposition 3.3 clearly holds for $\mathcal{O}\mathcal{T}$ and (B2) holds by Lemma 3.5. It follows that $H^i(\mathcal{O}\mathcal{T}^{\mathrm{op}}; M^j) = 0$ for all $i \geqslant 1$ as needed.

It remains to deal with the case $j = 0$. In this case $\mathcal{H}^0$ is the constant functor with value $\mathbb{F}_p$. Thus, $\mathrm{Out}_{\mathcal{F}}(P)$ acts trivially on $\mathbb{F}_p$ for any $P \in \mathcal{F}^c$. It follows from Lemma 3.10(b) that if $P = S$ then $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(S), \mathbb{F}_p) = 0$ for all $i > 0$, and if $P < S$ then $\mathrm{Out}_{\mathcal{F}}(P)$ contains an element of order p so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P), \mathbb{F}_p) = 0$. Now [8, Proposition 3.2] together with a filtration of $\mathcal{H}^0$ by atomic functors show that $\mathcal{H}^0$ is acyclic. □

4 | PUNCTURED GROUPS FOR $\mathcal{F}_{\mathrm{Sol}}(q)$

The Benson–Solomon systems were predicted to exist by Benson [5], and were later constructed by Levi and Oliver [35, 36]. They form a family of exotic fusion systems at the prime 2 whose isomorphism types are parameterized by the nonnegative integers. Later, Aschbacher and Chermak gave a different construction as the fusion system of an amalgamated free product of finite groups [1]. The main result of this section is the following theorem.

Theorem 4.1. *A Benson–Solomon system $\mathcal{F}_{\mathrm{Sol}}(q)$ over the 2-group S has a punctured group if and only if $q \equiv \pm 3 \pmod 8$. If $q \equiv \pm 3 \pmod 8$, there is a punctured group $\mathcal{L}$ for $\mathcal{F}_{\mathrm{Sol}}(q)$ that is unique up to rigid isomorphism with the following two properties:*

(1) $C_{\mathcal{L}}(Z(S)) = \mathrm{Spin}_7(3)$, *and*
(2) $\mathcal{L}|_\Delta$ *is a linking locality, where Δ is the set of $\mathcal{F}$-subcentric subgroups of S of 2-rank at least 2.*

4.1 | Notation for Spin_7 and Sol

It will usually be most convenient to work with a Lie theoretic description of Spin_7. The notational conventions that we use in this section for algebraic groups and finite groups of Lie type are summarized in the Appendix.

4.1.1 | The maximal torus and root system

Let p be an odd prime, and set

$$\overline{H} = \mathrm{Spin}_7(\overline{\mathbb{F}}_p).$$

Fix a maximal torus $\overline{T}$ of $\overline{H}$, let $X(\overline{T}) = \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_p^\times) \cong \mathbb{Z}^3$ be the character group (of algebraic homomorphisms), and denote by $V = \mathbb{R} \otimes_{\mathbb{Z}} X(\overline{T})$ the ambient Euclidean space that we regard as containing $X(\overline{T})$. Let $\Sigma(\overline{T}) \subseteq X(\overline{T})$ be the set of $\overline{T}$-roots. Denote a $\overline{T}$-root subgroup for the root α by

$$\overline{X}_\alpha = \{x_\alpha(\lambda) \mid \lambda \in \overline{\mathbb{F}}_p\}.$$

As $\overline{H}$ is semisimple, it is generated by its root subgroups [26, Theorem 1.10.1(a)]. We assume that the implicit parameterization $x_\alpha(\lambda)$ of the root subgroups is one like that given by Chevalley, so that the Chevalley relations hold with respect to certain signs $c_{\alpha,\beta} \in \{\pm 1\}$ associated to each pair of roots [26, Theorem 1.12.1].

We often identify $\Sigma(\overline{T})$ with the abstract root system

$$\Sigma = \{\pm e_i \pm e_j, \pm e_i \mid 1 \leqslant i, j \leqslant 3\} \subseteq \mathbb{R}^3$$

of type B_3, having base $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3,$$

where the e_i are standard vectors. Write $\Sigma^\vee = \{\alpha^\vee \mid \alpha \in \Sigma\}$ for the dual root system, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Instead of working with respect to the α_i, it is sometimes convenient to work instead with a different set of roots $\{\beta_i\} \subseteq \Sigma$:

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 = e_1 + e_2, \quad \beta_3 = \alpha_3.$$

This is an orthogonal basis of V with respect to the standard inner product $(\,,\,)$ on $\mathbb{R}^3$. An important feature of this basis is that for each i and j,

$$\Sigma \cap \{k\beta_i + l\beta_j \mid k, l \in \mathbb{Z}\} = \{\pm\beta_i, \pm\beta_j\}, \tag{4.1}$$

a feature not enjoyed, for example, by an orthogonal basis consisting of short roots. In particular, the β_j-root string through β_i consists of β_i only. This implies via Lemma A.1(4), that the corresponding signs involving the β_i that appear in the Chevalley relations for $\overline{H}$ are

$$c_{\beta_i,\beta_j} = 1 \text{ if } i \neq j, \text{ and } c_{\beta_i,\beta_i} = -1. \tag{4.2}$$

4.1.2 | The torus and the lattice of coroots

We next set up notation and state various relations for elements of $\overline{T}$. Let

$$h_\alpha(\lambda) \in \overline{T} \quad \text{and} \quad n_\alpha(\lambda) \in N_{\overline{H}}(\overline{T})$$

be as given in the Appendix as words in the generators $x_\alpha(\lambda)$. By Lemma A.2 and as $\overline{H}$ is of universal type, there is an isomorphism $\mathbb{Z}\Sigma^\vee \otimes \overline{\mathbb{F}}_p^\times \to \overline{T}$ that on simple tensors sends $\alpha^\vee \otimes \lambda$ to $h_\alpha(\lambda)$, and the homomorphisms $h_{\alpha_i} : \overline{\mathbb{F}}_p^\times \to \overline{T}$ are injective. In particular, as $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ is a basis for $\mathbb{Z}\Sigma^\vee$, we have $\overline{T} = h_{\alpha_1}(\overline{\mathbb{F}}_p^\times) \times h_{\alpha_2}(\overline{\mathbb{F}}_p^\times) \times h_{\alpha_3}(\overline{\mathbb{F}}_p^\times)$. Define elements z and $z_1 \in \overline{T}$ by

$$z_1 = h_{\alpha_1}(-1) \quad \text{and} \quad z = h_{\alpha_3}(-1).$$

Thus, z and z_1 are involutions. Similar properties hold with respect to the β_i's. Recall that $\beta_i = \alpha_i$ for $i = 1, 3$. As $\beta_2^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$, Lemma A.2(3) yields

$$h_{\beta_2}(-1) = h_{\alpha_1}(-1)h_{\alpha_2}((-1)^2)h_{\alpha_3}(-1) = z_1 z.$$

In particular,

$$h_{\beta_1}(-1)h_{\beta_2}(-1)h_{\beta_3}(-1) = z_1 z_1 z z = 1. \tag{4.3}$$

However, as the $\mathbb{Z}$-span of the $\beta_i^\vee$'s is of index 2 in $\mathbb{Z}\Sigma^\vee$ and every element of $\overline{\mathbb{F}}_p^\times$ is a square, we still have

$$\overline{T} = h_{\beta_1}(\overline{\mathbb{F}}_p^\times)h_{\beta_2}(\overline{\mathbb{F}}_p^\times)h_{\beta_3}(\overline{\mathbb{F}}_p^\times). \tag{4.4}$$

So, the $h_{\beta_i}(\overline{\mathbb{F}}_p)^\times$ generate $\overline{T}$, but the product is no longer direct.

4.1.3 | The normalizer of the torus and Weyl group

The subgroup

$$\widehat{W} := \langle n_{\alpha_1}(1), n_{\alpha_2}(1), n_{\alpha_3}(1)\rangle \leqslant N_{\overline{H}}(\overline{T})$$

projects onto the Weyl group

$$W = \langle w_{\alpha_1}, w_{\alpha_2}, w_{\alpha_3}\rangle \cong C_2 \wr S_3 \cong C_2 \times S_4$$

of type B_3 in which the w_{α_i} are fundamental reflections. Also, $\widehat{W} \cap \overline{T}$ is the 2-torsion subgroup $\{t \in \overline{T} \mid t^2 = 1\}$ of $\overline{T}$, see [26, Remark 1.12.11]. A subgroup similar to $\widehat{W}$ was denoted "W" in [1, Lemma 4.3]. It is sometimes called the Tits subgroup.

Let

$$\gamma = c_{\alpha_1,\alpha_2+\alpha_3} \in \{\pm 1\},$$

and fix a fourth root $i \in \mathbb{F}_p^\times$ of 1. (This notation will hopefully not cause confusion with the use of i as an index.) Define elements $w_0, \tau \in N_{\overline{H}}(\overline{T})$ by

$$w_0 = n_{\beta_1}(-\gamma)n_{\beta_2}(1)n_{\beta_3}(1) \quad \text{and} \quad \tau = n_{\alpha_2+\alpha_3}(1)h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i).$$

It will be shown in Lemma 4.2 that w_0 and τ are commuting involutions and that w_0 inverts $\overline{T}$.

4.1.4 | Three commuting SL_2's

Let

$$\overline{L}_i = \langle \overline{X}_{\beta_i}, \overline{X}_{-\beta_i} \rangle,$$

for $i = 1, 2, 3$. Thus, $\overline{L}_i \cong SL_2(\overline{\mathbb{F}}_p)$ for each i by the Chevalley relations, again using that $\overline{H}$ is of universal type when $i = 3$. A further consequence of (4.1) is that the Chevalley commutator formula [26, 1.12.1(b)] yields

$$[\overline{L}_i, \overline{L}_j] = 1 \quad \text{for all} \quad i \neq j.$$

For each i, $\overline{L}_i$ has unique involution $h_{\beta_i}(-1)$ that generates the center of $\overline{L}_i$. By (4.3), the center of the commuting product $\overline{L}_1\overline{L}_2\overline{L}_3$ is $\langle z, z_1 \rangle$, of order 4. By (4.4), $\overline{T} \leqslant \overline{L}_1\overline{L}_2\overline{L}_3$.

4.1.5 | The Steinberg endomorphism and $\mathrm{Spin}_7(q)$

We next set up notation for the Steinberg endomorphism we use to descend from $\overline{H}$ to the finite versions. Let $q = p^a$ be a power of p. Let $\epsilon \in \{\pm 1\}$ be such that $q \equiv \epsilon \pmod 4$, and let k be the 2-adic valuation of $q - \epsilon$.

The standard Frobenius endomorphism ζ of $\overline{H}$ is determined by its action $x_\alpha(\lambda)^\zeta = x_\alpha(\lambda^p)$ on the root groups, and so from the definition of the n_α and h_α in (A.1), also $n_\alpha(\lambda)^\zeta = n_\alpha(\lambda^p)$ and $h_\alpha(\lambda)^\zeta = h_\alpha(\lambda^p)$. Write c_{w_0} conjugation map induced by w_0, as usual, and define

$$\sigma = \begin{cases} \zeta^a & \text{if } \epsilon = 1 \\ \zeta^a c_{w_0} & \text{if } \epsilon = -1. \end{cases}$$

Then σ is a Steinberg endomorphism of $\overline{H}$ in the sense of [26, Definition 1.15.1], and we set

$$H := C_{\overline{H}}(\sigma) = \mathrm{Spin}_7(q).$$

Given that w_0 inverts $\overline{T}$, the action of σ on $\overline{T}$ is given for each $t \in \overline{T}$ by

$$t^{\sigma} = t^{\epsilon q},$$

and hence

$$C_{\overline{T}}(\sigma) = \{t \in \overline{T} \mid t^{\epsilon q} = t\} \cong (C_{q-\epsilon})^3.$$

Likewise,

$$C_{\overline{T}}(\sigma c_{w_0}) \cong (C_{q+\epsilon})^3.$$

Finally, let $\mu = \mu_q \in \overline{\mathbb{F}}_p^{\times}$ be a fixed element of 2-power order satisfying $\mu^{\epsilon q} = -\mu$ and powering to the fourth root i, and set

$$c = h_{\beta_1}(\mu) h_{\beta_2}(\mu) h_{\beta_3}(\mu) \in \overline{T}.$$

4.1.6 | A Sylow 2-subgroup

We next set up notation for Sylow 2-subgroups of $\overline{H}$ and H along with various important subgroups of them. Let

$$\overline{S} = \overline{T}_{2^\infty} \widehat{W}_{\overline{S}},$$

where $\overline{T}_{2^\infty}$ denotes the 2-power torsion in $\overline{T}$ and where $\widehat{W}_{\overline{S}} = \langle n_{\alpha_1}(1), n_{\alpha_2+\alpha_3}(1), n_{\alpha_3}(1)\rangle$.

Set

$$S = C_{\overline{S}}(\sigma).$$

Define subgroups

$$Z < U < E < A \leqslant S$$

by

$$Z = \langle z \rangle, \quad U = \langle z, z_1 \rangle, \quad E = \{t \in T \mid t^2 = 1\}, \quad \text{and } A = E\langle w_0 \rangle.$$

Then $Z = Z(S)$, U is the unique four subgroup normal in S, E is elementary abelian of order 8, and A is elementary abelian of order 16. It will be shown in Lemma 4.2 that $w_0 \in S$, and hence $A \leqslant S$.

We also write

$$T_S = T \cap S;$$

thus, $T_S = O_2(T) \cong (C_{2^k})^3$ is the 2^k-torsion in $\overline{T}$, a Sylow 2-subgroup of T.

4.2 | Conjugacy classes of elementary abelian subgroups of $\overline{H}$ and H

We state and prove here several lemmas on conjugacy classes of elementary abelian subgroups of $\overline{H}$ and H, and on the structure of various 2-local subgroups. Much of the material here is written down elsewhere, for example, in [35] and [1]. Our setup is a little different because of the emphasis on the Lie theoretic approach, so we aim to give more detail in order to make the treatment here as self-contained as possible.

The first lemma is elementary and records several initial facts about the elements we have defined in the previous section. Its proof is mainly an exercise in applying the various Chevalley relations defining $\overline{H}$.

Lemma 4.2. *Adopting the notation from Subsection 4.1, we have:*

(1) $Z(\overline{H}) = Z = \langle z \rangle$;
(2) *the elements* w_0 *and* τ *are involutions in* $N_S(\overline{T}) - \overline{T}$, *and* $c \in T_S$ *has order* 2^k, *powering into* $E - U$;
(3) w_0 *inverts* $\overline{T}$; *and*
(4) $[w_0, \tau] = [c, \tau] = 1$.

Proof.

(1): It is well-known that $Z(\overline{H})$ has order 2. We show here for the convenience of the reader that the involution generating $Z(\overline{H})$ is $z = h_{\alpha_3}(-1)$. We already observed in Subsection 4.1.3 that z is an involution. For each root $\alpha \in \Sigma$, the inner product of α with α_3 is an integer, and so $\langle \alpha, \alpha_3 \rangle = 2(\alpha, \alpha_3) \in 2\mathbb{Z}$. By Lemma A.2(1), $h_{\alpha_3}(-1)$ lies in the kernel of α. Thus, the centralizer in $\overline{H}$ of $h_{\alpha_3}(-1)$ contains all root groups by Proposition A.3, and hence $C_{\overline{H}}(h_{\alpha_3}(-1)) = \overline{H}$.

(2): We show that w_0 is an involution. Using Equations (A.6) and (4.2), we see that

$$[n_{\beta_i}(\pm 1), n_{\beta_j}(\pm 1)] = 1 \text{ for each } i, j \in \{1, 2, 3\}. \tag{4.5}$$

So, $w_0^2 = 1$ by (A.7) and (4.3).

We next prove that τ is an involution. Recall

$$\tau = n_{\alpha_2+\alpha_3}(1)h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i).$$

First, note that $n_{\alpha_2+\alpha_3}(1)^2 = z$. To see this, use (A.7) to get $n_{\alpha_2+\alpha_3}(1)^2 = h_{\alpha_2+\alpha_3}(-1)$. Then use $(\alpha_2 + \alpha_3)^\vee = 2\alpha_2 + 2\alpha_3 = 2\alpha_2^\vee + \alpha_3^\vee$ and Lemma A.2(3) to get

$$n_{\alpha_2+\alpha_3}(1)^2 = h_{\alpha_2+\alpha_3}(-1) = h_{\alpha_2}(-1)^2 h_{\alpha_3}(-1) = h_{\alpha_3}(-1) = z$$

as desired. Next, the fundamental reflection $w_{\alpha_2+\alpha_3}$ interchanges β_1 and β_2 and fixes β_3, so $n_{\alpha_2+\alpha_3}(1)$ inverts $h_{\beta_1}(-i)h_{\beta_2}(i)$ by conjugation and centralizes $h_{\beta_3}(i)$ by (A.5). Hence,

$$\begin{aligned}\tau^2 &= n_{\alpha_2+\alpha_3}(1)^2 (h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i))^{n_{\alpha_2+\alpha_3}(1)}(h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)) \\ &= n_{\alpha_2+\alpha_3}(1)^2 h_{\beta_3}(i)^2 = zz = 1.\end{aligned}$$

We show c is of order 2^k and powers into $E - U$. Recall that k is the 2-adic valuation of $q - \epsilon$, and that $C_{\overline{T}}(\sigma) = (C_{q-\epsilon})^3$. The latter has Sylow 2-subgroup of exponent 2^k. But $c \in C_{\overline{T}}(\sigma)$ because

$$c^\sigma = h_{\beta_1}(\mu^{\epsilon q})h_{\beta_2}(\mu^{\epsilon q})h_{\beta_3}(\mu^{\epsilon q}) = h_{\beta_1}(-\mu)h_{\beta_2}(-\mu)h_{\beta_3}(-\mu) \overset{(4.3)}{=} h_{\beta_1}(\mu)h_{\beta_2}(\mu)h_{\beta_3}(\mu) = c.$$

So, c has order at most 2^k. On the other hand,

$$c^{2^{k-1}} = h_{\beta_1}(i)h_{\beta_2}(i)h_{\beta_3}(i).$$

As in Subsection 4.1.4, we have $h_{\beta_2}(i) = h_{\alpha_1}(i)h_{\alpha_2}(i^2)h_{\alpha_3}(i)$, and so

$$c^{2^{k-1}} = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(-1).$$

As $\overline{H}$ is of universal type and $U = \langle h_{\alpha_1}(-1), h_{\alpha_3}(-1)\rangle$, it follows from Lemma A.2(2) that $c^{2^{k-1}} \in E - U$, and hence c has order 2^k as claimed. In particular, this shows $c \in T_S$.

It remains to show that $w_0, \tau \in S$ in order to complete the proof of (2). For each $\alpha \in \Sigma$, we have $[n_\alpha(\pm 1), \zeta] = 1$ by (A.1), while $[n_{\beta_i}(\pm 1), w_0] = 1$ for $i = 1, 2, 3$ by (A.6) and (4.2). Also, $h_{\beta_1}(\pm i)h_{\beta_2}(\pm i)h_{\beta_3}(\pm i) \in E \leqslant H$ by (4.3). These points combine to give $w_0 \in H$, $\tau^\zeta = \tau$, and $\tau \in \overline{S}$. As $[w_0, \tau] = 1$ by (4), we see $\tau \in H$, so indeed $\tau \in H \cap \overline{S} = S$. Finally,

$$n_{\beta_1}(1)n_{\beta_2}(\gamma)n_{\beta_3}(1) = n_{\beta_1}(1)n_{\beta_1}(1)^{n_{\alpha_2+\alpha_3}(1)}n_{\beta_3}(1) \in \overline{S}$$

and this element represents the same coset modulo E as w_0 does by (A.7) and (4.5). As $E \leqslant S$, it follows that $w_0 \in S$.

(3): As $\{\beta_1, \beta_2, \beta_3\}$ is an orthogonal basis of V, the image $w_{\beta_1}w_{\beta_2}w_{\beta_3}$ in W of w_0 acts as minus the identity on V. In particular, it acts as minus the identity on the lattice of coroots $\mathbb{Z}\Sigma^\vee \subseteq V$. This implies via Lemma A.2(4) that w_0 inverts $\overline{T}$, and so (3) holds.

(4): Showing $[w_0, \tau] = 1$ requires some information about the signs appearing in our fixed Chevalley presentation. First,

$$\langle \beta_1, \alpha_2 + \alpha_3\rangle = \frac{2(\alpha_1, \alpha_2 + \alpha_3)}{(\alpha_2 + \alpha_3, \alpha_2 + \alpha_3)} = -2.$$

So, by Lemma A.1(3),

$$c_{\beta_1,\alpha_2+\alpha_3}c_{\beta_2,\alpha_2+\alpha_3} = (-1)^{\langle\beta_1,\alpha_2+\alpha_3\rangle} = (-1)^{-2} = 1,$$

and hence $c_{\beta_2,\alpha_2+\alpha_3} = \gamma \overline{\text{def}} = c_{\beta_1,\alpha_2+\alpha_3}$. The root string of $\alpha_2 + \alpha_3 = e_2$ through $\beta_3 = e_3$ is $e_3 - e_2, e_3, e_3 + e_2$ so

$$c_{\beta_3,\alpha_2+\alpha_3} = (-1)^1 = -1$$

by Lemma A.1(4). So, $n_{\alpha_2+\alpha_3}(1)$ inverts each of $n_{\beta_1}(-\gamma)n_{\beta_2}(1)$ and $n_{\beta_3}(1)$, by (A.6). Using $[w_0, n_{\alpha_2+\alpha_3}(1)] \in E$, $h_{\beta_1}(\pm i)h_{\beta_2}(\pm i)h_{\beta_3}(\pm i) \in E$, and (4.5), we thus have

$$\begin{aligned}
[w_0, \tau] &= [w_0, h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)][w_0, n_{\alpha_2+\alpha_3}(1)]^{h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)} \\
&= [w_0, h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)][w_0, n_{\alpha_2+\alpha_3}(1)] \\
&= [w_0, n_{\alpha_2+\alpha_3}(1)] \\
&= [n_{\beta_1}(-\gamma)n_{\beta_2}(1), n_{\alpha_2+\alpha_3}(1)]^{n_{\beta_3}(1)}[n_{\beta_3}(1), n_{\alpha_2+\alpha_3}(1)]
\end{aligned}$$

$$= (n_{\beta_1}(-\gamma)^2 n_{\beta_2}(1)^2)^{n_{\beta_3}(1)} n_{\beta_3}(1)^2$$

$$= n_{\beta_1}(\gamma)^2 n_{\beta_2}(-1)^2 n_{\beta_3}(1)^2$$

$$= z_1 z_1 z z$$

$$= 1.$$

Finally, as $[c, n_{\alpha_2+\alpha_3}(1)] = 1$ by (A.5), we have $[c, \tau] = 1$. □

For any group X and nonnegative integer r, write $\mathscr{E}_r(X)$ for the elementary abelian subgroups of X of order 2^r and $\mathscr{E}_r(X, Y)$ for the subset of $\mathscr{E}_r(X)$ consisting of those members containing the subgroup Y.

We next record information about the conjugacy classes and normalizers of four subgroups containing Z.

Lemma 4.3. *Let $\overline{B} = N_{\overline{H}}(U)$ and $B = N_H(U)$. Write $\overline{B}^{\circ}$ for the connected component of $\overline{B}$.*

(1) *$\mathscr{E}_2(\overline{H}, Z) = U^{\overline{H}}$, and*

$$\overline{B} = (\overline{L}_1\overline{L}_2\overline{L}_3)\langle\tau\rangle \quad \textit{and} \quad \overline{B}^{\circ} = C_{\overline{H}}(U) = \overline{L}_1\overline{L}_2\overline{L}_3,$$

where τ interchanges $\overline{L}_1$ and $\overline{L}_2$ by conjugation and centralizes $\overline{L}_3$. Moreover $Z(\overline{B}^{\circ}) = U$.

(2) *$\mathscr{E}_2(H, Z) = U^H$, and*

$$B = (L_1L_2L_3)\langle c, \tau\rangle \text{ and } C_H(U) = (L_1L_2L_3)\langle c\rangle,$$

where $L_i = C_{\overline{L}_i}(\sigma)$, and where $c \in N_{\overline{T}}(L_1L_2L_3)$ acts as a diagonal automorphism on each L_i.

Proof. Viewing $\overline{H}$ classically, an involution in $\overline{H}/Z$ has involutory preimage in $\overline{H}$ if and only if it has -1-eigenspace of dimension 4 on the natural orthogonal module (see, for example, [1, Lemma 4.2] or [35, Lemma A.4(b)]). It follows that all noncentral involutions are $\overline{H}$-conjugate into U, and hence that all four subgroups containing Z are conjugate. Viewing $\overline{H}$ Lie theoretically gives another way to see this: let V be a four subgroup of $\overline{H}$ containing Z, and let $v \in V - Z$. By, for example, [50, 6.4.5(ii)], v lies in a maximal torus, and all maximal tori are conjugate. So, we may conjugate in $\overline{H}$ and take $v \in E$. Using Lemma 4.4(1), for example, $N_{\overline{H}}(\overline{T})/C_{N_{\overline{H}}(\overline{T})}(E) \cong S_4$ acts faithfully on E and centralizes Z, so as a subgroup of $GL(E)$ it is the full stabilizer of the chain $1 < Z < E$. This implies $N_{\overline{H}}(\overline{T})$ acts transitively on the nonidentity elements of the quotient E/Z, so v is $N_{\overline{H}}(T)$-conjugate into U.

We next use Proposition A.3 to compute $\overline{B}$. Recall that

$$U = \langle z, z_1\rangle = \langle h_{\alpha_3}(-1), h_{\alpha_1}(-1)\rangle$$

and that $z = h_{\alpha_3}(-1)$ is central in $\overline{H}$ by Lemma 4.2(1). So, $C_{\overline{H}}(U) = C_{\overline{H}}(h_{\alpha_1}(-1))$. By Proposition A.3 and inspection of Σ,

$$C_{\overline{H}}(U)^{\circ} = \langle \overline{T}, \overline{X}_\alpha \mid \langle\alpha, \alpha_1\rangle \text{ is even}\rangle$$

$$= \langle \overline{T}, \overline{X}_{\pm\alpha} \mid \alpha \in \{\beta_1, \beta_2, \beta_3\}\rangle.$$

Further, $\overline{T} \leqslant \overline{L}_1\overline{L}_2\overline{L}_3$ by (4.4), so

$$C_{\overline{H}}(U)^\circ = \langle \overline{X}_{\beta_i}, \overline{X}_{-\beta_i} \mid i \in \{1,2,3\}\rangle = \overline{L}_1\overline{L}_2\overline{L}_3 \tag{4.6}$$

as claimed.

We next prove that $C_{\overline{H}}(U)$ is connected. As $C_{\overline{H}}(U) = C_{\overline{H}}(z_1)$, this follows directly from a theorem of Steinberg to the effect that the centralizer of a semisimple element in a simply connected reductive group is connected, but it is possible to give a more direct argument in this special case. By Proposition A.3,

$$C_{\overline{H}}(U) = C_{\overline{H}}(U)^\circ C_{N_{\overline{H}}(\overline{T})}(U),$$

and we claim that $C_{N_{\overline{H}}(\overline{T})}(U) \leqslant C_{\overline{H}}(U)^\circ$. By (4.6), $N_{C_{\overline{H}}(U)^\circ}(\overline{T})/\overline{T}$ is elementary abelian of order 8. On the other hand, $C_{N_{\overline{H}}(\overline{T})}(U)/\overline{T}$ stabilizes the flag $1 < Z < U < E$, and so induces a group of transvections on E of order 4 with center Z and axis U. The element w_0 of $N_{\overline{H}}(\overline{T})$ inverts $\overline{T}$ and is trivial on E by Lemma 4.2(3). It follows that $|N_{C_{\overline{H}}(U)}(\overline{T})/\overline{T}| = |N_{C_{\overline{H}}(U)^\circ}(\overline{T})/\overline{T}|$, and so $N_{C_{\overline{H}}(U)}(\overline{T}) = N_{C_{\overline{H}}(U)^\circ}(\overline{T})$. Thus,

$$C_{N_{\overline{H}}(\overline{T})}(U) = N_{C_{\overline{H}}(U)}(\overline{T}) = N_{C_{\overline{H}}(U)^\circ}(\overline{T}) \leqslant C_{\overline{H}}(U)^\circ,$$

completing the proof of the claim. By (4.6)

$$C_{\overline{H}}(U) = \overline{L}_1\overline{L}_2\overline{L}_3. \tag{4.7}$$

For each $\lambda \in \overline{\mathbb{F}}_p$, we have

$$\begin{aligned} x_{\beta_3}(\lambda)^\tau &= x_{\beta_3}(\lambda)^{n_{\alpha_2+\alpha_3}(1)h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)} \\ &= x_{\beta_3}(-\lambda)^{h_{\beta_3}(i)} \\ &= x_{\beta_3}(i^{\langle\beta_3,\beta_3\rangle}(-\lambda)) \\ &= x_{\beta_3}(i^2(-\lambda)) \\ &= x_{\beta_3}(\lambda). \end{aligned}$$

Similarly, $x_{-\beta_3}(\lambda)^\tau = x_{-\beta_3}(i^{-2}(-\lambda)) = x_{-\beta_3}(\lambda)$. So, as $\overline{L}_3 = \langle x_{\pm\beta_3}(\lambda)\rangle$, we have $[\overline{L}_3, \tau] = 1$. Finally, as $w_{\alpha_2+\alpha_3}$ interchanges β_1 and β_2, and as $\overline{T}$ normalizes all root groups, τ interchanges $\overline{L}_1$ and $\overline{L}_2$. In particular, τ interchanges the central involutions $h_{\beta_1}(-1) = z_1$ and $h_{\beta_2}(-1) = zz_1$ of $\overline{L}_1$ and $\overline{L}_2$. This shows τ acts nontrivially on U, and hence

$$\overline{B} = (\overline{L}_1\overline{L}_2\overline{L}_3)\langle\tau\rangle,$$

completing the proof of (1).

By (1), $C_{\overline{H}}(U)$ is connected, so [26, Theorem 2.1.5] applies to give $\mathscr{E}_2(H,Z) = U^H$. Let $L_i = C_{\overline{L}_i}(\sigma)$ for $i = 1, 2, 3$, and set $B^\circ = L_1L_2L_3 \leqslant H$. As $w \in N_H(U) - C_H(U)$, we have $C_H(U) = C_{\overline{B}^\circ}(\sigma)$. Let $\tilde{B}$ denote the direct product of the $\overline{L}_i$, and let $\tilde{\sigma}$ be the Steinberg endomorphism lifting $\sigma|_{\overline{B}^\circ}$ along the isogeny $\tilde{B} \to \overline{B}^\circ$ given by quotienting by $\langle(-1,-1,-1)\rangle$ (see, e.g., [26, Lemma 2.1.2(d,e)]). Then $C_{\tilde{B}}(\tilde{\sigma}) = L_1 \times L_2 \times L_3$. So, by [26, Theorem 2.1.8] applied with the pair $\tilde{B}$, $\langle(-1,-1,-1)\rangle$ in the role of $\overline{K}, \overline{Z}$, we see that B° is of index 2 in $C_H(U)$ with $C_H(U) = B^\circ(C_H(U) \cap \overline{T}) = B^\circ T$. The element $c = h_{\beta_1}(\mu)h_{\beta_2}(\mu)h_{\beta_3}(\mu) \in T$ lifts to an element $\tilde{c} \in \tilde{B}$ with $[\tilde{c}, \tilde{\sigma}] = (-1,-1,-1)$ by definition of μ, and so $c \in C_H(U) - B^\circ$ by [26, Theorem 2.1.8]. Finally, as each L_i is generated by root groups on which c acts nontrivially, c acts as a diagonal automorphism on each L_i. $\square$

Next we consider the H-conjugacy classes of elementary abelian subgroups of order 8 that contain Z.

Lemma 4.4. *The following hold.*

(1) $N_{\overline{H}}(E) = N_{\overline{H}}(\overline{T})$ *and* $C_{\overline{H}}(E) = \overline{T}\langle w_0\rangle$.
(2) $N_H(E) = N_H(T)$ *and* $C_H(E) = T\langle w_0\rangle$.
(3) $N_{\overline{H}}(\overline{T})/\overline{T} \cong C_2 \times S_4 \cong N_H(T)/T$.

Proof. Given that w_0 inverts $\overline{T}$ (Lemma 4.2(3)) part (1) is proved in Proposition A.4.

By (1),

$$N_H(E) = N_{\overline{H}}(E) \cap H = N_{\overline{H}}(\overline{T}) \cap H = N_H(\overline{T}),$$

while $N_H(\overline{T}) \leqslant N_H(H \cap \overline{T}) = N_H(T)$. These combine to show the inclusion $N_H(E) \leqslant N_H(T)$. But $N_H(T) \leqslant N_H(E)$ because $E = \Omega_1(O_2(T))$ is characteristic in T. Next, by (1),

$$C_H(E) = C_{\overline{H}}(E) \cap H = \overline{T}\langle w_0\rangle \cap H = (\overline{T} \cap H)\langle w_0\rangle$$

with the last equality as $w_0 \in H$ by Lemma 4.2(2). This shows $C_H(E) = T\langle w_0\rangle$.

For part (3) in the case of $\overline{H}$, see Subsection 4.1.3. We show part (3) for H. First, by (1) and (2),

$$N_H(\overline{T}) = C_{N_{\overline{H}}(\overline{T})}(\sigma) = C_{N_{\overline{H}}(E)}(\sigma) = N_H(E) = N_H(T). \tag{4.8}$$

In the special case $\epsilon = 1$, σ centralizes $\widehat{W}$, which covers $W = N_{\overline{H}}(\overline{T})/\overline{T}$. Using (4.8), this shows $N_H(T) = N_H(\overline{T})$ projects onto W with kernel $\overline{T} \cap C_{N_{\overline{H}}(\overline{T})}(\sigma) = T$. So, $N_H(T)/T \cong W$ in this case.

In any case, $\overline{T}w_0$ generates the center of $N_{\overline{H}}(\overline{T})/\overline{T}$, so $g^\sigma g^{-1} \in \overline{T}$ for each $g \in N_{\overline{H}}(\overline{T})$. As $\overline{T}$ is connected, for each such g there is $t \in \overline{T}$ with $t^{-\sigma}t = g^\sigma g^{-1}$ by the Lang–Steinberg theorem, and hence $tg \in C_{N_{\overline{H}}(\overline{T})}(\sigma)$. This shows each coset $\overline{T}g$ contains an element centralized by σ, and so arguing as in the previous paragraph, we have $N_H(T)/T \cong W$. $\square$

Lemma 4.5. *Let $d = cw_0$ and $E' = U\langle d\rangle \leqslant S$. Then $\mathscr{E}_3(H,Z)$ is the disjoint union of E^H and E'^H. Moreover, there is a σ-invariant maximal torus T' of H with $E' = \{t \in T' \mid t^2 = 1\}$ such that the following hold.*

(1) $O_{2'}(C_H(E)) = O_{2'}(T) \cong (C_{(q-\epsilon)/2^k})^3$, *and* $N_H(T)/T \cong C_2 \times S_4$ *acts faithfully on the* r*-torsion subgroup of* T *for each odd prime* r *dividing* $q - \epsilon$.
(2) $O_{2'}(C_H(E')) = O_{2'}(T') \cong (C_{(q+\epsilon)/2})^3$, *and* $N_H(T')/T' \cong C_2 \times S_4$ *acts faithfully on the* r*-torsion subgroup of* T' *for each odd prime* r *dividing* $q + \epsilon$.
(3) $C_H(E') = T'\langle w_0' \rangle$ *for some involution* w_0' *inverting* T'.

Proof. By Lemma 4.2, w_0 is an involution inverting $\overline{T}$ and hence inverting c. So, d is an involution, and indeed, E' is elementary abelian of order 8.

Part of this lemma is proved by Aschbacher and Chermak [1, Lemma 7.8]. We give an essentially complete proof for the convenience of the reader. Let $\overline{X} \in \{\overline{B}^\circ, \overline{H}\}$, and write $X = C_{\overline{X}}(\sigma)$. The centralizer $C_{\overline{X}}(E) = \overline{T}\langle w_0 \rangle$ is not connected, but has the two connected components $\overline{T}$ and $\overline{T}w_0$. Thus, there are two $C_{\overline{X}}(\sigma)$-conjugacy classes of subgroups of X conjugate to E in $\overline{H}$ [26, 2.1.5]. A representative of the other X-class can be obtained as follows. As $\overline{X}$ is connected, we may fix by the Lang–Steinberg theorem $g \in \overline{X}$ such that $w_0 = g^\sigma g^{-1}$. Then $g^\sigma = w_0 g$. In the semidirect product $\overline{X}\langle \sigma \rangle$, we have $\sigma^g = \sigma w_0$. Now as $\overline{T}\langle w_0 \rangle$ is invariant under σw_0, it follows that $(\overline{T}\langle w_0 \rangle)^g$ is σ-invariant. Indeed by choice of g, we have $t^{g\sigma} = t^{\sigma w_0 g}$ for each $t \in \overline{T}$, that is, the conjugation isomorphism $\overline{T}\langle w_0 \rangle \xrightarrow{c_g} \overline{T}^g \langle w_0^g \rangle$ intertwines the actions of σw_0 on $\overline{T}\langle w_0 \rangle$ and σ on $\overline{T}^g \langle w_0^g \rangle$. Then E and E^g are representatives for the X-classes of subgroups of X conjugate in $\overline{X}$ to E, and

$$X \cap \overline{T}^g = C_{\overline{T}^g}(\sigma) \cong C_{\overline{T}}(\sigma w_0) = \{t \in T \mid t^{-\epsilon q} = t\} \cong (C_{q+\epsilon})^3. \tag{4.9}$$

The above argument shows we may take $g \in \overline{B}^\circ$ even when $\overline{X} = \overline{H}$. By Lemma 4.3, $\overline{B}^\circ$ is a commuting product $\overline{L}_1\overline{L}_2\overline{L}_3$ with $\overline{L}_i \cong SL_2(\overline{\mathbb{F}}_p)$ and $Z(\overline{B}^\circ) = U$. Also, $\overline{B}^\circ \cong \overline{J}/\langle j \rangle$ where $\overline{J}$ is a direct product of the $\overline{L}_i$'s and j the product of the unique involutions of the direct factors (Subsection 4.1.4). Thus, each involution in $\overline{B}^\circ - U$ is of the form $f_1 f_2 f_3$ for elements $f_i \in \overline{L}_i$ of order 4. But $\overline{L}_i$ is transitive on its elements of order 4. Hence, all elementary abelian subgroups of $\overline{B}^\circ$ of order 8 containing U are $\overline{B}^\circ$-conjugate. Now E is contained in the normal subgroup $L_1L_2L_3$ of $C_H(U)$, while E' is not because d lies in the coset $L_1L_2L_3c$. It follows that E^g is $C_H(U)$-conjugate to E'. Hence, E and E' are representatives for the X-conjugacy classes of elementary abelian subgroups of X of order 8 containing Z.

Fix $b \in C_H(U)$ with $E^{gb} = E'$. Set $\overline{T}' = \overline{T}^{gb}$, $T' = C_{\overline{T}^{gb}}(\sigma)$, and $w_0' = w_0^{gb}$. By (4.9), $O_{2'}(T')$ is as described in (a)(ii), and w_0' inverts T'. Now $N_H(T)/T \cong C_2 \times S_4$ by Lemma 4.4(3). As $\overline{T}w_0$ generates the center of $N_{\overline{H}}(\overline{T})/\overline{T}$, it follows by choice of g and [12, 3.3.6] that $N_H(T^g)/T^g \cong N_H(T)/T$, and hence $N_H(T')/T' \cong N_H(T)/T$ because $b \in H$.

Fix an odd prime r dividing $q - \epsilon$ (resp., $q + \epsilon$), and let T_r (resp., T_r') be the r-torsion subgroup of $\overline{T}$ (resp., $\overline{T}'$). Then $T_r \leqslant T$ (resp., $T_r' \leqslant T'$). As $N_{\overline{H}}(\overline{T})/\overline{T}$ (resp., $N_{\overline{H}}(\overline{T}')/\overline{T}'$) acts faithfully on T_r (resp., T_r') by Proposition A.4, it follows that the same is true for $N_H(T)/T$ (resp., $N_H(T')/T'$). This completes the proof of (1) and (2), and part (3) then follows. □

4.3 | Conjugacy classes of elementary abelian subgroups in a Benson–Solomon system

In this subsection, we look at the conjugacy classes and automizers of elementary abelian subgroups of the Benson–Solomon systems. We adopt the notation from the first part of this section, so S is a Sylow 2-subgroup of $H = \mathrm{Spin}_7(q)$, $Z = Z(S)$ is of order 2, U is the unique normal four subgroup of S, and E is the 2-torsion in the fixed maximal torus T of H, and $A = E\langle w_0\rangle$.

Lemma 4.6. *Let $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ be a Benson–Solomon fusion system over S. Then*

(1) *$\mathscr{E}_1(S) = Z^{\mathcal{F}}$, and $N_{\mathcal{F}}(Z) = C_{\mathcal{F}}(Z) \cong \mathcal{F}_S(H)$;*
(2) *$\mathscr{E}_2(S) = U^{\mathcal{F}}$;*
(3) *for $T_S = T \cap S$, $\mathrm{Out}_{\mathcal{F}}(T_S) = \mathrm{Aut}_{\mathcal{F}}(T_S) = C_2 \times GL_3(2)$, and $\mathrm{Out}_{\mathcal{F}}(T_S\langle w_0\rangle) \cong GL_3(2)$ acts naturally on $T_S/\Phi(T_S)$ and on E.*

Proof. Part (1) follows from the construction of $\mathcal{F}_{\mathrm{Sol}}(q)$. By part (1) and [2, Lemma II.3.1], every element of $\mathscr{E}_2(S)$ is $\mathcal{F}$-conjugate to a subgroup containing Z and thus by Lemma 4.3(2) to U.

The structure of $\mathrm{Out}_{\mathcal{F}}(T_S)$ in (3) follows from the construction of the Benson–Solomon systems; for example, see [1, Proposition 5.4(b), Lemma 7.13(e)] or [36, Proposition 1.5]. The structure of $\mathrm{Out}_{\mathcal{F}}(T_S\langle w_0\rangle)$ follows from that of $\mathrm{Out}_{\mathcal{F}}(T_S)$; for the details we refer the reader to [30, Lemma 2.38(c)]. As the actions in (3) are induced by the restriction map $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle) \to \mathrm{Aut}_{\mathcal{F}}(T_S)$, the remainder of (3) is clear. □

We saw in Lemma 4.5 that H has two conjugacy classes of elementary abelian subgroups of order 8 containing Z. As far as we can tell, Aschbacher and Chermak do not discuss the possible $\mathcal{F}$-conjugacy of E and E' explicitly, but such information can be deduced from their description of the conjugacy classes of elementary abelian subgroups of order 16. As we need to show later in Lemma 4.8 that E and E' are in fact not $\mathcal{F}$-conjugate, we provide an account of that description.

On page 935 of [1], T_S is denoted R_0. As on pages 935–936, write $R_1 = N_{\overline{T}}(T_S\langle w_0\rangle) \cong (C_{2^{k+1}})^3$. Thus, T_S has index 8 in R_1, and R_1/T_S is elementary abelian of order 8. Fix a set

$$\{x_e \mid e \in E\}$$

of coset representatives for T_S in R_1, with notation chosen so that $x_1 = 1$ and $x_e^{2^k} = e \in E$ for each $e \in E - \{1\}$, and set

$$A_e = A^{x_e}.$$

As w_0 inverts $\overline{T}$, we have $A_e = E\langle t_e w_0\rangle$ where $t_e := x_e^{-2} = [x_e, w_0] \in T_S$ also powers to e.

Denote by $\mathcal{A}$ the set of T_S-conjugacy classes of elementary abelian subgroups of $T_S\langle w_0\rangle$ of order 16. Then as $E \leqslant T_S$ and $[T_S, w_0] = \Phi(T_S)$, there are $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-equivariant bijections

$$\mathcal{A} \longrightarrow T_S/\Phi(T_S) \longrightarrow E$$
$$A_e^{T_S} \longmapsto t_e\Phi(T_S) \longmapsto e = t_e^{2^{k-1}}.$$

As $\mathrm{Inn}(T_S\langle w_0\rangle)$ acts trivially on these sets, by Lemma 4.6(3), $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$ has two orbits on $\mathscr{E}_4(T_S\langle w_0\rangle)$ with representatives $A = A_1$ and A_e with $e \neq 1$.

Lemma 4.7. *$\mathscr{E}_4(S)$ is the disjoint union of $A_1^{\mathcal{F}}$ and $A_e^{\mathcal{F}}$, where e is any nonidentity element of E. All A_e with $e \neq 1$ are $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-conjugate, and $\mathrm{Aut}_{\mathcal{F}}(A_e) = C_{\mathrm{Aut}(A_e)}(e)$ for each $e \in E$.*

Proof. This is [1, Lemma 7.12(c)], except for the statement on $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-conjugacy, which contained in the proof of 7.12(c) and observed above. An equivalent description of the $\mathcal{F}$-conjugacy classes of elementary abelian subgroups of S of rank 4 was given in [35, Proposition A.8, Lemma 3.1]. See also [36, p. 3018]. □

Lemma 4.8. *$\mathscr{E}_3(S)$ is the disjoint union of $E^{\mathcal{F}}$ and $E'^{\mathcal{F}}$, and we have* $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$ *and* $\mathrm{Aut}_{\mathcal{F}}(E') = \mathrm{Aut}(E')$.

Proof. In Lemma 4.5, we defined $E' = U\langle d\rangle$ where $d = cw_0$. It was shown there that E and E' are representatives for the two H-conjugacy classes of elementary abelian subgroups of order 8 containing Z. Recall that c was defined as $h_{\beta_1}(\mu)h_{\beta_2}(\mu)h_{\beta_3}(\mu)$ at the end of Subsection 4.1.5, and in Lemma 4.2(2) it was shown that $c \in T_S$ and $c^{2^{k-1}} \in E - U$. In particular, $c \in T_S - \Phi(T_S)$.

Take $e = c^{2^{k-1}}$ and consider $A_e = A^{x_e} = E\langle t_e w_0\rangle$. As both $t_e = [x_e, w_0]$ and c have 2^{k-1}st power e, there is $s \in \Phi(T_S) = \mho^1(T_S)$ with $c = t_e s$. Choose $t \in T_S$ with $t^{-2} = s$. Then $A_e^t = E^t\langle (t_e w_0)^t\rangle = E\langle (t_e w_0)^t\rangle$, and $(t_e w_0)^t = t_e w_0^t = t_e[t, w_0]w_0 = t_e t^{-2} w_0 = t_e s w_0 = cw_0$. This shows A_e is T_S-conjugate to the elementary abelian subgroup

$$A' := E\langle cw_0\rangle = EE' \tag{4.10}$$

of order 16. Alternatively, we could have chosen the coset representative x_e at the outset to satisfy $[x_e, w_0] = x_e^{-2} = c$ and in doing so arrange for $A_e = A'$, and thus for A_e to contain E'.

Assume to get a contradiction that E and E' are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate. As E is normal in S, it is fully $\mathcal{F}$-normalized, hence fully $\mathcal{F}$-centralized by saturation. So, there is a morphism $\varphi : C_S(E') \to C_S(E)$ in $\mathcal{F}$ with $E'^{\varphi} = E$ by [2, I.2.6]. By (4.10), φ is defined on A' and $A'^{\varphi} \leqslant C_S(E) = T_S\langle w_0\rangle$. By Lemma 4.7, we may choose $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$ with $A'^{\varphi\alpha} = A'$. Then as $\varphi\alpha \in \mathrm{Aut}_{\mathcal{F}}(A') = C_{\mathrm{Aut}_{\mathcal{F}}(A')}(e)$ by the same lemma, we have $e^{\varphi\alpha} = e$. On the other hand, as E is characteristic in $T_S\langle w_0\rangle$, we have $U^{\varphi\alpha} \leqslant E'^{\varphi\alpha} = E^{\alpha} = E$. Thus, $E^{\varphi\alpha} = (U\langle e\rangle)^{\varphi\alpha} = U^{\varphi\alpha}\langle e^{\varphi\alpha}\rangle \leqslant E$, a contradiction. Now we appeal to [1, Lemma 7.8] for the structure of the $\mathcal{F}$-automorphism groups. □

Remark 4.9. Lemma 4.8 says that there are two conjugacy classes of elementary abelian subgroups of order eight in a Benson–Solomon system, and is therefore incompatible with the part of [35, Lemma 3.1] which states that there is a single conjugacy class. We thank the referee for alerting us to this, and we refer the reader to [43].

4.4 | Proof of Theorem 4.1

We now turn to the proof of Theorem 4.1. As an initial observation, note that if $\mathcal{L}$ is a punctured group for $\mathcal{F}_{\mathrm{Sol}}(q')$ for some odd prime power q', then $C_{\mathcal{L}}(Z)$ is a group whose 2-fusion system is isomorphic to that of $\mathrm{Spin}_7(q')$. It is in this context that we use the following lemma.

By a field automorphism of $\mathrm{Spin}_7(q)$, we mean an automorphism acting on the root groups via $x_\alpha(\lambda) \mapsto x_\alpha(\lambda^\psi)$ where ψ is an automorphism of $\mathbb{F}_q$.

Lemma 4.10. *Let G be a finite group whose 2-fusion system is isomorphic to that of* $\mathrm{Spin}_7(q')$ *for some odd q'. Then $G/O_{2'}(G) \cong$* $\mathrm{Spin}_7(q)\langle\varphi\rangle$ *for some odd q with $v_2(q^2-1) = v_2(q'^2-1)$, and where φ induces a field automorphism of odd order.*

Proof. It was shown by Levi and Oliver in the course of proving $\mathcal{F}_{\mathrm{Sol}}(q)$ is exotic that $O^{2'}(G/O_{2'}(G))$ is isomorphic to $\mathrm{Spin}_7(q)$ for some odd q [35, Proposition 3.4]. If S' and S are the corresponding Sylow 2-subgroups, then S' and S are isomorphic by definition of an isomorphism of a fusion system. If k and k' are one less than the valuations of q^2-1 and q'^2-1, then the orders of S and S' are 2^{4+3k} and $2^{4+3k'}$, so $k=k'$. The description of $G/O_{2'}(G)$ follows, as $\mathrm{Out}(\mathrm{Spin}_7(q)) \cong C_a \times C_2$, where $q=p^a$ and C_a is generated by the class of a field automorphism. $\square$

The extension of $\mathrm{Spin}_7(q)$ by a group of field automorphisms of odd order has the same 2-fusion system as $\mathrm{Spin}_7(q)$, but we will not need this.

Lemma 4.11. *Let q be an odd prime power with the property that $GL_3(2)$ has a faithful 3-dimensional representation over $\mathbb{F}_r$ for each prime divisor r of q^2-1. Then each such r is a square modulo 7, and $q=3^{1+6a}$ for some $a \geqslant 0$. In particular, $q \equiv 3 \pmod 8$.*

Proof. Set $G=GL_3(2)$ for short. We first show that $GL_3(2)$ has a faithful 3-dimensional representation over $\mathbb{F}_r$ if and only if r is a square modulo 7. If $r=2$, 3, or 7, then as $|SL_3(3)|$ is not divisible by 7 and $G \cong PSL_2(7) \cong \Omega_3(7)$, the statement holds. So, we may assume that p does not divide $|G|$, so that $\mathbb{F}_r GL_3(2)$ is semisimple. Let V be a faithful 3-dimensional module with character φ, necessarily irreducible. From the character table for $GL_3(2)$, we see that φ takes values in $\mathbb{F}_r((1+\sqrt{-7})/2)$. By [21, I.19.3], a modular representation is writable over its field of character values, so this extension is a splitting field for V. Thus, V is writable over $\mathbb{F}_r$ if and only if -7 is a square modulo r, which by quadratic reciprocity is the case if and only if r is a square modulo 7.

Now fix an odd prime power q with the property that q^2-1 is divisible only by primes that are squares modulo 7. As $q(q-1)(q+1)$ is divisible by 3 and 3 is not a square, we have $q=3^l$ for some l. Now $q-1$ and $q+1$ are squares, so $q \equiv 1$ or $3 \pmod 7$. Assume the former. Then 6 divides l, so $q=3^l \equiv \pm 1 \pmod 5$. But then q^2-1 is divisible by the nonsquare 5, a contradiction. So, $q \equiv 3 \pmod 7$, $l=1+6a$ for some $a \geqslant 0$, and hence $q \equiv 3 \pmod 8$. $\square$

We write $G'=[G,G]$ for the commutator subgroup of a group G.

Lemma 4.12. *Let G be a finite group and $W \trianglelefteq M \trianglelefteq G$ such that $G/M \cong GL_3(2)$, $|W|=2$ and M/W is cyclic of odd order. Then $E(G)=G'$ is isomorphic to $GL_3(2)$ or $L_2(7)$. In the latter case, G has quaternion Sylow 2-subgroups.*

Proof. If we pick a generator xW of M/W, then $M=\langle x\rangle W$ is abelian as $W \leqslant Z(M)$. As M/W is of odd order and $|W|=2$, it follows indeed that M is cyclic. In particular, $\mathrm{Aut}(M)$ is abelian. As $C_G(F^*(G)) \leqslant F^*(G)$ and G/M is non-abelian simple, it follows that $F^*(G) \neq M = F(G)$ and so $E(G) \neq 1$. As $GL_3(2)$ is the only non-abelian composition factor in a composition series for G, it follows that $K := E(G)$ is quasisimple with $K/Z(K) \cong GL_3(2) \cong L_2(7)$. In particular, $G=KM$ where $M=F(G)$ is abelian and commutes with $K=E(G)=K'$. Hence, $G'=K=E(G)$.

As the Schur multiplier of $GL_3(2) \cong L_2(7)$ has order 2, we have $K \cong GL_3(2)$ or $K \cong SL_2(7)$. In the latter case, $Z(K)=W$ and K contains a Sylow 2-subgroup of G. As $SL_2(7)$ has quaternion Sylow 2-subgroups, the assertion follows. $\square$

If $\widehat{\Theta}$ is a partial normal p'-subgroup of a locality $(\mathcal{L},\Delta,S)$ at the prime p, then the restriction of the natural projection $\mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ to S is a monomorphism. Thus, we may identify S with its image in $\mathcal{L}/\widehat{\Theta}$. This is used to formulate the following proposition.

Proposition 4.13. *Suppose $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q')$ is a Benson–Solomon fusion system and $(\mathcal{L}, \Delta, S)$ is a punctured group over $\mathcal{F}$. Set $Z := Z(S)$ and $G = C_{\mathcal{L}}(Z)$.*

(1) *There exists a partial normal $2'$-subgroup $\widehat{\Theta}$ of $\mathcal{L}$ such that, identifying S with its image in $\mathcal{L}/\widehat{\Theta}$ in the natural way, $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a punctured group over $\mathcal{F}$ and $N_{\mathcal{L}/\widehat{\Theta}}(Z) \cong G/O_{2'}(G)$ is $2'$-reduced.*
(2) *$O^{2'}(G/O_{2'}(G)) \cong \mathrm{Spin}_7(q)$, where $q = 3^{1+6a}$ for some $a \geqslant 0$, and every prime divisor of $q^2 - 1$ is a square modulo 7. In particular, $q \equiv 3 \pmod 8$ and $q' \equiv \pm 3 \pmod 8$.*

Proof. Set $\overline{G} := G/O_{2'}(G)$ and $H := O^{2'}(\overline{G})$. As $\mathcal{L}$ is a locality on $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q')$, it follows from Lemmas 2.9(b) and 4.6(1) that $\mathcal{F}_{\overline{S}}(\overline{G}) \cong \mathcal{F}_S(G) = \mathcal{F}_S(N_{\mathcal{L}}(Z)) = N_{\mathcal{F}}(Z)$ is isomorphic to the 2-fusion system of $\mathrm{Spin}_7(q')$. Hence, by Lemma 4.10 there is an odd prime power q with $(q^2 - 1)_2 = (q'^2 - 1)_2$ such that H can be identified with $\mathrm{Spin}_7(q)$, and such that $\bar{G} = H\langle\varphi\rangle$ for φ a field automorphism of odd order. Where convenient, we adopt below the notation from Subsections 4.1–4.3.

(1) We will show the existence of a suitable signalizer functor on elements of order 2 (as introduced in Definition 1.2). For that we set

$$\theta(a) = O_{2'}(C_{\mathcal{L}}(a)) \text{ for each involution } a \in S.$$

By Lemma 2.7(b), θ is conjugacy invariant. Let $a, b \in S$ be two distinct commuting involutions. By Lemma 4.6(2) and conjugacy invariance, to verify the balance condition in Definition 1.2, we can assume $b = z$ and $a = u \in U - Z$. Set $X = O_{2'}(C_{\mathcal{L}}(u)) \cap G$ and note that X is an odd order normal subgroup of $C_{\mathcal{L}}(U) = C_G(U)$. By a Frattini argument $\overline{C_G(U)} = C_{\bar{G}}(\bar{U})$, so $\bar{X}$ is normal in the latter group. We use now that $\bar{G}$ is an extension of $H = \mathrm{Spin}_7(q)$ by a cyclic group generated by a field automorphism φ of odd order. As each component $L_i \cong SL_2(q)$ of $C_H(\bar{U})$ is generated by a root group and its opposite (Subsection 4.1.4 and Lemma 4.3(2)), it follows that φ acts nontrivially as a field automorphism on each such L_i, and hence $\bar{X} \leqslant O_{2'}(C_{\bar{G}}(\bar{U})) \leqslant O_{2'}(L_1L_2L_3) = 1$. Equivalently, $X = O_{2'}(C_{\mathcal{L}}(u)) \cap G \leqslant O_{2'}(G)$. This shows that the balance condition holds. For each $P \in \Delta$, set

$$\Theta(P) = \left(\bigcap_{x \in \mathcal{I}_2(P)} \theta(x)\right) \cap C_{\mathcal{L}}(P).$$

Then by Theorem 2.39, Θ defines a signalizer functor on objects. By Theorem 2.36, $\widehat{\Theta} = \bigcup_{P\in\Delta} \Theta(P)$ is a partial normal p'-subgroup of $\mathcal{L}$, and $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is again a punctured group for $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q')$ with $N_{\mathcal{L}/\widehat{\Theta}}(Z) \cong G/\Theta(Z)$. Writing $Z = \langle z\rangle$, note that $\mathcal{I}_2(Z) = \{z\}$ and $\Theta(Z) = \theta(z) \cap C_{\mathcal{L}}(Z) = O_{2'}(C_{\mathcal{L}}(z)) = O_{2'}(G)$. This proves (1).
(2) For the proof of (2), part (1) allows us to assume $O_{2'}(G) = 1$. Then $G = \overline{G}$, $H = O^{2'}(G) \cong \mathrm{Spin}_7(q)$, and G/H is cyclic of odd order. Recall that H has Sylow 2-subgroup S, $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \pmod 4$, and $E_1 := E$ and $E_{-1} := E'$ are the representatives for $\mathcal{F}$-conjugacy classes of elementary abelian subgroups of order 8 in S (Lemmas 4.5 and 4.8). For $\delta = \pm 1$, let T_δ be the maximal torus containing E_δ of Lemma 4.5. For each positive integer r dividing $q - \delta\epsilon$, write $T_{\delta,r}$ for the r-torsion in T_δ. Moreover, set $T_{\delta,S} = T_\delta \cap S$. Thus, $T_{1,S} = T_S = T_{1,2^k}$ is homocyclic of order 2^{3k}, and $T_{-1,S} = E_{-1}$.

Now fix δ and let $N = N_{\mathcal{L}}(T_{\delta,S})$. By Lemmas 4.4(2) and 4.5(3),

$$C_H(E_\delta) = T_\delta\langle w\rangle, \tag{4.11}$$

where w is an involution inverting T_δ. In particular, as

$$O_{2'}(T_\delta) = [O_{2'}(T_\delta), \langle w\rangle]$$

and $O^{2'}(G) = H$, we have

$$C_H(E_\delta) = O^{2'}(C_H(E_\delta)) = O^{2'}(C_G(E_\delta)).$$

Also, $C_{\mathcal{L}}(E_\delta) = C_G(E_\delta)$ as E_δ contains Z. It follows that $C_H(E_\delta) = O^{2'}(C_{\mathcal{L}}(E_\delta))$ is normal in $N_{\mathcal{L}}(E_\delta)$, so

$$C_H(E_\delta) \text{ and } O_{2'}(C_H(E_\delta)) \text{ are normal in } N_{\mathcal{L}}(E_\delta). \tag{4.12}$$

Next we show

$$N = N_{\mathcal{L}}(E_\delta). \tag{4.13}$$

We may assume $T_{\delta,S} > E_\delta$, and so $\delta = 1$, $T_{\delta,S} = T_S$, and $E_\delta = E$. Certainly $N_{\mathcal{L}}(T_S) \leqslant N_{\mathcal{L}}(E)$. For the other inclusion, note $N_{\mathcal{L}}(E)$ acts on $C_H(E)$ by (4.12) so it acts on T_S because T_S is the unique abelian 2-subgroup of maximum order in $C_H(E)$. Thus, $N_{\mathcal{L}}(E) \leqslant N_{\mathcal{L}}(T_S)$, completing the proof of (4.13).

Using (4.11) one observes easily that $T_\delta = T_{\delta,S}O_{2'}(T_\delta) = T_{\delta,S}O_{2'}(C_H(E_\delta))$. Hence, it follows from (4.12) and (4.13) that

$$T_\delta \trianglelefteq N. \tag{4.14}$$

Notice that $C_N(E_\delta)/C_H(E_\delta) \leqslant C_G(E_\delta)/C_H(E_\delta) \cong C_G(E_\delta)H/H \leqslant G/H$ and recall that G/H is cyclic of odd order. Hence, by (4.11), (4.12), (4.13), and (4.14), we are given a normal series

$$T_\delta \leqslant C_H(E_\delta) \leqslant C_N(E_\delta) = C_{\mathcal{L}}(E_\delta) \leqslant N_{\mathcal{L}}(E_\delta) = N,$$

where $C_H(E_\delta)/T_\delta \cong C_2$, $C_N(E_\delta)/C_H(E_\delta)$ is cyclic of odd order, and (by Lemma 4.8)

$$N/C_N(E_\delta) = N_{\mathcal{L}}(E_\delta)/C_{\mathcal{L}}(E_\delta) \cong \mathrm{Aut}_{\mathcal{F}}(E_\delta) = \mathrm{Aut}(E_\delta) \cong GL_3(2).$$

Set $\widetilde{N} := N/T_\delta$. By Lemma 4.5, $C_2 \times S_4 \cong \widetilde{N_H(T_\delta)} \leqslant \widetilde{N}$. In particular, $\widetilde{N}$ does not have quaternion Sylow 2-subgroups. Applying Lemma 4.12 with $(\widetilde{N}, \widetilde{C_N(E_\delta)}, \widetilde{C_H(E_\delta)})$ in place of (G, M, W), we see now that

$$\widetilde{N}' \cong GL_3(2).$$

Let r be a prime divisor of $q - \delta\epsilon$ and note that $\widetilde{N}$ acts on $T_{\delta,r}$. By Lemma 4.5, $\widetilde{N_H(T_\delta)} \cong C_2 \times S_4$ acts faithfully on $T_{\delta,r}$ and thus $A_4 \cong \widetilde{N_H(T_\delta)}' \leqslant \widetilde{N}'$ acts nontrivially on $T_{\delta,r}$. As $C_{\widetilde{N}'}(T_{\delta,r}) \trianglelefteq \widetilde{N}'$ and

$\widetilde{N}' \cong GL_3(2)$ is simple, it follows that $C_{\widetilde{N}'}(T_{\delta,r}) = 1$ and $\widetilde{N}' \cong GL_3(2)$ acts faithfully on $T_{\delta,r} \cong C_r^3$. As this holds for each $\delta = \pm 1$ and prime r, Lemma 4.11 implies that $q = 3^{1+6a}$ for some $a \geqslant 0$ and $q \equiv 3 \pmod 8$. As $q' \equiv \pm q \pmod 8$, this shows (2). □

Note that the conclusion of the following lemma does not hold if $H = \mathrm{Spin}_7(q)$ for some $q \neq 3$.

Lemma 4.14. *Let $H = \mathrm{Spin}_7(3)$ and $Z = Z(H)$. If $P \geqslant Z$ is a 2-subgroup of H of 2-rank at least 2, then $N_H(P)$ and $C_H(P)$ are of characteristic 2.*

Proof. Let $P \leqslant S$ with $Z \leqslant V \leqslant P$ and V a four group. By Lemma 4.3(2), we may conjugate in H and take $V = U$, and $C_H(U) = L_1L_2L_3\langle c\rangle$, where c induces a diagonal automorphism on each $L_i \cong SL_2(3)$. Thus, $O_2(C_H(U))$ is a commuting product of three quaternion subgroups of order 8 that contains its centralizer in $C_H(U)$, and hence $C_H(U)$ is of characteristic 2.

Recall that $N_H(P)$ is of characteristic 2 if and only if $C_H(P)$ is of characteristic 2 and that the normalizer of any 2-subgroup in a group of characteristic 2 is of characteristic 2 (see, e.g., [28, Lemma 2.2]). It follows that $N_{C_H(U)}(P)$ is of characteristic 2, so $C_H(P) = C_{C_H(U)}(P)$ is of characteristic 2, so $N_H(P)$ is of characteristic 2. □

Lemma 4.15. *Let $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(3)$ be a fusion system over S. Then every subgroup of S of 2-rank at least 2 is $\mathcal{F}$-subcentric.*

Proof. Set $Z = Z(S)$. By Lemma 4.6(1), we have $\mathcal{H} := N_{\mathcal{F}}(Z) = \mathcal{F}_S(H)$ where $H = \mathrm{Spin}_7(q)$ and S can be identified with the Sylow 2-subgroup of H defined in Subsection 4.1.6. Define U as before so that $\mathscr{E}_2(S) = U^{\mathcal{F}}$ by Lemma 4.6(2). As $\mathcal{F}^s$ is by [28, Proposition 3.3] closed under passing to $\mathcal{F}$-conjugates and overgroups, it is enough to prove that U is $\mathcal{F}$-subcentric. Indeed, as $Z \leqslant U$, we have $C_{\mathcal{F}}(U) = C_{\mathcal{H}}(U) = \mathcal{F}_{C_S(U)}(C_H(U))$. Hence, $C_{\mathcal{F}}(U)$ is constrained by Lemma 4.14 and so $U \in \mathcal{F}^s$ by [28, Lemma 3.1]. □

We may now prove the main theorem of this section.

Proof of Theorem 4.1. $(\Rightarrow)$: If $(\mathcal{L}, \Delta, S)$ is a punctured group over $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}(q)$ for some odd prime power q, then it follows from Proposition 4.13(2) (applied with q in place of q') that $q \equiv \pm 3 \pmod 8$.

$(\Longleftarrow)$: Now let $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(3)$ and $\mathcal{H} = C_{\mathcal{F}}(Z) = \mathcal{F}_S(H)$ with $H = \mathrm{Spin}_7(3)$. Set

$$\Delta = \{P \in \mathcal{F}^s \mid P \text{ is of 2-rank at least } 2\},$$

and $\Delta_Z = \{P \in \Delta \mid P \geqslant Z\}$. Then Δ is closed under $\mathcal{F}$-conjugacy and passing to overgroups by [28]. So, it is also closed under $\mathcal{H}$-conjugacy. We show now

$$\text{Every element of } \mathcal{H}^{cr} \cup \mathcal{F}^{cr} \text{ is of 2-rank at least 2.} \tag{4.15}$$

Indeed, assume there exists $Q \in \mathcal{H}^{cr} \cup \mathcal{F}^{cr}$ of 2-rank 1. Then $Q \neq S$ and so $C_S(Q) \leqslant Q$ implies $\mathrm{Inn}(Q) < \mathrm{Aut}_S(Q) \leqslant \mathrm{Aut}_{\mathcal{H}}(Q) \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$. Suppose first that Q is cyclic, or generalized quaternion of order at least 16. Then $\mathrm{Aut}(Q)$ is a 2-group and so $\mathrm{Out}_{\mathcal{H}}(Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q)$ are nontrivial 2-groups, which contradicts the assumption that Q is radical in $\mathcal{H}$ or $\mathcal{F}$. Assume now that Q is quaternion of order 8. As U is a normal subgroup of S, we have $[Q, U] \leqslant Z = Z(S) \leqslant Q$, so $U \leqslant N_S(Q)$. But $N_S(Q)$

is a 2-group containing Q self-centralizing with index at most 2, and so $N_S(Q)$ is quaternion or semidihedral of order 16. But neither of these groups has a normal four subgroup, a contradiction. This shows (4.15).

Each element of $\mathcal{F}^{cr} \cup \mathcal{H}^{cr}$ contains Z. It follows moreover from (4.15) and Lemma 4.15 that $\mathcal{F}^{cr} \cup \mathcal{H}^{cr} \subseteq \Delta$. Also $\mathcal{F}^s \subseteq \mathcal{H}^s$ by [28, Lemma 3.16]. Thus, we have shown

$$\mathcal{F}^{cr} \cup \mathcal{H}^{cr} \subseteq \Delta_Z \subseteq \Delta \subseteq \mathcal{F}^s \subseteq \mathcal{H}^s. \tag{4.16}$$

The hypotheses of [28, Theorem A] are thus satisfied, so we may fix a linking locality $\mathcal{L}$ on $\mathcal{F}$ with object set Δ, and this $\mathcal{L}$ is unique up to rigid isomorphism.

We shall verify the conditions (1)–(5) of [14, Hypothesis 5.3] with Z in the role of "T" and H in the role of "M". Conditions (1), (2) hold by construction. Condition (4) holds because Z is normal in H and $\mathcal{F}_S(N_{\mathcal{L}}(Z)) \cong \mathcal{H}$ by [35]. To see condition (3), first note that Z is fully normalized in $\mathcal{F}$ because it is central in S. Let Z', Z'' be distinct $\mathcal{F}$-conjugates of Z. Then $\langle Z', Z'' \rangle$ contains a four group V. By Lemma 4.6(2), V is $\mathcal{F}$-conjugate to U, and $O_2(N_{\mathcal{F}}(U)) \in \mathcal{F}^c$ is a commuting product of three quaternion groups of order 8. Thus, $V \in \Delta$, and hence $\langle Z', Z'' \rangle \in \Delta$. So, Condition (3) holds. It remains to verify Condition (5), namely that $N_{\mathcal{L}}(Z)$ and $\mathcal{L}_{\Delta_Z}(H)$ are rigidly isomorphic. By (4.16) and Lemma 4.14, $\mathcal{L}_{\Delta_Z}(H)$ is a linking locality over $\mathcal{H}$ with Δ_Z as its set of objects.

On the other hand, by [14, Lemma 2.19], $N_{\mathcal{L}}(Z)$ is a locality on $\mathcal{H}$ with object set Δ_Z, in which $N_{N_{\mathcal{L}}(Z)}(P) = C_{N_{\mathcal{L}}(P)}(Z)$ for each $P \in \Delta_Z$. As $\mathcal{L}$ a linking locality, $N_{\mathcal{L}}(P)$ is of characteristic 2, and hence the 2-local subgroup $N_{N_{\mathcal{L}}(Z)}(P)$ of $N_{\mathcal{L}}(P)$ is also of characteristic 2. So, again this together with (4.16) gives that $N_{\mathcal{L}}(Z)$ is a linking locality over $\mathcal{H}$ with object set Δ_Z. Thus, $\mathcal{L}_{\Delta_Z}(H)$ and $N_{\mathcal{L}}(Z)$ are linking localities over the same fusion system and with the same object set, thus rigidly isomorphic by [28, Theorem A]. This completes the proof of (5).

So, by [14, Theorem 5.14], there is a locality $\mathcal{L}^+$ over $\mathcal{F}$ with object set

$$\Delta^+ := \{P \leqslant S \mid Z^\varphi \leqslant P \text{ for some } \varphi \in \mathrm{Hom}_{\mathcal{F}}(Z, S)\},$$

such that $\mathcal{L}^+|_\Delta = \mathcal{L}$ and $N_{\mathcal{L}^+}(Z) = H$, and $\mathcal{L}^+$ is unique up to rigid isomorphism with this property. As each nontrivial subgroup of S contains an involution, and all involutions in S are $\mathcal{F}$-conjugate (by Lemma 4.6(1)), Δ^+ is the collection of all nontrivial subgroups of S. Thus, $\mathcal{L}^+$ is a punctured group for $\mathcal{F}$. $\square$

Remark 4.16. Theorem 4.1 leaves open the question whether there is a punctured group $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}_{\mathrm{Sol}}(3)$ such that, setting $Z := Z(S)$, the centralizer $C_{\mathcal{L}}(Z)$ is not isomorphic to $\mathrm{Spin}_7(3)$. Indeed, we show in Proposition 4.13 that always $O^{2'}(C_{\mathcal{L}}(Z)/O_{2'}(C_{\mathcal{L}}(Z))) \cong \mathrm{Spin}_7(q)$, where $q = 3^{1+6a}$ for some $a \geqslant 0$ with the property that $q^2 - 1$ is divisible only by primes that are squares modulo 7. Although there are at least several such nonnegative integers a with this property (the first few are $0, 1, 2, 3, 5, 7, 8, 13, 15, \ldots$), we are unable to determine whether a punctured group for $\mathcal{F}_{\mathrm{Sol}}(q)$ exists when $a > 0$.

5 | PUNCTURED GROUPS FOR EXOTIC FUSION SYSTEMS AT ODD PRIMES

In this section, we survey some of the known examples of exotic fusion systems at odd primes in the literature, and determine which ones have associated punctured groups.

Let $\mathcal{F}$ be a saturated fusion system over the p-group S. A subgroup Q of S is said to be *$\mathcal{F}$-subcentric* if Q is $\mathcal{F}$-conjugate to a subgroup P for which $O_p(N_{\mathcal{F}}(P))$ is $\mathcal{F}$-centric. Equivalently, by [28, Lemma 3.1], Q is $\mathcal{F}$-subcentric if, for any fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate P of Q, the normalizer $N_{\mathcal{F}}(P)$ is constrained. Write $\mathcal{F}^s$ for the set of subcentric subgroups of $\mathcal{F}$. Thus, $\mathcal{F}^s$ contains the set of nonidentity subgroups of S if and only if $\mathcal{F}$ is of characteristic p-type (and $\mathcal{F}^s$ is the set of *all* subgroups of S if and only if $\mathcal{F}$ is constrained).

A finite group G is said to be of *characteristic p* if $C_G(O_p(G)) \leqslant O_p(G)$. A *subcentric linking system* is a transporter system $\mathcal{L}^s$ associated to $\mathcal{F}$ such that $\mathrm{Obj}(\mathcal{L}^s) = \mathcal{F}^s$ and $\mathrm{Aut}_{\mathcal{L}^s}(P)$ is of characteristic p for every $P \in \mathrm{Obj}(\mathcal{L}^s)$. By a theorem of Broto, Castellana, Grodal, Levi, and Oliver [4], the constrained fusion systems are precisely the fusion systems of finite groups of characteristic p. The finite groups of characteristic p, which realize the normalizers of fully normalized subcentric subgroups, can be "glued together" to build a subcentric linking systems associated with $\mathcal{F}$. More precisely, building on the unique existence of centric linking systems, the first author [28, Theorem A] has used Chermak descent to show that each saturated fusion system has a unique associated subcentric linking system.

For each of the exotic systems $\mathcal{F}$ considered in this section, it will turn out that either $\mathcal{F}$ is of characteristic p-type, or S has a fully $\mathcal{F}$-normalized subgroup X of order p such that $N_{\mathcal{F}}(X)$ is exotic. In the latter case, there is the following elementary observation.

Lemma 5.1. *Let $\mathcal{F}$ be a saturated fusion system over S. Assume there is some nontrivial fully $\mathcal{F}$-normalized subgroup X such that $N_{\mathcal{F}}(X)$ is exotic. Then a punctured group for $\mathcal{F}$ does not exist.*

Proof. If there were a transporter system $\mathcal{L}$ associated with $\mathcal{F}$ having object set containing X, then $\mathrm{Aut}_{\mathcal{L}}(X)$ would be a finite group whose fusion system is $N_{\mathcal{F}}(X)$. □

We restrict attention here to the following families of exotic systems at odd primes, considered in order: the Ruiz–Viruel systems [48], the Oliver systems [42], the Clelland–Parker systems [17], and the Parker–Stroth systems [46]. The results are summarized in the following theorem.

Theorem 5.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(a) *If $\mathcal{F}$ is a Ruiz–Viruel system at the prime 7, then $\mathcal{F}$ is of characteristic 7-type, so has a punctured group.*
(b) *If $\mathcal{F}$ is an exotic Oliver system, then $\mathcal{F}$ has a punctured group if and only if $\mathcal{F}$ occurs in cases (a)(i), (a)(iv), or (b) of [42, Theorem 2.8].*
(c) *If $\mathcal{F}$ is an exotic Clelland–Parker system, then $\mathcal{F}$ has a punctured group if and only if each essential subgroup is abelian. Moreover, if so then $\mathcal{F}$ is of characteristic p-type.*
(d) *If $\mathcal{F}$ is a Parker–Stroth system, then $\mathcal{F}$ is of characteristic p-type, so has a punctured group.*

Proof. This follows upon combining Theorem 2.21 or Lemma 5.1 with Lemma 5.4, Proposition 5.7, Propositions 5.9 and 5.11, and Proposition 5.12, respectively. □

When showing that a fusion system is of characteristic p-type, we will often use the following elementary lemma.

Lemma 5.3. *Let X be a fully $\mathcal{F}$-normalized subgroup of S such that $C_S(X)$ is abelian. Then $N_{\mathcal{F}}(X)$ is constrained.*

Proof. Using Alperin's Fusion Theorem [2, Theorem I.3.6], one sees that $C_S(X)$ is normal in $C_{\mathcal{F}}(X)$. In particular, $C_{\mathcal{F}}(X)$ is constrained. Therefore, by [28, Lemma 2.13], $N_{\mathcal{F}}(X)$ is constrained. □

5.1 | The Ruiz–Viruel systems

Three exotic fusion systems at the prime 7 were discovered by Ruiz and Viruel, two of which are simple. The other contains one of the simple ones with index 2.

Lemma 5.4. *Let $\mathcal{F}$ be a saturated fusion system over an extraspecial p-group S of order p^3 and exponent p. Then $N_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(S)$. In particular, $\mathcal{F}$ is of characteristic p-type.*

Proof. Clearly $N_{\mathcal{F}}(S) \subseteq N_{\mathcal{F}}(Z(S))$. Note that $N_{\mathcal{F}}(Z(S))$ is a saturated fusion system over S as well. So, by [48, Lemma 3.2], if a subgroup of S is centric and radical in $N_{\mathcal{F}}(Z(S))$, then it is either elementary abelian of order p^2 or equal to S. Moreover, by [48, Lemma 4.1], an elementary abelian subgroup V of order p^2 is radical in $N_{\mathcal{F}}(Z(S))$ if and only if $\mathrm{Aut}_{\mathcal{F}}(V)$ contains $\mathrm{SL}_2(p)$. However, if $\mathrm{Aut}_{\mathcal{F}}(V)$ contains $\mathrm{SL}_2(p)$, then it does not normalize $Z(S)$. This implies that S is the only subgroup of S that is centric and radical in $N_{\mathcal{F}}(Z(S))$. Hence, by Alperin's Fusion Theorem [2, Theorem I.3.6], we have $N_{\mathcal{F}}(Z(S)) \subseteq N_{\mathcal{F}}(S)$ and thus $N_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(S)$. In particular, $N_{\mathcal{F}}(Z(S))$ is constrained. If X is a nontrivial subgroup of $\mathcal{F}$ with $X \neq Z(S)$, then $C_S(X)$ is abelian. So, it follows from Lemma 5.3 that $\mathcal{F}$ is of characteristic p-type. □

In Section 6, it is shown that for the three exotic Ruiz–Viruel systems, the subcentric linking system is the unique associated punctured group whose full subcategory on the centric subgroups is the centric linking system.

5.2 | Oliver's systems

A classification of the simple fusion systems $\mathcal{F}$ on p-groups with a unique abelian subgroup A of index p is given in [16, 42, 44]. Here we consider only those exotic fusion systems in which A is not essential in $\mathcal{F}$, namely those fusion systems appearing in the statement of [42, Theorem 2.8].

Whenever $\mathcal{F}$ is a saturated fusion system on a p-group S with a unique abelian subgroup A of index p, we adopt [42, Notation 2.2]. For example,

$$Z = Z(S), \quad Z_2 = Z_2(S), \quad S' = [S, S], \quad Z_0 = Z \cap S', \quad \text{and} \quad A_0 = ZS',$$

and also

$$\mathcal{H} = \{Z\langle x\rangle \mid x \in S - A\} \quad \text{and} \quad \mathcal{B} = \{Z_2\langle x\rangle \mid x \in S - A\}.$$

Lemma 5.5. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S having a unique abelian subgroup A of index p.*

(a) *If $P \leqslant S$ is $\mathcal{F}$-essential, then $P \in \{A\} \cup \mathcal{H} \cup \mathcal{B}$, $|N_S(P)/P| = p$, and each $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ extends to an automorphism of S.*

Assume now in addition that A is not essential in $\mathcal{F}$.

(b) *If $O_p(\mathcal{F}) = 1$, then $\mathcal{F}^e \cap \mathcal{H} \neq \emptyset$, $Z_0 = Z$ is of order p, $S' = A_0$ is of index p^2 in S, and S has maximal class.*
(c) *If $P \in \mathcal{H} \cup \mathcal{B}$ is $\mathcal{F}$-essential, then $P \cong C_p^2$ or p_+^{1+2} according to whether $P \in \mathcal{H}$ or $P \in \mathcal{B}$, and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$ acts naturally on $P/[P,P]$.*
(d) *If $P \in \mathcal{F}^e \cap \mathcal{H}$, then each $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(Z)$ extends to an automorphism of S.*
(e) *A subgroup $P \leqslant S$ is essential in $N_{\mathcal{F}}(Z)$ if and only if $P \in \mathcal{F}^e \cap \mathcal{B}$.*
(f) *There is $x \in S - A$ such that $A_0\langle x\rangle$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant.*

Proof. Parts (a), (b), and (f) are shown in [42, Lemma 2.3,2.4], and (c) follows from [42, Lemma 2.7]. Suppose as in (d) that $P \in \mathcal{F}^e \cap \mathcal{H}$. By (c), $\mathrm{Aut}_{\mathcal{F}}(P)$ is a subgroup of $GL_2(p)$ containing $SL_2(p)$, and the stabilizer of Z in this action normalizes $O^{p'}(C_{\mathrm{Aut}_{\mathcal{F}}(P)}(Z)) = \mathrm{Aut}_S(P)$. So, (d) follows from (a).

It remains to prove (e). If $P \in \mathcal{F}^e \cap \mathcal{B}$, then as $Z = [P,P]$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant in this case, $\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup, and so P is essential in $N_{\mathcal{F}}(Z)$. Conversely, suppose P is $N_{\mathcal{F}}(Z)$-essential. By (a) applied to $N_{\mathcal{F}}(Z)$, $P \in \{A\} \cup \mathcal{H} \cup \mathcal{B}$ and $\mathrm{Out}_S(P)$ is of order p, so by assumption $N_{\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)}(\mathrm{Out}_S(P))$ is strongly p-embedded in $\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)$ by [2, Proposition A.7]. Now each member of $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ extends to S by (a), so Z is $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$-invariant. Thus, $N_{\mathrm{Out}_{\mathcal{F}}(P)}(\mathrm{Out}_S(P)) = N_{\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)}(\mathrm{Out}_S(P))$ is a proper subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$, and hence strongly p-embedded by [2, Proposition A.7] again. So, P is essential in $\mathcal{F}$. By assumption $P \neq A$, and $P \notin \mathcal{H}$ by (d). So, $P \in \mathcal{B}$. □

For the remainder of this subsection, we let $\mathcal{F}$ be a saturated fusion system on a p-group S with a unique abelian subgroup A of index p. Further, we assume that $O_p(\mathcal{F}) = 1$ and A is not essential in $\mathcal{F}$.

We next set up some additional notation. Fix an element $x \in S - A$ such that $A_0\langle x\rangle$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, as in Lemma 5.5(f). As $O_p(\mathcal{F}) = 1$, S is of maximal class by Lemma 5.5(b). In particular, $Z = Z_0$ is of order p, A/A_0 is of order p, and $S' = A_0$, so we can adopt [42, Notation 2.5]. As in [42, Notation 2.5], let $a \in A\backslash A_0$, and define $\mathcal{H}_i$ and $\mathcal{B}_i$ to be the S-conjugacy classes of the subgroups $Z\langle xa^i\rangle$ and $Z_2\langle xa^i\rangle$ for $i = 0, 1, \dots, p-1$, and set

$$\mathcal{H}_* = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_{p-1} \quad \text{and} \quad \mathcal{B}_* = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_{p-1},$$

so that $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_*$.

Set

$$\Delta = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times \quad \text{and} \quad \Delta_i = \{(r, r^i) \mid r \in (\mathbb{Z}/p\mathbb{Z})^\times\}.$$

Define $\mu : \mathrm{Aut}_{\mathcal{F}}(S) \to \Delta$ and $\widehat{\mu} : \mathrm{Out}_{\mathcal{F}}(S) \to \Delta$ by $\widehat{\mu}([\alpha]) = \mu(\alpha) = (r, s)$, where

$$(xA_0)^\alpha = x^r A_0 \quad \text{and} \quad z^\alpha = z^s.$$

The following lemma looks at the image of homomorphisms analogous to μ and $\widehat{\mu}$ that are defined instead with respect to $N_{\mathcal{F}}(Z)/Z$ and $C_{\mathcal{F}}(Z)/Z$.

Lemma 5.6. *Assume $|S/Z| = p^m$ with $m \geqslant 4$. Let $\mathcal{E} \in \{N_{\mathcal{F}}(Z), C_{\mathcal{F}}(Z)\}$, and let $\mu_{\mathcal{E}}$ be the restriction of μ to $\mathrm{Aut}_{\mathcal{E}}(S)$. Let $\mu_{\mathcal{E}/Z} : \mathrm{Aut}_{\mathcal{E}/Z}(S/Z) \to \Delta$ be the map analogous to μ but defined instead with* *respect to S/Z. Then*

$$\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \{(r, sr^{-1}) \mid (r,s) \in \mathrm{Im}(\mu_{\mathcal{E}})\}.$$

In particular, if $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta$, *then* $\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \Delta$. *And if* $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta_i$ *for some i, then* $\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \Delta_{i-1}$, *where the indices are taken modulo $p-1$.*

Proof. This essentially follows from [16, Lemma 1.11(b)]. By assumption, $\mathcal{E}/Z$ is a fusion system over a p-group S/Z of order at least p^4. So A/Z is the unique abelian subgroup of S/Z of index p by [42, Lemma 1.9]. As S is of maximal class, so is the quotient S/Z. In particular, $Z(S/Z)$ is of order p, so we can define $\mu_{\mathcal{E}/Z}$ as suggested with xZ in the role of x and gZ in the role of z, where $g \in Z_2 - Z$ is a fixed element.

Let $\alpha \in \mathrm{Aut}_{\mathcal{E}}(S)$ with $\mu(\alpha) = (r,s)$, let $\bar{\alpha}$ be the induced automorphism of S/Z, and let $t \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ be such that $a^{\alpha}A_0 = a^t A_0$ (which exists because A and A_0 are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $|A/A_0| = |Z_0| = p$). By [16, Lemma 1.11(b)], α acts on the ith upper central quotient $Z_i(S)/Z_{i-1}(S)$ by raising a generator to the power tr^{m-i} for $i = 1, \dots, m-1$. Thus, $s = tr^{m-1}$ and $(gZ)^{\bar{\alpha}} = g^{\alpha}Z = g^{tr^{m-2}}Z$. Hence, $\mu_{\mathcal{E}/Z}(\bar{\alpha}) = (r, sr^{-1})$. Conversely if $\mu_{\mathcal{E}/Z}(\bar{\alpha}) = (r, \bar{s})$, then $\mu_{\mathcal{E}}(\alpha) = (r, \bar{s}r)$. □

In the following proposition, we refer to Oliver's systems according to the itemized list (a)(i–iv), (b) given in [42, Theorem 2.8].

Proposition 5.7. *Assume $\mathcal{F}$ is one of the exotic systems appearing in [42, Theorem 2.8]. Write $|S/Z| = p^m$ with $m \geqslant 3$.*

(a) *$\mathcal{F}$ is of characteristic p-type whenever $\mathcal{F}^e \subseteq \mathcal{H}$. In particular, this holds if $\mathcal{F}$ occurs in case (a)(i), (a)(iv), or (b).*
(b) *If $\mathcal{F}$ is in case (a)(ii) and $m \geqslant 4$, then $N_{\mathcal{F}}(Z)$ is exotic. Moreover, $\mathcal{F}$ is of component type, and $C_{\mathcal{F}}(Z)/Z$ is simple, exotic, and occurs in (a)(iv) in this case. If $\mathcal{F}$ is in case (a)(ii) with $m = 3$ (and hence $p = 5$), then $N_{\mathcal{F}}(Z)/Z$ is the fusion system of $5^2GL_2(5)$, and $\mathcal{F}$ is of characteristic 5-type.*
(c) *If $\mathcal{F}$ is in case (a)(iii), then $N_{\mathcal{F}}(Z)$ is exotic. Moreover, $\mathcal{F}$ is of component type where $C_{\mathcal{F}}(Z)/Z$ is simple, exotic, and of type (a)(i).*

Proof. Each of Oliver's systems is simple on S with a unique abelian subgroup A of index p that is not essential, so it satisfies our standing assumptions and the hypotheses of Lemmas 5.5 and 5.6, and we can continue the notation from above. In particular, $Z_0 = Z$ is of order p, $S' = A_0$, and S is of maximal class. Set

$$\mathcal{E} := C_{\mathcal{F}}(Z),\ \overline{S} = S/Z \text{ and } \overline{\mathcal{E}} = \mathcal{E}/Z.$$

If G is a group realizing $N_{\mathcal{F}}(Z)$, then $C_G(Z)$ realizes $\mathcal{E} = C_{N_{\mathcal{F}}(Z)}(Z)$ and so $C_G(Z)/Z$ realizes $\overline{\mathcal{E}}$. Hence

$$\text{if } \overline{\mathcal{E}} \text{ is exotic, then } N_{\mathcal{F}}(Z) \text{ is exotic.} \tag{5.1}$$

The $\mathcal{E}$-automorphism group of S is the centralizer of Z in $\mathrm{Aut}_{\mathcal{F}}(S)$. Thus, if $\mathrm{Im}(\mu) = \Delta$, then by definition of the map μ, we have

$$\mathrm{Im}(\mu_{\mathcal{E}}) = \{(r,1) \mid r \in (\mathbb{Z}/p\mathbb{Z})^{\times}\} = \Delta_0,$$

which implies $\mathrm{Im}(\mu_{\bar{\mathcal{E}}}) = \Delta_{-1}$ by Lemma 5.6. So,

$$\text{if } \mathrm{Im}(\mu) = \Delta, \text{ then } \mathrm{Im}(\mu_{\mathcal{E}}) = \Delta_0 \text{ and } \mathrm{Im}(\mu_{\bar{\mathcal{E}}}) = \Delta_{-1}. \tag{5.2}$$

For each fully $\mathcal{F}$-normalized subgroup $X \leqslant S$ of order p and not equal to Z, $C_S(X)$ is abelian: if $X \leqslant A$ this follows because $C_S(X) = A$ (X is not central), while if $X \nleqslant A$, this follows because $C_A(X) = Z$ by Lemma 5.5(b). Thus, $N_{\mathcal{F}}(X)$ is constrained in this case by Lemma 5.3. Hence,

$$\mathcal{F} \text{ is of characteristic } p\text{-type if and only if } N_{\mathcal{F}}(Z) \text{ is constrained.} \tag{5.3}$$

By Lemma 5.5(e), if $\mathcal{F}^e \subseteq \mathcal{H}$, then $N_{\mathcal{F}}(Z)$ has no essential subgroups. By the Alperin–Goldschmidt Fusion Theorem [2, I.3.5], each morphism in $N_{\mathcal{F}}(Z)$ extends in this case to S, and hence S is normal in $N_{\mathcal{F}}(Z)$. So,

$$\text{if } \mathcal{F}^e \subseteq \mathcal{H}, \text{ then } N_{\mathcal{F}}(Z) \text{ is constrained.} \tag{5.4}$$

In particular, the first part of (a) holds.

Case: $\mathcal{F}$ occurs in (a)(i), (a)(iv), or (b) of [42, Theorem 2.8]. We have $\mathcal{F}^e \subseteq \mathcal{H}$ precisely in these cases. So, $\mathcal{F}$ is of characteristic p-type by (5.4). This completes the proof of (a).

Case: $\mathcal{F}$ occurs in (a)(ii). Here, $m \equiv -1 \pmod{p-1}$, $\mathrm{Im}(\mu) = \Delta$, and $\mathcal{F}^e = \mathcal{B}_0 \cup \mathcal{H}_*$. By assumption $\mathcal{F}$ is exotic, so as $\mathcal{F}$ is the fusion system of ${}^3D_4(q)$ when $p = 3$, we have $p \geqslant 5$.

By Lemma 5.5(e), the set of $N_{\mathcal{F}}(Z)$-essential subgroups is $\mathcal{F}^e \cap \mathcal{B} = \mathcal{B}_0$. A straightforward argument shows now that the elements of $\mathcal{B}_0$ are also essential in $\mathcal{E} = C_{\mathcal{F}}(Z)$, and their images in $\overline{S}$ are essential in $\overline{\mathcal{E}}$. Thus, $\bar{\mathcal{B}}_0 \subseteq \bar{\mathcal{E}}^e$, where $\bar{\mathcal{B}}_0 = \{\bar{P} \mid P \in \mathcal{B}_0\}$.

Subcase: $m \geqslant 4$. By (5.1), it is sufficient to show that $\overline{\mathcal{E}}$ is simple, exotic and occurs in case (a)(iv) of Oliver's classification. As $\bar{S}$ has order p^m, we know that $\bar{A}$ is the unique abelian subgroup of $\bar{S}$ of index p by [42, Lemma 1.9]. As A is not $\mathcal{F}$-essential, it follows from the Alperin–Goldschmidt Fusion Theorem that every element of $\mathrm{Aut}_{\mathcal{F}}(A)$ extends to an $\mathcal{F}$-automorphism of S. From this one sees that $\overline{A}$ is not radical and thus not essential in $\overline{\mathcal{E}}$.

We will prove first that $\bar{\mathcal{E}}$ is reduced. As $O_p(\bar{\mathcal{E}})$ is contained in every $\bar{\mathcal{E}}$-essential subgroup, we have $O_p(\bar{\mathcal{E}}) \leqslant \bigcap \bar{\mathcal{B}}_0 = Z(\bar{S})$. By Lemma 5.5(c), Z_2 is not $\mathrm{Aut}_{\mathcal{E}}(P)$-invariant for any $P \in \mathcal{B}_0$, and hence $\bar{Z}_2 = Z(\bar{S})$ is not $\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})$-invariant for any $\bar{P} \in \bar{\mathcal{B}}_0$. So, $O_p(\bar{\mathcal{E}}) = 1$.

We next show that $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$. By [42, Proposition 1.3(c,d)], the focal subgroup of $\bar{\mathcal{E}}$ is generated by $[\bar{P}, \mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})]$ for $\bar{P} \in \bar{\mathcal{B}}_0 \cup \{\bar{S}\}$, and $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$ if and only if $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$. As $\bar{P}$ is a natural module for $O^{p'}(\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})) \cong SL_2(p)$ for each $\bar{P} \in \bar{\mathcal{B}}_0$ (Lemma 5.5(c)), the focal subgroup of $\bar{\mathcal{E}}$ contains $\langle \bar{\mathcal{B}}_0 \rangle = \bar{A}_0 \langle \bar{x} \rangle$. Thus, $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$ if $\bar{a} \in [\bar{S}, \mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{S})]$. By (5.2), $\mathrm{Im}(\mu_{\bar{\mathcal{E}}}) = \Delta_{-1}$. Further, if $\bar{\alpha}$ is an $\bar{\mathcal{E}}$-automorphism of $\bar{S}$ with $\mu_{\bar{\mathcal{E}}}(\bar{\alpha}) = (r, r^{-1})$, then for the class $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ with $(\bar{a}\bar{A}_0)^{\bar{\alpha}} = \bar{a}^t \bar{A}_0$, we have $r^{-1} = tr^{m-2}$ by [42, Lemma 2.6(a)], and hence $t = r^{-(m-1)}$. As $m + 1 \equiv 0 \pmod{p-1}$ and $p \geqslant 5$, we have $-(m-1) \not\equiv 0 \pmod{p-1}$. So $\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{S})$ acts nontrivially on $\bar{A}/\bar{A}_0$. Hence, $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$ and $\overline{\mathcal{E}} = O^p(\overline{\mathcal{E}})$.

We next show that $O^{p'}(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$ using [42, Lemma 1.4]. Set $\bar{P} = Z(\bar{S})\langle \bar{x} \rangle = \bar{Z}_2 \langle \bar{x} \rangle \in \bar{\mathcal{B}}_0$, and let $\bar{\alpha}$ be an $\bar{\mathcal{E}}$-automorphism of $\bar{S}$. Recall that x was chosen such that $A_0\langle x \rangle$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Moreover, $\mathcal{B}$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $\mathcal{B}_0$ consists of the elements of $\mathcal{B}$ that lie in $A_0\langle x \rangle$. Hence, $\mathcal{B}_0$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and so $\bar{\alpha}$ preserves the $\bar{S}$-class $\bar{\mathcal{B}}_0$ under conjugation. Thus, upon adjusting $\bar{\alpha}$ by an inner automorphism of $\bar{S}$ (which does not change the image of $\bar{\alpha}$ under $\mu_{\bar{\mathcal{E}}}$), we can assume that $\bar{\alpha}$ normalizes $\bar{P}$. The restriction of $\bar{\alpha}$ to $\bar{P}$ acts via an element of $SL_2(p)$ on $\bar{P}$ because $\mathrm{Im}(\mu_{\bar{\mathcal{E}}}) = \Delta_{-1}$, and so this restriction is contained in $O^{p'}(\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P}))$. Thus, $O^{p'}(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$ by [42, Lemma 1.4].

Thus, $\bar{\mathcal{E}}$ is reduced. Step 1 of the proof of [42, Theorem 2.8] then shows that $\bar{\mathcal{E}}$ is the unique reduced fusion system with the given data, and then Step 2 shows that $\bar{\mathcal{E}}$ is simple. So, $\bar{\mathcal{E}}$ is exotic and occurs in case (a)(iv) of Oliver's classification, as $m-1 \equiv -2 \not\equiv 0, -1 \pmod{p-1}$.

Subcase: $m = 3$. As $m \equiv -1 \pmod{p-1}$, we have $p = 5$. So, $\bar{S}$ is extraspecial of order 5^3 and exponent 5. We saw above that $N_{\mathcal{F}}(Z)^e = \mathcal{B}_0$, which is of size 1 in this case. That is $Z_2\langle x\rangle$ is the unique essential subgroup of $N_{\mathcal{F}}(Z)$, which is therefore invariant under $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(Z)}(S)$. By the Alperin–Goldschmidt Fusion Theorem, this subgroup is normal $N_{\mathcal{F}}(Z)$ and so $Z(\overline{S})\langle\overline{x}\rangle$ is normal in $N_{\mathcal{F}}(Z)/Z$. This implies that $N_{\mathcal{F}}(Z)$ and $N_{\mathcal{F}}(Z)/Z$ are constrained. In particular, $\mathcal{F}$ is of characteristic 5-type by (5.3). As $\mathrm{Im}(\mu_{N_{\mathcal{F}}(Z)}) = \mathrm{Im}(\mu) = \Delta$, it follows from Lemma 5.6 that $\mathrm{Im}(\mu_{N_{\mathcal{F}}(Z)/Z}) = \Delta$. Using this and Lemma 5.5(c), one sees that $N_{\mathcal{F}}(Z)/Z$ is indeed isomorphic to the fusion system of $5^2GL_2(5)$. This completes the proof of (b).

Case: $\mathcal{F}$ occurs in (a)(iii). Then $m \equiv 0 \pmod{p-1}$, $\mathcal{F}^e = \mathcal{H}_0 \cup \mathcal{B}_*$, and $\mathrm{Im}(\mu) = \Delta$. Again, by (5.1), it is sufficient to show that $\overline{\mathcal{E}}$ is simple, exotic and occurs in case (a)(i) of Oliver's classification. Similarly as in the previous case, by Lemma 5.5(e), the set of $N_{\mathcal{F}}(Z)$-essential subgroups is $\mathcal{F}^e \cap \mathcal{B} = \mathcal{B}_*$ and $\bar{\mathcal{B}}_* = \{\bar{P} \mid P \in \mathcal{B}_*\} \subseteq \overline{\mathcal{E}}^e$.

As $m \geqslant 3$, we have $p \geqslant 5$, and hence in fact $m \geqslant 4$. In particular, $\bar{A}$ is the unique abelian subgroup of $\bar{S}$. Moreover, $\bar{A}$ is not essential in $\bar{\mathcal{E}}$, and $O_p(\bar{\mathcal{E}}) = 1$ by a similar argument as in the previous case. Also as the previous case, the focal subgroup of $\bar{\mathcal{E}}$ contains $\langle\bar{\mathcal{B}}_*\rangle$, which this time is equal to $\bar{S}$. So, $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$.

Notice that $m \equiv 0 \pmod{p-1}$ implies $\Delta_0 = \Delta_m$ and thus $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta_m$ by (5.2). It follows therefore from [42, Lemma 2.6(b)] that $\mathcal{B}_i$ is $\mathrm{Aut}_{\mathcal{E}}(S)$-invariant for $i = 1, 2, \dots, p-1$. Hence, arguing as in the previous step (but with some $\mathcal{B}_i$ instead of $\mathcal{B}_0$), one sees that $O^{p'}(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$. Hence, $\bar{\mathcal{E}}$ is reduced.

It follows now from Steps 1 and 2 of the proof of [42, Theorem 2.8] that $\overline{\mathcal{E}}$ is simple and uniquely determined. As every essential subgroup of $\overline{\mathcal{E}}$ has order p^2 and $|\bar{S}/Z(\bar{S})| = p^{m-1}$ where $m-1 \equiv -1 \pmod{p-1}$, it follows moreover that $\bar{\mathcal{E}}$ occurs in case (a)(i) of Oliver's classification. In particular, $\bar{\mathcal{E}}$ is exotic. This completes the proof of (c) and thus the proof of the proposition. □

5.3 | The Clelland–Parker systems

We now describe the fusion systems constructed by Clelland and Parker in [17]. Throughout, we fix a power q of the odd prime p, a natural number $n \leqslant p-1$, and we set $k := \mathbb{F}_q$. Let $A := A(n,k)$ be the $(n+1)$-dimensional space of homogeneous polynomials of degree n in two variables with coefficients in k. The group $D := k^\times \times GL_2(k)$ acts on A via $f(x,y)\cdot(\lambda, [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = \lambda f(ax+b, cy+d)$. The subgroup $SL_2(k)$ of D acts irreducibly on A. Write G for the semidirect product DA. Let U be a Sylow p-subgroup of D and let $S := S(n,k) := UA$ be the semidirect product of A by U.

The center $Z := Z(S)$ is a 1-dimensional k-subspace of A and by [17, Lemma 4.2(iii)], we have

$$C_A(X) = Z(S) \text{ for each subgroup } X \text{ not contained in } A. \tag{5.5}$$

The second center $Z_2(S)$ is a 2-dimension k-subspace of A. Let $R = ZU$ and $Q = Z_2(S)U$. Then $R \cong q^2$ and Q is special of shape q^{1+2}. Let H_R be the stabilizer in $GL_3(k)$ of a 1-dimensional subspace, and identify its unipotent radical with R. Let H_Q be the stabilizer in $GSp_4(k)$ of a 1-dimensional subspace and identify the corresponding unipotent radical with Q. It is shown in [17] that $N_G(R)$ is isomorphic to a Borel subgroup $GL_3(k)$, and that $N_G(Q)$ is isomorphic to a Borel subgroup of $GSp_4(k)$. This allows to form the free amalgamated products

$$F(1,n,k,R) := G *_{N_G(R)} H_R$$

and

$$F(1,n,k,Q) := G *_{N_G(Q)} H_Q.$$

Set

$$\mathcal{F}(1,n,k,R) := \mathcal{F}_S(F(1,n,k,R))$$

and

$$\mathcal{F}(1,n,k,Q) := \mathcal{F}_S(F(1,n,k,Q)).$$

More generally, for each $X \in \{R,Q\}$ and each divisor r of $q-1$, subgroup $F(r,n,k,X)$ of $F(1,n,k,X)$ of index r, which contains $O^{p'}(G)$ and $O^{p'}(H_X)$. They set then

$$\mathcal{F}(r,n,k,X) = \mathcal{F}_S(F(r,n,k,X)).$$

As they show, distinct fusion systems are only obtained for distinct divisors r of $(n+2,q-1)$ when $X = R$, and for distinct divisors r of $(n,q-1)$ when $X = Q$. By [17, Theorem 4.9], for all $n \geqslant 1$ and each divisor r of $(n+2,q-1)$, $\mathcal{F}(r,n,k,R)$ is saturated. Similarly, $\mathcal{F}(r,n,k,Q)$ is saturated for each $n \geqslant 2$ and each divisor r of $(n,q-1)$. It is determined in [17, Theorems 5.1 and 5.2 and Lemma 5.3] which of these fusion systems are exotic. It turns out that $\mathcal{F}(r,n,k,R)$ is exotic if and only if either $n > 2$ or $n = 2$ and $q \notin \{3,5\}$. Furthermore, $\mathcal{F}(r,n,k,Q)$ is exotic if and only if $n \geqslant 3$, in which case $p \neq 3$ as $n \leqslant p-1$.

For the remainder of this subsection, except in Lemma 5.10, we use the notation introduced above.

For the problems we will consider here, we will sometimes be able to reduce to the case $r = 1$ using the following lemma.

Lemma 5.8. *For any divisor r of $q-1$, the fusion system $\mathcal{F}(r,n,k,R)$ is a normal subsystem of $\mathcal{F}(1,n,k,R)$ of index prime to p, and the fusion system $\mathcal{F}(r,n,k,Q)$ is a normal subsystem of $\mathcal{F}(1,n,k,Q)$ of index prime to p.*

Proof. For $X \in \{R,Q\}$, the fusion systems $\mathcal{F}(r,n,k,X)$ and $\mathcal{F}(1,n,k,X)$ are both saturated by the results cited above, As $F(r,n,k,X)$ is a normal subgroup of $F(1,n,k,X)$, it is easy to check that $\mathcal{F}(r,n,k,X)$ is $\mathcal{F}(1,n,k,X)$-invariant. As both $\mathcal{F}(1,n,k,X)$ and $\mathcal{F}(r,n,k,X)$ are fusion systems over S, the claim follows. □

Proposition 5.9. *$\mathcal{F}(r,n,k,R)$ is of characteristic p-type for all $1 \leqslant n \leqslant p-1$ and for all divisors r of $(n+2,q-1)$.*

Proof. Fix $1 \leqslant n \leqslant p-1$ and a divisor r of $(n+2, q-1)$. Set $\mathcal{F} = \mathcal{F}(1, n, k, R)$. By Lemma 5.8, $\mathcal{F}(r, n, k, R)$ is a normal subsystem of $\mathcal{F}$ of index prime to p. So, by [28, Proposition 2(c)], it suffices to show that $\mathcal{F}$ is of characteristic p-type. By [17, Lemma 5.3(i,ii)], $\mathcal{F}$ is of realizable and of characteristic p-type when $n = 1$, so we may and do assume $n \geqslant 2$.

Using the notation above, set $\mathcal{F}_1 = \mathcal{F}_S(G)$, $S_2 = N_S(R)$, and $\mathcal{F}_2 = \mathcal{F}_{S_2}(H_R)$. The fusion system $\mathcal{F}$ is generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ by [17, Theorem 3.1], and so as $\mathcal{F}_1$ and $\mathcal{F}_2$ are both constrained with $O_p(\mathcal{F}_1) = A$ and $O_p(\mathcal{F}_2) = R$, it follows that $\mathcal{F}$ is in turn generated by $\mathrm{Aut}_{\mathcal{F}_1}(A)$, $\mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}_2}(R)$, and $\mathrm{Aut}_{\mathcal{F}_2}(S_2)$. However, the last automorphism group is redundant, as $N_{H_R}(S_2) = N_G(R)$ induces fusion in $\mathcal{F}_1$. Hence,

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(R)\rangle. \tag{5.6}$$

Observe also that the following property is a direct consequence of (5.5):

$$\text{If } X \leqslant S \text{ with } X \nleqslant Z, \text{ then either } X \leqslant A \text{ and } C_S(X) = A, \text{ or } |C_S(X)| \leqslant q^2. \tag{5.7}$$

We can now show that $\mathcal{F}$ is of characteristic p-type. Let first $X \in \mathcal{F}^f$ such that $X \nleqslant Z$. We show that $N_{\mathcal{F}}(X)$ is constrained. If X is not $\mathcal{F}$-conjugate into A or into R, then every morphism in $N_{\mathcal{F}}(X)$ extends by (5.6) to an automorphism of S. So, $N_{\mathcal{F}}(X) = N_{N_{\mathcal{F}}(S)}(X)$. As $N_{\mathcal{F}}(S)$ is constrained, it follows thus from [28, Lemma 2.11] that $N_{\mathcal{F}}(X)$ is constrained. So, we may assume that there exists an $\mathcal{F}$-conjugate Y of X with $Y \leqslant A$ or $Y \leqslant R$. We will show that $C_S(X)$ is abelian so that $N_{\mathcal{F}}(X)$ is constrained by Lemma 5.3. Note that $|C_S(X)| \geqslant |C_S(Y)|$ as X is fully normalized and thus fully centralized in $\mathcal{F}$. As X is not contained in $Z = Z(S)$, we have in particular $Y \nleqslant Z$. If $Y \leqslant A$, then $A \leqslant C_S(Y)$ and, as X is fully centralized and $n \geqslant 2$, $|C_S(X)| \geqslant |C_S(Y)| \geqslant |A| > q^2$. So, by (5.7), $C_S(X) = A$ is abelian. Similarly, by (5.7), if $X \leqslant A$ then $C_S(X) = A$ is abelian. Thus, we may assume $Y \leqslant R$ and $X \nleqslant A$. Then $R \leqslant C_S(Y)$ and (5.7) implies $q^2 \geqslant |C_S(X)| \geqslant |C_S(Y)| \geqslant |R| = q^2$. So, the inequalities are equalities, $C_S(Y) = R$ and $|C_S(X)| = q^2$. By the extension axiom, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(Y), C_S(X))$. So, it follows that $C_S(X) \in R^{\mathcal{F}}$ is abelian. This completes the proof that $N_{\mathcal{F}}(X)$ is constrained for every $X \in \mathcal{F}^f$ with $X \nleqslant Z$.

Let now $1 \neq X \leqslant Z$. It remains to show that $N_{\mathcal{F}}(X)$ is constrained. If $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$, then again by [28, Lemma 2.11], $N_{\mathcal{F}}(X) = N_{N_{\mathcal{F}}(S)}(X)$ is constrained because $N_{\mathcal{F}}(S)$ is constrained. We will finish the proof by showing that indeed $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$. Assume by contradiction that $N_{\mathcal{F}}(X) \nsubseteq N_{\mathcal{F}}(S)$. Then there exists an essential subgroup E of $N_{\mathcal{F}}(X)$. Observe that $Z < E$, as E is $N_{\mathcal{F}}(X)$-centric. As $\mathrm{Aut}_S(E)$ is not normal in $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$, there exists an element of $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$ that does not extend to an $\mathcal{F}$-automorphism of S. So, by (5.6), E is $\mathcal{F}$-conjugate into A or into R. Assume first that there exists an $\mathcal{F}$-conjugate $\widehat{E}$ of E such that $\widehat{E} \leqslant A$. Property (5.5) yields that $R \cap A = Z$. So E is conjugate to $\widehat{E} \leqslant A$ via an element of $\mathrm{Aut}_{\mathcal{F}_1}(S)$ by (5.6). Thus, as $C_S(E) \leqslant E$, we have $A \leqslant C_S(\widehat{E}) \leqslant \widehat{E}$. Hence, $A = \widehat{E}$ by (5.7). As A is $\mathrm{Aut}_{\mathcal{F}_1}(S)$-invariant, it follows $E = A$. Looking at the structure of G, we observe now that $N_G(X) = N_G(S)$ and so $\mathrm{Aut}_S(A)$ is normal in $\mathrm{Aut}_{N_{\mathcal{F}}(X)}(A) = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(X) = N_{\mathrm{Aut}_{\mathcal{F}_1}(A)}(X)$. Hence, A cannot be essential in $N_{\mathcal{F}}(X)$ and we have derived a contradiction. Thus, E is not $\mathcal{F}$-conjugate into A. Therefore, again by (5.6), E is conjugate into R under an element of $\mathrm{Aut}_{\mathcal{F}_1}(S)$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}_1}(S)$ such that $E^\alpha \leqslant R$. As $C_S(E) \leqslant E$, we have then $C_S(E^\alpha) \leqslant E^\alpha$. As R is abelian, it follows $E^\alpha = R$. Thus, we have

$$\mathrm{Aut}_{N_{\mathcal{F}}(X)}(E)^\alpha = N_{\mathrm{Aut}_{\mathcal{F}}(E)}(X)^\alpha = N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha).$$

As $1 \neq X^\alpha \leqslant Z^\alpha = Z \leqslant R$ and $\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{F}_2}(R)$ acts k-linearly on R, $N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha)$ has a normal Sylow p-subgroup. Thus, $\mathrm{Aut}_{N_{\mathcal{F}}(X)}(E) \cong N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha)$ has a normal Sylow p-subgroup, contradicting the fact that E is essential in $N_{\mathcal{F}}(X)$. This final contradiction shows that $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$ is constrained. This completes the proof of the assertion. □

Our next goal will be to show that $\mathcal{F} := \mathcal{F}(r, n, k, Q)$ does not have a punctured group for $n \geqslant 3$ (i.e., in the case that $\mathcal{F}$ is exotic). For that we prove that, using the notation introduced at the beginning of this subsection, $N_{\mathcal{F}}(Z)/Z$ is exotic. The structure of $N_{\mathcal{F}}(Z)/Z$ resembles the structure of $\mathcal{F}(r, n-1, k, R)$ except that the elementary abelian normal subgroup of index q is not essential. Indeed, it will turn out that the problem of showing that $N_{\mathcal{F}}(Z)/Z$ is exotic reduces to the situation treated in the following lemma, whose proof of part (c) depends on the classification of finite simple groups.

Lemma 5.10. *Fix a power q of p as before. Let S be an arbitrary p-group such that $S = U \ltimes A$ splits as a semidirect product of an elementary abelian subgroup A with an elementary abelian subgroup U. Assume $|U| = q$, and $|A| = q^n$ for some $3 \leqslant n \leqslant p-1$. Set $P := Z(S)U$, $T := [S,S]U$, and let $\mathcal{F}$ be a saturated fusion system over S. Assume the following conditions hold.*

(i) *$Z(S)$ has order q, $[S,S] \nleqslant Z(S)$, and $Z(S) = C_A(u)$ for every $1 \neq u \in U$.*
(ii) *$O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(q)$ and P is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$.*
(iii) *$\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_{\mathcal{F}}(S)$.*
(iv) *$\mathrm{Aut}_{\mathcal{F}}(S)$ acts irreducibly on $A/[S,S]$, $|A/[S,S]| \geqslant q$.*
(v) *there is a complement to $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ that normalizes U.*

Then the following hold.

(a) *The nontrivial strongly closed subgroups of $\mathcal{F}$ are precisely S and T.*
(b) *Neither S nor T can be written as the direct product of two nontrivial subgroups.*
(c) *$\mathcal{F}$ is exotic.*

Proof. Observe first that (iii) implies that P is fully normalized. In particular, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. As $Z := Z(S)$ has order q, it follows from (ii) that $Z(S) = C_P(N_S(P)) = [P, N_S(P)] \leqslant [S,S]$. In particular, $P \leqslant T$. We note also that $C_S(P) = P$ as $C_A(U) = Z(S)$ by (i).

(a) We argue first that T is strongly closed. Observe that T is normal in S, as T contains $[S,S]$. As $[S,S]$ is characteristic in S, it follows thus from (v) that T is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Thus, as $P \leqslant T$, (iii) implies that T is strongly closed in $\mathcal{F}$. Let now S_0 be a nontrivial proper subgroup of S strongly closed in $\mathcal{F}$. As S_0 is normal in S, it follows $1 \neq S_0 \cap Z(S) \leqslant P$. By (ii), $\mathrm{Aut}_{\mathcal{F}}(P)$ acts irreducibly on P. So, $P \leqslant S_0$. Hence, $[S,S] = [A,U] \leqslant [S,P] \leqslant [S,S_0] \leqslant S_0$ and thus $T = [S,S]U = [S,S]P \leqslant S_0$. Suppose $T < S_0$. As $U \leqslant S_0 \leqslant S = AU$, we have $S_0 = (S_0 \cap A)U$ and thus $[S,S] < S_0 \cap A < A$. So $\mathrm{Aut}_{\mathcal{F}}(S)$ does not act irreducibly on $A/[S,S]$, contradicting (iv). This shows (a).
(b) Let $S^* \in \{S,T\}$ and assume by contradiction that $S^* = S_1 \times S_2$ where S_1 and S_2 are nontrivial subgroups of S^*. Notice that in either case $Z = Z(S^*)$ by (i). Moreover, again using (i), we note that $[S_1,S_1] \times [S_2,S_2] = [S,S] \nleqslant Z = Z(S)$. So, there exists in either case $i \in \{1,2\}$ with $[S_i,S_i] \nleqslant Z$ and thus $S_i \cap A \nleqslant Z$. We assume without loss of generality that $\overline{S_1 \cap A} \nleqslant Z$. Setting $\overline{S^*} = S^*/Z$, we note that $\overline{S_1 \cap A}$ is a nontrivial normal subgroup of $\overline{S^*}$, and intersects thus nontrivially with $Z = Z(\overline{S^*})$. Hence, $((S_1 \cap A)Z) \cap Z_2(S^*) \nleqslant Z$ and so $S_1 \cap$

$A \cap Z_2(S^*) \nleqslant Z$. Choosing $s \in (S_1 \cap A \cap Z_2(S^*))\backslash Z$, we have $s \in N_S(P)\backslash P$ as $A \cap P = Z$ and $[s, P] \leqslant [Z_2(S^*), P] \leqslant Z \leqslant P$. Using (ii) and $C_S(P) \leqslant P$, it follows $Z = [P, s]$. So, $Z = [P, s] \leqslant [P, S_1] \leqslant S_1$ as $P \leqslant S^*$ and S_1 is normal in S^*. As S_2 is a nontrivial normal subgroup of S^*, we have $S_2 \cap Z = S_2 \cap Z(S^*) \neq 1$. This contradicts $S_1 \cap S_2 = 1$. Thus, we have shown that S^* cannot be written as a direct product of two nontrivial subgroups, that is, property (b) holds.

(c) Part (c) follows now using the classification of finite simple groups. Most notably, we use knowledge of the automorphism groups of finite simple groups, Oliver's work on fusion systems over p-groups with an abelian subgroup of index p [42], and the work of Flores–Foote [22] determining the simple groups having a Sylow p-subgroup with a proper nontrivial strongly closed subgroup. To argue in detail, assume that $\mathcal{F}$ is realizable. By (b), neither S nor T can be written as a direct product of two nontrivial subgroups. By (a), S and T are the only nontrivial strongly closed subgroups. The subgroup T is $\mathcal{F}$-centric because T is strongly closed and $P \leqslant T$ is self-centralizing in S. Clearly, S is $\mathcal{F}$-centric. So, as $\mathcal{F}$ is realizable, it follows from [19, Proposition 2.19] that $\mathcal{F} = \mathcal{F}_S(G)$ for some almost simple group G with $S \in \mathrm{Syl}_p(G)$. Set

$$G_1 := F^*(G) \text{ and } \mathcal{F}_1 := \mathcal{F}_{S \cap G_1}(G_1).$$

If $S \leqslant G_1$, then note that T is a proper subgroup of S that is strongly closed in S with respect to G and thus with respect to G_1. Hence, it follows work of Flores–Foote [22] that $p = |T| = 3$, which contradicts our assumption. (We refer the reader to [2, Theorem II.12.12], which summarizes for us the relevant part of the work of Flores–Foote.) Hence, $S \neq S \cap G_1$. As $S \cap G_1$ is strongly closed in $\mathcal{F}$, it follows thus from (a) that

$$S \cap G_1 = T,$$

that is, $\mathcal{F}_1$ is a fusion system over T. In particular, as $T < S$, the prime p divides G/G_1 and thus the outer automorphism group of the simple group G_1. As $p \geqslant 5$, it follows that G_1 is not alternating or a sporadic simple group. Hence, by the classification of finite simple groups, G_1 is of Lie type. We identify G now in the natural way with a subgroup of $\mathrm{Aut}(G_1)$; in particular, we identify G_1 with $\mathrm{Inn}(G_1)$. Write D for the subgroup of $\mathrm{Aut}(G_1)$ generated by G_1 and the diagonal automorphisms of G_1, and let E be the subgroup of $\mathrm{Aut}(G_1)$ generated by D and the group of field automorphisms of G_1 with respect to some fixed maximal torus and root structure. By [26, Theorem 2.5.12], D and E are normal in $\mathrm{Aut}(G_1)$, $|\mathrm{Aut}(G_1) : E|$ is not divisible by $p \geqslant 5$, E/D is cyclic, and D/G_1 is either cyclic or of order 4. In particular, $G_0 := O^{p'}(G) \leqslant E \cap G$, $G_0/D \cap G_0$ is cyclic, and $D \cap G_0/G_1$ is cyclic or of order 4.

If p divides the order of $G_0/D \cap G_0$, then this group has a unique subgroup of index p whose preimage is then a normal subgroup of G that has index p in G_0. Otherwise, $G_0 = O^{p'}(G) \leqslant D$ and p must divide the order of $G_0/G_1 \leqslant D/G_1$. Thus, in this case G_0/G_1 is cyclic and so there is a unique subgroup of G_0/G_1 of index p. So, in either case we find a normal subgroup N of G that has index p in $G_0 = O^{p'}(G)$. As $N \cap S$ is strongly closed in $\mathcal{F} = \mathcal{F}_S(G)$, it follows now from (a) that $N \cap S = T$ and $p = |S/T|$. Using (iv) we see now that $p = |S/T| = |A/[S, S]| \geqslant q$ and hence $q = p$. In particular, A has index p in S, and similarly, $A_0 := [S, S]$ is an abelian subgroup of T of index p.

Assume now first $n \geqslant 4$ and thus $|A_0| \geqslant p^3$. Our goal is to apply [42, Lemma 1.6] with (G_1, T) in place of (G, S), so we verify now the hypotheses of this lemma. It follows from the last condition in (i) that any abelian subgroup of T is either contained in $A_0 = [S, S] = A \cap T$ or has order at most p^2. Hence, A_0 is the unique abelian subgroup of T of index p. Moreover, by (i), $Z(T) = Z$ has order p and $|[T, T]| = |[A_0, U]| = |A_0/C_{A_0}(U)| = |A_0/Z|$. This implies that $[T, T]$ has index p in A_0 and thus index p^2 in T. As $|A_0| \geqslant p^3 > |P|$, it follows from (iii) that every $\mathcal{F}$-automorphism of A_0 lifts to an $\mathcal{F}$-automorphism of S. In particular, $\mathrm{Aut}_T(A_0)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(A_0)$ and thus also in $\mathrm{Aut}_{\mathcal{F}_1}(A_0)$. Therefore, A_0 is not essential in $\mathcal{F}_1$. Now [42, Lemma 1.6] implies $p = 3$, contradicting our assumption.

We have thus $n = 3$. So, $|A_0| = p^2$ and T is extraspecial of order p^3 and exponent p. In particular, P is normal in T and so (ii) together with $C_S(P) \leqslant P$ implies $T = N_S(P)$. As T is strongly $\mathcal{F}$-closed, every $\mathcal{F}$-conjugate of P is in T. If $Q \in P^{\mathcal{F}}$ is fully normalized, then P is conjugate to Q under some element of $\mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$, and thus by (iii) under some element of $\mathrm{Aut}_{\mathcal{F}}(S)$. So, P itself is fully normalized and a similar argument yields $P^{\mathcal{F}} = P^{\mathrm{Aut}_{\mathcal{F}}(S)}$. In particular, for every $P^* \in P^{\mathcal{F}}$, the fact that T is strongly closed implies that $N_S(P^*) = T$ and so

$$O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P^*)) = \langle \mathrm{Aut}_T(P^*)^{\mathrm{Aut}_{\mathcal{F}}(P)} \rangle \leqslant \mathrm{Aut}_{\mathcal{F}_1}(P^*).$$

In particular, $\mathrm{Aut}_{\mathcal{F}_1}(P^*)$ is isomorphic to a subgroup of $GL_2(p)$ containing $SL_2(p)$ and has thus a strongly p-embedded subgroup. Hence, the elements of $P^{\mathcal{F}}$ are essential and thus centric radical in $\mathcal{F}_1$. As A_0 is normal in S, it follows moreover that A_0 is not conjugate to P. As $|A_0| = |P|$, property (iii) implies thus that every element of $\mathrm{Aut}_{\mathcal{F}}(A_0)$ extends to an $\mathcal{F}$-automorphism of S. In particular, A_0 is not radical in $\mathcal{F}_1$. As T is extraspecial of order p^3 and exponent p, T has exactly $p + 1$ subgroups of order p^2. Moreover, as $T = N_S(P)$ has index p in S, the conjugacy class P^S has p elements. This shows that there are exactly p subgroups of T of order p^2 that are centric and radical in $\mathcal{F}_1$, namely the elements of $P^S = P^{\mathcal{F}}$. However, by the classification of Ruiz and Viruel [48, tables 1.1 and 1.2], there is no saturated fusion system over T with exactly p essential subgroups of order p^2. This contradiction completes the proof of (c) and the lemma. □

Recall that $\mathcal{F}(r, n, k, Q)$ is realizable in the case $n = 2$ and thus has a punctured group. So, the case $n \geqslant 3$, which we consider in the following proposition, is actually the only interesting remaining case.

Proposition 5.11. *Let $3 \leqslant n \leqslant p - 1$ (and thus $p \geqslant 5$), let r be a divisor of $(n, q - 1)$, and set $\mathcal{F} = \mathcal{F}(r, n, k, Q)$. Then $N_{\mathcal{F}}(Z)$ and $N_{\mathcal{F}}(Z)/Z$ are exotic. In particular, $\mathcal{F}$ does not have a punctured group.*

Proof. By Lemma 5.1, $\mathcal{F}$ does not have a punctured group if $N_{\mathcal{F}}(Z)$ is exotic. Moreover, if $N_{\mathcal{F}}(Z)$ is realized by a finite group H, then $N_{\mathcal{F}}(Z)$ is also realized by $N_H(Z)$, and $N_{\mathcal{F}}(Z)/Z$ is realized by $N_H(Z)/Z$. So, it is sufficient to show that $N_{\mathcal{F}}(Z)/Z$ is exotic.

Recall from above that $S = S(n, k)$, $A = A(n, k)$ and $Z := Z(S)$. Set $\mathcal{F}_1 = \mathcal{F}_S(G)$ and $\mathcal{F}_2 = \mathcal{F}_{S_2}(H_Q)$ with $S_2 = N_S(Q)$. Suppose first $r = 1$. Then one argues similarly as in the proof of Proposition 5.9 that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(Q) \rangle$. Namely, $\mathcal{F}$ is generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ by [17, Theorem 3.1], and so as $\mathcal{F}_1$ and $\mathcal{F}_2$ are both constrained with $O_p(\mathcal{F}_1) = A$ and $O_p(\mathcal{F}_2) = Q$, it follows that $\mathcal{F}$ is in turn generated by $\mathrm{Aut}_{\mathcal{F}_1}(A)$, $\mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}_2}(Q)$, and $\mathrm{Aut}_{\mathcal{F}_2}(S_2)$. However, the last automorphism group is redundant, as $N_{H_R}(S_2) = N_G(Q)$ induces fusion in $\mathcal{F}_1$. So, indeed $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(Q) \rangle$ if $r = 1$. This implies $\mathrm{Aut}_{\mathcal{F}_1}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$, $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_{\mathcal{F}_1}(A)$ and (as $N_G(Q) = N_{H_Q}(S_2)$) $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}_2}(Q)$. Moreover, the set of $\mathcal{F}$-essential subgroups comprises A and all $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugates of Q. One easily checks that, for any saturated fusion system $\mathcal{G}$, a normal subsystem of $\mathcal{G}$ of index prime to p has the same essential subgroups as $\mathcal{G}$ itself. Note moreover that, for arbitrary r, $\mathcal{F}$ is a normal subsystem of $\mathcal{F}(1, n, k, Q)$ of index prime to p by Lemma 5.8. Hence, in any case, the $\mathcal{F}$-essential subgroups are A and the $\mathrm{Aut}_{\mathcal{F}_1}(S)$-conjugates of Q. As there is a complement to S in $N_G(S)$ that normalizes U and thus Q, the $\mathrm{Aut}_{\mathcal{F}_1}(S)$-conjugates of Q are precisely the S-conjugates of Q. So, for arbitrary r, we have

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(A), \mathrm{Aut}_{\mathcal{F}}(Q) \rangle. \tag{5.8}$$

Moreover, $\mathrm{Aut}_{\mathcal{F}}(S) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}}(A) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(A)$, and $\mathrm{Aut}_{\mathcal{F}}(Q) \leqslant \mathrm{Aut}_{\mathcal{F}_2}(Q)$. Recall also that $O^{p'}(H_Q) \leqslant F(r, n, k, Q)$ and thus $\mathrm{SL}_2(q) \cong O^{p'}(\mathrm{Aut}_{\mathcal{F}_2}(Q)) \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$.

Note that $\mathrm{Aut}_{\mathcal{F}}(Q)$ normalizes Z and lies thus in $N_{\mathcal{F}}(Z)$. We will show next that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$. By the Alperin–Goldschmidt Fusion Theorem, it suffices to show that every essential subgroup of $N_{\mathcal{F}}(Z)$ is an $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate of Q. So, fix an essential subgroup E of $N_{\mathcal{F}}(Z)$ and assume that $E \notin Q^{\mathrm{Aut}_{\mathcal{F}}(S)}$. As $C_S(E) \leqslant E$, we have $Z < E$. If $E \leqslant A$ then $E = A$. However, $\mathrm{Aut}_{\mathcal{F}}(A) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(A) = \mathrm{Aut}_G(A)$ and one observes that S is normal in $N_G(Z)$. So, $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(A) = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(Z)$ has a normal Sylow p-subgroup, which contradicts E being essential. Assume now that $E \leqslant Q$. Suppose first $Z < Z(E)$. The images of the maximal abelian subgroups of Q are precisely the 1-dimensional k-subspaces of Q/Z. As $\mathrm{Aut}_{\mathcal{F}}(Q)$ fixes Z and acts transitively on the 1-dimensional k-subspaces of Q/Z, we see that $Z(E)$ is conjugate into $Z_2(S) = A \cap Q$ under an element of $\mathrm{Aut}_{\mathcal{F}}(Q)$. So, replacing E by a suitable $\mathrm{Aut}_{\mathcal{F}}(Q)$-conjugate, we may assume $Z(E) \leqslant A \cap Q$. As $Z < Z(E)$, it follows then from (5.5) that $E \leqslant A$. As $C_S(E) \leqslant E$ and $A \nleqslant Q$, this is a contradiction. So, we have $Z = Z(E)$. As $[E, Q] \leqslant [Q, Q] \leqslant Z$, it follows $\mathrm{Aut}_Q(E) \leqslant C := C_{\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)}(E/Z(E)) \cap C_{\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)}(Z(E))$. However, C is a normal p-subgroup of $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$. Thus, as E is radical in $N_{\mathcal{F}}(Z)$, we have $\mathrm{Aut}_Q(E) \leqslant C \leqslant \mathrm{Inn}(E)$. As $C_S(E) \leqslant E$, it follows $E = Q$ contradicting the choice of E. So, we have shown that E lies neither in A nor in Q. As the choice of E was arbitrary, this means that E is not $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate into A or Q. So, by (5.8), every $\mathcal{F}$-automorphism of E extends to an $\mathcal{F}$-automorphism of S. This implies that $\mathrm{Aut}_S(E)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(E)$ and thus in $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$. Again, this contradicts E being essential. So, we have shown that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Set $\overline{S} = S/Z$ and $\overline{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$. We will check that the hypotheses of Lemma 5.10 are fulfilled with $\overline{\mathcal{F}}$, $\overline{S}$, $\overline{A}$, $\overline{U}$ and $\overline{Q}$ in place of $\mathcal{F}$, S, A, U and P. Part (c) of this lemma will then imply that $N_{\mathcal{F}}(Z)/Z$ is exotic as required. Notice that $|\overline{U}| = |U| = q$, $|A| = q^{n+1}$ and $|\overline{A}| = q^n$. As $Q = Z_2(S)U$, we have $\overline{Q} = Z(\overline{S})\overline{U}$. By [17, Lemma 4.2(i),(iii)], hypothesis (i) of Lemma 5.10 holds. Recall that $O^{p'}(\mathrm{Aut}_{H_Q}(Q)) \cong \mathrm{SL}_2(q)$ lies in $N_{\mathcal{F}}(Z)$. In particular, hypothesis (ii) in Lemma 5.10 holds with $\overline{\mathcal{F}}$ and $\overline{Q}$ in place of $\mathcal{F}$ and P. As we have shown above that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$, it follows that $\overline{\mathcal{F}}$ fulfills hypothesis (iii) of Lemma 5.10. Observe that there exists a complement K of S in $N_G(S)$ that normalizes U. Then $\mathrm{Aut}_{\mathcal{F}}(S) \leqslant \mathrm{Aut}_G(S) = \mathrm{Inn}(S)\mathrm{Aut}_K(S)$. Thus, $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)(\mathrm{Aut}_K(S) \cap \mathrm{Aut}_{\mathcal{F}}(S))$ and $\mathrm{Aut}_K(S) \cap \mathrm{Aut}_{\mathcal{F}}(S)$ is a complement to $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ that normalizes U. This implies that hypothesis (v) of Lemma 5.10 holds for $\overline{\mathcal{F}}$.

It remains to show hypothesis (iv) of Lemma 5.10 for $\overline{\mathcal{F}}$. Notice that $[\overline{S}, \overline{S}] = [\overline{A}, \overline{U}]$ is a proper $\mathbb{F}_q$-subspace of $\overline{A}$ and has thus index at least q in $\overline{A}$. So, it remains to show the first condition in (iv). Equivalently, we need to show that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(Z)}(S)$ acts irreducibly on $A/[S, S]$. For the proof, we use the representations Clelland and Parker give for G and H_Q, and the way they construct the free amalgamated product; see [17, pp. 293, 296]. Let ξ be a generator of $k^\times$. We have

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi^{-1} & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O^{p'}(H_Q) \leqslant N_{F(r,n,k,Q)}(Z).$$

In the free amalgamated product $F(1,n,k,Q)$, the element $g \in H_Q$ is identified with

$$\left(1, \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, 0_{A(n,k)}\right) \in N_G(Q),$$

and this element can be seen to act by scalar multiplication with ξ^n on $y^n \in A = A(n,k)$ and thus on $A/[S,S]$. As $n \leqslant p-1$ and ξ has order $q-1$, the action of g on $A(n,k)/[S,S]$ is thus irreducible. Hence, the action of $\mathrm{Aut}_{\mathcal{F}}(S)$ on $A/[S,S]$ is irreducible. This shows that the hypothesis of Lemma 5.10 is fulfilled with $\overline{\mathcal{F}}$ in place of $\mathcal{F}$, and thus $\overline{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$ is exotic as required. □

5.4 | The Parker–Stroth systems

Let $p \geqslant 5$ be a prime and $m = p-4$. Let $A = A(m, \mathbf{F}_p)$ and D be as in Subsection 5.3. The Parker–Stroth systems are fusion systems over the Sylow subgroup S of a semidirect product $P = Q \rtimes D$, where Q is extraspecial of order $p^{1+(p-3)}$ and of exponent p, and where $Q/Z(Q) \cong A$ as an $\mathbf{F}_pD$-module. Then $Z := Z(S) = Z(Q)$ is of order p, while $Z_2(S) \leqslant Q$ is elementary abelian of order p^2.

It turns out that $C_D(Q)$ has order $p-1$ (cf. [46, Lemma 2.3(i)] where our D is called L) and so S can be identified with its image in $P_1 := P/C_D(Q)$. Parker and Stroth find then a subgroup W of S such that W is elementary abelian of order p^2 and

$$W \nleqslant Q. \tag{5.9}$$

We refer to [46, p. 317] for more details on the embedding of W in S, where our W is denoted W_0. Choose a finite group K with $K \cong p^2 : SL_2(p)$, and let C be the normalizer in K of a Sylow p-subgroup of K (cf. [46, p. 315]). It turns out that $N_{P_1}(W)$ can be identified with C in such a way that W is identified with $O_p(K)$ (cf. [46, p. 319]). The exotic Parker–Stroth system $\mathcal{F}$ at the prime p is then the fusion system over S of the free amalgamated product $P_1 *_C K$, where we identify $S \in \mathrm{Syl}_p(P)$ with its image in P_1 as before. Identifying further $N_{P_1}(W)$ with C (and thus $N_S(W)$ with a Sylow p-subgroup of K), it is shown in [46, Lemma 3.1] that $\mathcal{F}$ is generated by $\mathcal{F}_S(P_1) = \mathcal{F}_S(P)$ and $\mathcal{F}_{N_S(W)}(K)$. Here $\mathrm{Aut}_K(N_S(W)) = \mathrm{Aut}_C(N_S(W)) = \mathrm{Aut}_{N_P(W)}(N_S(W))$ because of the identification in the free amalgamated product. Note that $\mathcal{F}_S(P)$ is generated by $\mathrm{Aut}_P(Q)$ and $\mathrm{Aut}_P(S)$, and that $\mathcal{F}_{N_S(W)}(K)$ is generated by $\mathrm{Aut}_K(W) \cong SL_2(p)$ and $\mathrm{Aut}_K(N_S(W))$. Hence, we obtain that

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q), \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(W) \rangle, \tag{5.10}$$

where $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_P(Q)$, $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_P(S)$, and $\mathrm{Aut}_{\mathcal{F}}(W) \cong SL_2(p)$. In particular, as $Q \trianglelefteq P$, we have that

$$Q \text{ is } \mathrm{Aut}_{\mathcal{F}}(S)\text{-invariant.} \tag{5.11}$$

Proposition 5.12. *Each Parker–Stroth system is of characteristic p-type, and so has a punctured group in the form of its subcentric linking system.*

Proof. We use the notation from above. Let Y be a subgroup of order p in S that is fully $\mathcal{F}$-centralized. We need to show that $C_{\mathcal{F}}(Y)$ is constrained. For that purpose fix a subgroup $E \in C_{\mathcal{F}}(Y)^{cr}$.

Case: $Y \nleq Q$. Then $C_{Q/Z}(Y)$ is of order p, so $C_Q(Y)$ is elementary abelian (of order p^2). Hence, $C_S(Y) = C_Q(Y)Y$ is abelian in this case, and so $C_{\mathcal{F}}(Y)$ is constrained.

Case: $Y \leqslant Q$ **but** $Y \nleq Z$. Then $C_S(Y)$ is abelian when $p = 5$ as then $m = 1$ and YZ/Z is its own orthogonal complement with respect to the symplectic form on Q/Z. We may therefore assume $p \geqslant 7$.

We consider now the possibilities for E.

Subcase: $E \nleqslant Q$. As $E \cap Q$ contains $YZ(S) \cong C_p \times C_p$, we have then $|E| \geqslant p^3$ and so E is not $\mathcal{F}$-conjugate into W. Thus, it follows in this case that from (5.10) and (5.11) that E is also not $\mathcal{F}$-conjugate into Q. Hence, again by (5.10), every $\mathcal{F}$-automorphism of E extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$. In particular, every element of $\mathrm{Aut}_{C_{\mathcal{F}}(Y)}(E)$ extends to an element of $\mathrm{Aut}_{C_{\mathcal{F}}(Y)}(C_S(Y))$. As $E \in C_{\mathcal{F}}(Y)^{cr}$, this implies $E = C_S(Y)$. It follows from (5.11) that $C_Q(Y)$ is invariant under $\mathrm{Aut}_{C_{\mathcal{F}}(Y)}(E) = \mathrm{Aut}_{C_{\mathcal{F}}(Y)}(C_S(Y))$.

Subcase: $E \leqslant Q$. As $W \cong C_p \times C_p \cong ZY \leqslant E$, it follows now from (5.9) and (5.11) that E is not $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate into W. So, by (5.10), every morphism in a decomposition of $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}}(Y)}(E)$ lies in $\mathrm{Aut}_{\mathcal{F}}(Q)$ or $\mathrm{Aut}_{\mathcal{F}}(S)$. Hence, using again (5.11), we conclude that α extends to Q and thus to an element of $\mathrm{Aut}_{C_{\mathcal{F}}(Y)}(C_Q(Y))$. So, $C_Q(Y) \leqslant E$ because α was chosen arbitrarily and $E \in C_{\mathcal{F}}(Y)^{cr}$. Our assumption yields $E \leqslant Q \cap C_S(Y) = C_Q(Y)$ and so $E = C_Q(Y)$.

In the case $Z \neq Y \leqslant Q$, we have thus shown that $C_S(Y)$ and $C_Q(Y)$ are the only candidates for subgroups that are centric and radical in $C_{\mathcal{F}}(Y)$, and that $C_Q(Y)$ is $\mathrm{Aut}_{C_{\mathcal{F}}(Y)}(C_S(Y))$-invariant. Thus, by Alperin's Fusion Theorem, $C_Q(Y)$ is normal in $C_{\mathcal{F}}(Y)$. As $p \geqslant 7$ and so $m \geqslant 3$, it follows from the construction of Q and P that $C_Q(Y) \neq Z_2(S)$ and so $C_Q(Y)$ is self-centralizing in $C_S(Y)$ (see [46, Lemma 2.2(i) and Lemma 2.3(iii)]). Therefore, $C_{\mathcal{F}}(Y)$ is constrained.

Case: $Y = Z$. Assume first that E is $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate into W. Note that E has order at least p^2. Hence, E is in fact $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate to $W \cong C_p \times C_p$. We may therefore assume that $E = W$. By (5.9) and (5.11), W is not $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate into Q and so, by (5.10), every $\mathcal{F}$-conjugate of W is $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate to W. In particular, W is fully $\mathcal{F}$-normalized. As $\mathrm{Aut}_{\mathcal{F}}(W) \cong SL_2(p)$, we have $\mathrm{Aut}_{C_{\mathcal{F}}(Z)}(W) = C_{\mathrm{Aut}_{\mathcal{F}}(W)}(Z) = N_{\mathrm{Aut}_{\mathcal{F}}(W)}(\mathrm{Aut}_S(W))$ and hence every element of $\mathrm{Aut}_{C_{\mathcal{F}}(Z)}(W)$ extends by the saturation axioms to an element of $\mathrm{Aut}_{\mathcal{F}}(N_S(W))$. As $W \nleq Q$, it follows from (5.10) and (5.11) that every element of $\mathrm{Aut}_{\mathcal{F}}(N_S(W))$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$. This contradicts the assumption that $W = E \in C_{\mathcal{F}}(Y)^{cr} = C_{\mathcal{F}}(Z)^{cr}$. So, E is not $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate into W. Using again (5.10) and (5.11), one sees now that $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}}(Y)}(E)$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$ if $E \nleqslant Q$, and to an element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ if $E \leqslant Q$. As $E \in C_{\mathcal{F}}(Y)^{cr}$, it follows that $E \in \{Q, S\}$. As E was arbitrary, it follows from (5.11) that $Q \trianglelefteq C_{\mathcal{F}}(Y) = C_{\mathcal{F}}(Z)$. As $C_S(Q) \leqslant Q$, it follows that $C_{\mathcal{F}}(Y)$ is constrained.

Thus, in all cases, $C_{\mathcal{F}}(Y)$ is constrained. We conclude that the Parker–Stroth systems are of characteristic p-type and therefore have a punctured group. □

6 | PUNCTURED GROUPS OVER p_+^{1+2}

The main purpose of this section is to illustrate that there can be several punctured groups associated to the same fusion system, and that the nerves of such punctured groups (regarded as transporter systems) might not be homotopy equivalent to the nerve of the centric linking system. Indeed, working in the language of localities, we will see that there can be several punctured groups extending the centric linking locality. This is the case even though we consider examples of fusion systems of characteristic p-type, and so in each case, the subcentric linking locality exists as the "canonical" punctured group extending the centric linking locality. On the other hand, we will see that in many cases, the subcentric linking locality is indeed the only p'-reduced punctured group over a given fusion system. Thus, "interesting" punctured groups seem still somewhat rare.

More concretely, we will look at fusion systems over a p-group S that is isomorphic to p_+^{1+2}. Here p_+^{1+2} denotes the extraspecial group of order p^3 and exponent p if p is an odd prime, and (using a somewhat nonstandard notation) we write p_+^{1+2} for the dihedral group of order 8 if $p = 2$. Note that every subgroup of order at least p^2 is self-centralizing in S and thus centric in every fusion system over S. Thus, if $\mathcal{F}$ is a saturated fusion system over S with centric linking locality $(\mathcal{L}, \Delta, S)$, we just need to add the cyclic groups of order p as objects to obtain a punctured group. We will again use Chermak's iterative procedure, which gives a way of expanding a locality by adding one $\mathcal{F}$-conjugacy class of new objects at the time. If all subgroups of order p are $\mathcal{F}$-conjugate, we thus only need to complete one step to obtain a punctured group. Conversely, we will see in this situation that a punctured group extending the centric linking locality is uniquely determined up to a rigid isomorphism by the normalizer of an element of order p. Therefore, we will restrict attention to this particular case. More precisely, we will assume the following hypothesis.

Hypothesis 6.1. Assume that p is a prime and S is a p-group such that $S \cong p_+^{1+2}$ (meaning here $S \cong D_8$ if $p = 2$). Set $Z := Z(S)$. Let $\mathcal{F}$ be a saturated fusion system over S such that all subgroups of S of order p are $\mathcal{F}$-conjugate.

It turns out that there is a fusion system $\mathcal{F}$ fulfilling Hypothesis 6.1 if and only if $p \in \{2, 3, 5, 7\}$; for odd p this can be seen from the classification theorem by Ruiz and Viruel [48]. More precisely, we obtain the following lemma.

Lemma 6.2. *Hypothesis 6.1 holds if and only if we are in one of the following cases.*

- $p = 2$ *and* $\mathcal{F}$ *is realized by* A_6.
- $p = 3$ *and* $\mathcal{F}$ *is realized by* ${}^2F_4(2)'$ *or* J_4.
- $p = 5$ *and* $\mathcal{F}$ *is realized by* Th.
- $p = 7$ *and* $\mathcal{F}$ *is one of the three exotic fusion systems discovered by Ruiz and Viruel [48].*

Proof. Suppose first that $p = 2$ so that $S \cong D_8$. Then there are precisely two elementary abelian subgroups of order 4, which are the only candidates for $\mathcal{F}$-essential subgroups. Indeed, all involutions are $\mathcal{F}$-conjugate if and only if both of them are essential, in which case $\mathcal{F}$ is the 2-fusion system of A_6.

Suppose now that p is odd. In this case, the claim follows essentially from the list provided by Ruiz–Viruel [48, table 1.1]. To see this it should be noted that all elements of order p are $\mathcal{F}$-conjugate if and only if all of the $p+1$ elementary abelian subgroups of S of order p^2 are in the set $\mathcal{F}^{ec}$-rad of $\mathcal{F}$-centric $\mathcal{F}$-radical elementary abelian subgroups. □

Assume Hypothesis 6.1. One easily observes that the 2-fusion system of A_6 is of characteristic 2-type. Therefore, it follows from Lemma 5.4 that the fusion system $\mathcal{F}$ is always of characteristic p-type and thus the associated subcentric linking locality is a punctured group. As discussed in Remark 2.25, this leads to a host of examples for punctured groups $\mathcal{L}^+$ over $\mathcal{F}$ that are modulo a partial normal p'-subgroup isomorphic to a subcentric linking locality over $\mathcal{F}$. One can ask whether there are more examples. Indeed, the next theorem tells us that this is the case if and only if $p=3$. For $p\in\{5,7\}$ our two theorems below depend on the classification of finite simple groups.

Theorem 6.3. *Under Hypothesis 6.1, there exists a punctured group $(\mathcal{L}^+,\Delta^+,S)$ over $\mathcal{F}$ such that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is not a subcentric linking locality if and only if $p=3$.*

It seems that for $p=3$, the number of $3'$-reduced punctured groups over $\mathcal{F}$ is probably also severely limited. However, as we do not want to get into complicated and lengthy combinatorial arguments, we will not attempt to classify them all. Instead, we will prove the following theorem, which leads already to the construction of interesting examples.

Theorem 6.4. *Assume Hypothesis 6.1. Suppose that $\mathcal{L}^+$ is a punctured group over $\mathcal{F}$ such that $\mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking locality over $\mathcal{F}$. Then $\mathcal{L}^+$ is p'-reduced. Moreover, up to a rigid isomorphism, $\mathcal{L}^+$ is uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$, and one of the following holds:*

(a) *$\mathcal{L}^+$ is the subcentric linking system for $\mathcal{F}$; or*
(b) *$p=3$, $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $N_{\mathcal{L}^+}(Z)\cong 3S_6$; or*
(c) *$p=3$, $\mathcal{F}$ is the 3-fusion system of Ru and of J_4, and $N_{\mathcal{L}^+}(Z)\cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.*

Conversely, each of the cases listed in (a)-(c) occurs in an example for $\mathcal{L}^+$.

Here the notation $A\#B$ is as in [26, p. 261], namely it describes a group X with normal subgroup $N\cong A$ and quotient $X/N\cong B$, and such that $N\not\leq Z(X)$ and X does not split over N. In the case $3\#\mathrm{Aut}(A_6)$, this is the unique extension of the quasisimple group $3A_6$ by $\mathrm{Out}(3A_6)\cong C_2\times C_2$.

Before beginning the proof, we make some remarks. The 3-fusion systems of Ru and J_4 are isomorphic. For $G=Ru$ and S a Sylow 3-subgroup of G, one has $N_G(Z(S))\cong 3\#\mathrm{Aut}(A_6)$ [26, table 5.3r], so the punctured group $\mathcal{L}^+$ in Theorem 6.4(c) is the punctured group of Ru at the prime 3 (for example, as our theorem tells us that $\mathcal{L}^+$ is uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$). Using the classification of finite simple groups, this can be shown to be the only punctured group in (b) or (c) that is isomorphic to the punctured group of a finite group. For example, when $G=J_4$, one has $N_G(Z(S))\cong(6M_{22})\cdot 2$. The 3-fusion system of $6M_{22}$ is constrained and isomorphic to that of $3M_{21}=3L_3(4)$ and also that of $3M_{10}=3(A_6.2)$, where the extension $A_6.2$ is nonsplit (see [26, table 5.3c]). If we are in the situation of Theorem 6.4(c) and $N_{\mathcal{L}^+}(Z(S))$ is an extension of $3L_3(4)$ by a field automorphism, then $N_{\mathcal{L}^+}(Z(S))$ is a section of $N_G(Z(S))$. Also, for $G={}^2F_4(2)'$, the normalizer in G of a subgroup of order 3 is solvable [37, Proposition 1.2].

By Lemma 6.2, for $p \in \{2, 5, 7\}$, there are also saturated fusion systems over S, in which all subgroups of order p are conjugate. Moreover, for $p = 5$, the only such fusion system is the fusion system of the Thompson sporadic group. It should be noted here that the Thompson group is of local characteristic 5, and thus its punctured group is just the subcentric linking locality. The three exotic fusion systems at the prime 7, which were discovered by Ruiz and Viruel, are of characteristic 7-type. As our theorem shows, for each of these fusion systems, the subcentric linking locality is the only associated punctured group extending the centric linking locality.

We will now start to prove Theorems 6.3 and 6.4 in a series of lemmas. If Hypothesis 6.1 holds and $\mathcal{L}^+$ is a punctured group over $\mathcal{F}$, then $M_0 := N_{\mathcal{L}^+}(Z)$ is a finite group containing S as a Sylow p-subgroup. Moreover, Z is normal in M_0. These properties are preserved if we replace M_0 by $M := M_0/O_{p'}(M_0)$ and identify S with its image in M. Moreover, we have $O_{p'}(M) = 1$. We analyze the structure of such a finite group M in the following lemma. Most of our arguments are elementary. However, for $p \geqslant 5$, we need the classification of finite simple groups in the form of knowledge about the Schur multipliers of finite simple groups to show in case (b) that $p = 3$.

Lemma 6.5. *Let M be a finite group with a Sylow p-subgroup $S \cong p_+^{1+2}$. Assume that $Z := Z(S)$ is normal in M and $O_{p'}(M) = 1$. Then one of the following holds:*

(a) *$S \trianglelefteq M$ and $C_M(S) \leqslant S$, or*
(b) *$p = 3$, $S \leqslant F^*(M)$, and $F^*(M)$ is quasisimple with $Z(F^*(M)) = Z$.*

Proof. Assume first that $S \trianglelefteq M$. In this case we have $[S, E(M)] = 1$ and thus $S \cap E(M) \leqslant Z(E(M))$. So, by [3, 33.12], $E(M)$ is a p'-group. As we assume $O_{p'}(M) = 1$, this implies $E(M) = 1$ and $F^*(M) = O_p(M) = S$. Therefore, (a) holds.

Thus, for the remainder of the proof, we will assume that S is not normal in M, and we will show (b). First we prove

$$E(M) \neq 1. \tag{6.1}$$

Suppose $E(M) = 1$ and set $P = O_p(M)$. Note that $Z \leqslant P$. As $O_{p'}(M) = 1$, we have $P = F^*(M)$, so $C_M(P) \leqslant P$ and $P \neq Z$. As we assume that S is not normal in M, we have moreover $P \neq S$. If P is elementary abelian of order p^2, then M/P acts on P and normalizes Z, thus it embeds into a Borel subgroup of $GL_2(p)$. If $p = 2$ and P is cyclic of order 4 then $\mathrm{Aut}(P)$ is a 2-group. So, S is in any case normal in M and this contradicts our assumption. Thus, (6.1) holds.

We can now show that

$$p \text{ divides } |Z(K)| \text{ for some component } K \text{ of } M. \tag{6.2}$$

First note that p divides $|K|$ for each component K of M. For otherwise, if p does not divide $|K|$ for some K, then $1 < K \leqslant O_{p'}(E(M)) \leqslant O_{p'}(M) = 1$, a contradiction.

Supposing (6.2) is false, $Z(E(M))$ is a p'-group and thus by assumption trivial. Hence, $E(M)$ is a direct product of simple groups. As Z is normal in M, $[Z, E(M)] = 1$ and thus $Z \cap E(M) = 1$. As the p-rank of M is two and p divides $|K|$ for each component K, there can be at most one component, call it J, which is then simple and normal in M. As p divides $|J|$ and J is normal in M, it follows that $S \cap J \neq 1$. But then $[S \cap J, S] \leqslant J \cap Z = 1$ and so $S \cap J = Z$ is normal in J, a contradiction. Thus, (6.2) holds.

Next we will show that

$$K = F^*(M) \text{ is quasisimple with } S \leqslant K \text{ and } Z(K) = Z. \tag{6.3}$$

To prove this fix a component K of M such that p divides $|Z(K)|$. Then p divides $|K|/|Z(K)|$ by [3, 33.12]. If S is not a subgroup of K, then $K/Z(K)$ is a perfect group with cyclic Sylow p-subgroups, so $Z(K)$ is a p'-group by [3, 33.14], a contradiction. Therefore, $S \leqslant K$. If there were a component L of M different from K, then we would have $[S \cap L, L] \leqslant [K, L] = 1$, that is, L would have a central Sylow p-subgroup. However, we have seen above that p divides the order of each component, so we would get a contradiction to [3, 33.12]. Hence, $K = F^*(G)$ is the unique component of M. Note that $O_{p'}(Z(K)) \leqslant O_{p'}(M) = 1$ and thus $Z(K)$ is a p-group. As $[Z, K] = 1$, this implies $Z = Z(K)$. Thus, (6.3) holds.

To prove (b), it remains to show that $p = 3$. Assume first that $p = 2$ so that $S \cong D_8$. Then $\mathrm{Aut}(S)$ is a 2-group and thus $N_K(S) = SC_K(S)$. Hence, with $\bar{K} = K/Z$, we have $N_{\bar{K}}(\bar{S}) = C_{\bar{K}}(\bar{S})$. Therefore, $\bar{K}$ has a normal p-complement by Burnside's theorem (see, e.g., [34, 7.2.1]), a contradiction that establishes $p \neq 2$.

For $p \geqslant 5$, we appeal to the account of the Schur multipliers of the finite simple groups given in [26, chapter 6] to conclude that, by the classification of the finite simple groups, $K/Z(K)$ must be isomorphic to $L_m(q)$ with p dividing $(m, q-1)$, or to $U_m(q)$ with p dividing $(m, q+1)$. But each group of this form has Sylow p-subgroups of order at least p^4, a contradiction. □

Lemma 6.6. *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$. Then the following hold.*

(a) *If $P \in \Delta^+$ with $|P| \geqslant p^2$, then $N_{\mathcal{L}^+}(P)$ is Sylow p-constrained and thus p-constrained.*
(b) *If $p \neq 3$ then, upon identifying S with its image in $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$, the triple $(\mathcal{L}^+/O_{p'}(\mathcal{L}^+), \Delta^+, S)$ is a subcentric linking locality over $\mathcal{F}$.*

Proof. If $P \in \Delta^+$ with $|P| \geqslant p^2$, then $S = N_S(P)$ is a Sylow p-subgroup of $N_{\mathcal{L}^+}(P)$. As P is normal in $N_{\mathcal{L}^+}(P)$ and $C_S(P) \leqslant P$, it follows that $N_{\mathcal{L}^+}(P)$ is Sylow p-constrained. Thus, (a) holds by Lemma 2.37.

Assume now $p \neq 3$. As all subgroups of order p are by assumption $\mathcal{F}$-conjugate, we have by Lemma 2.7(b) and Lemma 2.9(a) that $N_{\mathcal{L}^+}(P) \cong M := N_{\mathcal{L}^+}(Z)$ for every $P \in \Delta^+$ with $|P| = p$. Moreover, by Lemma 6.5, $M/O_{p'}(M)$ has a normal Sylow p-subgroup and is thus in particular Sylow p-constrained. Hence, using (a) and Lemma 2.37, we can conclude that $N_{\mathcal{L}^+}(P)$ is p-constrained for every $P \in \Delta^+$. Therefore, by Proposition 2.36, the triple $(\mathcal{L}^+/O_{p'}(\mathcal{L}^+), \Delta^+, S)$ is a locality over $\mathcal{F}$ of objective characteristic p. As $\Delta^+ = \mathcal{F}^s$ by Lemma 5.4, part (b) follows. □

Note that Lemma 6.6 proves one direction of Theorem 6.3, whereas the other direction would follow from Lemma 6.2 and Theorem 6.4. Therefore, we will focus now on the proof of Theorem 6.4 and thus consider punctured groups that restrict to the centric linking system. If $\mathcal{L}^+$ is such a punctured group, then we will apply Lemma 6.5 to $N_{\mathcal{L}^+}(Z)$. To do this, we need the following two lemmas.

Lemma 6.7. *Let M be a finite group with a Sylow p-subgroup $S \cong p_+^{1+2}$. Assume that $Z := Z(S)$ is normal in M and $C_M(V) \leqslant V$ for every subgroup V of S of order at least p^2. Then $O_{p'}(M) = 1$.*

Proof. Set $U = O_{p'}(M)$. As Z is normal in M, it centralizes U. So, $\bar{S} = S/Z$ acts on U. Let $x \in S - Z(S)$. Then setting $V = \langle x, z\rangle$, the centralizer $C_M(V)$ contains the p'-group $C_U(\bar{x})$. So, our hypothesis implies $C_U(\bar{x}) = 1$. Hence, by [34, 8.3.4(b)], $U = \langle C_U(\bar{x}) : \bar{x} \in \bar{S}^{\#}\rangle = 1$. □

Lemma 6.8. *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$ such that $\mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking locality over $\mathcal{F}$. If we set $M := N_{\mathcal{L}^+}(Z)$ the following conditions hold:*

(a) *S is a Sylow p-subgroup of M and Z is normal in M,*
(b) *$\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, and*
(c) *$C_M(V) \leqslant V$ for each subgroup V of S of order p^2.*

Proof. Property (a) is clearly true. Moreover, by Lemmas 2.9(b) and 5.4, we have $\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, so (b) holds. Set $\Delta = \mathcal{F}^c$. By assumption $\mathcal{L} := \mathcal{L}^+|_{\Delta}$ is a centric linking locality. So, by [28, Proposition 1(d)], we have $C_{\mathcal{L}}(V) \subseteq V$ for every $V \in \Delta$. Hence, for every subgroup $V \in \Delta$, we have $C_M(V) \subseteq C_{\mathcal{L}^+}(V) = C_{\mathcal{L}}(V) \subseteq V$, where the equality follows from the definition of $\mathcal{L} = \mathcal{L}^+|_{\Delta}$. As every subgroup of S of order at least p^2 contains its centralizer in S, each such subgroup is $\mathcal{F}$-centric. Therefore, (c) holds. □

Lemma 6.9. *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$ such that $\mathcal{L} := \mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking locality over $\mathcal{F}$. Set $M := N_{\mathcal{L}^+}(Z)$. Then one of the following conditions holds:*

(a) *$S \trianglelefteq M$, the group M is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, and $(\mathcal{L}^+, \Delta^+, S)$ is a subcentric linking locality over $\mathcal{F}$;*
(b) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $M \cong 3S_6$; or*
(c) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of Ru and of J_4, and $M \cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.*

Moreover, in either of the cases (b) and (c), $N_{\mathrm{Out}(S)}(\mathrm{Out}_{\mathcal{F}}(S))$ is a Sylow 2-subgroup of $\mathrm{Out}(S) \cong GL_2(3)$, and every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an automorphism of M.

Proof. Set $\Delta = \mathcal{F}^c$. By Lemmas 6.7 and 6.8, we have $O_{p'}(M) = 1$, $\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and $C_M(S) \leqslant S$. In particular, if $S \trianglelefteq M$, then M is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. For any $P \in \Delta$, the group $N_{\mathcal{L}^+}(P) = N_{\mathcal{L}}(P)$ is of characteristic p. As $\Delta^+ = \Delta \cup Z^{\mathcal{F}}$, if $S \trianglelefteq M$, the punctured group $(\mathcal{L}^+, \Delta^+, S)$ is of objective characteristic p and thus (a) holds.

So, assume now that S is not normal in M. By Lemma 6.5, we have then $p = 3$, $K := F^*(M)$ is quasisimple, $S \leqslant K$ and $Z = Z(K)$. Set $\overline{M} := M/Z$ and $G := \overline{K}$. Let $1 \neq \overline{x} \in \overline{S}$. Then the preimage V of $\langle \overline{x}\rangle$ in S has order at least 3^2. Thus, by Lemma 6.8(c), we have $C_M(V) \leqslant V$. A $3'$-element in the preimage of $C_G(\overline{x}) = C_G(\overline{V})$ in K acts trivially on $\overline{V}$ and $Z = Z(K)$. Thus, it is contained in $C_M(V) \leqslant V$ and therefore trivial. Hence, we have

$$C_G(\overline{x}) = \overline{S} \text{ for every } 1 \neq \overline{x} \in \overline{S}. \tag{6.4}$$

Notice also that G is a simple group with Sylow 3-subgroup $\overline{S}$, which is elementary abelian of order 3^2. Moreover, $\mathrm{Aut}_G(\overline{S})$ is contained in a Sylow 2-subgroup of $\mathrm{Aut}(\overline{S}) \cong GL_2(3)$, and such a Sylow 2-group is semidihedral of order 16. In particular, if $\mathrm{Aut}_G(\overline{S})$ has 2-rank at least 2, then $\mathrm{Aut}_G(\overline{S})$ contains a conjugate of every involution in $\mathrm{Aut}(\overline{S})$, which is impossible because of (6.4). Hence, $\mathrm{Aut}_G(\overline{S})$ has 2-rank one, and is thus either cyclic of order at most 8 or quaternion of order 8 (and certainly nontrivial by [34, 7.2.1]). By a result of Smith and Tyrer [51], $\mathrm{Aut}_G(\overline{S})$ is not cyclic of order 2. Using (6.4), it follows from [29, Theorem 13.3] that $G \cong L_2(9) \cong A_6$ if $\mathrm{Aut}_G(\overline{S})$ is cyclic of order 4, and from a result of Fletcher [23, Lemma 1] that $G \cong L_3(4)$ (and thus $\mathrm{Aut}_G(\overline{S})$ is quaternion) if $\mathrm{Aut}_G(\overline{S})$ is of order 8.

It follows from Lemma 6.8(b) that $\mathrm{Aut}_M(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. As $C_M(S) = Z$ and $C_G(\overline{S}) = \overline{S}$ by (6.4), we have $\mathrm{Aut}_G(\overline{S}) \cong N_G(\overline{S})/C_G(\overline{S}) = \overline{N_K(S)}/\overline{S} \cong \mathrm{Aut}_K(S)/\,\mathrm{Inn}(S) = \mathrm{Out}_K(S)$. Hence,

$$\mathrm{Out}_G(\overline{S}) \cong \mathrm{Aut}_G(\overline{S}) \cong \mathrm{Out}_K(S) \leqslant \mathrm{Out}_M(S) = \mathrm{Out}_{\mathcal{F}}(S).$$

As $p = 3$, it follows from Lemma 6.2 that $\mathcal{F}$ is the 3-fusion system of the Tits group or the 3-fusion system of J_4.

Consider first the case that $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)$, which has $\mathrm{Out}_{\mathcal{F}}(S) \cong D_8$. Then $\mathrm{Out}_G(\overline{S})$ cannot be quaternion, that is, we have $\mathrm{Out}_G(\overline{S}) \cong C_4$ and $G = A_6$. So, conclusion (b) of the lemma holds, as S_6 is the only twofold extension of A_6 whose Sylow 3-normalizer has dihedral Sylow 2-subgroups. By [48, Lemma 3.1], we have $\mathrm{Out}(S) \cong GL_2(3)$. It follows from the structure of this group that $N_{\mathrm{Out}(S)}(\mathrm{Out}_{\mathcal{F}}(S)) \cong SD_{16}$ is a Sylow 2-subgroup of $\mathrm{Out}(S)$. As $M \cong 3S_6$ has an outer automorphism group of order 2, it follows that every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an automorphism of M.

Assume now that $\mathcal{F}$ is the 3-fusion system of J_4, so that $\mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$. An extension of $3A_6$ with these data must be $3\#\mathrm{Aut}(A_6)$. Suppose now $\mathrm{Aut}_G(\overline{S}) \cong Q_8$ and $G \cong L_3(4)$. Then $\overline{M}$ must be a twofold extension of $L_3(4)$. However, a graph-field automorphism centralizes a Sylow 3-subgroup, and so M must be an extension of $L_3(4)$ by a field or a graph automorphism. Hence, (c) holds in this case. If (c) holds, then $\mathrm{Out}_M(S) = \mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$ is always a self-normalizing Sylow 2-subgroup in $\mathrm{Out}(S) \cong GL_2(3)$. In particular, every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an inner automorphism of M. This proves the assertion. $\square$

Note that the previous lemma shows basically that, for any punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ that restricts to a centric linking locality, one of the conclusions (a)–(c) in Theorem 6.4 holds. To give a complete proof of Theorem 6.4, we will also need to show that each of these cases actually occurs in an example. To construct the examples, we will need the following two lemmas. The reader might want to recall the definition of $\mathcal{L}_\Delta(M)$ from Example 2.6

Lemma 6.10. *Let M be a finite group isomorphic to $3S_6$ or $3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism. Let S be a Sylow 3-subgroup of M. Then $S \cong 3^{1+2}_+$ and, writing Δ for all subgroups of S of order 3^2, we have $\mathcal{L}_\Delta(M) = N_M(S)$. Moreover, $\mathcal{F}_S(M) = \mathcal{F}_S(N_M(S))$.*

Proof. It is well-known that M has in all cases a Sylow 3-subgroup isomorphic to 3^{1+2}_+. By definition of $\mathcal{L}_\Delta(M)$, clearly $N_M(S) \subseteq \mathcal{L}_\Delta(M)$. Moreover, if $g \in \mathcal{L}_\Delta(M)$, then there exists $P \in \Delta$ such that $P^g \leqslant S$. Note that $Z := Z(S) \trianglelefteq M$ and $\overline{M} := M/Z$ has a normal subgroup $\overline{K}$ isomorphic to A_6 or $L_3(4)$. Denote by K the preimage of $\overline{K}$ in M. Then $S \leqslant K$ and by a Frattini argument, $M = KN_M(S)$. Hence, we can write $g = kh$ with $k \in K$ and $h \in N_M(S)$. To prove that $g \in N_M(S)$ and thus $\mathcal{L}_\Delta(M) \subseteq N_M(S)$, it is sufficient to show that $k \in N_M(S)$. Note that $P^k = (P^g)^{h^{-1}} \leqslant S$. As $\overline{S}$ is abelian, fusion in $\overline{K}$ is controlled by $N_{\overline{K}}(\overline{S})$. So, there exists $x \in K$ such that $\overline{kx^{-1}} \in C_{\overline{K}}(\overline{P})$. As $\overline{K} \cong A_6$ of $L_3(4)$ and $\overline{P}$ is a nontrivial 3-subgroup of $\overline{K}$, one sees that $C_{\overline{K}}(\overline{P}) = \overline{S}$. Hence, $kx^{-1} \in S$ and $k \in N_M(S)$. This shows $\mathcal{L}_\Delta(M) = N_M(S)$. By Alperin's Fusion Theorem, we have $\mathcal{F}_S(M) = \mathcal{F}_S(\mathcal{L}_\Delta(M)) = \mathcal{F}_S(N_M(S))$. $\square$

Lemma 6.11. *Assume Hypothesis 6.1. If $(\mathcal{L}, \Delta, S)$ is a centric linking locality over $\mathcal{F}$, then $N_{\mathcal{L}}(Z) = N_{\mathcal{L}}(S)$. In particular, $N_{\mathcal{L}}(Z)$ is a group that is a model for $N_{\mathcal{F}}(S)$.*

Proof. By Lemma 5.4, we have $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. So, $Z \trianglelefteq S$ is a fully $\mathcal{F}$-normalized subgroup such that every proper overgroup of Z is in Δ and $O_p(N_{\mathcal{F}}(Z)) = S \in \Delta$. Hence, by [28, Lemma 7.1], $N_{\mathcal{L}}(Z)$ is a subgroup of $\mathcal{L}$ that is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. As $N_{\mathcal{L}}(S) \subseteq N_{\mathcal{L}}(Z)$ is by Lemma 2.9(b) a model for $N_{\mathcal{F}}(S)$, and a model for a constrained fusion system is by [2, Theorem III.5.10] unique up to isomorphism, it follows that $N_{\mathcal{L}}(Z) = N_{\mathcal{L}}(S)$. □

We are now in a position to complete the proof of Theorem 6.4.

Proof of Theorem 6.4. Assume Hypothesis 6.1. By Lemma 6.9, for every punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ that restricts to a centric linking locality, one of the cases (a)–(c) of Theorem 6.4 holds. It remains to show that each of these cases actually occurs in an example and that moreover the isomorphism type of $N_{\mathcal{L}^+}(Z)$ determines $(\mathcal{L}^+, \Delta^+, S)$ uniquely up to a rigid isomorphism.

By Lemma 5.4, we have $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and $\mathcal{F}^s$ is the set of nontrivial subgroups of S. Hence, the subcentric linking locality $(\mathcal{L}^s, \mathcal{F}^s, S)$ over $\mathcal{F}$ is always a punctured group over S. Moreover, it follows from Lemma 2.9(b) that $N_{\mathcal{L}^s}(Z)$ is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and thus S is normal in $N_{\mathcal{L}^s}(Z)$ by [2, Theorem III.5.10]. So, case (a) of Theorem 6.4 occurs in an example. Moreover, if $(\mathcal{L}^*, \Delta^+, S)$ is a punctured group such that $\mathcal{L}^*|_\Delta$ is a centric linking locality and $N_{\mathcal{L}^*}(Z) \cong N_{\mathcal{L}^s}(Z)$, then $N_{\mathcal{L}^*}(Z)$ has a normal Sylow p-subgroup and is thus by Lemma 6.9 a subcentric linking locality. Hence, by Theorem 2.21, $(\mathcal{L}^*, \Delta^+, S)$ is rigidly isomorphic to $(\mathcal{L}^s, \mathcal{F}^s, S)$.

We are now reduced to the case that $p = 3$ and we are looking at punctured groups in which the normalizer of Z does not have a normal Sylow 3-subgroup. So, assume now $p = 3$. By Lemma 6.2, $\mathcal{F}$ is the 3-fusion system of the Tits group or of J_4. Let M always be a finite group containing S as a Sylow 3-subgroup and assume that one of the following holds:

(b') $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $M \cong 3S_6$; or
(c') $\mathcal{F}$ is the 3-fusion system of J_4, and $M \cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.

In either case, one checks that $C_M(S) \leqslant S$. Moreover, if (b') holds, then $\mathrm{Out}_{\mathcal{F}}(S) \cong D_8$ and $N_M(S) \cong 3^{1+2}_+ : D_8$. As $\mathrm{Out}(S) \cong GL_2(3)$ has Sylow 2-subgroups isomorphic to SD_{16} and moreover, SD_{16} has a unique subgroup isomorphic to D_8, it follows that $\mathrm{Out}_M(S)$ and $\mathrm{Out}_{\mathcal{F}}(S)$ are conjugate in $\mathrm{Out}(S)$. Similarly, if (c') holds, then $\mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$ and $\mathrm{Out}_M(S)$ are both Sylow 2-sugroups of $\mathrm{Out}(S)$ and thus conjugate in $\mathrm{Out}(S)$. Hence, $N_M(S)$ is always isomorphic to a model for $N_{\mathcal{F}}(S)$ and, replacing M by a suitable isomorphic group, we can and will always assume that $N_M(S)$ is a model for $N_{\mathcal{F}}(S)$. We have then in particular that $N_{\mathcal{F}}(S) = \mathcal{F}_S(N_M(S))$.

Pick now a centric linking locality $(\mathcal{L}, \Delta, S)$ over S. By Lemma 6.11, $N_{\mathcal{L}}(Z)$ is a model for $N_{\mathcal{F}}(S)$. Hence, by the model theorem [2, Theorem III.5.10(c)], there exists a group isomorphism $\lambda : N_{\mathcal{L}}(Z) \to N_M(S)$ that restricts to the identity on S. By Lemma 6.10, we have $N_M(S) = \mathcal{L}_\Delta(M)$ and $\mathcal{F}_S(M) = \mathcal{F}_S(N_M(S)) = N_{\mathcal{F}}(S) = N_{\mathcal{F}}(Z)$. Note that $N_M(S)$ and $\mathcal{L}_\Delta(M)$ are actually equal as partial groups and the group isomorphism λ can be interpreted as a rigid isomorphism from $N_{\mathcal{L}}(Z)$ to $\mathcal{L}_\Delta(M)$. So, [14, Hypothesis 5.3] holds with Z in place of T. As $\Delta = \mathcal{F}^c$ is the set of all subgroups of S of order at least 3^2 and as all subgroups of S of order 3 are $\mathcal{F}$-conjugate, the set Δ^+ of non-identity subgroups of S equals $\Delta \cup Z^{\mathcal{F}}$. So, by [14, Theorem 5.14], there exists a punctured group

$(\mathcal{L}^+(\lambda), \Delta^+, S)$ over $\mathcal{F}$ with $N_{\mathcal{L}^+(\lambda)}(Z) \cong M$. Thus we have shown that all the cases listed in (a)-(c) of Theorem 6.4 occur in an example.

Let now $(\mathcal{L}^*, \Delta^+, S)$ be any punctured group over $\mathcal{F}$ such that $\mathcal{L}' := \mathcal{L}^*|_\Delta$ is a centric linking locality and $N_{\mathcal{L}^*}(Z) \cong M$. Pick a group isomorphism $\varphi : M \to M^* := N_{\mathcal{L}^*}(Z)$ such that $S^\varphi = S$. Then $\varphi|_S$ is an automorphism of S with $(\varphi|_S)^{-1}\mathrm{Aut}_M(S)\varphi|_S = \mathrm{Aut}_{M^*}(S)$. Recall that $\mathcal{F}_S(M) = N_\mathcal{F}(S)$, Moreover, by Lemma 2.9(b), we have $\mathcal{F}_S(M^*) = N_\mathcal{F}(Z) = N_\mathcal{F}(S)$. Hence, $\mathrm{Aut}_M(S) = \mathrm{Aut}_\mathcal{F}(S) = \mathrm{Aut}_{M^*}(S)$ and $\varphi|_S \in N_{\mathrm{Aut}(S)}(\mathrm{Aut}_\mathcal{F}(S))$. So, by Lemma 6.9, there exists $\psi \in \mathrm{Aut}(M)$ such that $\psi|_S = \varphi|_S$. Then $\mu := \psi^{-1}\varphi$ is an isomorphism from M to $M^* = N_{\mathcal{L}^*}(Z)$ that restricts to the identity on $S = N_S(Z)$. Moreover, by Theorem 2.20, there exists a rigid isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$. Therefore, by [14, Theorem 5.15(a)], there exists a rigid isomorphism from $(\mathcal{L}^+(\lambda), \Delta^+, S)$ to $(\mathcal{L}^*, \Delta^+, S)$. This shows that a punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$, which restricts to a centric linking locality, is up to a rigid isomorphism uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$. $\square$

Proof of Theorem 6.3. Assume Hypothesis 6.1. If $p \neq 3$, then it follows from Lemma 6.6 that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is a subcentric linking locality for every every punctured group $(\mathcal{L}^+, \Delta^+, S)$ over S. On the other hand, if $p = 3$, then Theorem 6.4 together with Lemma 6.2 gives the existence of a punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ such that $O_{p'}(\mathcal{L}^+) = 1$ and $N_{\mathcal{L}^+}(Z)$ is not of characteristic p, that is, such that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is not a subcentric linking locality. $\square$

APPENDIX A: NOTATION AND BACKGROUND ON GROUPS OF LIE TYPE

We record here some generalities on algebraic groups and finite groups of Lie type that are needed in Section 4. Our main references are [11, 26], and [9], as these references contain proofs for all of the background lemmas we need.

Fix a prime p and a semisimple algebraic group $\overline{G}$ over $\overline{\mathbb{F}}_p$. Let $\overline{T}$ be a maximal torus of $\overline{G}$, $W = N_{\overline{G}}(\overline{T})/\overline{T}$ the Weyl group, and let $X(\overline{T}) = \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_p^\times)$ be the character group. Let $\overline{X}_\alpha = \{x_\alpha(\lambda) \mid \lambda \in \overline{\mathbb{F}}_p\}$ denote a root subgroup, namely a closed $\overline{T}$-invariant subgroup isomorphic $\overline{\mathbb{F}}_p$. The root subgroups are indexed by the roots of $\overline{T}$, the characters $\alpha \in X(\overline{T})$ with $x_\alpha(\lambda)^t = x_\alpha(\alpha(t)\lambda)$ for each $t \in \overline{T}$. The character group $X(\overline{T})$ is written additively: for each $\alpha, \beta \in X(\overline{T})$ and each $t \in \overline{T}$, we write $(\alpha + \beta)(t) = \alpha(t)\beta(t)$. For each $n \in N_{\overline{G}}(\overline{T})$, $\alpha \in X(\overline{T})$, and $t \in \overline{T}$ we write $({}^n\alpha)(t) = \alpha(t^n)$ for the induced action of $N_{\overline{G}}(\overline{T})$ action on $X(\overline{T})$.

Let $\Sigma(\overline{T})$ be the set of $\overline{T}$-roots $\alpha \in X(\overline{T})$, and let $V = \mathbb{R} \otimes_\mathbb{Z} X(\overline{T})$ be the associated real inner product space with W-invariant inner product $(,)$. We regard $X(\overline{T})$ as a subset of V, and write $w_\alpha \in W$ for the reflection in the hyperplane $\alpha^\perp$.

For each root $\alpha \in \Sigma(\overline{T})$ and each $\lambda \in \overline{\mathbb{F}}_p^\times$, let $n_\alpha(\lambda), h_\alpha(\lambda) \in \langle \overline{X}_\alpha, \overline{X}_{-\alpha}\rangle$ be the images of the elements $\left[\begin{smallmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right]$ under the homomorphism $SL_2(\overline{\mathbb{F}}_p) \to G$ that sends $\left[\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right]$ to $x_\alpha(u)$ and $\left[\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right]$ to $x_{-\alpha}(v)$. Thus,

$$n_\alpha(\lambda) = x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda) \quad \text{and} \quad h_\alpha(\lambda) = n_\alpha(1)^{-1}n_\alpha(\lambda), \tag{A.1}$$

and $n_\alpha(1)$ represents w_α for each $\alpha \in \Sigma$. We assume throughout that parameterizations of the root groups have been chosen so that the Chevalley relations of [26, 1.12.1] hold.

Although $\Sigma(\overline{T})$ is defined in terms of characters of the maximal torus $\overline{T}$, it will be convenient to identify $\Sigma(\overline{T})$ with an abstract root system Σ inside some standard Euclidean space

$\mathbb{R}^l$, $(,)$, via a W-equivariant bijection that preserves sums of roots [26, 1.9.5]. We'll write also V for this Euclidean space. The symbol Π denotes a fixed but arbitrary base of Σ.

The maps $h_\beta : \overline{\mathbb{F}}_p^\times \to \overline{T}$, defined above for each $\beta \in \Sigma$, are algebraic homomorphisms lying in the group of cocharacters $X^\vee(\overline{T}) := \mathrm{Hom}(\overline{\mathbb{F}}_p^\times, \overline{T})$. Composition induces a W-invariant perfect pairing $X(\overline{T}) \otimes_{\mathbb{Z}} X^\vee(\overline{T}) \to \mathbb{Z}$ defined by $\alpha \otimes h \mapsto \langle \alpha, h\rangle$, where $\langle \alpha, h\rangle$ is the unique integer such that $\alpha(h(\lambda)) = \lambda^{\langle \alpha, h\rangle}$ for each $\lambda \in \overline{\mathbb{F}}_p^\times$. As Σ contains a basis of V, we can identify V^* with $\mathbb{R} \otimes_{\mathbb{Z}} X^\vee(\overline{T})$, and view $X^\vee(\overline{T}) \subseteq V^*$ via this pairing. Under the identification of V with V^* via $v \mapsto (-, v)$, for each $\beta \in \Sigma$ there is $\beta^\vee \in V$ such that $(-, \beta^\vee) = \langle -, h_\beta\rangle$ in V^*, namely the unique element such that $(\beta, \beta^\vee) = 2$ and such that w_β is reflection in the hyperplane $\ker((-, \beta^\vee))$. Thus, when viewed in V in this way (as opposed to in the dual space V^*), $\beta^\vee = 2\beta/(\beta, \beta)$ is the abstract coroot corresponding to β. Write $\Sigma^\vee = \{\beta^\vee \mid \beta \in \Sigma\} \subseteq V$ for the dual root system of Σ.

If we set $\langle \alpha, \beta\rangle = (\alpha, \beta^\vee) = 2(\alpha, \beta)/(\beta, \beta)$ for each pair of roots $\alpha, \beta \in \Sigma$, then

$$\langle \alpha, \beta\rangle = \langle \alpha, h_\beta\rangle, \tag{A.2}$$

where the first is computed in Σ, and the second is the pairing discussed above. Equivalently,

$$x_\alpha(\mu)^{h_\beta(\lambda)} = x_\alpha(\lambda^{\langle \alpha, \beta\rangle}\mu) \tag{A.3}$$

for each $\alpha, \beta \in \Sigma$, each $\mu \in \overline{\mathbb{F}}_p$, and each $\lambda \in \overline{\mathbb{F}}_p^\times$.

Additional Chevalley relations we need are

$$x_\alpha(\lambda)^{n_\beta(1)} = x_{w_\beta(\alpha)}(c_{\alpha,\beta}\lambda), \tag{A.4}$$

$$h_\alpha(\lambda)^{n_\beta(1)} = h_{w_\beta(\alpha)}(\lambda), \tag{A.5}$$

$$n_\alpha(\lambda)^{n_\beta(1)} = n_{w_\beta(\alpha)}(c_{\alpha,\beta}\lambda), \tag{A.6}$$

$$n_\alpha(1)^2 = h_\alpha(-1), \tag{A.7}$$

where

$$w_\beta(\alpha) = \alpha - \langle \alpha, \beta\rangle\beta$$

is the usual reflection in the hyperplane $\beta^\perp$, and where the $c_{\alpha,\beta} \in \{\pm 1\}$, in the notation of [26, Theorem 1.12.1], are certain signs that depend on the choice of the Chevalley generators. This notation is related to the signs $\eta_{\alpha,\beta}$ in [11, chapter 6] by $c_{\alpha,\beta} = \eta_{\beta,\alpha}$.

Important tools for determining the signs $c_{\alpha,\beta}$ in certain cases are proved in [11, Propositions 6.4.2 and 6.4.3], and we record several of those results here.

Lemma A.1. *Let $\alpha, \beta \in \Sigma$ be linearly independent roots.*

(1) $c_{\alpha,\alpha} = -1$ *and* $c_{-\alpha,\alpha} = -1$.
(2) $c_{-\alpha,\beta} = c_{\alpha,\beta}$.
(3) $c_{\alpha,\beta}c_{w_\beta(\alpha),\beta} = (-1)^{\langle \alpha, \beta\rangle}$.

(4) *If the β-root string through α is of the form*

$$\alpha - s\beta, \dots, \alpha, \dots, \alpha + s\beta$$

for some $s \geqslant 0$, that is, if α and β are orthogonal, then $c_{\alpha,\beta} = (-1)^s$.

Proof. The first three listed properties are proved in [11, Proposition 6.4.3]. By the proof of that proposition, there are signs $\epsilon_i \in \{\pm 1\}$ such that $c_{\alpha,\beta} = (-1)^s \frac{\epsilon_0 \cdots \epsilon_{s-1}}{\epsilon_0 \cdots \epsilon_{r-1}}$, whenever the β-root string through α is of the form $\alpha - s\beta, \dots, \alpha, \dots, \alpha + r\beta$. When α and β are orthogonal, we have $r - s = \langle \alpha, \beta \rangle = 0$, and hence $c_{\alpha,\beta} = (-1)^s$. □

Lemma A.2. *The following hold.*

(1) *For each $\alpha, \beta \in \Sigma$, we have*

$$\alpha(h_\beta(\lambda)) = \lambda^{\langle \alpha, \beta \rangle}.$$

(2) *The maximal torus $\overline{T}$ is generated by the $h_\alpha(\lambda)$ for $\alpha \in \Sigma$ and $\lambda \in \overline{\mathbb{F}}_p^\times$. If $\overline{G}$ is simply connected, and $\lambda_\alpha \in \mathbb{F}_p^\times$ are such that $\prod_{\alpha \in \Pi} h_\alpha(\lambda_\alpha) = 1$, then $\lambda_\alpha = 1$ for all $\alpha \in \Pi$. Thus,*

$$\overline{T} = \prod_{\alpha \in \Pi} h_\alpha(\overline{\mathbb{F}}_p^\times),$$

and h_α is injective for each α.

(3) *If $\beta, \alpha_1, \dots, \alpha_k \in \Sigma$ and $n_1, \dots, n_k \in \mathbb{Z}$ are such that $\beta^\vee = n_1\alpha_1^\vee + \cdots + n_k\alpha_k^\vee$, then*

$$h_\beta(\lambda) = h_{\alpha_1}(\lambda^{n_1}) \cdots h_{\alpha_k}(\lambda^{n_k}).$$

(4) *Define*

$$\Phi : \mathbb{Z}\Sigma^\vee \times \overline{\mathbb{F}}_p^\times \longrightarrow \overline{T} \quad \text{by} \quad \Phi(\alpha^\vee, \lambda) = h_\alpha(\lambda).$$

Then Φ is bilinear and $\mathbb{Z}[W]$-equivariant. It induces a surjective $\mathbb{Z}[W]$-module homomorphism $\mathbb{Z}\Sigma^\vee \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p^\times \to \overline{T}$ that is an isomorphism if $\overline{G}$ is of universal type.

Proof. (1) is the statement in (A.2) and is part of [26, Remark 1.9.6]. We refer to [9, Lemma 2.4(c)] for a proof, which is based on the treatment in Carter [11, pp. 97–100]. Part (2) is proved in [9, Lemma 2.4(b)], and part (3) is [9, Lemma 2.4(d)]. Finally, part (4) is proved in [9, Lemma 2.6]. □

Proposition A.3. *For each subgroup $X \leqslant \overline{T}$,*

$$C_{\overline{G}}(X) = C_{\overline{G}}(X)^\circ C_{N_{\overline{G}}(\overline{T})}(X).$$

The connected component $C_{\overline{G}}(X)^\circ$ is generated by $\overline{T}$ and the root groups $\overline{X}_\alpha$ for those roots $\alpha \in \Sigma$ whose kernel contains X. In particular, if $X = \langle h_\beta(\lambda) \rangle$ for some $\beta \in \Sigma$ and some $\lambda \in \overline{\mathbb{F}}_p^\times$ having multiplicative order r, then

$$C_{\overline{G}}(X)^\circ = \langle \overline{T}, \overline{X}_\alpha \mid \alpha \in \Sigma,\ r \text{ divides } \langle \alpha, \beta \rangle \rangle.$$

Proof. See [9, Proposition 2.5], which is based on [12, Lemma 3.5.3]. The referenced result covers all but the last statement, which then follows from the previous parts and Lemma A.2(1), given the definition of r. □

Proposition A.4. *Let $\overline{G}$ be a simply connected, simple algebraic group over $\overline{\mathbb{F}}_p$, let $\overline{T}$ be a maximal torus of $\overline{G}$, and let $T_r = \{t \in \overline{T} \mid t^r = 1\}$ with $r > 1$ prime to p. Then one of the following holds.*

(1) *$C_{\overline{G}}(T_r) = \overline{T}$ and $N_{\overline{G}}(T_r) = N_{\overline{G}}(\overline{T})$.*
(2) *$r = 2$, $C_{\overline{G}}(T_r) = \overline{T}\langle w_0\rangle$ for some element $w_0 \in N_{\overline{G}}(\overline{T})$ inverting $\overline{T}$, and $N_{\overline{G}}(T_r) = N_{\overline{G}}(\overline{T})$.*
(3) *$r = 2$, and $\overline{G} = Sp_{2n}(\overline{\mathbb{F}}_p)$ for some $n \geqslant 1$.*

Proof. By Lemma A.2(2) and as $\overline{G}$ is simply connected, the torus is direct product of the images of the coroots for fundamental roots:

$$\overline{T} = \prod_{\alpha\in\Pi} h_\alpha(\overline{\mathbb{F}}_p^\times). \tag{A.8}$$

Thus, if $\lambda \in \overline{\mathbb{F}}_p^\times$ is a fixed element of order r, then T_r is the direct product of $\langle h_\alpha(\lambda)\rangle$ as α ranges over Π.

We first look at $C_{\overline{G}}(T_r)^\circ$, using Proposition A.3. By Lemma A.2(1), T_r is contained in the kernel of a root β if and only if $\beta(h_\alpha(\lambda)) = \lambda^{\langle\beta,\alpha\rangle} = 1$ for all simple roots α, that is, if $\langle\beta,\alpha\rangle$ is divisible by r for each fundamental root α. Let Σ_r be the set of all such roots β. For each $\alpha \in \Pi$, the reflection w_α sends a root β to $\beta - \langle\beta,\alpha\rangle\alpha$. Hence, $\beta \in \Sigma_r$ if and only if $w_\alpha(\beta) \in \Sigma_r$ because $\langle -, -\rangle$ is linear in the first component. As the Weyl group is generated by w_α, $\alpha \in \Pi$, it follows that Σ_r is invariant under the Weyl group. By [31, Lemma 10.4C], and as $\overline{G}$ is simple, W is transitive on all roots of a given length, and so either $\Sigma_r = \varnothing$, or Σ_r contains all long roots or all short ones. Thus, by [31, table 1], we conclude that either $\Sigma_r = \varnothing$, or $r = 2$, each root in $\Pi \cap \Sigma_r$ is long, and each $\alpha \in \Pi$ not orthogonal to β is short and has angle $\pi/4$ or $3\pi/4$ with β. Now by inspection of the Dynkin diagrams corresponding to irreducible root systems, we conclude that the latter is possible only if $\Sigma = A_1 = C_1, C_2$, or C_3. Thus, either $C_{\overline{G}}(T_r)^\circ = \overline{T}$ or (3) holds.

So, we may assume that $C_{\overline{G}}(T_r)^\circ = \overline{T}$. Now $N_{\overline{G}}(\overline{T}) \leqslant N_{\overline{G}}(T_r)$ because T_r is characteristic in T. As $C_{\overline{G}}(T_r)^\circ = \overline{T}$, also $\overline{T}$ is normalized by $N_{\overline{G}}(T_r)$, so $N_{\overline{G}}(\overline{T}) = N_{\overline{G}}(T_r)$. For $r \geqslant 3$, it follows from [9, Lemma 2.7] that $C_{N_{\overline{G}}(\overline{T})}(T_r) = \overline{T}$, completing the proof of (1) in this case.

Assume now that $r = 2$ and (1) does not hold. Let $B := C_W(T_2) \leqslant W = N_{\overline{G}}(\overline{T})/\overline{T}$. To complete the proof, we need to show $B = \langle -1_V\rangle$ or else (3) holds. Here we argue as in Case 1 of the proof of [9, Proposition 5.13].

Let $\Lambda = \mathbb{Z}\Sigma^\vee$ be the lattice of coroots, and fix $\lambda \in \overline{\mathbb{F}}_p^\times$ of order 4. The map $\Phi_\lambda : \Lambda \to \overline{T}$ defined by $\Phi_\lambda(\alpha^\vee) = h_\alpha(\lambda)$ is a W-equivariant homomorphism by Lemma A.2(3). As $\overline{G}$ is simply connected, this homomorphism has kernel 4Λ, image T_4, and it identifies $\Lambda/2\Lambda$ with T_2, by Lemma A.2(2).

As B acts on T_4 and centralizes T_2, we have $[T_4, B] \leqslant T_2 \leqslant C_{\overline{T}}(B)$, so B acts quadratically on T_4. As B acts faithfully on T_4 by (1), it follows that B is a 2-group.

Assume that $B \neq \langle -1_V\rangle$. If B is of 2-rank 1 with center $\langle -1_V\rangle$ then by assumption there is some $b \in B$ with $b^2 = -1_V$. In this case, b endows V with the structure of a complex vector space, and so b does not centralize $\Lambda/2\Lambda$, a contradiction. Thus, there is an involution $b \in B$ that is not -1_V. Let $V = V_+ \oplus V_-$ be the decomposition of V into the sum of the eigenspaces for b, and set $\Lambda_\pm = \Lambda \cap V_\pm$. Fix $v \in \Lambda$, and write $v = v_+ + v_-$ with $v_\pm \in V_\pm$. Then $2v_- = v - v^b = [v, b] \in$

$V_- \cap 2\Lambda = 2\Lambda_-$. So, $v_- \in \Lambda_-$, and then $v_+ \in \Lambda_+$. This shows that $\Lambda = \Lambda_+ \oplus \Lambda_-$ with $\Lambda_\pm \neq 0$. The hypotheses of [9, Lemma 2.8] thus hold, and so $G = Sp_{2n}(\overline{\mathbb{F}}_p)$ for some $n \geqslant 2$ by that lemma. □

ACKNOWLEDGMENTS

It was Andy Chermak who first asked the question in 2011 (arising out of his proof of existence and uniqueness of linking systems) of which exotic systems have localities on the set of nonidentity subgroups of a Sylow group. We thank him for comments on an earlier version of this paper and many helpful conversations, including during a visit to Rutgers in 2014, where he and the third author discussed the possibility of constructing punctured groups for the Benson–Solomon systems. We are grateful to George Glauberman for communicating to us several comments, corrections, and suggestions for improvement. We are especially grateful to the referee for pointing out that Lemma 4.8 conflicts with a lemma of Levi and Oliver, for alerting us to errors too numerous to mention, and for other suggestions. We would like to thank the Centre for Symmetry and Deformation at the University of Copenhagen for supporting a visit of the third-named author, where some of the early work on this paper took place. Finally, we thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program "Groups, representations and applications," where work on this paper was undertaken and supported by EPSRC Grant Number: EP/R014604/1. Justin Lynd was partially supported by NSF Grant DMS-1902152.

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The *Transactions of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. M. Aschbacher and A. Chermak, *A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver*, Ann. of Math. (2) **171** (2010), no. 2, 881–978.
2. M. Aschbacher, R. Kessar, and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834
3. M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 1993, Corrected reprint of the 1986 original. MR 1264416 (95b:20002)
4. C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, *Subgroup families controlling p-local finite groups*, Proc. Lond. Math. Soc. (3) **91** (2005), no. 2, 325–354.
5. D. Benson, *Conway's group Co_3 and the Dickson invariants*, Manuscripta Math. **85** (1994), no. 2, 177–193.
6. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer, Berlin-New York, 1972. MR 0365573
7. C. Broto, R. Levi, and B. Oliver, *Homotopy equivalences of p-completed classifying spaces of finite groups*, Invent. Math. **151** (2003), no. 3, 611–664.
8. C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).
9. C. Broto, J. Møller, and B. Oliver, *Automorphisms of fusion systems of finite simple groups of Lie type*, Mem. Amer. Math. Soc. **262** (2019), no. 1267, iii+117. MR 4044462
10. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, New York-Berlin, 1982. MR 672956
11. R. W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics, vol. 28, John Wiley & Sons, London-New York-Sydney, 1972. MR 0407163

12. R. W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication. MR 794307
13. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, NJ, 1956. MR 0077480 (17,1040e)
14. A. Chermak, *Fusion systems and localities*, Acta Math. **211** (2013), no. 1, 47–139. MR 3118305
15. A. Chermak, *Finite localities I*, Forum Math. Sigma **10** (2022), Paper No. e43, 31. MR 4439780
16. D. A. Craven, B. Oliver, and J. Semeraro, *Reduced fusion systems over p-groups with abelian subgroup of index p: II*, Adv. Math. **322** (2017), 201–268. MR 3720798
17. M. Clelland and C. Parker, *Two families of exotic fusion systems*, J. Algebra **323** (2010), no. 2, 287–304. MR 2564839 (2011e:20025)
18. A. Diaz and S. Park, *Mackey functors and sharpness for fusion systems*, Homology Homotopy Appl. **17** (2015), no. 1, 147–164. MR 3338545
19. A. Díaz, A. Ruiz, and A. Viruel, *All p-local finite groups of rank two for odd prime p*, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1725–1764 (electronic).
20. W. G. Dwyer, *Sharp homology decompositions for classifying spaces of finite groups*, Group representations: cohomology, group actions and topology (Seattle, WA, 1996), Proc. Sympos. Pure Math., vol. 63, Amer. Math. Soc., Providence, RI, 1998, pp. 197–220. MR 1603159
21. W. Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, vol. 25, North-Holland Publishing Co., Amsterdam, 1982.
22. R. J. Flores and R. M. Foote, *Strongly closed subgroups of finite groups*, Adv. Math. **222** (2009), no. 2, 453–484. MR 2538017 (2011b:20075)
23. L. R. Fletcher, *A characterisation of PSL(3, 4)*, J. Algebra **19** (1971), 274–281. MR 0304482 (46 #3617)
24. G. Glauberman and J. Lynd, *Control of fixed points and existence and uniqueness of centric linking systems*, Invent. Math. **206** (2016), no. 2, 441–484. MR 3570297
25. G. Glauberman and J. Lynd, *Rigid automorphisms of linking systems*, Forum Math. Sigma **9** (2021), 1–20, e23.
26. D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, Amer. Math. Soc., Providence, RI, 1998, Almost simple K-groups. MR 1490581 (98j:20011)
27. E. Henke, *Direct and central products of localities*, J. Algebra **491** (2017), 158–189. MR 3699092
28. E. Henke, *Subcentric linking systems*, Trans. Amer. Math. Soc. **371** (2019), no. 5, 3325–3373. MR 3896114
29. G. Higman, *Odd characterizations of finite simple groups*, Lecture Notes of University of Michigan, Ann Arbor, MI, 1968, 76 p.
30. E. Henke and J. Lynd, *Fusion systems with Benson-Solomon components*, Duke Math. J. **171** (2022), no. 3, 673–737. MR 4382978
31. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer, New York-Berlin, 1972. MR 0323842
32. S. Jackowski and J. McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), no. 1, 113–132. MR 1153240 (92k:55026)
33. S. Jackowski, J. McClure, and B. Oliver, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. **147** (1995), no. 2, 99–126. MR 1341725
34. H. Kurzweil and B. Stellmacher, *The theory of finite groups – An introduction*, Universitext, Springer, New York, 2004, An introduction, translated from the 1998 German original. MR 2014408 (2004h:20001)
35. R. Levi and B. Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson*, Geom. Topol. **6** (2002), 917–990 (electronic).
36. R. Levi and B. Oliver, *Correction to: "Construction of 2-local finite groups of a type studied by Solomon and Benson" [Geom. Topol. **6** (2002), 917–990 (electronic); mr1943386]*, Geom. Topol. **9** (2005), 2395–2415 (electronic).
37. G. Malle, *The maximal subgroups of ${}^2F_4(q^2)$*, J. Algebra **139** (1991), no. 1, 52–69. MR 1106340
38. B. Oliver, *Equivalences of classifying spaces completed at odd primes*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 2, 321–347.
39. B. Oliver, *Equivalences of classifying spaces completed at the prime two*, Mem. Amer. Math. Soc. **180** (2006), no. 848, vi+102pp.
40. B. Oliver, *Extensions of linking systems and fusion systems*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5483–5500. MR 2657688 (2011f:55032)

41. B. Oliver, *Existence and uniqueness of linking systems: Chermak's proof via obstruction theory*, Acta Math. **211** (2013), no. 1, 141–175. MR 3118306
42. B. Oliver, *Simple fusion systems over p-groups with abelian subgroup of index p: I*, J. Algebra **398** (2014), 527–541. MR 3123784
43. B. Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson: correction*, Geom. Topol. (2023), to appear.
44. B. Oliver and A. Ruiz, *Reduced fusion systems over p-groups with abelian subgroup of index p: III*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 3, 1187–1239. MR 4091058
45. B. Oliver and J. Ventura, *Extensions of linking systems with p-group kernel*, Math. Ann. **338** (2007), no. 4, 983–1043. MR 2317758 (2008k:55029)
46. C. Parker and G. Stroth, *A family of fusion systems related to the groups* $\mathrm{Sp}_4(p^a)$ *and* $\mathrm{G}_2(p^a)$, Arch. Math. (Basel) **104** (2015), no. 4, 311–323. MR 3325764
47. L. Puig, *Frobenius categories*, J. Algebra **303** (2006), no. 1, 309–357.
48. A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), no. 1, 45–65.
49. R. Solomon, *Finite groups with Sylow 2-subgroups of type .3*, J. Algebra **28** (1974), 182–198.
50. T. A. Springer, *Linear algebraic groups*, 2nd ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. MR 2458469
51. S. D. Smith and A. P. Tyrer, *On finite groups with a certain Sylow normalizer. I, II*, J. Algebra **26** (1973), 343–365; ibid. 26 (1973), 366–367. MR 0323888 (48 #2241)