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On Reading Timbre and Tempo from the Score

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Abstract: Typically, a musical score alludes only briefly to the ways in which timbre and tempo vary through the piece, leaving it to the performer to answer the question of how to interpret those elements in detail. If a musical piece is programmed with a computer, timbre and tempo must be specified throughout the piece. This leaves us with the problem of how to program tempo and timbre if we are given a musical score. We describe here some systematic techniques for reading timbre and tempo from a musical score. The basic idea is to associate an algebraic structure to our score, and, by associating certain parameters to that algebraic structure, derive the timbre and tempo of the piece. Our first approach is to associate a consonance structure to the score, and reflect that in the timbre and tempo of the piece. This is what we do in sections 2-5 and section 7. Here, our approach relates to the problem of tuning the notes of our piece in a consistent way. A second approach is to reflect higher dimensional arrows implied by temporal subdivisions of the piece. This is what we do in section 6. In section 8 we mention a third crude approach, where we just count through the notes of a piece successively, so our notes are indexed by elements of an interval in $\mathbb{Z}$, which we reflect in the timbre of our piece using a function.

Keywords: Timbre, Tempo, Consonance

1. Introduction

When programming a computer to play an unornamented musical score, it is simplest to set notes to have a constant amplitude and to be played with a constant tempo. The result is a sound which can be quite plain, compared to the sound made if the same piece is interpreted artistically by humans. This plainness means that when we hear the music, it faithfully reflects the score, without embellishment. However, the ear can yearn for some variation of timbre and tempo. Here, we describe some systematic approaches to introducing variation in timbre and tempo into computer generated interpretations of a musical score. To do this, in sections 2 to 5, and section 7, we reflect a consonance structure implied by the score in the timbre and tempo of an interpretation. Alternatively, in section 6, we reflect higher dimensional arrows, implied by temporal subdivisions of the piece, in the timbre and tempo of an interpretation. Throughout, a consonance structure implied by the score is reflected in the consonance structure of pitches of an interpretation.

Let S denote the set of finite subsets of $\mathbb{R}^\times \times \mathbb{R}_{>0}$ with distinct second coordinates. For an element $s \in S$, let

$$\xi_s(t) = \sum_{(a,f)\in s} a\sin(2\pi f t).$$

Thus ξ_s is a function from $\mathbb{R}$ to $\mathbb{R}$ which, when played through loudspeakers, determines a sound. We have the diagonal action of $\mathbb{R}^\times \times \mathbb{R}_{>0}$ on S, in which the groups $\mathbb{R}^\times$ and $\mathbb{R}_{>0}$ act by multiplication. We say ξ_s and ξ_t have the same timbre if s and t belong to the same orbit under this action on S.

Some remarks:

We say two musical notes are consonant if they share a common harmonic. For example, ξ_s and ξ_t as above are consonant if s and t contain elements with a common second coordinate.

If we allow the timbre of notes to vary throughout an interpretation of a piece, but in doing so, preserve some of the consonances of the original, then the fundamental pitches may be forced to diverge from their original values.

We construct our sounds by additive synthesis. Music constructed like this goes back some way, eg. [12].

Our manipulation of consonance structures is a sort of retuning. Less complicated tuning systems are explored elsewhere, eg. [3]. The use of tunings which vary with time is not new, see eg. [11].

We derive music by similarity, cf. [7]. However, our approach differs from an approach that derives music by similarity from a large body of work (eg. [5]) - we only work with one piece at a time.

We make tunings using consonances, as defined above. We do not in general limit ourselves to 'consonant' frequency ratios involving the primes 2, 3, and 5, as happens in just intonation approximating 12 tone equal temperament. However we do limit ourselves in this way in Examples 3.1, 4.1, 6.1, and 6.2.

We have given a more detailed analysis of consonance structures elsewhere [14].

There are sound files to accompany the paper [15].

2. Substituting for Equal Temperament

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let n be a fixed natural number with $5 \leq n \leq 10$. Let $g : \{1, 2, ..., n\} \to \mathbb{Z}$ be given by $g(1) = 0$, $g(2) = 12$, $g(3) = 19$, $g(4) = 24$, $g(5) = 28$, $g(6) = 31$, $g(7) = 34$, $g(8) = 36$, $g(9) = 38$, $g(10) = 40$. We extend g to a function from $\{1, 2, ..., n\} \cdot \{1, 2, ..., n\}^{-1}$ to $\mathbb{Z}$ via $g(\frac{a}{b}) = g(a) - g(b)$. We associate to a natural number m a function ϵ_m from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\epsilon_m(t) = \sum_{i=1}^{n} \sin(2\pi f(m + g(i))t).$$

To a musical score, we associate an interpretation in which the note m semitones above the note an octave below middle C is given by the function ϵ_m. Consonances in the score correspond to pairs of notes represented by integers m and m' where $m - m' = g(\frac{a}{b})$ for some $1 \leq a, b \leq n$. These correspond to consonances in our interpretation, since we have $m + g(b) = m' + g(a)$, and therefore the b^{th} harmonic of the note represented by m is equal to the a^{th} harmonic of the note represented by m'.

In case $f(t) = 110 \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{t}{12}}$, our interpretations correspond to tunings in equal temperament.

Example 2.1. Let $n = 6$. Our examples are interpretations of Bach's Two Part Invention No. 9 [1]. For $d \in \mathbb{R}$, let

$$f(t) = 110 \cdot 2^{\frac{1}{4}} \cdot (15 \cdot (\frac{t}{60})^d + 1).$$

Our two examples are the cases when $d = 1$ and where $d = 3$.

3. Quivers Associated to Two Part Compositions

Here we generalise the tunings of previous work on consonance structures [14], to incorporate variations in timbre that reflect those consonance structures.

Let $\mathcal{M}$ denote the subgroup of $\mathbb{Q}^\times$ generated by $2, 3, 5, 7$. Consider the group homomorphism $m : \mathcal{M} \to \mathbb{Z}$ sending $2, 3, 5, 7$ to $12, 19, 28, 34$ respectively. Suppose we have a fixed natural number n with $5 \leq n \leq 10$. We define g to be the restriction of m to $\{1, 2, ..., n\} \cdot \{1, 2, ..., n\}^{-1}$. Let $\mathcal{G}$ denote the image of g. Suppose we have a fixed section s of g.

Suppose we are given a two part composition on the stave, such as a Bach Two-Part Invention. We denote one of the parts 1 and the other part 2. We have a linear order of the notes of our composition, where notes are ordered by start time, and given two notes starting at the same time we precede the note in part 2 by the note in part 1. We denote by N the number of notes of our composition.

For $1 \leq x \leq y \leq N$ define the sequence $S(x, y)$ of elements of $\{1, 2, ..., N\}$ to be

$$(x + 1, x + 2, x + 3, ..., y, x - 1, x - 2, x - 3, ..., 1).$$

We denote by $i(x, y) \in \mathbb{Z}$ the number of semitones required to ascend from the x^{th} note to the y^{th} note.

We define a quiver Q whose vertices are given by the notes of our composition, and whose arrows are labelled with elements of $\mathcal{G}$. This is the consonance structure which, when represented, defines an interpretation of our two part composition.

Our algorithm to define Q begins with a quiver with a single vertex, corresponding to the first note of the composition, and no arrows; it adds vertices and arrows successively. We run through elements y of $\{1, 2, ..., N\}$ consecutively, in standard order. For a fixed y we run through the elements x with $1 \leq x \leq y$ in reverse order. For a fixed x and y we search through $S(x, y)$ for vertices in our quiver to connect to x. If x already belongs to our quiver, we abandon our search through $S(x, y)$ straightaway. Otherwise we run through the elements z of $S(x, y)$ in sequence. If $i(x, z) \in \mathcal{G}$ and z belongs to our quiver, we add x to our quiver, draw an arrow from x to z, labelled with $i(x, z)$, and discontinue the search through $S(x, y)$.

The underlying graph of Q is a tree, since our algorithm involves adding leaves successively. We will assume that the vertex set of Q is the set of all notes of our composition, although there do exist examples where this is not the case. The idea here is to construct Q from consonances between notes which are close in the score.

Let $l : \mathbb{Q}^\times_{>0} \to \mathbb{N} \times \mathbb{N}$ be the function which sends a fraction $\frac{a}{b}$ in its lowest terms to (a, b).

Suppose we have an element $\theta_v = (\theta_{v,1}, \theta_{v,2}, \theta_{v,3}, ..., \theta_{v,n})$ of $\mathbb{R}^n_{>0}$, where $\theta_{v,1} = 1$, for every vertex v of Q.

We label the arrows of the double quiver of Q as follows. Given an arrow from v_1 to v_2 in Q, labelled with γ, we label the corresponding arrow in the double quiver with the real number $\theta_{v_2,l(s(\gamma))_1}{\theta_{v_1,l(s(\gamma))_2}}^{-1}$. We label the corresponding reverse arrow in the double quiver with the inverse of this real number.

A path in the underlying graph of Q determines a path in the double of Q, and thus a real number, via the above representation: this real number is the product of the real numbers labelling the arrows in the path. Choose an initial frequency $F_0 \in \mathbb{R}$. Every vertex v of our quiver is connected by a unique path in the underlying graph of Q from the first note of the composition, and thus multiplying the real number given by this path by F_0 determines a frequency, which gives the frequency F_v of v.

Suppose we have an element $\sigma_v = (\sigma_{v,1}, \sigma_{v,2}, ..., \sigma_{v_n})$ of $\mathbb{R}^n$ for every vertex v of Q.

To the vertex v in Q we assign the function η_v from $\mathbb{R}$ to $\mathbb{R}$, sending t to the sum $\sum_{i=1}^{n} \sigma_{v,i} \sin(2\pi F_v \theta_{v,i} t)$. We call $F_v\theta_{v,i}$ the i^{th} harmonic of this function.

By construction, an arrow in Q directed from v_1 to v_2 corresponds to at least one common harmonic of the functions assigned to v_1 and v_2. Indeed, if $l(s(i(v_1, v_2))) = (\alpha_1, \alpha_2)$, for $1 \leq \alpha_1, \alpha_2 \leq n$, then the frequencies of v_1 and v_2 differ by the factor $\theta_{v_2,\alpha_1}{\theta_{v_1,\alpha_2}}^{-1}$, which implies the α_2^{th} harmonic of the function assigned to v_1 is equal to the α_1^{th} harmonic of the function assigned to v_2.

We obtain a piece by playing, for every vertex v, the function η_v assigned to v, for the duration of the note associated to v in our score. This piece is a representation of the consonance structure Q, and an interpretation of the original two part composition. It is a retuning of the original two part composition.

The above construction relies on selecting a tuple $((\sigma_{v,1}, \theta_{v,1}), ..., (\sigma_{v,n}, \theta_{v,n}))$, for v a vertex of Q. We next use Q to derive such data from the consonance structure of our two part composition.

Suppose we are given $S_0 \in \mathbb{R}^n$ and $S_2, S_3, S_5, S_7 \in \mathbb{R}^n$. These determine n homomorphisms $S(1), ..., S(n) : \mathcal{M} \to \mathbb{R}$, where $S(i)$ sends $2, 3, 5, 7$ to $S_{2i}, S_{3i}, S_{5i}, S_{7i}$, for $1 \leq i \leq n$.

Fix i with $1 \leq i \leq n$. Consider the double of Q. We label the arrows of the double of Q as follows: given an arrow in our quiver labelled by γ, we label the corresponding arrow in our double quiver with $S(i)(s(\gamma))$ and the corresponding reverse arrow with $-S(i)(s(\gamma))$.

A path in the underlying graph of Q determines a path in the double of Q, and thus a real number, via the above representation: this real number is the sum of the real numbers labelling the arrows in the path. Every vertex v of our quiver is connected by a unique path in the underlying graph of Q from the first note of the composition, and thus adding the real number given by this path to S_{0i} determines an amplitude, which we call $\sigma_{v,i}$.

Suppose we are given $T_0 \in \mathbb{R}^n_{>0}$ with $T_{0i \cdot j} = T_{0i} \cdot T_{0j}$ for $1 \leq i, j, i \cdot j \leq n$. Suppose we are also given $T_2, T_3, T_5, T_7 \in \mathbb{R}^n_{>0}$ with $T_{ki \cdot j} = T_{ki} \cdot T_{kj}$ for $1 \leq i, j, i \cdot j \leq n$ and $k = 2, 3, 5, 7$. These determine $n-1$ homomorphisms $T(2), ..., T(n) : \mathcal{M} \to \mathbb{R}_{>0}$, where $T(i)$ sends $2, 3, 5, 7$ to $T_{2i}, T_{3i}, T_{5i}, T_{7i}$, for $1 \leq i \leq n$.

Fix i with $2 \leq i \leq n$. Consider the double of Q. We label the arrows of the double of Q as follows: given an arrow in our quiver labelled by γ, we label the corresponding arrow in our double quiver with $T(i)(s(\gamma))$ and the corresponding reverse arrow with $T(i)(s(\gamma))^{-1}$.

A path in the underlying graph of Q determines a path in the double of Q, and thus a real number, via the above representation: this real number is the product of the real numbers labelling the arrows in the path. Every vertex v of our quiver is connected by a unique path in the underlying graph of Q from the first note of the composition, and thus multiplying the real number given by this path by T_{0i} determines a scalar, which we call $\theta_{v,i}$.

Our examples are all interpretations of Bach's Two Part Invention No. 9.

Example 3.1. Let $n = 5$. We set $S_0 = (1, 1, 1, 1, 1)$, and $S_2 = (0.1, 0, 0, 0.1, 0)$, $S_3 = (0, 0.1, 0, 0, 0.1)$, $S_5 = (0, 0, 0.1, 0, 0)$. We set $T_0 = (1, 2, 3, 4, 5)$, and $T_2 = T_3 = T_5 = (1, 1, 1, 1, 1)$, and thus do not allow θ_v to vary throughout the interpretation. We set $F_0 = 1056$.

Example 3.2. Let $n = 5$. In this example, we set $S_0 = (1, 1, 1, 1, 1)$ and $S_2 = S_3 = S_5 = (0, 0, 0, 0, 0)$, and thus do not allow amplitudes to vary through the interpretation. We set $T_0 = (1, 2, 3, 4, 5)$, and $T_2 = T_5 = (1, 1, 1, 1, 1)$, $T_3 = (1, 1, 1, 1, (6/5)^{\frac{1}{27}})$. We set $F_0 = 2112$. When v is the last note of our piece, $\theta_v = (1, 2, 3, 4, 6)$. Part way through our piece, θ_v is approximately $(1, 2, 3, 4, \frac{16}{3})$. This is audible as a section in which nearby intervals are approximately given by unisons, or stacks of fourths and fifths.

Example 3.3. Let $n = 5$. We set $S_0 = (1, 1, 1, 1, 1)$, and $S_2 = (-0.1, 0, 0, -0.1, 0)$, $S_3 = (0, -0.1, 0, 0, -0.1)$, $S_5 = (0, 0, -0.1, -0.1, -0.1)$. We set $T_0 = (1, \frac{3}{2}, 2, \frac{9}{4}, 3)$, and $T_2 = (1, 1.001, 1, 1.002001, 1)$, $T_3 = (1, 1, 0.999, 1, 1)$, $T_5 = (1, 1, 1, 1, 1.001)$. Let $F_0 = 1056$.

Example 3.4. Let $n = 5$. We set $S_0 = (1, 1, 1, 1, 1)$, and $S_2 = (-0.1, 0, 0, -0.1, 0)$, $S_3 = (0, -0.1, 0, 0, -0.1)$, $S_5 = (0, 0, -0.1, -0.1, -0.1)$. We set $T_2 = (1, 1.03, 1, 1.0609, 1)$, $T_3 = (1, 1, 0.985, 1, 1)$, and $T_5 = (1, 1, 1, 1, 1.03)$. Our two examples are given by $T_0 = (1, 2, 3, 4, 5)$, $F_0 = 1056$, and $T_0 = (1, 1, 1, 1, 1)$, $F_0 = 2112$.

4. Ckeleta

A note of a piece associated to a two part composition in the preceding section is somewhat fleshy, in the sense that it possesses harmonics which do not contribute to consonances given by Q. We define the *consonance skeleton*, or *ckeleton*, of this piece to be that which eliminates this excess in the following way:

An arrow in Q directed from v_1 to v_2 defines a pair of harmonics of the same frequency. Indeed, if $l(s(i(v_1, v_2))) = (\alpha_1, \alpha_2)$, for $1 \leq \alpha_1, \alpha_2 \leq n$, then the frequencies of v_1 and v_2 differ by the factor $\theta_{v_2,\alpha_1}{\theta_{v_1,\alpha_2}}^{-1}$, which implies the α_2^{th} harmonic of the function assigned to v_1 is equal to the α_1^{th} harmonic of the function assigned to v_2. Such harmonics sound, as sine waves with amplitude 1, in the ckeleton for the duration of the corresponding note in our two part composition. These are the only sounds in the ckeleton. In this way, each arrow in Q defines a call and a response of a certain sine wave in the corresponding ckeleton.

Example 4.1. Let $n = 5$. We set $T_0 = (1, 2, 3, 4, 5)$, and $T_2 = T_3 = T_5 = (1, 1, 1, 1, 1)$, and thus do not allow θ_v to vary throughout. We set $F_0 = 528$, and take the resulting ckeleton (4.1a).

5. Rubato

There is another parameter in which we can represent our consonance structure, namely tempo. Let us describe how to do this.

Suppose we are given $U_0 \in \mathbb{R}_{>0}$, and $U_2, U_3, U_5, U_7 \in \mathbb{R}_{>0}$. This determines a homomorphism $U : \mathcal{M} \to \mathbb{R}_{>0}$ that sends $2, 3, 5, 7$ to U_2, U_3, U_5, U_7.

Consider the double of Q. We label the arrows of the double of Q as follows: given an arrow in our quiver labelled by γ, we label the corresponding arrow in our double quiver with $U(s(\gamma))$ and the corresponding reverse arrow with $U(s(\gamma))^{-1}$.

A path in the underlying graph of Q determines a path in the double of Q, and thus a real number, via the above representation: this real number is the product of the real numbers labelling the arrows in the path. Every vertex v of our quiver is connected by a unique path in the underlying graph of Q from the first note of the composition, and thus multiplying the real number given by this path by U_0 determines a tempo, which we call υ_v. When we play our interpretation, we play the note v with tempo υ_v.

Example 5.1. We use the data from Example 3.4a: we set $T_2 = (1, 1.03, 1, 1.0609, 1)$, $T_3 = (1, 1, 0.985, 1, 1)$, and $T_5 = (1, 1, 1, 1, 1.03)$, $T_0 = (1, 2, 3, 4, 5)$, $F_0 = 1056$. We take the ckeleton of the resulting interpretation, and set U_0 to give one semiquaver a duration of $0.4s$. We set $(U_2, U_3, U_5) = (1.12, 1.15, 1.18)$.

6. Higher Dimensional Arrows

We have derived our parameters σ_v, θ_v, and υ_v from the quiver Q, which is in turn derived from the consonance structure of our piece. Here we present an alternative derivation of such parameters from the score, this time from higher dimensional arrows derived from the rhythmical structure of the score (note: there is a significant body of previous work on higher dimensional arrows, some of it musical, eg. [2, 4, 8, 9, 10] and references therein).

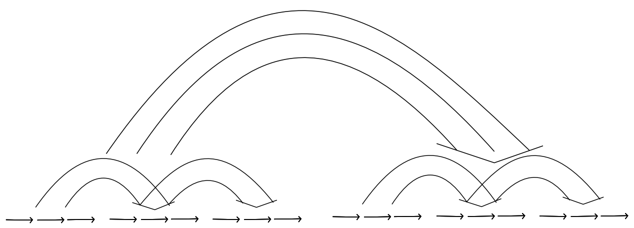

Figure 1. *A* 3-*arrow.*

Let S denote a set. We define a 0-path to be an element of S. We define the source and target of a 0-path to be $\circ$. For $d \geq 1$ we define a d-path P to be a sequence $(a_0, a_1, ..., a_l)$ of $d-1$-paths with the same source and target. We then define the source of P to be a_l and the target of P to be a_0. We define the d^{th} coordinate of the $d-1$-path a_r to be $\frac{r}{l}$.

For $i = 0, ..., d-1$, we define the set of i-gens of P to be the disjoint union of the sets of i-gens of $a_0, a_1, ..., a_l$. We define the set of d-gens of P to be the singleton set $\{P\}$. For $1 \leq i \leq d$, each 0-gen x of P is a 0-gen of a unique $i-1$-gen, and the i^{th} coordinates of these $i-1$-gens associate a sequence of coordinates $\tilde{x} = (x_1, ..., x_d) \in [0,1]^d$ to x.

Suppose we have a map ϕ from the notes of a piece written on the stave to the set of 0-gens of a d-path P. For example, consider Bach's Two Part Invention No. 9. Let $S = \{\bullet\}$. Let $s = \bullet$, a 0-path to symbolise a semiquaver of our piece. Let $q = (s, s)$, a 1-path to symbolise a quaver of our piece. Let $c = (q, q)$, a 2-path to symbolise a crotchet of our piece. Let $b = (c, c, c)$, a 3-path to symbolise a bar of our piece. Let $f = (b, b, b, b)$, a 4-path to symbolise four bars of our piece, and $t = (b, b)$, a 4-path to symbolise two bars of our piece. Let $P = (f, f, f, f, f, f, f, f, t)$, a 5-path to symbolise our piece, partitioned into eight four bar sections followed by a two bar section. The 0-gens of P correspond in this way to the quavers of our piece. We have partitioned the piece using i-gens to reflect the rhythmical structure of the score. A note corresponds to the semi-quaver at its start, which has coordinates in $[0,1]^5$.

Suppose we now have an element η of $\mathbb{R}^n$, and an $n \times d$ real matrix ζ. Associated to each note p of our piece, we have a vector $\eta + \zeta\tilde{\phi(p)}$ in $\mathbb{R}^n$. This determines a set of amplitudes for harmonics of the notes of the piece.

Suppose we have an element u of $\mathbb{R}_{>0}$ and an element v of $\mathbb{R}^d$. Associated to each note p of our piece, we have an element $u\exp(v.\tilde{\phi(p)}))$. This gives us a set of real numbers, which, when multiplied by the number of semiquavers in the relevant notes, determine durations for the notes of our piece.

Example 6.1. We take $n = d = 5$, with Bach's Two Part Invention No. 9 represented as a 5-path P as above. We take $\eta = (0.1, 0.1, 0.1, 0.1, 0.1)$, and

$$\zeta = \begin{pmatrix} 0.01 & 0.02 & 0.06 & 0.12 & -0.4 \\ -0.02 & -0.03 & -0.08 & 0.15 & -0.08 \\ 0.03 & 0.04 & -0.1 & 0.03 & 0.16 \\ -0.04 & 0.05 & -0.02 & -0.06 & -0.24 \\ -0.05 & 0.01 & 0.04 & 0.09 & 0.32 \end{pmatrix}.$$

We take $u = 0.2$ and $v = (0.075, -0.06, 0.09, -0.09, 0.12)$.

We can allow the timbre of a note to vary for the duration of the note, and do this in such a way to represent the d-path P, as follows:

Let η_1, η_2, η_3 be elements of $\mathbb{R}^n$, and $\zeta_1, \zeta_2, \zeta_3$ be $n \times d$ real matrices. Associated to each note p of our piece, we have a vector $\eta_i + \zeta_i\tilde{\phi(p)}$ in $\mathbb{R}^n$, for $i = 1, 2, 3$. This determines three amplitudes for harmonics of a note of the piece, which we take to represent the amplitudes of our harmonics at the beginning, at m, and at the end of the note (for $i = 1, 2, 3$ respectively). Here, m is some point between the beginning and the end of our note. We linearly interpolate amplitudes for harmonics in between the beginning of the note, and m, and in between m and the end of the note.

How do we define m?

Suppose we have elements u_1, u_2 of $\mathbb{R}_{>0}$ and elements v_1, v_2 of $\mathbb{R}^d$. Associated to each note p of our piece, we have an element $u_i \exp(v_i.\tilde{\phi(p)})$), for $i = 1, 2$. Suppose for $i = 2$ and every note p this element is < 1. The elements $u_1 \exp(v_1.\tilde{\phi(p)})$). give a set of real numbers, which, when multiplied by the number of semiquavers in the relevant notes, determine durations for the notes of our piece. The element $u_2 \exp(v_2.\tilde{\phi(p)}))u_1 \exp(v_1.\tilde{\phi(p)}))$ is a real number less than the $u_1 \exp(v_1.\tilde{\phi(p)})$), which, when multiplied by the number of semiquavers in the relevant note, determines a point in time m between the beginning of the note and the end of the note.

Example 6.2. We take $n = d = 5$, with Bach's Two Part Invention No. 9 represented as a 5-path P as above. We take $\eta_1 = (0.1, 0.1, 0.1, 0.1, 0.1)$, and $\eta_2 = \eta_3 = (0, 0, 0, 0, 0)$. We take

$$\zeta_1 = \begin{pmatrix} 0.01 & 0.02 & 0.06 & 0.12 & -0.4 \\ -0.02 & -0.03 & -0.08 & 0.15 & -0.08 \\ 0.03 & 0.04 & -0.1 & 0.03 & 0.16 \\ -0.04 & 0.05 & -0.02 & -0.06 & -0.24 \\ -0.05 & 0.01 & 0.04 & 0.09 & 0.32 \end{pmatrix},$$

$$\zeta_2 = \begin{pmatrix} -0.1 & 0.04 & 0.06 & 0.06 & 0.08 \\ -0.04 & -0.03 & -0.04 & -0.03 & 0.8 \\ 0.03 & 0.02 & 0.02 & -0.3 & 0.32 \\ -0.02 & -0.01 & 0.2 & -0.12 & -0.24 \\ 0.01 & -0.1 & 0.08 & 0.09 & 0.16 \end{pmatrix},$$

$$\zeta_3 = \begin{pmatrix} -0.05 & 0.01 & 0.02 & 0.03 & 0.08 \\ -0.01 & -0.01 & -0.02 & -0.03 & 0.4 \\ 0.01 & 0.01 & 0.02 & -0.15 & 0.08 \\ -0.01 & -0.01 & 0.1 & -0.03 & -0.08 \\ 0.01 & -0.05 & 0.02 & 0.03 & 0.08 \end{pmatrix}.$$

We take $u_1 = 0.2$ and $v_1 = (0.075, -0.06, 0.09, -0.09, 0.12)$. We take $u_2 = \frac{3}{5}$ and $v_2 = (-0.2, 0.2, -0.4, 0.6, -0.16)$.

7. Reordering

When we retune a two part composition using a quiver, as in section 3, the resulting interpretation can have lines with quite a different structure from those in the original composition. For example, two successive notes in the original, the second of which is higher in pitch than the first, can become two successive notes in the retuned interpretation, the first of which is higher in pitch than the second. We can correct for this effect as follows:

Take a two part composition, interpreted as in section 3. For each bar, take the fundamental frequencies F_v of the notes $n_1, n_1 + 1, n_1 + 2, ..., n_1 + n_2 - 1$ of that bar in order, and take a permutation σ that permutes these notes so that these fundamental frequencies lie in increasing order. Take the fundamental frequencies of the notes $n_1, n_1 + 1, n_1 + 2, ..., n_1 + n_2 - 1$ of the original composition, and take a permutation τ that permutes these notes, so these fundamental frequencies lie in increasing order. Apply the permutation $\tau^{-1}\sigma$ to the notes of the bar of the retuned interpretation, and reorder the frequencies of the piece correspondingly. In this way, we obtain a reordered composition which, within bars at least, has fundamental frequencies that occur in the same order as those in the original composition.

We have not stated how we should permute notes in a bar so that fundamental frequencies lie in increasing order. To do this, we begin with our list of fundamental frequencies: $f_0, f_1, f_2, ..., f_{n_2-1}$. We let i run from 1 to $n_2 - 1$, and if $f_{i-1} > f_i$ we swap f_{i-1} and f_i. We then repeat this run $n_2 - 1$ times, at which point our frequencies will be in ascending order. This algorithm determines a permutation as required.

Our reason for restricting our reordering to within bars is the following: given two vertices of our quiver Q which are connected by an arrow, the corresponding notes of the retuned composition of section 3 will be close together in time, by construction, and consequently the corresponding notes of the reordered retuned composition will be fairly close together in time.

Example 7.1. We take example 3.4a, and reorder its frequencies as stated above.

We can also merely reorder notes in a bar so that fundamental frequencies lie in increasing order, as in the following example:

Example 7.2. Let $n = 5$. We set $S_0 = (1, 1, 1, 1, 1)$, and $S_2 = (-0.1, 0, 0, -0.1, 0)$, $S_3 = (0, -0.1, 0, 0, -0.1)$, $S_5 = (0, 0, -0.1, -0.1, -0.1)$. We set $T_0 = (1, \frac{63}{16}, 2, \frac{15}{2}, \frac{31}{2})$, and $T_2 = T_3 = T_5 = (1, 1, 1, 1, 1)$. Let $F_0 = 264$.

The piece here is Bach Invention No. 9, but the reordered retuned composition is perhaps structured more like a Bach prelude.

8. Enumeration of Notes

A general way to vary the timbre through a piece is to first define a bijection ϕ from a set $\{1, 2, 3, ..., N\}$ to the set of notes of the piece, where given a note $\phi(i)$ that is earlier in the piece than $\phi(j)$ we have $i < j$. We then specify functions $a_1, ..., a_n : \mathbb{N} \to \mathbb{R}$, and set the amplitude of the k^{th} harmonic of the note $\phi(i)$ to be $a_k(i)$, for $k = 1, ..., n$.

Example 8.1. For a piece written in 31 tone equal temperament ([13], section 9, cf. also [6]), we take $a_k(i) = \frac{500-i}{500}(1 - \frac{i}{100k})^3$.

9. Conclusion

We have introduced a number of different methods for varying timbre and tempo when programming a computer to play a musical score. Our techniques involve algebraic structures, with a topological combinatorial flavour (quivers, higher dimensional arrows) This should not be too surprising - associations are expressed algebraically, and musical data is often combinatorial. In some of our constructions we relate the problem of choosing timbre and tempo to the problem of tuning the piece - in both cases we are organising the notes of the score in a given way.

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