Quantifying strange property of attractors in quasiperiodically forced systems

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Abstract

Quasiperiodically forced systems are an important class of dynamical systems exhibiting quasiperiodic, strange nonchaotic, and chaotic attractors. A major concern is the identification of the parameter range in which each one of the attractors is present. In this work, based on the phase sensitivity proposed by Pikovsky and Feudel, we define a measure to quantitatively distinguish quasiperiodic attractors, strange nonchaotic attractors, and chaotic attractors. Particularly, we can determine the boundary points of these three attractors in parameter space. The reliability of this measure is verified in smooth and non-smooth systems.

Keywords: quasiperiodically forced system, strange nonchaotic attractors, phase sensitivity.

1. Introduction

There are a variety of ways to characterise attractors in dynamical systems. The primary concern of this work is to understand and distinguish how the nonlinear dynamical system responds to quasiperiodic forcing. Chiefly, as the parameter varies, the system can evolve from quasiperiodic to strange nonchaotic to chaotic attractor [1]. In particular, the study of strange nonchaotic attractors (SNAs) has attracted much attention from researchers [2–5]. The nonchaotic property of SNA can be asserted by the Lyapunov exponent, but it is more difficult in terms of the strange property. In this work we define a measure which allows not only to characterise the SNAs but it also quantifies the parameter range in which each of the three attractors occur in quasiperiodically forced systems.

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For the identification of the strange property of attractors, two methods are particularly effective, namely rational approximation and phase sensitivity [6]. For the rational approximation method, the frequency of the quasiperiodic system is usually taken as the reciprocal of the golden mean, which can be used to distinguish SNA by using Fibonacci number [5, 7]. The phase sensitivity can detect the sensitive to the initial phase, and the strange property of SNA can be verified by the stairstep diagram and the phase sensitivity exponent [4, 6, 8, 9]. In addition, recurrence quantification analysis, distribution of finite-time Lyapunov exponents, spectral distribution function, singular continuous power spectrum can also be used to verify the strange property of SNA [10–15]. Recurrence quantification analysis refers to the time when the state of dynamical system reappears. Different types of attractors can be distinguished on recursive matrix by the structure of recurrence plots. The distribution of the finite time Lyapunov exponent can also distinguish different types of attractors [16–19], and the distribution is plotted by counting the number of Lyapunov exponents that are greater and less than zero at a fixed time. This method is also used to describe certain mechanisms of SNAs, characterised by the differences in the distribution diagram. By counting the number of peaks larger than a threshold value in the power spectrum, the scale power law relation can be obtained. Then it can be used to distinguish the quasiperiodic attractors from SNAs [10]. In fact the main difficulty in the study of SNAs is to distinguish SNAs from quasiperiodic and chaotic attractors. Generally, two power spectra (discrete and continuous) are observed in dynamic systems. When the system presents periodic or quasiperiodic motion, the corresponding power spectrum is discrete. For a chaotic motion, a continuous power spectrum appears. When the attractor is an SNA, it is characterised by a singular continuous spectrum, which is between discrete and continuous [5]. However, the above methods cannot distinguish quantitatively quasiperiodic attractors, SNAs and chaotic attractors.

The remaining of this paper is organised as follows. In Sec. 2, based on the phase sensitivity proposed by Pikovsky and Feudel [5], we introduce a measure to characterise the strange property of attractors, and the basic idea of it is explained. In Sec. 3, we illustrate the effectiveness of this measure in smooth and non-smooth systems, and it is verified that this measure can distinguish the three types of attractors in quasiperiodic dynamics, namely, quasiperiodic, strange nonchaotic, and chaotic attractors. In Sec. 4, the measure is used to establish the range of SNA, and SNA is identified globally through the two-parameter transition diagram. Finally, in Sec. 5, we give the conclution.

2. Characterizing strange property of attractors

Consider following two-dimensional skew product systems:

$$\begin{aligned} x_{n+1} &= f\left(x_n, \theta_n\right), \\ \theta_{n+1} &= \theta_n + 2\pi\omega (\operatorname{mod} 2\pi), \end{aligned} \tag{1}$$

where ω is an irrational number. The Lyapunov exponent in the x direction is given by

$$\lambda_x = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{\partial f\left(x_k, \theta_k\right)}{\partial x_k} \right|. \tag{2}$$

If $\lambda_x > 0$, the attractor is chaotic. If $\lambda_x \leq 0$, the attractor is either a quasiperiodic attractor or an SNA. To verify the strange property of the attractors, Pikovsky and Feudel [6] proposed the method of phase sensitivity. We briefly recall the definition. From the map (1), we have the recurrence relation

$$\frac{\partial x_{n+1}}{\partial \theta_n} = f_\theta\left(x_n, \theta_n\right) + f_x\left(x_n, \theta_n\right) \frac{\partial x_n}{\partial \theta_n}. \tag{3}$$

Thus, starting from the initial derivative $\frac{\partial x_n}{\partial \theta_0}$, the derivatives at all points of the trajectory are

$$\frac{\partial x_n}{\partial \theta_0} = \sum_{k=1}^{n} f_\theta\left(x_{k-1}, \theta_{k-1}\right) R_{n-k}\left(x_k, \theta_k\right) + R_n\left(x_0, \theta_0\right) \frac{\partial x_0}{\partial \theta_0}, \tag{4}$$

where

$$R_M\left(x_m, \theta_m\right) = \prod_{i=0}^{M-1} f_x\left(x_{m+i}, \theta_{m+i}\right), \tag{5}$$

$R_0 = 1$, and n is the number of iterations. According to (2), $R_n \approx \pm\exp\left(\lambda_x n\right)$. If the attractor is not chaotic, then $\lambda_x \leq 0$. In such a case, $R_n(x_0, \theta_0)\frac{\partial x_0}{\partial \theta_0} \to 0$ as $n \to +\infty$. Then the equation (4) can be expressed as

$$\frac{\partial x_n}{\partial \theta_0} \approx S_n = \sum_{k=1}^{n} f_\theta(x_{k-1}, \theta_{k-1}) R_{n-k}(x_k, \theta_k). \tag{6}$$

If S_n tends to infinite as $n \to +\infty$ then the attractor is strange.

The maximum value of S_n after n iterations is denoted by

$$\tau_n = \max_{1 \leq i \leq n} \{S_i\}. \tag{7}$$

Take $\gamma_n = \min_{(x,\theta)} \tau_n(x, \theta)$ in some set of (x, θ), which is called the phase sensitivity. As the number of iterations increases, the value of τ_n increases accordingly. If S_n tends to infinite as $n \to +\infty$ then the attractor has infinite derivative with respect to the phase θ. In such a case, the attractor is strange.

The phase sensitivity can distinguish quasiperiodic attractors and SNAs, but it can not distinguish effectively SNAs and chaotic attractors. To overcome this difficulty, inspired by the phase sensitivity [5], we consider the following measure, the average

$$L = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\left|\frac{\partial x_{k+1}}{\partial \theta_0}\right|. \tag{8}$$

which characterises the strange property of attractors. In particular, this quantity yields to be more intuitive. When $L < 0$, the attractor is quasiperiodic. When $L > 0$, the attractor has the strange property. When L is not convergent, the attractor is chaotic. Thus, using the quantity L, the quasiperiodic attractor, SNA, and chaotic attractor can be distinguished. In Sec. 3, we will verify the reliability of this measure and compare it with the phase sensitivity through concrete examples.

Now, we give a brief explanation of the physical meaning of the measure L. For map (1), if $f(x_k, \theta_k) = p(x_k)g(\theta_k)$, it has an invariant curve $x = \phi(\theta)$, then

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{\partial \left[p\left(x_k\right) g\left(\theta_k\right) \right]}{\partial \theta_0} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{d \left[p\left(\varphi\left(\theta_k\right)\right) g\left(\theta_k\right) \right]}{d \theta_0} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{d \left[p\left(\varphi\left(\theta_0 + k\omega\right)\right) g\left(\theta_0 + k\omega\right) \right]}{d \theta_0} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| p'\left(\varphi\left(\theta_0 + k\omega\right)\right) \varphi'\left(\theta_0 + k\omega\right) g\left(\theta_0 + k\omega\right) + p\left(\varphi\left(\theta_0 + k\omega\right)\right) g'\left(\theta_0 + k\omega\right) \right| .
\end{aligned} \tag{9}
$$

By Birkhoff's ergodic theorem [5], we have

$$
\int_{S^1} \ln \left| p'\left(\varphi\left(\theta\right)\right) \varphi'\left(\theta\right) g\left(\theta\right) + p\left(\varphi\left(\theta\right)\right) g'\left(\theta\right) \right| d\theta. \tag{10}
$$

If the invariant curve is an SNA, the absolutely value of $p'\left(\varphi\left(\theta\right)\right) \varphi'\left(\theta\right) g\left(\theta\right)$ is larger than that of $p\left(\varphi\left(\theta\right)\right) g'\left(\theta\right)$. Thus,

$$
L \approx \int_{S^1} \ln \left| p'(\varphi(\theta))\varphi'(\theta)g(\theta) \right| d\theta = \int_{S^1} \ln \left| p'(\varphi(\theta)) \right| d\theta + \int_{S^1} \ln \left| \varphi'(\theta) \right| d\theta + \int_{S^1} \ln |g(\theta)| d\theta. \tag{11}
$$

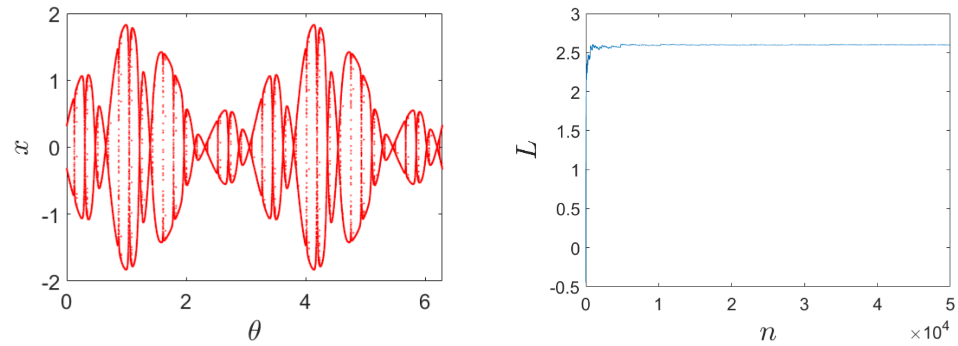
Since $p(x)$ and $g(\theta)$ are smooth, the main contribution to L is $\int_{S^1} \ln |\varphi'(\theta)|$. Therefore, L describes the oscillation of the invariant curve.

3. Examples

Example 1: The GOPY model (Grebogi et al [2], the seminal work on SNA),

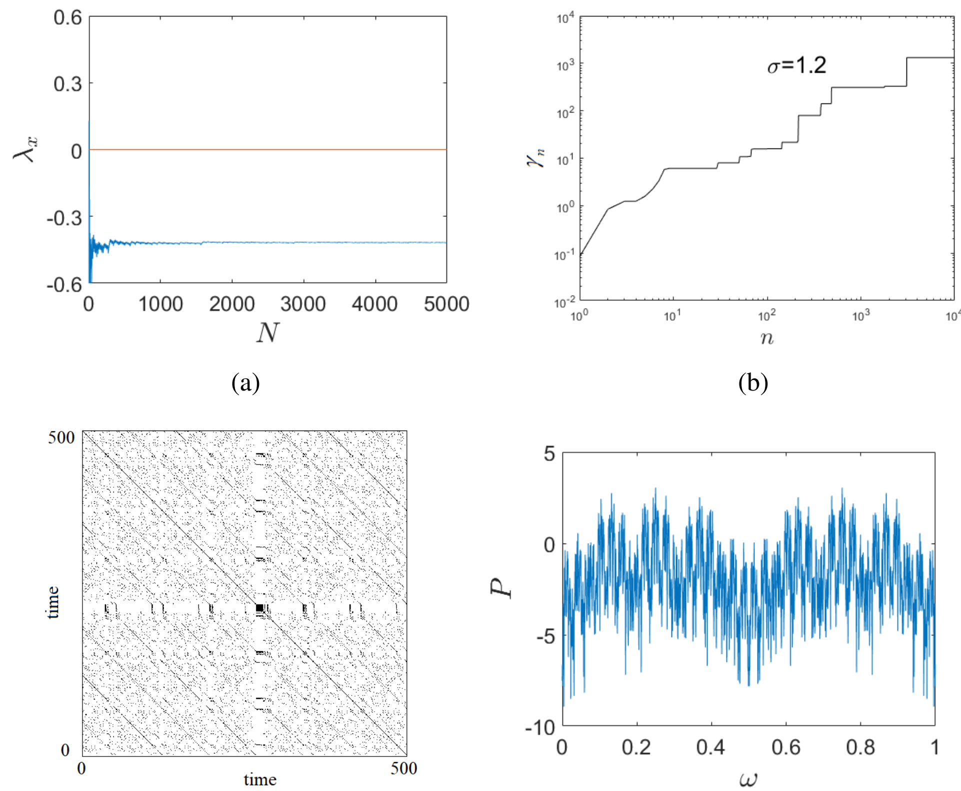
$$
\begin{aligned}
x_{n+1} &= 2\sigma(\tanh x_n) cos\theta_n, \\
\theta_{n+1} &= \theta_n + 2\pi\omega (\operatorname{mod} 2\pi).
\end{aligned} \tag{12}
$$

where ω is an irrational number. SNA is widespread in dynamical systems with quasiperiodic excitation. In Ref. [2], ω is the inverse of the golden mean ($\omega = (\sqrt{5} - 1)/2$). According to the analysis in Ref. [2], there is an SNA for $|\sigma| > 1$. Here, we take $\sigma = 1.2$ as an example to verify the strange property of SNA using the definition (8). In Fig. 1(a), the SNA is plotted. Because of the ergodicity in the θ direction, the orbit is the uniform distribution along the θ-axis, so the attractor is a dense set of points in θ direction [20]. The numerical results show that L is equal to 2.6, a positive constant, indicating that the attractor has the strange property, as shown in Fig. 1(b). The corresponding to the Lyapunov exponent λ_x is negative (λ_x=-0.417), as shown in Fig. 2(a). Then we know that the SNA is non-chaotic, and the strange property can be verified by phase sensitivity [6], recurrence plots [21], and singular continuous power spectrum [7]. For phase sensitivity, the value of γ_n increases with the number of iterations, and it does not tend to be a fixed value, as shown in Fig. 2(b). For recurrence plots, it has the complex texture and the signature of disruptions on the recursive matrix, and the continuity of the diagonal can verify that the process is non-chaotic, as shown in Fig. 2(c). For singular continuous power spectrum, it exhibits a combination of continuous and discrete components, as shown in Fig. 2(d).



(a) (b)

Figure 1: For $\sigma = 1.2$, (a) the phase diagram; (b) the measure L.



(c) (d)

Figure 2: Verifying strange property of SNA for $\sigma = 1.2$: (a) Lyapunov exponent in the x direction; (b) phase sensitivity; (c) recurrence plots; (d) singular continuous power spectrum.

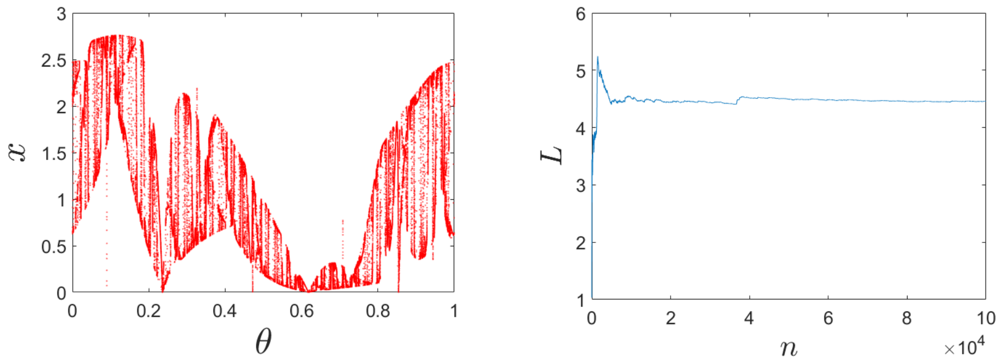
Example 2: The Ricker family with quasiperiodic excitation is considered [22], the skew product system $F : \mathbb{S}^1 \times \mathbb{R}^+ \to \mathbb{S}^1 \times \mathbb{R}^+$ is defined by

$$(\theta, x) \mapsto (\theta + \omega, f(x)g(\theta)), \tag{13}$$

where ω is still the inverse of the golden mean. $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ denotes the unit circle, $g(\theta) = sin(\pi\theta)$, and the function $f(x) = f_{\alpha,\beta}(x) = \alpha x e^{-\beta x}$. Then we get

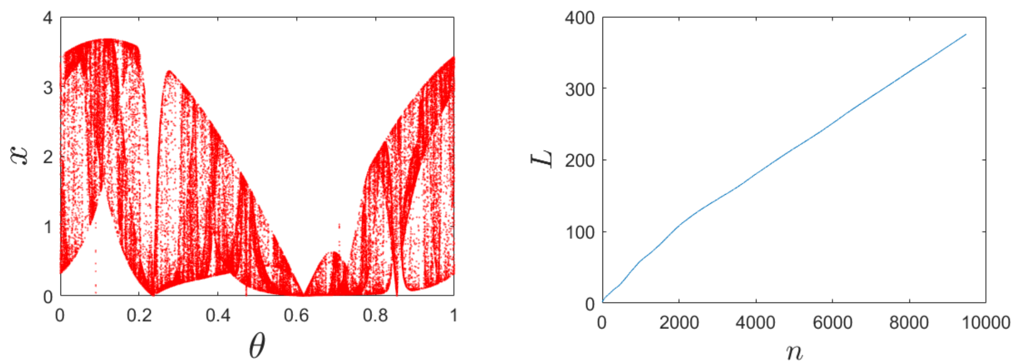
$$f(x)g(\theta) = f_{\alpha,\beta}(x)g(\theta) = \alpha x e^{-\beta x} sin(\pi\theta), \alpha > 0, \beta > 0. \tag{14}$$

We take $\beta = 2$, α is the control parameter. By combining theory and numerical simulation, the attractor is an SNA for $2 < \alpha < 16.5$. Here, $\alpha = 15$ is taken as an example to calculate the measure L of the attractor. The result shows that SNA has fractal geometric structure, as shown in Fig. 3(a). In addition, we can see that L eventually converges to 4.46. In the initial stage of calculation (the first $40,000$ iterations), the value of $\frac{1}{n}\sum_{k=0}^{n-1} \ln \left|\frac{\partial x_{k+1}}{\partial \theta_0}\right|$ is not stable, exhibiting a transient, but it tends to be stable in the following $60,000$ iterations, as shown in Fig. 3(b). The corresponding Lyapunov exponent λ_x is negative (λ_x=-0.09), as shown in Fig. 5(a). For $\alpha = 20$, the attractor is chaotic, the calculation result increases linearly, the value of L lends to infinity with increasing number of iterations, as shown in Figs. 4(a) and 4(b). λ_x is negative (λ_x=0.07), as shown in Fig. 5(b). Then the reliability of the definition (8) is verified again. This measure is convenient, it only needs to know the final stable value to judge the strange property of attractors. Now, we use the method of phase sensitivity to study these two sets of parameters. When the attractors are SNAs ($\alpha = 15$) and chaotic attractors ($\alpha = 20$), as the number of iterations increases, the value of γ_n increases continuously, as shown in Figs. 6(a) and 6(b). Then according to this concept by itself we cannot distinguish between these two kinds of attractors. In addition, in order to determine the strange property of SNA more accurately, we use the methods of recurrence plots and singular continuous power spectrum. The vertical and horizontal bands in Fig. 7(a) represent the recurrence of states at different epochs and are displayed at constant time periods. The result shows that the texture is complex. However, the diagonal lines in the recurrence plot is continuous, the SNA is nonchaotic. Moreover, it has a singular continuum spectrum, which is represented by δ-peaks at certain frequencies, but it is also continuous, as shown in Fig. 7(b).



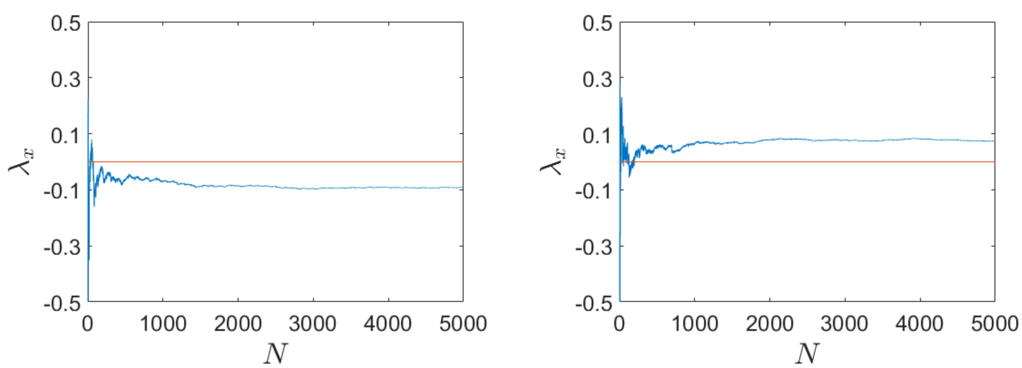
(a) (b)

Figure 3: For $\alpha = 15$, (a) the phase diagram; (b) the measure L.



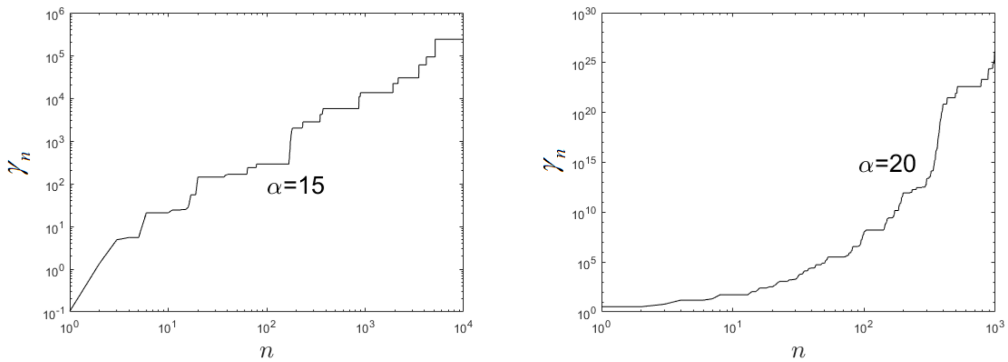
(a) (b)

Figure 4: For $\alpha = 20$, (a) the phase diagram; (b) the measure L.



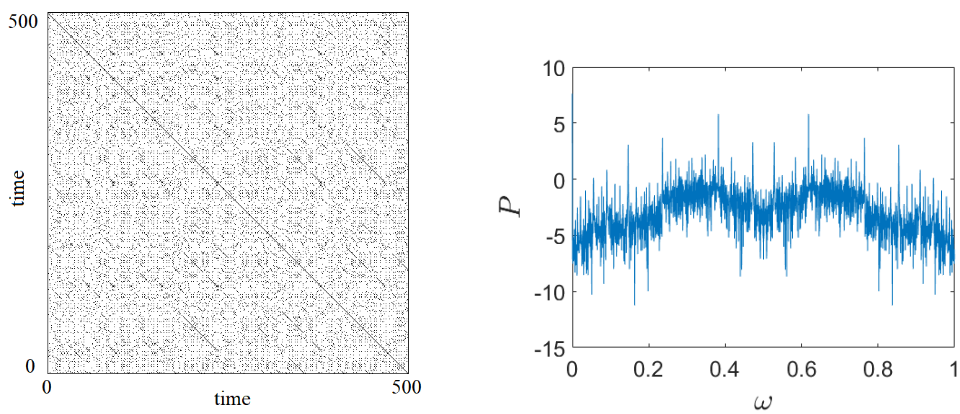
(a) (b)

Figure 5: Lyapunov exponent in the x direction, (a) $a = 15$; (b) $a = 20$.



(a) (b)

Figure 6: Phase sensitivity, (a) SNA; (b) chaotic attractor.



(a) (b)

Figure 7: Verifying strange property of SNA for $\alpha = 15$, (a) recurrence plots; (b) singular continuous power spectrum.

In order to verify the universality of the definition (8), we next investigate the properties of SNA in a non-smooth system. In the process of numerical calculation, we can successfully determine the properties of SNA by the measure L and the Lyapunov exponent.

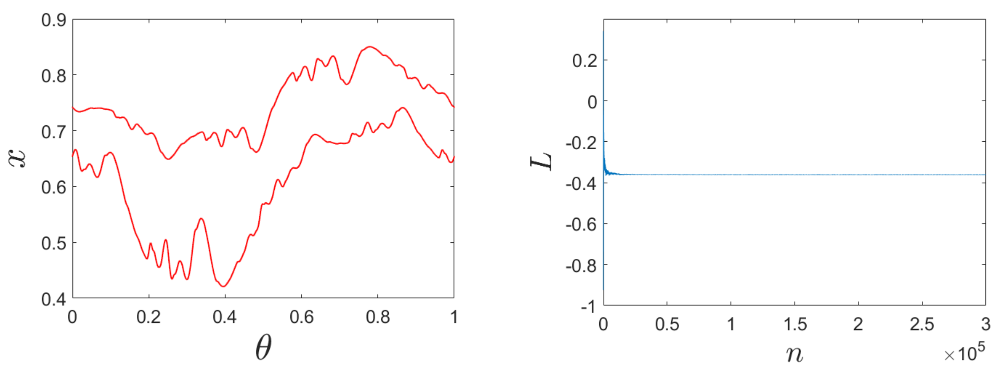
Example 3: The piecewise linear logistic map

$$x_{n+1} = \begin{cases} (r + \varepsilon \cos 2\pi\phi_n)\, x_n, & 0 \leq x_n < \bar{x}, \\ (a + \varepsilon \cos 2\pi\phi_n)\, x_n\, (1 - x_n)\,, & \bar{x} \leq x_n \leq 1, \end{cases} \quad \bar{x} - \frac{r}{a}, \qquad (15)$$
$$\phi_{n+1} = \phi_n + \omega (\mathrm{mod} 1).$$

We take $r = 1.95$ and $\varepsilon = 0.3$, a is to be considered as a control parameter. In Ref. [23], we learned that a quasiperiodic attractor can evolve into an SNA through the fractal route. For $a = 3.25$, the attractor is 2-tori quasiperiodic attractor which has no strange property, and $L < 0$, as shown in Figs. 8(a) and 8(b). When a is increased to 3.257, the stability of the attractor changes, becoming unstable. The local structure of the attractor shows a wrinkling shape, but the attractor still does not have the strange property, and the corresponding $L = -0.076$, but it is very close to zero, as shown in Figs. 9(a) and 9(b). When a is increased to 3.27, the attractor loses smoothness, as shown in Fig. 10(a). The measure L tends to a stable value 3.042 after 150,000 iterations, indicating that the attractor has strange property, as shown in Fig. 10(b). The Lyapunov exponent in the x direction is $\lambda_x = -0.01$, indicating that the attractor has nonchaotic property, as shown in 12(a). Therefore, the attractor is an SNA. When x is increased to 3.28, the attractor is chaotic, the Lyapunov exponent λ_x is equal to 0.006, as shown in Figs. 11(a) and 12(b). The measure L of chaotic attractors shows different characteristics from that of quasiperiodic attractors and SNAs. The measure L of quasiperiodic attractors and SNAs tend to a stable value, but that of the chaotic attractors increases linearly. If the number of iterations is infinity, the value of L corresponding chaotic attractors will go to infinity as in Fig. 11(b). Although both SNAs and chaotic attractors have the strange property, it can be concluded that chaotic attractors are stronger than SNAs by calculating the measure L.

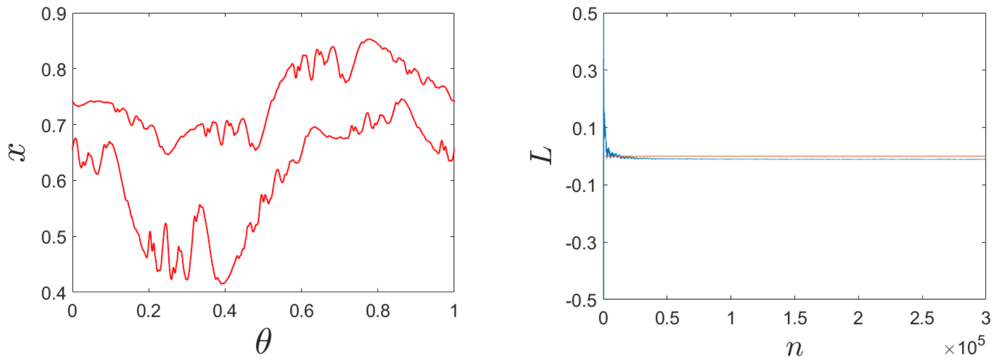
However, we study these three sets of parameters by the phase sensitivity, and the quasiperiodic attractor and SNA can be clearly distinguished, because the quasiperiodic attractor is s-mooth, γ_n is bounded. SNA is nonsooth, γ_n tends to infinite, as shown in Fig. 13(a). But for chaotic attractors, γ_n also tends to infinity, as shown in Fig. 13(b). We still cannot distinguish between SNAs and chaotic attractors. However, the measure L can do this, and three kinds of attractors can be distinguished.

For this set of parameters, recurrence plots has the continuity of diagonal lines and complex texture, and singular continuous power spectrum has continuous discrete components, which can indicate that SNA has the strange property, as shown in Figs. 14(a) and 14(b).



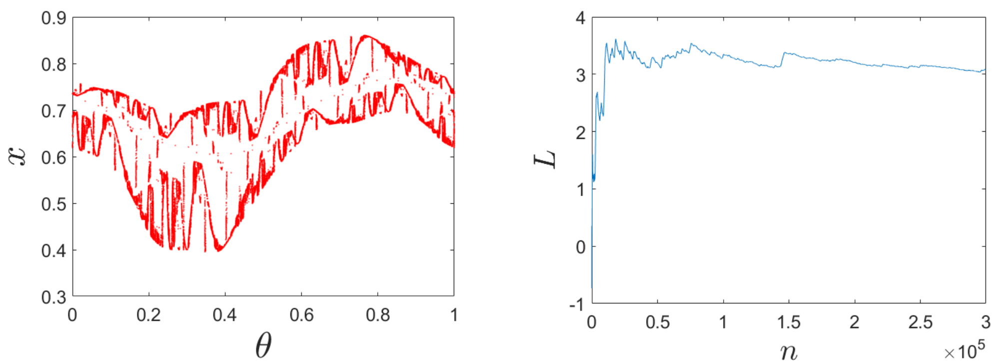
(a) (b)

Figure 8: For $a = 3.25$, (a) the phase diagram; (b) the measure L.



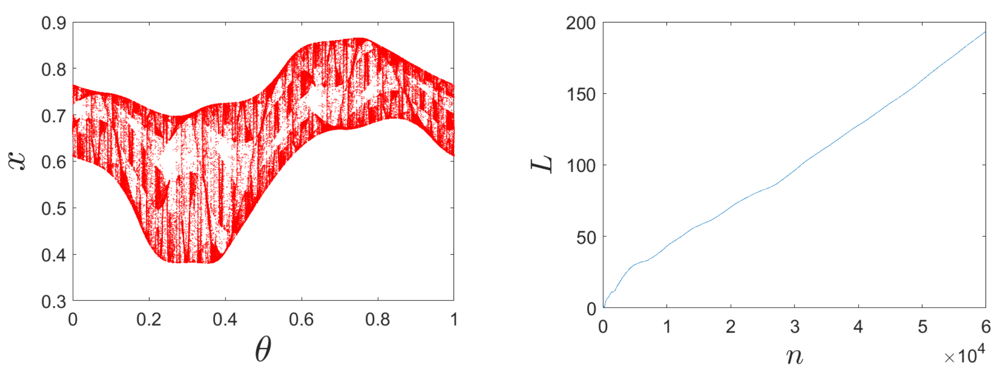
(a) (b)

Figure 9: For $a = 3.257$, (a) the phase diagram; (b) the measure L.



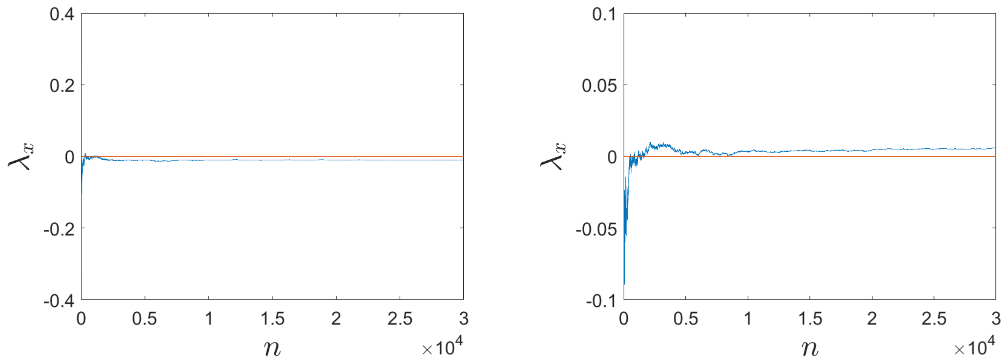
(a) (b)

Figure 10: For $a = 3.27$, (a) the phase diagram; (b) the measure L.



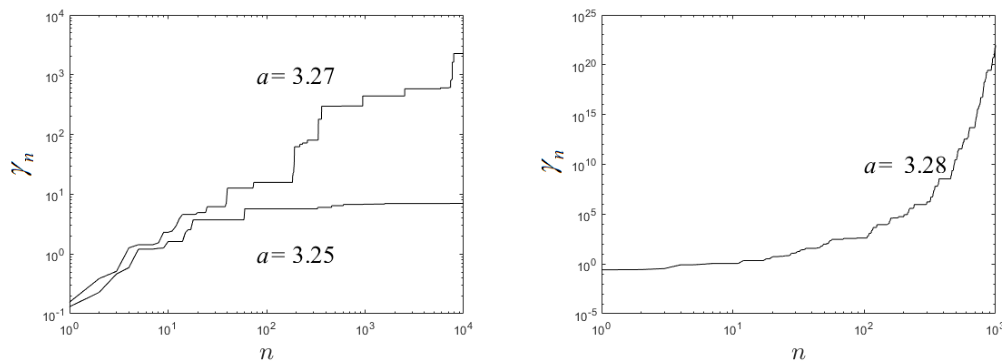
(a) (b)

Figure 11: For $a = 3.28$, (a) the phase diagram; (b) the measure L.



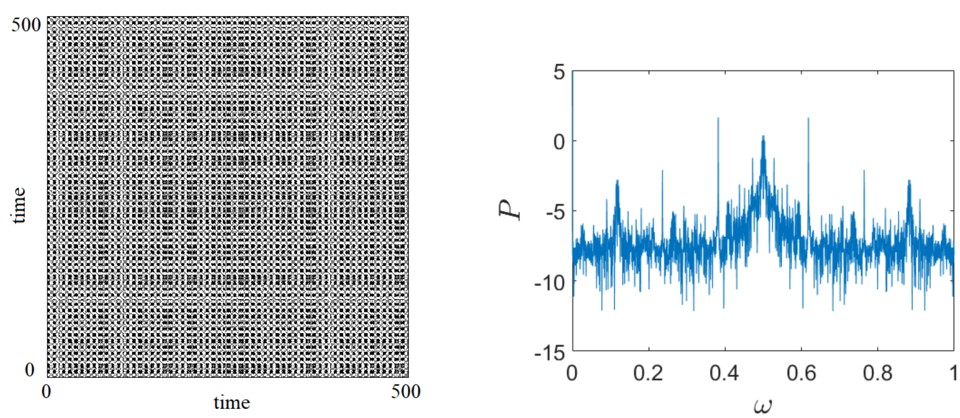
(a) (b)

Figure 12: Lyapunov exponent in the x direction, (a) $a = 3.27$; (b) $a = 3.28$.



(a) (b)

Figure 13: Phase sensitivity, (a) $a = 3.25$ and $a = 3.27$; (b) $a = 3.28$.



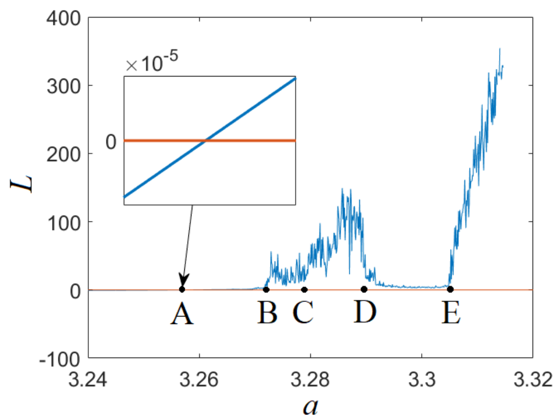
(a) (b)

Figure 14: Verifying strange property of SNA for $a = 3.27$, (a) recurrence plots; (b) singular continuous power spectrum.

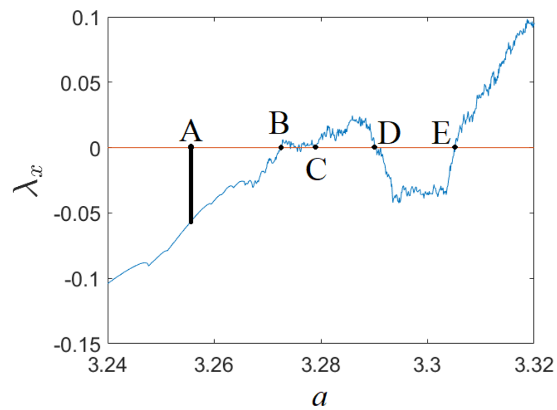
4. Determining the parameter ranges of the SNA

Verifying the strange property of SNA has been a difficult problem. Some numerical methods can be used to determine it, such as phase sensitivity, power spectrum, finite-time Lyapunov exponents, rational approximations, and so on, but these methods can not accurately find the boundary point among quasiperiodic attractors, SNAs, and chaotic attractors, that is, the parameter range corresponding SNA can not be determined accurately. The relevant numerical results will be demonstrated by concrete examples in this Section. The boundary between quasiperiodic attractor and SNA is found by calculating $L = 0$, while the boundary between SNA and chaotic attractor is found by determining the Lyapunov exponent $\lambda_x = 0$. Thus, the parameter interval corresponding to SNAs can be determined accurately.

In this Section, we take example 3 to demonstrate how to determine the parameter range corresponding to SNAs. We also take $r = 1.95$ and $\varepsilon = 0.3$, a is the control parameter. We calculate the measure L varying with the parameter a and find the parameter values with $L = 0$. L varying with the parameter a is shown in Fig. 15(a). We investigate the strange property of the attractor in the interval $[3.24, 3.32]$. For $a \in [3.24, 3.257)$, we have $L < 0$, the attractor does not have the strange property and is a quasiperiodic attractor. When $a = 3.257$, L is equal to 0, which is the boundary point (the point A is shown in Fig. 15) between the quasiperiodic attractors and SNAs. For $3.257 < a < 3.32$, we have $L > 0$, there are two kinds of attractors in this parameter interval, namely, SNAs and chaotic attractors. They can be distinguished by the definition (8). For $a = 3.272$, the Lyapunov exponent λ_x is equal to 0, and $a = 3.272$ is the boundary point (the point B is shown in Fig. 15) between the SNAs and chaotic attractors. Then we get that in the interval they are SNAs (between points A and B in Fig. 15). Therefore, there exists a transition region in which SNAs alternate with chaotic attractors for $a \in [3.272, 3.279]$, and the interval is between points B and C in Fig. 15. For $a \in (3.279, 3.29)$, there are chaotic attractors, which are between points C and D in Fig. 15. But the chaotic attractors become SNAs in the range of $a \in (3.29, 3.305)$, which appears between two chaotic regions. On the one hand, the result shows that the SNA appears not only between the quasiperiodic region and the chaotic region, but also between two chaotic regions. On the other hand, we know that the strange property of chaotic attractors is stronger than SNAs, the values of L is markedly different, as show in Fig. 15(a). In addition, the boundary points of these attractors can be accurately matched in Figs. 15(a) and 15(b), indicating that the definition (8) is feasible. In such cases, it is concluded that the measure L is an effective and reliable method to verify the strange property of attractors, which can distinguish among quasiperiodic attractors and SNAs. Lyapunov exponent, as also L, can distinguish SNAs and chaotic attractors, so the range of SNAs can be obtained.



(a)



(b)

Figure 15: For $r = 1.5$ and $\varepsilon = 0.3$, (a) L with the variation of a; (b) the Lyapunov exponent in the x direction with the variation of a.

In addition, we study the range of SNAs in co-dimension two parameter diagram. We take $r = 1.9312$, a and ε are control parameters. The global dynamics is shown in Fig. 16. In numerical the calculation, we use the measure L and the Lyapunov exponent to determine the kinds of attractors. The range corresponding to quasiperiodic attractors (QA) is denoted by white, satisfying $L < 0$ and Lyapunov exponent $\lambda_x < 0$. The gray region corresponds to SNAs, and the criterion of SNAs is that the Lyapunov exponent λ be less or equal to zero, and $L > 0$. Chaotic attractors are in the purple range, which has the criterion of L increasing linearly with the number of iterations and $\lambda_x > 0$. The escape regions are shown in black. We know that SNAs exist between quasiperiodic attractors and chaotic attractors, as shown in Fig. 16. Therefore, it is shown that the measure L is effective in verifying the strange property of attractors.

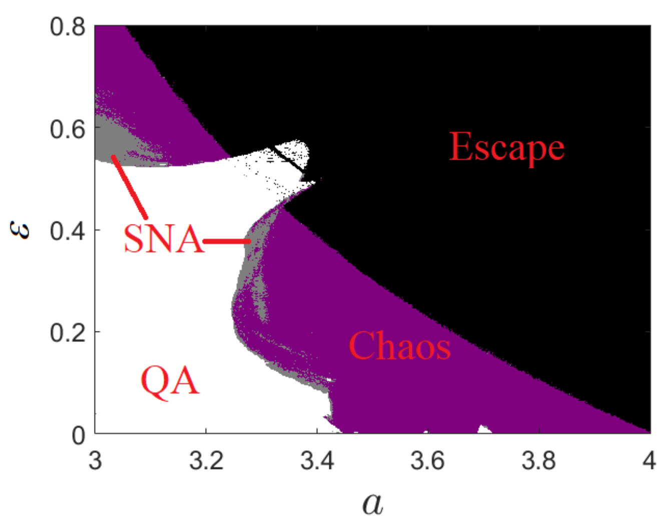


Figure 16: The two-parameter transition diagram.

5. Conclusions

In this paper, we propose a measure to distinguish three kinds of attractors, namely, quasiperiodic attractors, SNAs, and chaotic attractors. The effectiveness of definition (8) is illustrated in smooth and non-smooth systems. The measure L of SNAs is positive, that of quasiperiodic attractors is negative, and the measure of chaotic attractors tend to infinity linearly. It is shown that the strange property of chaotic attractors is larger than that of SNAs. Moreover, for a non-smooth system, the parameter range of SNAs is determined by the measure L and the Lyapunov exponent, and it is seen globally in the two-parameter transition diagram.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Nos. 11832009, 12002300, 12072291 and 12362002), and the Natural Science Foundation of Hebei Province (Grant No. A2021203013).

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