ABSTRACT WEIGHTED BASED GRADUAL SEMANTICS IN ARGUMENTATION THEORY

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ABSTRACT

Weighted gradual semantics provide an acceptability degree to each argument representing the strength of the argument, computed based on factors including background evidence for the argument, and taking into account interactions between this argument and others.

We introduce four important problems linking gradual semantics and acceptability degrees. First, we reexamine the inverse problem, seeking to identify the argument weights of the argumentation framework which lead to a specific final acceptability degree. Second, we ask whether the function mapping between argument weights and acceptability degrees is injective or a homeomorphism onto its image. Third, we ask whether argument weights can be found when preferences, rather than acceptability degrees for arguments are considered. Fourth, we consider the topology of the space of valid acceptability degrees, asking whether "gaps" exist in this space. While different gradual semantics have been proposed in the literature, in this paper, we identify a large family of weighted gradual semantics, called *abstract weighted based gradual semantics*. These generalise many of the existing semantics while maintaining desirable properties such as convergence to a unique fixed point. We also show that a sub-family of the weighted gradual semantics, called *abstract weighted* (L^p, λ, μ, A)*-based gradual semantics* and which include well-known semantics, solve all four of the aforementioned problems.

***K*eywords** Argumentation · Gradual semantics · Inverse problems

1 Introduction

In the context of Dung's abstract argumentation [1], we consider a set of abstract arguments and a binary attack relation between them, encoding these as a directed graph. Argumentation semantics then identify which sets of arguments are justified together by considering inter-argument interactions [2, 3, 4, 5]. Within the argumentation community, there has been increasing interest in so-called *ranking-based* semantics [6, 7, 8, 9]. These aim to identify a ranking over the arguments, with higher ranked arguments considered more justified (i.e., "less attacked") than arguments ranked lower (i.e., "more attacked"). One approach to creating such a ranking, called *gradual semantics* [10, 11, 12], involves associating a numerical *acceptability degrees* to all arguments within the system, with the final ranking computed according to the numeric ordering. Furthermore, some ranking-based semantics compute the acceptability degree of an argument based not only on the topology of the argumentation framework, but also based on some *initial weight* assigned to each argument. These *weighted gradual semantics*, exemplified by the weighted *max-based*, *card-based* and *h-categorizer* semantics, are widely studied and shown to have various desirable properties [13, 14, 10].

While a specific weighted gradual semantics takes an argumentation system and initial weights as input, and outputs the arguments' acceptability degrees, the *inverse problem* seeks to identify the initial weights for a given semantics, argumentation system and acceptability degrees [14]. Previous research used an iterative numeric technique to identify such initial weights for the three weighted gradual semantics mentioned above. However, no guarantee was made that such a solution always exists. This paper continues investigation into the inverse problem and in doing so obtains some critical insights. Namely, we ask ourselves the questions: Can we always find a solution to the inverse problem for the three weighted gradual semantics? If yes, can we generalise this result to a more general family of gradual semantics and how is this family caracterised? For any preference ordering on arguments (rather than simply numerical acceptability degrees), can we always find some weights on arguments so that the degree obtained will follow the preference ordering? Our work makes the following contributions.

- We establish a general family of "abstract weighted based gradual semantics". In doing so, we generalise the result of Pu et al. [15] to this large, general family of semantics. This result is used across the remainder of the paper.
- We introduce a sub-family of abstract weighted based gradual semantics, called abstract weighted (L^p, λ, μ, A)-based gradual semantics, which includes, among others, the three weighted gradual semantics mentioned above.
- We show that the inverse problem is solved for any semantics in this sub-family.
- We show that any preference ordering between the arguments is realised by some acceptability degree in all the semantics in this sub-family.
- We describe some topological properties of the acceptability degree space of semantics in this sub-family.

The importance of this sub-family of abstract weighted (L^p, λ, μ, A)-based gradual semantics goes beyond giving a uniform treatment of the three semantics above. It allows us, among other things, to define new gradual semantics for which the above inverse-type problems are solved. In turn, inverse problems can be used — for example — to automatically elicit and identify users' preferences based on the acceptability degrees over arguments, and as part of an argument strategy to help decide which arguments to advance based on the implicit weights attached to arguments by other dialogue participants. Such applications are not however the focus of the current paper, and are left for future work. Instead, we focus on the formal underpinnings of inverse problems and gradual semantics.

While inverse-type problems for weighted gradual semantics are new, we also want to highlight that there has been some recent research in this area. Skiba et al. [16] have studied whether, given a ranking and a ranking-based semantics, we can find an argumentation framework (without weights) such that the selected ranking-based semantics induces the ranking when applied to the framework. Kido and Liao [17] have studied whether, given noisy sets of acceptable arguments, we can find attack relations that explain the sets well in terms of acceptability semantics. This is similar to the work of Oren and Yun [18], where the authors identify the complexity class for the problem of identifying a set of attacks that will yield the desired acceptability degrees for a set of weighted arguments with respect to three weighted gradual semantics. Mailly [19] also studied realization problems for extension semantics but made used of auxiliary arguments, i.e., given a set of extensions S and using k auxiliary arguments, can we find m argumentation frameworks such that the union of their extensions is exactly equal to S? The existing body of research demonstrates the growing interest and diverse approaches in exploring various aspects of inverse problems in the argumentation domain.

The remainder of this paper is structured as follows. In Section 2, we briefly recap the background notions and formalism needed in this paper. In Section 3, we introduce the notion of scoring schemes and rephrase weighted gradual semantics as collections of functions mapping adjacency matrices to scoring schemes. Then, we formalise four inverse-type problems in terms of scoring schemes, define the family of abstract weighted based gradual semantics, and show that they always converge to a unique fixed point. In Section 4, we provide conditions for based scoring schemes to solve the four inverse-type problems. This leads us to define the sub-family of abstract weighted (L^p, λ, μ, A)-based gradual semantics which always solve the four problems in Section 5. In Section 6, we show multiple examples of concrete semantics in this class and introduce new semantics. We summarize and conclude our work in Section 7.

2 Background

This section introduces the necessary argumentation notions used in the rest of the paper. Our departure point in this paper is the three weighted gradual semantics described in [13]. As input, these semantics take in a *weighted argumentation framework* [20, 21, 22]. A weighted argumentation framework is a Dung's abstract framework (with arguments and binary attacks) augmented with a weighting function which assigns a number between 0 and 1 to each argument.

Definition 2.1 (WAF). A **weighted argumentation framework** (WAF) is a triple $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$ where $\mathcal{A}$ is a finite set of arguments; $\mathcal{D} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation; and $w : \mathcal{A} \to [0, 1]$ is a **weighting function** assigning an initial weight to each argument. The **underlying argumentation framework** (of $\mathcal{F}$) is $\mathcal{G} = \langle \mathcal{A}, \mathcal{D} \rangle$. The set of attackers of an argument $a \in \mathcal{A}$ is denoted $\texttt{Att}(a) = \{b \in \mathcal{A} | (b, a) \in \mathcal{D}\}$.

Upon ordering $\mathcal{A} = \{a_1, \ldots, a_n\}$, the **adjacency matrix** of $\mathcal{G}$ is the $n \times n$-matrix A with $A_{i,j} = 1$ if $(j, i) \in \mathcal{D}$ and $A_{i,j} = 0$ otherwise.

A gradual semantics is a function that takes as input a weighted argumentation framework and produces a *scoring function*. Note that the latter function is sometimes called a *weighting* [13], but we use our terminology to distinguish it from w.

Definition 2.2 (Gradual Semantics)**.** A **gradual semantics** Σ is a function that for each weighted argumentation framework $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$, associates a **scoring function** $\Sigma^{\mathcal{F}} : \mathcal{A} \to [0, 1]$, whose values are referred to as the **acceptability degrees** of the arguments.

Due to various desirable properties, the argumentation literature focuses on three weighted gradual semantics (see for example [13]), which are specific instantiations of the gradual semantics of Definition 2.2 above. We thus introduce these three semantics as concrete examples of gradual semantics.

Example 2.3. The **weighted max-based** gradual semantics $\Sigma_{\mathbb{MB}}$. Given $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$, the acceptability degree of an argument $a \in \mathcal{A}$ is defined by

$$\Sigma^{\mathcal{F}}_{\mathbb{MB}}(a) = \lim_{k \to \infty} \mathrm{MB}_k(a)$$

for the sequence of functions $\mathrm{MB}_k \colon \mathcal{A} \to [0, 1]$ defined recursively by setting $\mathrm{MB}_0(a) = w(a)$ for all $a \in \mathcal{A}$ and

$$\mathrm{MB}_{k+1}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}(a)} \mathrm{MB}_k(b)}$$

Example 2.4. The **weighted card-based** gradual semantics $\Sigma_{\mathbb{CB}}$. Given $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$, the acceptability degree of an argument $a \in \mathcal{A}$ is

$$\Sigma^{\mathcal{F}}_{\mathbb{CB}}(a) = \lim_{k \to \infty} \mathrm{CB}_k(a)$$

for the sequence of functions $\mathrm{CB}_k \colon \mathcal{A} \to [0, 1]$ defined recursively by $\mathrm{CB}_0(a) = w(a)$ for all $a \in \mathcal{A}$ and

$$\mathrm{CB}_{k+1}(a) = \frac{w(a)}{1 + |\mathtt{Att}^*(a)| + \frac{1}{|\mathtt{Att}^*(a)|} \cdot \sum\limits_{b \in \mathtt{Att}^*(a)} \mathrm{CB}_k(b)}$$

Here, $\mathtt{Att}^*(a) = \{b \in \mathtt{Att}(a) \mid w(b) > 0\}$ and if $\mathtt{Att}^*(a) = \emptyset$ the term involving division by $|\mathtt{Att}^*(a)|$ vanishes.

Example 2.5. The **weighted h-categorizer** gradual semantics $\Sigma_{\mathbb{HC}}$. Given $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$, the acceptability degree of an argument $a \in \mathcal{A}$ is

$$\Sigma^{\mathcal{F}}_{\mathbb{HC}}(a) = \lim_{k \to \infty} \mathrm{HC}_k(a)$$

for the sequence of functions $\mathrm{HC}_k \colon \mathcal{A} \to [0, 1]$ defined recursively by $\mathrm{HC}_0(a) = w(a)$ and

$$\mathrm{HC}_{k+1}(a) = \frac{w(a)}{1 + \sum\limits_{b \in \mathtt{Att}(a)} \mathrm{HC}_k(b)}$$

A crucial issue to address with these definitions is the need to prove that the limits exist and yield scoring functions. The convergence has been shown to exist (and be unique) for the three semantics mentioned above by Amgoud et al. (see in particular Theorems 7, 12, and 17 in [13]).

Using gradual semantics, one can obtain a ranking on arguments by using their acceptability degrees. With the three gradual semantics above, it is the case that a is at least as preferred as b iff a's acceptability degree is at least as high as b's. We use the following notation to describe preferences over arguments, where $a \trianglerighteq b$ denotes that a is at least as preferred as b, $a \simeq b$ iff $a \trianglerighteq b \wedge b \trianglerighteq a$, $a \triangleright b$ iff $a \trianglerighteq b \wedge a \not\trianglelefteq b$, and $a \trianglelefteq b$ iff $a \not\trianglerighteq b$.

Example 2.6. (taken from [14]). Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{D}, w \rangle$ be a WAF depicted in Figure 1 with weights $w(a_0) = 0.43, w(a_1) = 0.39, w(a_2) = 0.92$, and $w(a_3) = 0.3$. Table 1 lists the acceptability degrees and the associated rankings on arguments for the semantics $\Sigma_{\mathbb{MB}}, \Sigma_{\mathbb{CB}}$ and $\Sigma_{\mathbb{HC}}$ described in Examples 2.3, 2.4 and 2.5.

We now make a historical note about the problem which inspired this paper. The *inverse problem* seeks a set of initial weights which, when applied to a specific argumentation framework under the chosen semantics will result in a desired preference ordering, derived from the numeric acceptance degree computed for each argument.

In [14] the authors examined how weights could be computed for a given acceptance degree. The approach described there consists of two phases. In the first phase, a target acceptance degree is computed for each argument. In the second phase, a numerical method (the bisection method) is used to find the initial weights which lead to this acceptance degree. Since the bisection method is designed to find the zeros of a function with only one variable, and since changing

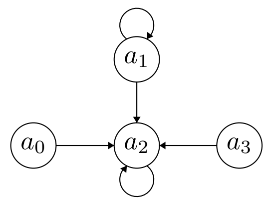


Figure 1: Graphical representation of a WAF

	a_0	a_1	a_2	a_3	
$w(a_i)$	0.43	0.39	0.92	0.3	Argument ranking
$\Sigma^{\mathcal{F}}_{\mathbb{MB}}(a_i)$	0.43	0.30	0.58	0.30	$a_1 \simeq a_3 \lhd a_0 \lhd a_2$
$\Sigma^{\mathcal{F}}_{\mathbb{HC}}(a_i)$	0.43	0.30	0.38	0.30	$a_1 \simeq a_3 \lhd a_2 \lhd a_0$
$\Sigma^{\mathcal{F}}_{\mathbb{CB}}(a_i)$	0.43	0.18	0.17	0.30	$a_2 \lhd a_1 \lhd a_3 \lhd a_0$

Table 1: Acceptability degrees of the arguments from Figure 1

the initial weight of one argument can affect the acceptance degree of other arguments, repeated applications of the bisection method are often necessary. The authors of [14] identify several strategies for selecting the argument to which the bisection method should be applied, and demonstrate that selecting the argument whose current acceptance degree is furthest away from its target acceptance degree works well in practice.

The inverse problem for $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ was the focus of previous research [14]. In Theorem 5.6 below we give a complete characterisation of the inverse problem for these semantics. In fact we give a positive answer to several other related problems, specifically the *projective preference ordering problem*, two variants of the *reflection problem*, and the *radiality problem*. These are described in detail next.

3 Rephrasing Weighted Gradual Semantics with Scoring schemes

We start by presenting a slightly different approach to weighted gradual semantics. For any A and B, let $\mathrm{Func}(A, B)$ denote the set of all functions $f: A \to B$ and $\mathrm{End}(A)$ the set of endomorphisms (i.e., self functions) $f: A \to A$. Given $T \in \mathrm{End}(A)$, a sequence $a_0, a_1, \dots \in A$ is called a **T-sequence** if $a_{k+1} = T(a_k)$ for all $k \geq 0$. Given a directed graph $\mathcal{G} = \langle \mathcal{A}, \mathcal{D} \rangle$, we may, and will, assume throughout that $\mathcal{A} = \{1, \dots, n\}$ and $n \geq 1$.

A scoring function $s: \mathcal{A} \to [0, 1]$ is identified with a vector $(x_1, \dots, x_n)$ in $X = [0, 1]^n$ which we call the **space of scores**. Similarly the **space of weights**, i.e., functions $w: \mathcal{A} \to [0, 1]$, can be identified with vectors in $W = [0, 1]^n$. We also set $K = [0, 1]^n$ which we call the **space of vertices**. We regard X, W, and K as subspaces of the Euclidean space $\mathbb{R}^n$, which we equip with the partial order $x \preceq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. Thus, they also inherit this order.

Definition 3.1. A **scoring scheme** is a function $\sigma: W \to X$. The space of all scoring schemes is $\mathrm{Scheme} = \mathrm{Func}(W, X)$.

The space of **acceptability degrees** of $\sigma \in \mathrm{Scheme}$ is $D_\sigma = \{\sigma(w) | \forall w \in W\} = \sigma(W)$, the image of W.

Let Σ be a weighted gradual semantics. For any argumentation framework $\mathcal{G} = \langle \mathcal{A}, \mathcal{D} \rangle$, we obtain a scoring scheme $\sigma^{\mathcal{G}}: W \to X$

$$\sigma^{\mathcal{G}}(w) = \Sigma^{\langle \mathcal{A}, \mathcal{D}, w \rangle}.$$

This means that a gradual semantics, which takes a weighted argumentation framework and returns a scoring function, can be seen as a function which takes an argumentation framework (without weights) and return a particular scoring scheme. We formalise this in the next explanatory comment.

Scholium. A weighted gradual semantics Σ can be viewed as a function

$$\Sigma: \{\text{argumentation frameworks } \mathcal{G}\} \xrightarrow{\mathcal{G} \mapsto \sigma^{\mathcal{G}}} \{\text{scoring schemes}\}.$$

By partitioning argumentation frameworks according to the number of their arguments $n = |\mathcal{A}|$ and using the notation above, a weighted gradual semantics is a collection of functions, one for each $n \geq 1$

$$\Sigma: \{\text{argumentation frameworks } \mathcal{G} \text{ with } |\mathcal{A}| = n\} \xrightarrow{\mathcal{G} \mapsto \sigma^{\mathcal{G}}} \mathrm{Scheme}.$$

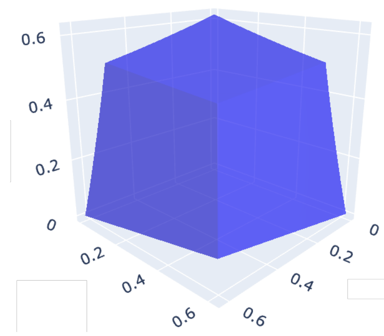


Figure 2: Representation (in blue) of the acceptability degree space of $\Sigma_{\mathbb{HC}}$ for a complete argumentation graph with 3 arguments. Note that the surfaces in the figure are non-linear.

The acceptability degree space of the semantics Σ for a graph $\mathcal{G}$ is $D_{\sigma^{\mathcal{G}}}$.

Since every argumentation framework $\mathcal{G}$ is determined by its adjacency matrix, a weighted gradual semantics is the same as a collection of functions, one for each $n \geq 1$,

$$\Sigma\colon \{\ n \times n \text{ adjacency matrices}\} \to \text{Scheme}\,.$$

If $A \in \text{Mat}_{n\times n}(\mathbb{R})$ we write $A \geq 0$ if $a_{i,j} \geq 0$. Denote

$$\text{Mat}^{+}_{n\times n}(\mathbb{R}) = \{A \in \text{Mat}_{n\times n}(\mathbb{R}) : A \geq 0\}.$$

Definition 3.2. An **abstract weighted gradual semantics** is a collection of functions, one for each $n \geq 1$,

$$\Sigma\colon\ \text{Mat}^{+}_{n\times n}(\mathbb{R}) \to \text{Scheme}\,.$$

Thus, by restriction to $n \times n$ adjacency matrices (those whose entries are only zeroes and ones) any *abstract* weighted gradual semantics restricts to an ordinary weighted gradual semantics.

Example 3.3. Let $\mathcal{G}$ be a complete graph with arguments $\mathcal{A} = \{1, 2, 3\}$. Then, the acceptability degree space of semantics $\Sigma_{\mathbb{HC}}$ for $\mathcal{G}$ is $D_{\sigma^{\mathcal{G}}}$, where $\sigma^{\mathcal{G}}$ is the associated scoring scheme and for every $w \in W$, $\sigma^{\mathcal{G}}(w) = \Sigma_{\mathbb{HC}}^{\langle\mathcal{A},\mathcal{D},w\rangle}$. A representation of $D_{\sigma^{\mathcal{G}}}$ is shown in Figure 2. Note that the boundaries of the acceptability degree space are not linear.

3.1 The inverse problems

We generalise the inverse problem introduced in [14] to a family of *inverse-type problems*. These revolve around moving from acceptability degrees to weights, but consider different facets in doing so. Thus, we are — for example — interested in finding conditions on scoring schemes which guarantee that acceptability degrees satisfying certain properties are realised, and in this case calculate the weights that realise them. We formulate four inverse-type problems that are the focus of this paper.

The first problem asks whether, for a given scoring scheme σ, we can find an *inverse function* inv such that for any element $x \in X$, we can determine whether x is in the space of acceptability degrees of σ by testing whether $\text{inv}(x) \in W$, and in this case $w = \text{inv}(x)$ is a weight such that $\sigma(w) = x$.

Problem 3.4. The **inverse problem** for a scoring scheme $\sigma \in \text{Scheme}$ seeks a (computable) function $\text{inv}\colon X \to \mathbb{R}^n$ such that

(i) $x \in D_\sigma \iff \text{inv}(x) \in W$, and in this case

(ii) $\sigma(\text{inv}(x)) = x$.

The second problem (reflection) ask whether the scoring scheme is injective or a homeomorphism onto its image. This allows us to answer questions such as: "Given an argumentation framework and a gradual semantics Σ, is it possible to find two distinct weighting functions such that the acceptability degrees will be the same?"

Problem 3.5. The **reflection problem** for $\sigma \in \text{Scheme}$ asks whether $\sigma\colon W \to X$ is injective, i.e., every acceptability degree is obtained by a *unique* weight $w \in W$.

The closely related **topological reflection problem** asks whether $\sigma\colon W \to D_\sigma$ is a homeomorphism onto its image.

The third problem asks whether any element y of the space of scores can be projected into an element $x \in \mathcal{D}_\sigma$ by re-scaling it. Note that the acceptability degrees x have the same arguments' preference ordering and ratio between them as the acceptability degrees y. This will allow us to answer questions such as: "Given an argumentation framework and a gradual semantics Σ, is it possible to find a weighting function such that the acceptability degree of argument a is n times as much as the one of b?".

Problem 3.6. The **projective preference ordering problem** for a scoring scheme σ asks whether for any $y \in X$ there exists some $t > 0$ such that $t \cdot y \in D_\sigma$.

The fourth problem ask whether for any element x of the acceptability degree space of σ (D_σ), all the elements on the line from 0^n to x are in D_σ. In other words, a positive answer to this question means there are no "gaps" in the acceptability degree space along any line starting at the origin and a point in the acceptability degree space. This is important as it allows us to scale down any element along the line (via Problem 3.6).

Problem 3.7. The **radiality problem** for a scoring scheme σ asks whether for any $x \in D_\sigma$, the line segment $[0, x] = \{tx : 0 \le t \le 1\}$ in $\mathbb{R}^n$ is contained in D_σ. In this case we call D_σ *radial*.

Definition 3.8. We say that a weighted gradual semantics Σ has a solution for the inverse problem 3.4, the (topological) reflection problem 3.5, the projective preference ordering problem 3.6 or the radiality problem 3.7 if for any argumentation framework $\mathcal{G}$ the scoring scheme $\sigma^{\mathcal{G}} = \Sigma(\mathcal{G}) \in \mathrm{Scheme}$ has a solution for these problems.

One of the achievements of this paper is the identification of a large family of abstract weighted gradual semantics which includes the semantics $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ from Examples 2.3, 2.4, and 2.5 and for which all four inverse-type problems above are solved. Moreover, we give explicit and easily computable inverse functions $\mathrm{inv}_{\mathbb{MB}}, \mathrm{inv}_{\mathbb{CB}}$ and $\mathrm{inv}_{\mathbb{HC}}$ for these three semantics. See Theorem 5.6 below for a detailed statement of the result.

3.2 Based scoring schemes

In this section, we introduce the important notion of *based scoring schemes* as particular scoring schemes which are obtained through a weighted scoring base (described in Definition 3.18). This formalism is crucial to structure the following properties and discussion.

We first start with the necessary definitions to introduce the notion of a scoring base.

Definition 3.9. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called **increasing** if for any $x, y \in \mathbb{R}^n$

$$x \preceq y \implies f(x) \le f(y).$$

It is called **homogeneous** if for all $x \in \mathbb{R}^n$ and all $t \ge 0$

$$f(t \cdot x) = t \cdot f(x).$$

A function $f\colon X \to \mathbb{R}^n$ is called homogeneous (resp. increasing) if each component $f_i\colon X \to \mathbb{R}$ is homogeneous (resp. increasing). We say it is *non-negative*, written $f \ge 0$, if $f_i(x) \ge 0$ for all $1 \le i \le n$. It is *bounded* if there is some $M > 0$ such that $f_i(x) \le M$ for all $1 \le i \le n$. Let

$$\mathcal{B}^+\mathcal{HI}_n(X)$$

denote the set of all functions $f\colon X \to \mathbb{R}^n$ which are bounded, non-negative, homogeneous and increasing. For $n = 1$, we write $\mathcal{B}^+\mathcal{HI}(X)$.

The following two propositions show that the collection $\mathcal{B}^+\mathcal{HI}(X)$ is very rich. Their proofs are straightforward and left to the reader.

Proposition 3.10. *$\mathcal{B}^+\mathcal{HI}(X)$ contains all projections $\pi_i\colon X \xrightarrow{x \mapsto x_i} [0, \infty)$.*

Proposition 3.11. *Let $\psi_1, \dots, \psi_k \in \mathcal{B}^+\mathcal{HI}(X)$. Then the following functions obtained from $\psi_1, \dots, \psi_k$ are also in $\mathcal{B}^+\mathcal{HI}(X)$.*

(i) $\sum_j a_j \psi_j$ *for any* $a_1, \dots, a_k \ge 0$.

(ii) $\max\{\psi_1, \dots, \psi_k\}$.

(iii) $\sqrt[k]{\psi_1 \cdots \psi_k}$ *(geometric mean).*

(iv) $\sqrt[p]{\psi_1^p + \cdots + \psi_k^p}$ *for any* $p > 0$.

We can now introduce the fundamental notion of a scoring base.

Definition 3.12. A **scoring base** is a pair $(c, f) \in K \times \mathcal{B}^+\mathcal{HI}_n(X)$. That is,

- $c \in K$ and
- $f : X \to [0, \infty)^n$ is bounded, homogeneous, non-negative, and increasing.

The set of all scoring bases is denoted

$$\mathrm{Base}(X) = K \times \mathcal{B}^+\mathcal{HI}_n(X) \qquad (\subseteq K \times \mathrm{Func}(X, \mathbb{R}^n)).$$

From a scoring base, one can construct a particular endomorphism from X to X.

Definition 3.13. Let (c, f) be a scoring base, and $i \in \{1 \dots |n|\}$, be an index w.r.t arguments. Define $T_{(c,f)} \in \mathrm{End}(X)$ by

$$T_{(c,f)}(x)_i = \frac{c_i}{1 + f(x)_i}.$$

$T_{(c,f)}$ is well defined because $0 \leq c_i \leq 1$ and $f(x)_i \geq 0$ so $T_{(c,f)}(x) \in X$ for any $x \in X$.

A crucial question is: "Given an arbitrary element $x \in X$, will the successive application of $T_{(c,f)}$ lead to convergence and will it always converge to the same fix point?". To formalise this, we introduce the following notion of scoring dynamics so that the previous question can thus be rephrased as: "Is $T_{(c,f)}$ a scoring dynamics?".

Definition 3.14. A **scoring dynamics** is a function $T \in \mathrm{End}(X)$ such that

(i) T has a unique fixed point denoted $\mathrm{fix}(T) \in X$, and

(ii) Every T-sequence $x_0, x_1, \dots \in X$ is convergent to $\mathrm{fix}(T)$.

Denote by $\mathcal{D}yn(X)$ the collection of all scoring dynamics. The assignment $T \mapsto \mathrm{fix}(T)$ gives a function

$$\mathrm{fix}\colon \mathcal{D}yn(X) \to X.$$

To answer the question "Is $T_{(c,f)}$ a scoring dynamics?", we start by showing, in Theorem 3.16, that many order reversing functions of $\mathrm{End}(X)$ are scoring dynamics.

While our proof for Theorem 3.16 is inspired by the work of Pu, Zhang and Luo ([23, Theorems 1 and 2]) who showed that a unique fixed-point exists for the h-categorizer semantics, i.e., in the case where all arguments have an initial weight of 1; we are able to prove such a result for a much larger class of functions.

Definition 3.15. A function $f : X \to \mathbb{R}^n$ is called **order preserving** if $x \preceq x' \implies f(x) \preceq f(x')$ for any $x, x' \in X$. It is called **order reversing** if $x \preceq x' \implies f(x') \preceq f(x)$ for any $x, x' \in X$.

Theorem 3.16. *Let $f : X \to X$ be a function such that*

(a) f is order reversing, and

(b) there is some $0 < \alpha \leq 1$ such that for any $x \in X$ and any $0 \leq t \leq 1$

$$f(tx) \preceq \frac{1}{t + \alpha(1 - t)} f(x).$$

Then f has a unique fixed point $y \in X$. Furthermore, $y = \lim_{k\to\infty} x^{(k)}$ where $x^{(k)}$ is any sequence defined recursively by choosing $x^{(0)} \in X$ arbitrarily and $x^{(k+1)} = f(x^{(k)})$.

The next theorem shows that this generalisation applies to $T_{(c,f)}$ (obtained via a scoring base), and we will show below how this therefore applies to a very large class of gradual semantics (see Definition 3.19).

Theorem 3.17. *Let $(c, f) \in \mathrm{Base}(x)$ be a scoring base. Then $T_{(c,f)} \in \mathcal{D}yn(X)$.*

A weighted scoring base is a function which associate a scoring base to each element of W by making use of two functions which instantiate each of the elements of the scoring base.

Definition 3.18. A **weighted scoring base** is a function $b\colon W \to \mathrm{Base}(X)$. More specifically, $b = (\kappa, \varphi)$ where κ and φ are functions

$$\kappa\colon W \to K$$
$$\varphi\colon W \to \mathrm{Func}(X, \mathbb{R}^n)$$

such that $(\kappa(w), \varphi(w)) \in \mathrm{Base}(X)$ for all $w \in W$, that is $\varphi\colon W \to \mathcal{B}^+\mathcal{HI}_n(X)$. The set of all weighted scoring bases is denoted w-Base(X).

The associated scoring scheme $\sigma_b \in \mathrm{Func}(W, X)$ is defined by

$$\sigma_b(w) = \mathrm{fix}(T_{(\kappa(w),\varphi(w))}).$$

That is, σ_b is the composition from X to X via a scoring base and scoring dynamics.

$$\sigma_b\ :\ W \xrightarrow{b} \mathrm{Base}(X) \xrightarrow{T} \mathcal{D}yn(X) \xrightarrow{\mathrm{fix}} X.$$

We will also write $\sigma_{(\kappa,\varphi)}$ for σ_b. We can thus obtain a function

$$\text{w-Base}(X) \xrightarrow{(\kappa,\varphi)\mapsto\sigma_{(\kappa,\varphi)}} \text{Scheme}\,.$$

A scoring scheme is **based** if it has the form $\sigma_{(\kappa,\varphi)}$ for some weighted scoring base $b = (\kappa, \varphi)$. Its acceptability degree space is denoted $D_{(\kappa,\varphi)}$.

Figure 3 summarises the notions and the results of Section 3.2.

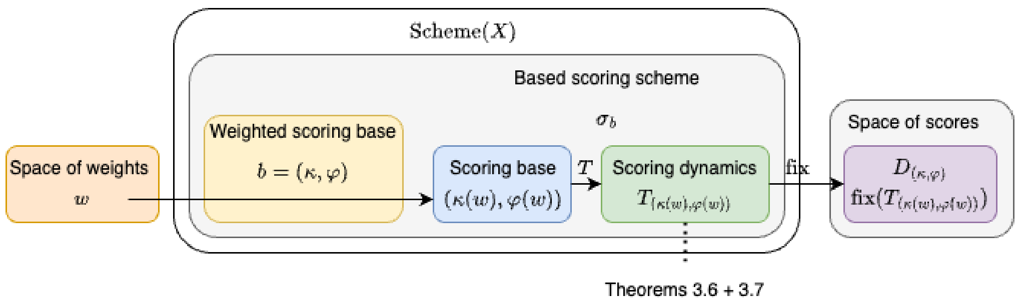


Figure 3: Representation of a based scoring scheme σ_b and the contributions of Section 3.2.

Definition 3.19. An **abstract weighted based gradual semantics** is one which is obtained by composing a function

$$\mathrm{Mat}^+_{n\times n}(\mathbb{R}) \xrightarrow{\beta} \text{w-Base}(X)$$

with the map w-Base$(X) \xrightarrow{b\mapsto\sigma_b}$ Scheme in Definition 3.18.

By construction, if Σ is an abstract weighted based gradual semantics then for any A the scoring scheme $\sigma(A) = \sigma_{\beta(A)}$ (Definition 3.18) is based.

4 Solving Inverse Problems

In this section, we give conditions on weighted scoring bases b so that the associated scoring scheme σ_b admits a solution for the inverse problems described in Section 3.1.

We start by introducing the necessary definitions used to describe our constraints.

The **support** of $x \in \mathbb{R}^n$ is the subset of $\{1, \dots, n\}$ defined as $\mathrm{supp}(x) = \{1 \le i \le n : x_i \neq 0\}$.

Definition 4.1. We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ **preserves supports** if for every $x \in \mathbb{R}^n$

$$\mathrm{supp}(x) = \mathrm{supp}(f(x))$$

We say that $f\colon \mathbb{R}^n \to Y$, for some set Y, is **independent of supports** if

$$\mathrm{supp}(x) = \mathrm{supp}(y) \implies f(x) = f(y).$$

In this case, if $I \in \wp(\{1, \dots, n\})$ we write $f^I \in Y$ for the value of f on all $x \in X$ with support I.

As we will see in the following theorems, some of the essential constraints for a weighted scoring base (κ, φ) to have a solution for the aforementioned inverse problems are that κ preserves supports and that φ is independent of supports.

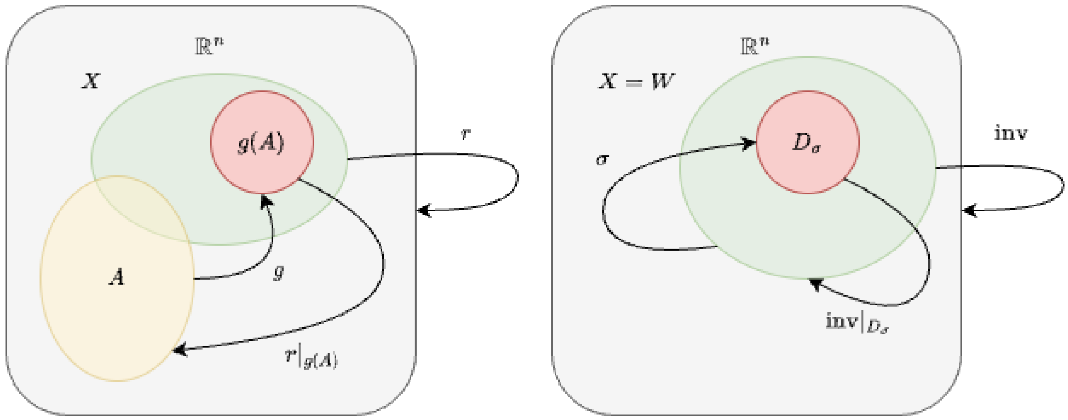


Figure 4: Representation of the discerning right inverse r of g (left) and a discerning right inverse of a scoring scheme σ (right).

Definition 4.2. Let A be a subset of $\mathbb{R}^n$. A **discerning right inverse** for $g\colon A \to X$ is a function $r\colon X \to \mathbb{R}^n$ such that $r(x) \in A \iff x \in g(A)$ and in this case $g(r(x)) = x$. That is, $g \circ r|_{g(A)} = \mathrm{id}_{g(A)}$.

Remark. The inverse problem 3.4 for a scoring scheme σ seeks a computable discerning right inverse inv for σ (see Figure 4).

The next theorem gives conditions on a weighted scoring base $(\kappa, \varphi) \in$ w-Base(X) so that the associated based scoring scheme $\sigma_{(\kappa,\varphi)}$ admits a solution to the inverse and the reflection problems (3.4, 3.5). It also gives conditions for it to admit a solution to the topological reflection problem.

Theorem 4.3. *Let $b = (\kappa, \varphi)$ be a weighted scoring base. Assume that*

(i) $\kappa\colon W \to K$ is the restriction of $\tilde{\kappa}\colon \mathbb{R}^n \to \mathbb{R}^n$ which is bijective and preserves supports.

(ii) $\varphi\colon W \to \mathrm{Func}(X, \mathbb{R}^n)$ is independent of supports.

Then $\sigma_{(\kappa,\varphi)}\colon W \to X$ solves the inverse and reflection problems. That is, $\sigma_{(\kappa,\varphi)}$ is injective and admits a discerning right inverse inv$\colon X \to \mathbb{R}^n$ *which is defined as follows. Recall the notation of* supp *from Definition 4.1 and set*

$$\overline{\mathrm{inv}}(x)_i = x_i(1 + \varphi^{\mathrm{supp}(x)}(x)_i)$$
$$\mathrm{inv}(x) = \tilde{\kappa}^{-1} \circ \overline{\mathrm{inv}}.$$

If $\tilde{\kappa}$ is a homeomorphism and $\varphi\colon W \to \mathrm{Base}(X)$ is constant with value (c, f) such that $f\colon X \to [0, \infty)^n$ is continuous, then $\sigma_{(\kappa,\varphi)}\colon W \to X$ is a homeomorphism onto its image. That is, $\sigma_{(\kappa,\varphi)}$ solves the topological inverse problem.

Our next goal is to find conditions on a weighted scoring base (κ, φ) so that the projective preference ordering problem 3.6 admits a solution. We view $K = [0, 1]^n$ as a subspace of $\mathbb{R}^n$. Let $0 \in K$ denote the origin in $\mathbb{R}^n$. Recall that a neighbourhood U of 0 in K is the intersection of K with an open subset of $\mathbb{R}^n$ containing $0 \in \mathbb{R}^n$. Thus, there exists $\epsilon > 0$ such that U contains any $x \in \mathbb{R}^n$ such that $\|x\| < \epsilon$ and $x_i \geq 0$ for all $1 \leq i \leq n$.

Theorem 4.4. *Let $b = (\kappa, \varphi)$ be a weighted scoring base. Suppose that*

(i) $\kappa\colon W \to K$ preserves supports.

(ii) $\kappa(W) \subseteq K$ contains a neighbourhood U of $0 \in K$.

(iii) φ is independent supports.

Then the projective preference ordering problem 3.6 has a solution for the scoring scheme $\sigma_{(\kappa,\tau)}$.

Finally, we find conditions on a weighted scoring base (κ, φ) which guarantees that the scoring scheme $\sigma_{(\kappa,\varphi)}$ admits a solution to the radiality problem (Problem 3.7), i.e $D_{(\kappa,\varphi)}$ is radial subset of X. Recall the partial order $\preceq$ on $\mathbb{R}^n$.

Definition 4.5. A subset A of $K = [0, 1]^n$ is **$\preceq$-closed** in K, respectively **$(\mathrm{supp}, \preceq)$-closed** in K, if for any $a \in A$ and any $x \in K$,

$$x \preceq a \implies x \in A$$
$$x \preceq a \text{ and } \mathrm{supp}(x) = \mathrm{supp}(a) \implies x \in A.$$

Theorem 4.6. *Let $b = (\kappa, \varphi)$ be a weighted scoring base. Suppose that*

(i) $\kappa \colon W \to K$ *preserves supports.*

(ii) $\kappa(W)$ *is* $(\mathrm{supp}, \preceq)$*-closed in* K.

(iii) $\varphi \colon W \to [0,\infty)^n$ *is independent of supports.*

Then σ_b solves the radiality problem, i.e $D_{(\kappa,\varphi)} = \sigma_{(\kappa,\varphi)}(W)$ is radial.

By Theorems 4.3, 4.4 and 4.6 all four Problems 3.4–3.7 have positive solution for particular abstract weighted based gradual semantics. Now that we have defined the necessary constraints on weighted scoring bases for all the inverse problems, can we find a concrete family of based scoring schemes (scoring schemes obtained via a weighted scoring base) that have a positive solution to all the inverse problems?

5 (L^p, λ, μ, A)-based scoring schemes

In this section we construct a family of based scoring schemes which answers our four problems in the positive, and link this family back to the max-based, h-categoriser and card-based semantics.

We fix

- $1 \leq p \leq \infty$,
- $\mu \colon \mathbb{R}^n \to (0,1]^n$ independent of supports,
- $\lambda \colon \mathbb{R}^n \to \mathrm{End}([0,\infty)^n)$ independent of supports,
- $n \times n$ matrix $A = (a_{i,j})$ such that $A \geq 0$, i.e $a_{i,j} \geq 0$.

Recall that the L^p-norm on $\mathbb{R}^n$ is defined by

$$\|(x_1,\dots,x_n)\|_p = \begin{cases} (x_1^p + \cdots x_n^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \max\{x_1,\dots,x_n\} & \text{if } p = \infty \end{cases} \tag{1}$$

Given vectors $x, y \in \mathbb{R}^n$ we denote

$$x \wedge y = (x_1 y_1, \dots, x_n y_n). \tag{2}$$

We denote

$$a_{i,*} \ = \ \text{the } i^{\text{th}} \text{ row of } A.$$

With this data and notation, define $\kappa \colon W \to K$ and $\varphi \colon W \to \mathrm{Func}(X, [0,\infty)^n)$ as follows.

$$\begin{aligned} \kappa(w)_i &= \mu(w)_i \cdot w_i \\ \varphi(w)(x)_i &= \|\lambda(w)(a_{i,*}) \wedge x\|_p. \end{aligned} \tag{3}$$

Proposition 5.1. *The pair (κ, φ) defined in* (3) *is a weighted scoring base.*

Definition 5.2. The associated scoring scheme $\sigma_{(\kappa,\varphi)}$ defined in (3) is called the (L^p, λ, μ, A)-based scoring scheme $\sigma_{(L^p,\lambda,\mu,A)}$.

Theorem 5.3. *Any (L^p, λ, μ, A)-based scoring scheme has positive solution to the inverse problem, the reflection problem, the projective preference ordering problem and the radiality problem.*

The discerning right inverse function $\mathrm{inv} \colon X \to \mathbb{R}^n$ *has the form*

$$\mathrm{inv}(x)_i = \tfrac{1}{\mu(x)_i} x_i (1 + \|\lambda(x)(a_{i*}) \wedge x\|_p).$$

If μ and λ are constant functions then the topological reflection problem is answered positively.

Theorem 5.4. *Suppose that $p \leq q$ then*

$$D_{(L^p,\lambda,\mu,A)} \subseteq D_{(L^q,\lambda,\mu,A)}.$$

The weighted gradual semantics $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ in Examples 2.3–2.5, which are our prime source of interest, are examples of the framework we have developed.

Proposition 5.5. *The weighted gradual semantics $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ are abstract weighted (L^p, λ, μ, A)-based gradual semantics.*

As a result we can deduce the following theorem about these important gradual semantics.

Theorem 5.6. *The gradual semantics $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ solve the inverse, reflection, projective preference ordering, and the radiality problems. Moreover $\Sigma_{\mathbb{HC}}, \Sigma_{\mathbb{MB}}$ solve the topological reflection problem.*

It is easy to see that the discerning right inverses $\mathrm{inv}\colon X \to \mathbb{R}^n$ of these gradual semantics are given by the following formulae. We let A denote the adjacency matrix an argumentation framework $\mathcal{G} = \langle \mathcal{A}, \mathcal{D} \rangle$, whose individual elements are denoted $a_{i,j}$. Then

$$\mathrm{inv}^{\mathcal{G}}_{\mathbb{HC}}(x)_i = x_i(1 + \sum_j a_{i,j} x_j).$$

$$\mathrm{inv}^{\mathcal{G}}_{\mathbb{MB}}(x)_i = x_i(1 + \max_j \{a_{i,j} x_j\})$$

$$\mathrm{inv}^{\mathcal{G}}_{\mathbb{CB}}(x)_i = x_i \Big(1 + S(x)_i + \tfrac{1}{S(x)_i}(1 + \sum_{j \in \mathrm{supp}(x)} a_{i,j} x_j)\Big).$$

In the last formula $S(x)_i = \sum_{j \in \mathrm{supp}(x)} a_{i,j}$ and if $S(x)_i = 0$ then by convention $\frac{1}{S(x)_i} = 0$.

Remark 5.7. It is useful to identify $\wp(n)$ with $\{0,1\}^n$ as a subset of $\mathbb{R}^n$, thus we view $\mathrm{supp}(x)$ as a vector of zeros and ones. If $v \in \mathbb{R}^n$ we write $\mathrm{diag}(v)$ for the diagonal matrix with $\mathrm{diag}(v)_{ii} = v_i$. In matrix form, one can write $\mathrm{inv}_{\mathbb{HC}}, \mathrm{inv}_{\mathbb{CB}}$ and $\mathrm{inv}_{\mathbb{MB}}$ as follows. Set

$$S(x) = A \cdot \mathrm{supp}(x)$$

and write $S(x)^{-1} := (\frac{1}{S(x)_1}, \dots, \frac{1}{S(x)_n})$ where $\frac{1}{S(x)_i} = 0$ whenever $S(x)_i = 0$. Also let $max_c(M)$ return a column-wise operation which retains the maximum element in a column, but sets all other entries in the column to 0 (breaking ties at random).

Then

$$\mathrm{inv}^{A}_{\mathbb{HC}}(x) = (I + \mathrm{diag}(x) \cdot A) \cdot x$$

$$\mathrm{inv}^{A}_{\mathbb{CB}}(x) = \Big(I + \mathrm{diag}(S(x)) + \mathrm{diag}(S(x)^{-1}) \cdot A \cdot \mathrm{diag}(\mathrm{supp}(x))\Big) \cdot x$$

$$\mathrm{inv}^{A}_{\mathbb{MB}}(x) = \Big(I + max_c(A \cdot diag(x))\Big) \cdot x$$

This result allows us to easily and precisely compute the initial weights for an argumentation framework and associated acceptability degrees, without resorting to numerical methods as was done in [14].

6 New gradual semantics

With the machinery we have developed, there is a large collection of desirable weighted gradual semantics that we can define inspired by the content of Proposition 5.5. In this section we provide two new example semantics built using our work.

6.1 The Weighted Euclidean-based gradual semantics

The weighted Euclidean-based gradual semantics $\Sigma_{\mathbb{EB}}$ is the (L^p, λ, μ, A)-based semantics with $p = 2$ and $\mu = 1$ and λ constant with value $\mathrm{id} \in \mathrm{End}([0,\infty)^n)$. Thus, $\Sigma^{\mathcal{G}}_{\mathbb{EB}}(w)$ is obtained as the limit of the sequence of vectors $\mathrm{EB}_k \in X$ where EB_0 is chosen arbitrarily and

$$\mathrm{EB}_{k+1}(x) = \frac{w_i}{1 + \sqrt{\sum_{j \in \mathtt{Att}(i)} x_j^2}}.$$

From our results, this semantics positively answers the inverse problem, topological reflection problem, the projective preference order problem and the radiality problem. The discerning right inverse function $\mathrm{inv}_{\mathbb{EB}}\colon X \to \mathbb{R}^n$ is given by

$$\mathrm{inv}^{\mathcal{G}}_{\mathbb{EB}}(x)_i = x_i \left(1 + \sqrt{\sum_{j \in \mathtt{Att}(i)} x_j^2}\right).$$

As one can observe, it is easy to create basic variations of existing weighted gradual semantics while preserving important properties such as convergence or satisfactions to inverse problems.

6.2 Gradual semantics with remote attacks

In most weighted gradual semantics considered to date (including $\Sigma_{\mathbb{HC}}$, $\Sigma_{\mathbb{MB}}$, and $\Sigma_{\mathbb{CB}}$), only the acceptability degree of direct attackers on arguments in the graph $\mathcal{G}$ were considered. The intuition given by this is that the acceptability degrees of direct attackers already "compile" the acceptability degree of indirect attackers. However, there may be cases where it is useful to access the degree of these indirect attackers.

That is, given $i \in \mathcal{A}$, attacks on i are considered only by arguments $j \in \mathcal{A}$ such that $(j, i) \in \mathcal{D}$, i.e., j is a neighbour at distance 1 from i within the underlying argumentation framework. The "amount" of attack on i is formulated by means of a function α_i which only depends on x_j with $j \in \texttt{Att}(i)$. For example, in $\Sigma_{\mathbb{HC}}$ we get $\alpha_i(x) = \sum_{j \in \texttt{Att}(j)} x_j$ and in $\Sigma_{\mathbb{MB}}$ it is $\alpha_i(x) = \max_{j \in \texttt{Att}(i)} x_j$. More generally, if Σ is associated to a weighted scoring base (κ, φ) then the attack is formulated by means of the functions $\varphi_i \colon X \to [0, \infty)$.

Instead, it is conceivable to consider "secondary level" attacks on i by arguments j at distance greater than 1. Such attacks are loosely analogous to indirect defeaters in bipolar argumentation frameworks [24], but will have "weaker" strength which is achieved by multiplying the function φ by a discount factor $\delta < 1$.

Recall that if A is the adjacency matrix of A then the $(i, j)^{\text{th}}$ entry of A^k is equal to the number of (directed) paths of length k from j to i, thus the number of "secondary attacks" of j on i via a sequence of $k+1$ arguments (including i and j). In order to encode such "remote attacks" one should consider the matrix

$$A' = A + \delta_2 A^2 + \delta_3 A^3 + \cdots$$

where $0 \leq \delta_k \leq 1$ indicate the multiplicative strength factors of attacks from distance k.

As a concrete example, let us show how we modify $\Sigma_{\mathbb{HC}}$ to a new weighted gradual semantics $\Sigma_{\mathbb{HC},r}$ which takes into account remote attacks from distance 2. Choose some $0 < \delta < 1$. Suppose that $\mathcal{G}$ is an argumentation framework and let A be its adjacency matrix. Set

$$A' = A + \delta \cdot A^2.$$

Consider the abstract (L^p, λ, μ, A')-based scoring scheme with $p = 1$ and $\lambda = \mu = 1$. Then attacks on i are made using the formula

$$\frac{w_i}{1 + (A'x)_i} = \frac{w_i}{1 + (Ax)_i + \delta \cdot (A^2 x)_i}$$

and observe that $Ax_i = \sum_{j \in \texttt{Att}(i)} x_j$ and $(A^2x)_i = \sum_{\gamma \in \texttt{Att}_2(i)} x_j$ where $\texttt{Att}_2(i)$ is the set of all paths γ of length 2 in $\mathcal{G}$ ending at i, and the term x_j in the sum refers to the argument starting the path γ. We highlight that here, the degree of indirect attackers may be counted multiple times if there are multiple paths.

To generalise this result to remote attacks further away, it is reasonable to use the matrix

$$A' = \sum_{k=1}^{\infty} \delta^{k-1} A^k.$$

This second example shows the richness of the family of abstract weighted (L^p, λ, μ, A)-based gradual semantics.

7 Conclusions

We began this paper by rephrasing the notion of a weighted gradual semantics in terms of scoring schemes. This allowed us to formulate four problems, including the inverse problem of [14] around such scoring schemes. These four problems revolve around the uniqueness of solutions, and continuity of solutions, for weighted gradual semantics, and thus encode highly desirable properties of any semantics.

We introduced a very general class of semantics, the $(L^p, \gamma, \mu, A)-$based scoring schemes. Any semantics which can be encoded as such as scoring scheme (including the weighted h-categoriser, max-based and card-based semantics) answer the four problems we describe in the positive. We also demonstrated that an analytical solution to the original inverse problem can be derived, in contrast to the numeric approach used in [14].

Finally, building on the generality of our semantics and the desirable properties of our problems, we described two new semantics.

Our paper addresses several important theoretical gaps in argumentation theory, and facilitates computationally efficient solutions to the inverse problem. In the context of future work, the new semantics we propose in Section 6 are very interesting. One obvious avenue of such future work involves determining which of the properties described in [13] this family of new semantics comply with. In addition, the σ-acceptability-maximal semantics and ratio semantics differ significantly from existing semantics, and offer the possibility of making elicitation from expert knowledge easier, reducing knowledge engineering effort. Thus, exploring the properties of these semantics, and how closely they mirror human reasoning, is an avenue of work we intend to pursue.

A Appendix

This appendix contains the proofs of the results presented in this paper. We kept the order in which we prove our results in the same order in which we present them. All definitions, lemmas, theorems, and propositions in this appendix are intermediary results which are used in the proofs.

We will use the following notation: Given $a, b \in X$ such that $a \preceq b$, the **interval** between a and b in X is $[a, b]_X \stackrel{\text{def}}{=} \{x \in X \,:\, a \preceq x \preceq b\}$.

Proof of Theorem 3.16. Define a sequence $u^{(n)}$ in X by recursion by $u^{(0)} = 0$ and $u^{(n+1)} = f(u^{(n)})$ for every $n \geq 0$. Then $u^{(0)} \preceq u^{(1)}$ since $0 \in X$ is the minimum element in $(X, \preceq)$. Since f is order reversing, we get $u^{(0)} \preceq u^{(2)} \preceq u^{(1)}$. By induction one easily shows

(i) $u^{(2k)} \preceq u^{(2k-1)}$ for all $k \geq 1$,

(ii) $u^{(2k)} \preceq u^{(2k+2)}$ for all $k \geq 0$,

(iii) $u^{(2k+1)} \preceq u^{(2k-1)}$ for all $k \geq 1$.

Thus, the sequence $u^{(2k)}$ is increasing in the sense that $u_i^{(2k)}$ is an increasing sequence in $\mathbb{R}$ for every $i = 1, \ldots, n$, and bounded above by $1 \in X$. Similarly, $u^{(2k-1)}$ is decreasing and bounded below by $0 \in X$. We may therefore define

$$u^{(\text{ev})} = \sup_k u^{(2k)} = \lim_k u^{(2k)}$$
$$u^{(\text{odd})} = \inf_k u^{(2k-1)} = \lim_k u^{(2k-1)}.$$

It follows from (i) that $u^{(\text{ev})} \preceq u^{(\text{odd})}$.

Our next goal is to show that $u^{(\text{ev})} = u^{(\text{odd})}$. For every $k \geq 1$, set:

$$\pi_k = \sup\,\{0 \leq t \leq 1 \,:\, t \cdot u^{(2k-1)} \preceq u^{(2k)}\}.$$

The set on the right is not empty since $0 \cdot u^{(2k-1)} = 0 \preceq u^{(2k)}$, so:

$$0 \leq \pi_k \leq 1.$$

It is clear from the construction that for every $k \geq 1$

$$\pi_k \cdot u^{(2k-1)} \preceq u^{(2k)}.$$

By applying hypotheses (a) and (b) to this inequality and then using (ii), for every $k \geq 1$

$$\begin{aligned} u^{(2k+1)} &= f(u^{(2k)}) \\ &\preceq f(\pi_k \cdot u^{(2k-1)}) \\ &\preceq \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot f(u^{(2k-1)}) \\ &= \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot u^{(2k)} \\ &\preceq \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot u^{(2k+2)}. \end{aligned}$$

Hence $(\pi_k + \alpha(1 - \pi_k)) \cdot u^{(2k+1)} \preceq u^{(2k+2)}$, so by the definition of π_{k+1}

$$\pi_k + \alpha(1 - \pi_k) \leq \pi_{k+1}.$$

It follows that $1 - \pi_{k+1} \leq (1 - \alpha)(1 - \pi_k)$ for all $k \geq 1$. Therefore

$$1 - \pi_{k+1} \leq (1 - \pi_1)(1 - \alpha)^k \xrightarrow{k \to \infty} 0.$$

But $\pi_k \leq 1$ for all k, so $\lim_k \pi_k = 1$.

Consider some $\epsilon > 0$. Then $\pi_k > 1 - \epsilon$ for all $k \gg 0$. Equation (i) implies

$$(1 - \epsilon) \cdot u^{(2k-1)} \preceq \pi_k \cdot u^{(2k-1)} \preceq u^{(2k)} \preceq u^{(2k-1)}.$$

Letting $k \to \infty$, this implies $(1-\epsilon)u^{(\text{odd})} \preceq u^{(\text{ev})} \preceq u^{(\text{odd})}$. Since $\epsilon > 0$ was arbitrary,

$$u^{(\text{ev})} = u^{(\text{odd})}$$

as needed. We will denote $u^* := u^{(\text{ev})} = u^{(\text{odd})}$.

Write $f^n \colon X \to X$ for the n-fold composition of f with itself, i.e $f^n = f \circ \cdots \circ f$. Since $0 \in X$ is minimal, $u^{(0)} \preceq x$ for any $x \in X$. Since f is order reversing, $u^{(0)} \preceq f(x) \preceq u^{(1)}$. It then follows by induction that for all $k \geq 1$

$$\begin{aligned} &f^{2k-1}(X) \subseteq [u^{(2k-2)}, u^{(2k-1)}]_X \\ &f^{2k}(X) \subseteq [u^{(2k)}, u^{(2k-1)}]_X. \end{aligned}$$

It follows from Equation (ii) that

$$\begin{aligned} \bigcap_{n\geq 1} f^n(X) &= \bigcap_{k\geq 1} f^{2k-1}(X) \cap f^{2k}(X) \\ &\subseteq \bigcap_{k\geq 1} [u^{(2k-2)}, u^{(2k-1)}]_X \cap [u^{(2k)}, u^{(2k-1)}]_X \\ &= \bigcap_{k\geq 1} [u^{(2k)}, u^{(2k-1)}]_X \\ &= [u^{(\text{ev})}, u^{(\text{odd})}]_X = \{u^*\}. \end{aligned}$$

Notice that $u^* \in [u^{(2k)}, u^{(2k-1)}]_X$ for all $k \geq 1$. Since f is order reversing $f(u^*) \in [u^{(2k)}, u^{(2k+1)}]_X$ for all $k \geq 1$. Therefore $f(u^*) \in \{u^*\}$, so u^* is a fixed point of f.

If $y \in X$ is a fixed point of f then by induction $y \in f^n(X)$ for all n, so $y \in \cap_n f^n(X) = \{u^*\}$ and it follows that $y = u^*$. Therefore u^* is the unique fixed point of f.

Finally, choose some $x \in X$ and define a sequence by recursion $x^{(0)} = x$ and $x^{(n+1)} = f(x^{(n)})$. Then $u^{(0)} = 0 \preceq x$ and since f is order reversing, $u^{(0)} \preceq x^{(1)} \preceq u^{(1)}$. Then one proves by induction that $u^{(2k-2)} \preceq x^{(2k-1)} \preceq u^{(2k-1)}$ and $u^{(2k)} \preceq x^{(2k)} \preceq u^{(2k-1)}$ for all $k \geq 1$. By the sandwich rule $\lim_n x^{(n)} = u^*$. □

Proof of Theorem 3.17. Let $(c, f) \in \text{Base}(X)$. Thus $c \in K$ and $f \colon X \to [0,\infty)^n$ is bounded, homogeneous and increasing. Set $F = T_{(c,f)}$, see Definition 3.13. Recall that $F \in \text{End}(X)$.

Claim 1: F is order reversing.

Proof of Claim 1: Suppose that $x \preceq y$, i.e $x_i \leq y_i$ for all $i = 1, \ldots, n$. Since $f_i \colon X \to [0, \infty)$ is increasing

$$F(y)_i = \frac{c_i}{1 + f(y)_i} \leq \frac{c_i}{1 + f(x)_i} = F(x)_i.$$

It follows that $F(y) \preceq F(x)$. This proves Claim 1.

Claim 2: There exists $0 < \alpha \leq 1$ such that for any $x \in X$ and any $0 \leq t \leq 1$

$$F(tx) \preceq \frac{1}{t + \alpha \cdot (1-t)} \cdot F(x).$$

Proof of Claim 2: Say that the functions $f_i \colon X \to [0, \infty)$ are bounded above by $M > 0$. Set $\alpha = \frac{1}{1+M}$. Then for any $x \in X$

$$\frac{1}{1 + f(x)_i} \geq \frac{1}{1+M} = \alpha.$$

Since f_i is homogeneous, for any $x \in X$ and any $0 \leq t \leq 1$,

$$\begin{aligned}
F(t \cdot x)_i &= \frac{c_i}{1 + f_i(t \cdot x)} \\
&= \frac{c_i}{1 + t \cdot f_i(x)} \\
&= \frac{c_i}{(1-t) + t(1 + f_i(x))} \\
&= \frac{c_i}{1 + f_i(x)} \cdot \frac{1 + f_i(x)}{(1-t) + t(1 + f_i(x))} \\
&= F(x)_i \cdot \frac{1}{(1-t)\frac{1}{1+f_i(x)} + t} \\
&\leq \frac{1}{(1-t)\alpha + t} \cdot F(x)_i.
\end{aligned}$$

The theorem follows by applying Theorem 3.16 to F. □

Definition A.1. Let Y be a set. Let U be a subset of W. A function $f \colon W \to Y$ is called **independent of weights** on U if $f|_U \colon U \to Y$ is a constant function. The constant value of $f|_U$, i.e the element $y \in Y$ such that $f|_U \equiv y$, is called the **U-value** of f.

We say that f is independent of weights if it is independent of weights on W, i.e it is constant.

Recall that a topological space X is called *locally compact* if every $x \in X$ has a compact neighbourhood, i.e a neighbourhood whose closure is compact.

Lemma A.2. *Let X, Y be metric spaces, X compact. Let $D \subseteq X$ be a subspace and $C \subseteq Y$ locally compact subspace. Let $k \colon X \to Y$ be a continuous function such that*

(a) $k(D) = C$ and the restriction $k|_D \colon D \to C$ is a bijective function with inverse $f \colon C \to D$.

(b) For every $x \in X$,
$$k(x) \in C \iff x \in D.$$

Then $k|_D$ and f are homeomorphisms.

Proof. Since $k|_D$ is continuous, it remains to show that its inverse f is continuous. To do this it suffices to show that f is continuous at any $c \in C$.

Let $c \in C$. Since C is locally compact, choose a neighbourhood $c \in U \subseteq C$ whose closure $\overline{U}$ in C is compact. Set

$$E := f(\overline{U}) \subseteq D.$$

We will next show that E is a closed subset of X. Let (e_n) be a convergent sequence in X contained in E and set $x = \lim_{n\to\infty} e_n$. We must show that $x \in E$.

By construction $e_n = f(u_n)$ for some $u_n \in \overline{U}$. Since k is continuous, the sequence $k(e_n)$ is a convergent sequence in Y and

$$\lim_{n\to\infty} k(e_n) = k(x).$$

By the assumption (b) $k(e_n) \in C$ (because $e_n \in E \subseteq D$), and assumption (a) implies that

$$f(k(e_n)) = e_n = f(u_n).$$

Since f is injective, $k(e_n) = u_n \in \overline{U}$. Since $\overline{U}$ is compact it is closed in Y, hence $k(x) \in \overline{U} \subseteq C$. Assumption (b) implies that $x \in D$, and by assumption (a)

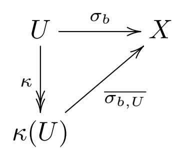
$$x = f(k(x)) \in f(\overline{U}) = E.$$

This completes the proof that E is a closed subset of X.

Since X is compact, so is E. Assumption (a) now implies that $k|_E \colon E \to \overline{U}$ is a bijective continuous function between compact metric spaces, hence it is a homeomorphism. Therefore its inverse $f|_{\overline{U}}$ is continuous and in particular, since U is open, f is continuous at c. This completes the proof. □

Theorem A.3. *Let $b\colon W \to \mathrm{Base}(X)$ be a weighted scoring base. Write $b = (\kappa, \varphi)$ as in Definition 3.18. Assume that φ is independent of weights on some $U \subseteq W$ with U-value $\psi\colon X \to [0,\infty)^n$. Then*

(a) *There exists a bijective function*
$$\overline{\sigma_{b,U}}\colon \kappa(U) \xrightarrow{\cong} \sigma_b(U) \subseteq X$$
which renders the following triangle commutative



(b) *The inverse of $\overline{\sigma_{b,U}}$ is given by the restriction to $\sigma_b(U) \subseteq X$ of the function*
$$\overline{\mathrm{inv}_{b,U}}\colon X \to \mathbb{R}^n$$
defined on each components by
$$\overline{\mathrm{inv}_{b,U}}(x)_i = x_i(1+\psi(x)_i).$$

(c) *For any $x \in X$,*
$$\overline{\mathrm{inv}_{b,U}}(x) \in \kappa(U) \iff x \in \sigma_b(U).$$

(d) *If $\kappa(U)$ is locally compact then $\overline{\sigma_{b,U}}$ is a homeomorphism.*

Proof. (a) and (b). Define $\overline{\sigma_{b,U}}\colon \kappa(U) \to \sigma_b(U)$ as follows. Given $c \in \kappa(U)$ choose $u \in U$ such that $\kappa(u) = c$ and set
$$\overline{\sigma_{b,U}}(c) = \sigma_b(u)$$
We need to show that this is independent of the choices. If $u' \in U$ is another preimage of c then by the definition of σ_b in Definition 3.18 and since φ is independent of weights on U and $\kappa(u) = c = \kappa(u')$
$$\sigma_b(u') = \mathrm{fix}(T_{(\kappa(u'),\varphi(u'))}) = \mathrm{fix}(T_{(\kappa(u),\varphi(u))})) = \sigma_b(u).$$
Hence $\overline{\sigma_{b,U}}(c)$ is independent of the choice of the preimages of c in U. By construction, $\overline{\sigma_{b,U}}(c) \in \sigma_b(U)$ and the diagram is commutative, that is
$$\overline{\sigma_{b,U}} \circ \kappa|_U = \sigma_b|_U.$$
In particular $\overline{\sigma_{b,U}}$ is onto $\sigma_b(U)$ which we henceforce denote $D_{b,U}$.

Look at $\overline{\mathrm{inv}_{b,U}}\colon X \to \mathbb{R}^n$ defined in the statement of the theorem and recall that $\psi \in \mathcal{B}^+\mathcal{HI}_n(X)$ (Definition 3.9). Consider some $c \in \kappa(U)$ and set $y = \overline{\sigma_{b,U}}(c)$. Then $c = \kappa(u)$ for some $u \in U$ and by construction
$$y = \overline{\sigma_{b,U}}(c) = \sigma_b(u) \in D_{b,U}.$$
By the definition of σ_b,
$$y = \mathrm{fix}(T_{b(u)})$$
and by Definition 3.13, $T_{b(u)} = T_{(c,\psi)}$ and
$$y_i = \frac{c_i}{1+\psi(y)_i} \qquad (1 \le i \le n).$$
Then $c_i = y_i(1+\psi(y)_i)$ and notice that $0 \le y_i \le 1$ so $y \in X$. It follows that $c = \overline{\mathrm{inv}_{b,U}}(y)$. We have shown that
$$\overline{\mathrm{inv}_{b,U}}|_{D_{b,U}} \circ \overline{\sigma_{b,U}} = \mathrm{id}_{\kappa(U)}.$$
It follows that $\overline{\sigma_{b,U}}$ is injective, hence bijective onto its image $D_{b,U}$ and $\overline{\mathrm{inv}_{b,U}}|_{D_{b,U}}$ is its inverse.

(c). Let $x \in X$. Suppose that $\overline{\mathrm{inv}_{b,U}}(x) \in \kappa(U)$. Then $\overline{\mathrm{inv}_{b,U}}(x) = \kappa(u)$ for some $u \in U$. Thus, $\kappa(u)_i = x_i(1+\psi(x)_i)$ for all $i = 1, \dots, n$. Definition 3.13
$$x_i = \frac{\kappa(u)_i}{1+\psi(x)_i} = T_{(\kappa(u),\psi)}(x)_i.$$
It follows that $T_{(\kappa(u),\psi)}(x) = x$. Since $\psi = \varphi(u)$ and $b(u) = (\kappa(u), \varphi(u))$ is a scoring base, it follows that $\mathrm{fix}(T_{b(u)}(x) = x$ so $\sigma_b(u) = x$, see Definition 3.18. Hence $x \in \sigma_b(U) = D_{b,U}$.

Conversely, suppose that $x \in D_{b,U}$. By construction there exists $u \in U$ such that $x = \sigma_b(u) = \overline{\sigma_{b,U}}(\kappa(u))$. By definition, $x = \text{fix}(T_{\kappa(u),\varphi(u)})$, hence

$$x_i = T_{(\kappa(u),\varphi(u))}(x)_i = T_{(\kappa(u),\psi)}(x)_i = \frac{\kappa(u)_i}{1+\psi(x)_i}.$$

It follows that

$$\kappa(u)_i = x_i(1+\psi(x)_i) = \overline{\text{inv}_{b,U}}(x)_i \qquad (1 \le i \le n).$$

Thus, $\overline{\text{inv}_{b,U}}(x) = \kappa(u) \in \kappa(U)$.

(d). Apply Lemma A.2 with the spaces $X = [0,1]^n = K$, $C = \kappa(U) \subseteq K$, $Y = \mathbb{R}^n$ and $D = D_{b,U}$, and with $k = \overline{\text{inv}_{b,U}}$ and $f = \overline{\sigma_{b,U}}$. Condition (a) of that lemma is guaranteed by parts (a) and (b) of this theorem which we have proven above, and condition (b) of Lemma A.2 is part (c). ☐

Recall the concept of preservation of support and independence of supports from Definition 4.1.

Proposition A.4. *The function $\overline{\text{inv}_{b,U}}$ defined in Theorem A.3 preserves supports.*

Proof. $x_i = 0 \iff x_i(1+\psi^I(x)_i) = 0 \iff \overline{\text{inv}_{b,U}} = 0.$ ☐

Lemma A.5. *Let (c, f) be a scoring base (Definition 3.12). Consider $T_{(c,f)}\colon X \to X$ (Definition 3.13 and Theorem 3.17). Then*

$$\text{fix}(T_{c,f})_i = 0 \iff c_i = 0.$$

Proof. Set $x = \text{fix}(T_{(c,f)})$. By construction of $T_{(c,f)}$ we get $c_i = x_i(1 + f(x)_i)$ and the result follows since $f \ge 0$ by hypothesis. ☐

Theorem A.6. *Let $b = (\kappa, \varphi)$ be a weighted scoring base. Then $\sigma_b\colon W \to X$ preserves supports if and only if $\kappa\colon W \to K$ preserves supports.*

Proof. Consider some $w \in W$. By Lemma A.5 and the definition of σ_b

$$\sigma_b(w)_i = 0 \iff \text{fix}(T_{(\kappa(u),\varphi(u)})_i = 0 \iff \kappa(w)_i = 0.$$

We have shown that $\text{supp}(\sigma_b(w)) = \text{supp}(\kappa(w))$ for all $w \in W$. So σ_b preserves supports (i.e $\text{supp}(\sigma_b(w)) = \text{supp}(w)$ for all w) if and only if κ preserves supports (i.e $\text{supp}(\kappa(w)) = \text{supp}(w)$ for all w). ☐

Let $\wp(n)$ be the power set of $\{1, \dots, n\}$. The support gives a function

$$\text{supp}\colon \mathbb{R}^n \to \wp(n)$$

Given $I \in \wp(n)$ write

$$\mathbb{R}^n(I) = \text{supp}^{-1}(I) = \{v \in \mathbb{R}^n : \text{supp}(v) = I\}.$$

More generally, if $A \subseteq \mathbb{R}^n$ we write $A(I) := A \cap \mathbb{R}^n(I)$, the set of all $x \in A$ with support I.

With the terminology of Definition 4.1, the statement that a function $f\colon W \to Y$, where Y is some set, is independent of supports is equivalent to the assertion that f is independent of weights on $W(I)$, see Definition A.1, for all $I \in \wp(n)$.

Proof of Theorem 4.3. Write $b = (\kappa, \varphi)$. Clearly $\{W(I)\}_{I\in\wp(n)}$ forms a partition of W. Since κ preserves supports, it easily follows that $\kappa(W(I)) = \kappa(W) \cap \mathbb{R}^n(I)$ for every $I \in \wp(n)$. In particular, $\{\kappa(W(I))\}_{I\in\wp(n)}$ forms a partition of $\kappa(W)$.

Set $D_b = \sigma_b(W)$. By Theorem A.6 $\sigma_b\colon W \to X$ preserves supports, hence $\sigma_b(W(I)) = D_b(I)$. Therefore $\{\sigma_b(W(I))\}_{I\in\wp(n)}$ forms a partition of D_b.

By Theorem A.3 for every $I \in \wp(n)$ there is an injective function

$$\overline{\sigma_{b,W(I)}}\colon \kappa(W(I)) \to X$$

with image $\sigma_b(W(I))$ such that $\overline{\sigma_{b,W(I)}} \circ \kappa|_{\kappa(W(I))} = \sigma_b|_{W(I)}$. There is also a discerning right inverse for $\overline{\sigma_{b,W(I)}}$,

$$\overline{\text{inv}_{b,W(I)}}\colon X \to \mathbb{R}^n.$$

Since $\{\kappa(W(I))\}_I$ form a partition of $\kappa(W)$, we define $\overline{\sigma_b}\colon \kappa(W) \to X$ by the requirement

$$\overline{\sigma_b}|_{\kappa(W(I))} = \overline{\sigma_{b,W(I)}}.$$

Since the image of $\overline{\sigma_{b,W(I)}}$ is $D_b(I)$ and since $\{D_b(I)\}_I$ forms a partition of D_b, it follows that $\overline{\sigma_b}$ is injective with image $D_b = \sigma_b(W)$. For any $w \in W$ set $I = \operatorname{supp}(\kappa(w))$. Since κ preserves supports $\operatorname{supp}(w) = I$ and we get $\overline{\sigma_b} \circ \kappa(w) = \overline{\sigma_{b,W(I)}} \circ \kappa(w) = \sigma_b|_{W(I)}(w)$. We deduce that

$$\overline{\sigma_b} \circ \kappa = \sigma_b.$$

Since $\{X(I)\}_{I \in \wp(n)}$ is clearly a partition of X, we define $\overline{\operatorname{inv}_b}\colon X \to \kappa(W)$ by the requirement

$$\overline{\operatorname{inv}_b}|_{X(I)} = \overline{\operatorname{inv}_{b,W(I)}}|_{X(I)}.$$

By the defining formula for $\overline{\operatorname{inv}_{b,W(I)}}|_{X(I)}$ in Theorem A.3(b), for any $x \in X$, if we set $I = \operatorname{supp}(x)$

$$\overline{\operatorname{inv}_{b,W(I)}}(x)_i = x_i(1 + \psi^I(x)_i).$$

Thus, $\overline{\operatorname{inv}_b}$ we have just defined coincides with $\overline{\operatorname{inv}}$ in the statement of the theorem. Notice that by this formula it is clear that $\overline{\operatorname{inv}_b}$ preserves supports since $\overline{\operatorname{inv}_b}_i = 0 \iff x_i = 0$.

Next we claim that $x \in \sigma_b(W) \iff \overline{\operatorname{inv}_b}(x) \in \kappa(W)$ and that in this case $\overline{\sigma_b} \circ \overline{\operatorname{inv}_b}(x) = x$. Indeed, suppose that $x \in \sigma_b(W)$ and set $I = \operatorname{supp}(x)$. Since σ_b preserves supports, $x = \sigma_b(w)$ for some $w \in W(I)$. Thus, $x \in \sigma_b(W(I))$. Since $\overline{\sigma_{b,W(I)}}$ is a discerning right inverse for $\overline{\sigma_{b,W(I)}}$ it follows that $\overline{\operatorname{inv}_b}(x) = \overline{\operatorname{inv}_{b,W(I)}}(x) \in \kappa(W(I))$ as needed. Furthermore, by the definition of $\overline{\sigma_b}$

$$\overline{\sigma_b}(\overline{\operatorname{inv}_b}(x)) = \overline{\sigma_b}(\overline{\operatorname{inv}_{b,W(I)}}(x)) = \overline{\sigma_{b,W(I)}}(\overline{\operatorname{inv}_{b,W(I)}}(x)) = x.$$

Conversely, suppose that $\overline{\operatorname{inv}_b}(x) \in \kappa(W)$. Set $I = \operatorname{supp}(x)$. Since $\overline{\operatorname{inv}_b}$ preserves supports, $\overline{\operatorname{inv}_b}(x) \in \kappa(W) \cap \mathbb{R}^n(I) = \kappa(W(I))$. By definition, $\overline{\operatorname{inv}_b}(x) = \overline{\operatorname{inv}_{b,W(I)}}(x)$ so $\overline{\operatorname{inv}_{b,W(I)}}(x) \in \kappa(W(I))$. Since $\overline{\operatorname{inv}_{b,W(I)}}$ is a discerning right inverse for $\overline{\sigma_{b,W(I)}}$, it follows that $x \in \sigma_b(W(I)) \subseteq \sigma_b(W)$. We have thus shown that $\overline{\operatorname{inv}_b}$ is a discerning right inverse for $\overline{\sigma_b}$.

Define $\operatorname{inv}_b\colon X \to \mathbb{R}^n$ by $\operatorname{inv}_b = \tilde{\kappa}^{-1} \circ \overline{\operatorname{inv}_b}$. Then inv_b coincides with inv in the statement of the theorem. Since $\sigma_b = \overline{\sigma_b} \circ \kappa$ we get

$$x \in \sigma_b(W) \iff x \in \overline{\sigma_b}(\kappa(W)) \iff \overline{\operatorname{inv}_b}(x) \in \sigma_b(W).$$

Furthermore, since $\overline{\operatorname{inv}_b}$ is a discerning right inverse of $\overline{\sigma_b}$

$$\sigma_b \circ \operatorname{inv}_b|_{D_b} = (\overline{\sigma_b} \circ \kappa) \circ (\tilde{\kappa}^{-1} \circ \overline{\operatorname{inv}_b}|_{D_b}) = \overline{\sigma_b} \circ \overline{\operatorname{inv}_b}|_{D_b} = \operatorname{id}_{D_b}.$$

Thus, we have shown that inv_b is a discerning right inverse for σ_b as needed.

Finally, suppose that φ is independent of weights with values $\psi\colon X \to [0,\infty)^n$ which is continuous. Since $\tilde{\kappa}$ is a homeomorphism and W is compact, $\kappa(W) = \tilde{\kappa}(W)$ is compact. Theorem A.3(d) shows that $\sigma_b = \overline{\sigma_b} \circ \kappa$ is a homemorphism onto its image. ☐

Proof of Theorem 4.4. Set $I = \operatorname{supp}(y)$. By the hypothesis φ is independent of weights on $W(I)$. Let $\psi\colon X \to [0,\infty)^n$ be the $W(I)$-value of φ.

Let $\overline{\sigma_{b,W(I)}}\colon \kappa(W(I)) \to X$ and $\overline{\operatorname{inv}_{b,W(I)}}\colon X \to \mathbb{R}^n$ be as in Theorem A.3. Thus, $\overline{\sigma_{b,W(I)}}$ is bijective onto $\sigma_b(W(I))$. By Theorem A.6 σ_b preserves supports, and it easily follows that $\sigma_b(W(I)) = D_b(I)$ where $D_b = \sigma_b(W)$. Similarly, since κ preserves supports,

$$\kappa(W(I)) = \mathbb{R}^n(I) \cap \kappa(W).$$

Clearly, $t \cdot y \in X(I)$ for any $0 < t \leq 1$. By definition of $\overline{\operatorname{inv}_{b,W(I)}}$ in Theorem A.3 and since by hypothesis ψ_i are homogeneous

$$\overline{\operatorname{inv}_{b,W(I)}}(t \cdot y)_i = ty_i(1 + \psi(ty)_i) = ty_i(1 + t \cdot \psi(y)_i) \xrightarrow{t \to 0} 0.$$

It follows that there exists $s > 0$ such that $\overline{\operatorname{inv}_{b,W(I)}}(ty) \in U$ for every $0 < t \leq s$. Moreover, for any $t > 0$ it is clear that $\overline{\operatorname{inv}_{b,W(I)}}(ty)_i = 0$ if and only if $y_i = 0$, hence $\operatorname{supp}(\overline{\operatorname{inv}_{b,W(I)}}(ty)) = \operatorname{supp}(y) = I$ for any $t > 0$. Since κ preserves supports, for any $0 < t \leq s$.

$$\overline{\operatorname{inv}_{b,W(I)}}(t \cdot y) \in \mathbb{R}^n(I) \cap U \subseteq \mathbb{R}^n(I) \cap \kappa(W) = \kappa(W(I)).$$

Part (c) of Theorem A.3 implies that $ty \in \sigma_b(W(\alpha)) \subseteq D_{(\kappa,\varphi)}$. ☐

Proof of Theorem 4.6. Consider some $x \in D_b = \sigma_b(W)$. We need to prove that the interval $[0, x] \subseteq D_b$, i.e $tx \in D_b$ for any $0 \leq t \leq 1$. By Theorem A.6 σ_b preserves supports. In particular, $\sigma_b(0) = 0$ (where $0 \in W$ and $0 \in X$). Thus, for the rest of the proof we assume that $0 < t \leq 1$.

Set $I = \operatorname{supp}(x)$. Since $x = \sigma_b(w)$ for some $w \in W$ and Since σ_b preserves supports, it follows that $w \in W(I)$. By hypothesis φ is independent of supports on $W(I)$. Let $\psi \colon X \to [0, \infty)^n$ denote the $W(I)$-value of φ. Then $\psi \in \mathcal{B}^+\mathcal{HI}_n(X)$.

Consider $\overline{\sigma_b, W(I)} \colon \kappa(W(I)) \to X$ and $\overline{\operatorname{inv}_{b,W(I)}} \colon X \to \mathbb{R}^n$ from Theorem A.3. Recall that the image of $\overline{\sigma_b}$ is $\sigma_b(W(I))$. Since $x = \sigma_b(w) = \overline{\sigma_{b,W(I)}}(\kappa(w))$, it follows from Theorem A.3 that $\overline{\operatorname{inv}_{b,W(I)}}(x) = \kappa(w)$. By the construction of $\overline{\operatorname{inv}_{b,W(I)}}$ and since ψ is homogeneous

$$\begin{aligned}
\overline{\operatorname{inv}_{b,W(I)}}(tx)_i &= tx_i(1 + \psi(tx)_i) \\
&= tx_i(1 + t\psi(x)_i) \\
&\leq x_i(1 + \psi(x)_i) \\
&= \overline{\operatorname{inv}_{b,W(I)}}(x)_i \\
&= \kappa(x)_i.
\end{aligned}$$

Thus, $\overline{\operatorname{inv}_{b,W(I)}}(tx) \preceq \kappa(w) \in \kappa(W(I))$. It is clear from the defining formula of $\overline{\operatorname{inv}_{b,W(I)}}$ that it preserves supports, so $\operatorname{supp}(\overline{\operatorname{inv}_{b,W(I)}}(tx)) = \operatorname{supp}(tx) = \operatorname{supp}(x) = I$. Since $\kappa(W)$ is supp - $\preceq$ closed in K and since $\kappa(W(I)) = \kappa(W) \cap \mathbb{R}^n(I)$, we deduce that $\overline{\operatorname{inv}_{b,W(I)}}(tx) \in \kappa(W(I))$. By Theorem A.3(c) implies that $tx \in \sigma_b(W(I)) \subseteq D_b$. ☐

Since λ and μ are independent of weights, given $I \in \wp(n)$ we write $\lambda^I, \mu^I \in \mathbb{R}^n$ for the value of these functions on $\mathbb{R}^n(I)$.

Proof of Proposition 5.1. Since $0 \leq \mu(w)_i \leq 1$ it is clear that $\kappa(w) \in K$ for all $w \in W$ so $\kappa \colon W \to K$ is well defined.

Choose some $w \in W$ and set $\psi = \varphi(w)$. We need to show that $\psi \in \mathcal{B}^+\mathcal{HI}_n(X)$, see Definition 3.9. It is clear that $\psi \geq 0$ since $\| \, \|_p \geq 0$. It is also clear that ψ is continuous since $\| \, \|_p$ is continuous as well as $x \mapsto v \wedge x$ for a fixed $v \in \mathbb{R}^n$. Then ψ is bounded since Since X is compact, ψ is bounded. For any $t \geq 0$ it is clear that

$$\psi(tx)_i = \|\lambda(w)(a_{i*}) \wedge tx\|_p = t\|\lambda(w)(a_{i*}) \wedge x\|_p = t \cdot \psi(x)_i.$$

So ψ is homogeneous. Suppose that $x \preceq x'$ in X, i.e $x_i \leq x'_i$. Since $\lambda(w)(a_{i,*}) \geq 0$ it follows that $0 \leq \lambda(w)(a_{i,*})_j x_j \leq \lambda(w)(a_{i,*})_j x_j$. By the definition of the L^p-norms (1) it is easy to check that $\|\lambda(w)(a_{i,*})(a_{i,*}) \wedge x\|_p \leq \|\lambda(w)(a_{i,*}) \wedge x'\|_p$, so $\psi(x) \preceq \psi(x')$. Hence, ψ is increasing. ☐

Proposition A.7. *Let (κ, φ) be the weighted scoring base defined in* (3)*. Then*

- *(i)* *κ is the restriction of a bijective support-preserving function $\tilde{\kappa} \colon \mathbb{R}^n \to \mathbb{R}^n$. If μ is independent of weights then $\tilde{\kappa}$ is a homeomorphism.*
- *(ii)* *$\kappa(W)$ contains a neighbourhood of 0 in K.*
- *(iii)* *$\kappa(W)$ is $(\operatorname{supp}, \preceq)$-closed in K.*
- *(iv)* *φ is independent of supports.*
- *(v)* *$\varphi(w) \colon X \to [0, \infty)^n$ is continuous for all $w \in W$.*

Proof. Define $\tilde{\kappa} \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{\kappa}(u)_i = \mu(u)_i u_i.$$

Then $\tilde{\kappa}$ preserves supports because $\mu(u)_i \neq 0$ so $\tilde{\kappa}(u)_i = \iff u_i = 0$. If μ is independent of weights, i.e constant with value $d \in (0,1)^n$, then $\tilde{\kappa}$ is the linear transformation given by the diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$. In particular, it is a homeomorphism. This proves (i).

Let $\mu^I \in (0,1)^n$ denote the constant value of μ on $W(I)$. Then $\tilde{\kappa}|_{\mathbb{R}^n(I)}$ is multiplication by the matrix $D^I = \operatorname{diag}(\mu^I_1, \dots, \mu^I_n)$ which preserves supports. It follows that

$$\kappa(W) = \coprod_{I \in \wp(n)} \left(\prod_{j=1}^{n} [0, \mu^I_j] \right).$$

It is now clear that $\kappa(W)$ is $(\text{supp}, \preceq)$-closed and that it contains the cube $[0, m]$ where $m = \min\{\mu_j^I\}$. This proves (ii) and (iii).

Since λ is independent of supports it is clear that φ is independent of supports with $\varphi^{\mathbb{R}^n(I)}(x)_i = \|\lambda^{\mathbb{R}^n(I)}(a_{i,*}) \wedge x\|_p$. It also follows that $\varphi^{\mathbb{R}^n(I)}$ is continuous. Thus, (iv) and (v) follow. □

Proof of Theorem 5.3. By Proposition A.7(i) and (iv), all the conditions of Theorem 4.3 are satisfied. We deduce that $\sigma_{\kappa,\varphi}$ admits a solution to the inverse and reflection problems 3.4, 3.5. By that theorem, the inverse function is computed as follows. Define $\overline{\text{inv}}\colon X \to \mathbb{R}^n$ by

$$\overline{\text{inv}}(x)_i = x_i(1 + \varphi^{\text{supp}(x)}(x)_i) = x_i(1 + \|\lambda(x)(a_{i,*}) \wedge x\|_p).$$

Set $I = \text{supp}(x)$. Since $\tilde{\kappa}$ is given on $\mathbb{R}^n(I)$ by the matrix $\text{diag}(\mu_1^I, \dots, \mu_n^I)$, and since $\mu^I = \mu(x)$ and $\lambda^I = \lambda(x)$,

$$\text{inv}(x)_i = \tilde{\kappa}^{-1}(\overline{\text{inv}(x)})_i = \tfrac{1}{\mu(x)_i} x_i(1 + \|\lambda(x)(a_{i,*}) \wedge x\|_p).$$

If λ, μ are independent of weights, i.e they are constant, then by Proposition A.7(i) and (v) the conditions for a solution of the topological reflections problem hold.

Proposition A.7(i),(ii),(iii) and (iv) show that all the conditions of Theorems 4.4 and 4.6 are satisfied so $\sigma_{\kappa,\varphi}$ solve the projective preference ordering problem 3.6 and the radiality problem 3.7. □

Proposition A.8. *Let $b = (\kappa, \varphi)$ and $b' = (\kappa', \varphi')$ be weighted scoring bases. Suppose that*

(i) κ and κ' preserve supports.

(ii) $\kappa(W)$ and $\kappa'(W)$ are $(\text{supp}, \preceq)$-closed in K.

(iii) φ and φ' are independent of supports. Let ψ^I and ψ'^I denote their $W(I)$-value where $I \in \wp(n)$.

(iv) For every $I \in \wp(n)$ there exists $0 < \alpha^I \leq 1$ such that that

- $\psi'^I \leq \alpha^I \cdot \psi^I$
- $\kappa|_{W(I)} \leq \alpha^I \cdot \kappa'|_{W(I)}$.

Then $D_{(\kappa,\varphi)} \subseteq D_{(\kappa',\varphi')}$.

Proof. Consider some $x \in D_{(\kappa,\varphi)}$ and set $I = \text{supp}(x)$. Then $x = \sigma_b(w)$ for some $w \in W$. By Theorem A.6 σ_b preserve supports. Therefore, $w \in W(I)$ and $x \in \sigma_b(W(I)) = \sigma_b(W) \cap \mathbb{R}^n(I)$. In particular $\kappa(W(I)) \subseteq K(I)$.

Apply Theorem A.3 to obtain functions

$$\overline{\sigma_{b,W(I)}}\colon \kappa(W(I)) \to X$$
$$\overline{\text{inv}_{b,W(I)}}\colon X \to \mathbb{R}^n$$

where $\overline{\sigma_{b,W(I)}}$ is bijective onto $\sigma_b(W(I))$ and $\overline{\text{inv}_{b,W(I)}}$ is a discerning right inverse. Then

$$\overline{\text{inv}_{b,W(I)}}(x) = \overline{\text{inv}_{b,W(I)}}(\sigma_b(w)) = \overline{\text{inv}_{b,W(I)}}(\overline{\sigma_{b,W(I)}}\kappa(w)) = \kappa(w) \in \kappa(W(I)).$$

By Proposition A.4 $\overline{\text{inv}_{b,W(I)}}$ preserves supports. Since $0 < \alpha^I \leq 1$ we get $\alpha^I x \in X(I)$, so

$$\overline{\text{inv}_{b,W(I)}}(\alpha^I x) \in \mathbb{R}^n(I).$$

Clearly $\alpha^I x \preceq x$ and since ψ^I is increasing

$$0 \leq \overline{\text{inv}_{b,W(I)}}(\alpha^I x) = \alpha^I x_i(1 + \psi^I(\alpha^I x)_i) \leq$$
$$x_i(1 + \psi^I(x)_i) = \overline{\text{inv}_{b,W(I)}}(x)_i = \kappa(w)_i \leq 1.$$

It follows that $\overline{\text{inv}_{b,W(I)}}(\alpha^I x) \in K \cap \mathbb{R}^n(I) = K(I)$ and $\overline{\text{inv}_{b,W(I)}}(\alpha^I x) \preceq \kappa(w) \in \kappa(W(I))$.

By assumption $\kappa(W)$ is $(\mathrm{supp}, \preceq)$-closed in K, so $\overline{\mathrm{inv}_{b,W(I)}}(\alpha^I x) \in \kappa(W) \cap \mathbb{R}^n(I) = \kappa(W(I))$. Therefore $\overline{\mathrm{inv}_{b,W(I)}}(\alpha^I x) = \kappa(U)$ for some $u \in W(I)$. It follows that

$$\begin{aligned}
\overline{\mathrm{inv}_{b',W(I)}}(x)_i &= x_i(1 + \psi'^I(x)_i) \\
&\le x_i(1 + \alpha^I \psi^I(x)_i) \\
&= \tfrac{1}{\alpha^I}(\alpha^I x_i)(1 + \psi^I(\alpha^I x)_i) \\
&= \tfrac{1}{\alpha^I}\overline{\mathrm{inv}_{b,W(I)}}(\alpha^I x)_i \\
&= \tfrac{1}{\alpha^I}\kappa(u) \\
&\le \tfrac{1}{\alpha^I}\alpha^I \kappa'(u)_i \\
&= \kappa'(u)_i.
\end{aligned}$$

Thus, we have shown that

$$\overline{\mathrm{inv}_{b',W(I)}}(x) \preceq \kappa'(u).$$

By Proposition A.4 $\mathrm{supp}(\overline{\mathrm{inv}_{b',W(I)}}(x)) = \mathrm{supp}(x) = I$ and also $\mathrm{supp}(\kappa'(u)) = \mathrm{supp}(u) = I$ and $0 \le \overline{\mathrm{inv}_{b',W(I)}}(x)_i \le \kappa'(u)_i \le 1$. Since $\kappa'(W)$ is $(\mathrm{supp}, \preceq)$-closed in K we get $\overline{\mathrm{inv}_{b,W(I)}}(x) \in \kappa'(W) \cap \mathbb{R}^n(I) = \kappa'(W(I))$. By Theorem A.3(c) $x \in \sigma_{b'}(W(I)) \subseteq \sigma_{b'}(W) = D_{(\kappa',\varphi')}$. □

Proof of Theorem 5.4. Let $b = (\kappa, \varphi)$ be the (L^p, λ, μ, A) weighted scoring base and $b' = (\kappa', \varphi')$ be the (L^q, λ, μ, A) weighted scoring base. By construction $\kappa = \kappa'$.

Conditions (i), (ii) and (iii) of Proposition A.8 hold by Proposition A.7. Condition (iv) also holds by setting $\alpha^I = 1$ since $\| \, \|_q \le \| \, \|_p$ and $\kappa \le \kappa'$ (in fact, equality). □

Proof of Proposition 5.5. Let $\bar{1} \in \mathbb{R}^n$ denote the vector $(1, \ldots, 1)$. We first highlight the following three items (and their proofs).

1. The gradual semantics $\Sigma_{\mathbb{HC}}$ is an abstract (L^p, λ, μ)-based gradual semantics with $p = 1$ and $\mu\colon \mathbb{R}^n \to (0,1]^n$ and $\lambda\colon \mathbb{R}^n \to \mathrm{End}([0,\infty)^n)$ given by

$$\begin{aligned}
&\mu(v) = \bar{1} \qquad (v \in \mathbb{R}^n) \\
&\lambda(v) = \mathrm{id} \in \mathrm{End}([0,\infty)^n) \qquad (v \in \mathbb{R}^n).
\end{aligned}$$

Proof. By construction $\tilde{\kappa}\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity because $\mu(v)_i = 1$ for all $v \in \mathbb{R}^n$. Hence $\kappa\colon W \to K$ is the identity of $[0,1]^n$. The function $\varphi\colon W \to \mathrm{Func}(X, [0,\infty)^n)$ is

$$\varphi(w)(x)_i = \|a_{i,*} \wedge x\|_1 = \sum_j a_{i,j} x_j.$$

By definition, $\sigma_{(L^p,\lambda,\mu,A)}(w) = \mathrm{fix}(T_{(\kappa(w),\varphi(w))})$ and it is the limit of any $T_{(\kappa(w),\varphi(w))}$-sequence in X (Definition 3.14 and Theorem 3.17). Now,

$$T_{(\kappa(w),\varphi(w))}(x)_i = \frac{\kappa(w)_i}{1 + \varphi(w)(x)_i} = \frac{w_i}{1 + \|a_{i,*} \wedge x\|_1}.$$

Since $A = (a_{i,j})$ is the adjacency matrix of a graph $\mathcal{G}$,

$$\|a_{i,*} \wedge x\|_1 = \sum_j a_{i,j} x_j = \sum_{j \in \mathtt{Att}(i)} x_j.$$

From the construction of $\Sigma_{\mathbb{HC}}$, $\Sigma_{\mathbb{HC}}^{A,w} = \mathrm{fix}(T_{(\kappa(w),\varphi(w))}) = \sigma_{(L^p,\lambda,\mu,A)}(w)$. Thus, $\Sigma_{\mathbb{HC}} = \Sigma_{(L^p,\lambda,\mu)}$. □

2. The gradual semantics $\Sigma_{\mathbb{MB}}$ is an abstract (L^p, λ, μ)-based gradual semantics with $p = \infty$ and $\mu\colon \mathbb{R}^n \to (0,1]^n$ and $\lambda\colon \mathbb{R}^n \to \mathrm{End}([0,\infty)^n)$ given by

$$\begin{aligned}
&\mu(v) = \bar{1} \qquad (v \in \mathbb{R}^n) \\
&\lambda(v) = \mathrm{id} \in \mathrm{End}([0,\infty)^n) \qquad (v \in \mathbb{R}^n).
\end{aligned}$$

Proof. The proof of item 1 can be read verbatim upon replacing $p = 1$ with $p = \infty$ and $\Sigma_{\mathbb{HC}}$ with $\Sigma_{\mathbb{MB}}$ in Example 2.3 and observing that

$$\|a_{i,*} \wedge x\|_\infty = \max_j \{a_{i,j} x_j\} = \max_{j \in \mathtt{Att}(i)} x_j.$$

The details are left to the reader. □

3. The gradual semantics $\Sigma_{\mathbb{CB}}$ is an abstract (L^p, λ, μ)-based gradual semantics with $p = 1$ and $\mu \colon \mathbb{R}^n \to (0,1]^n$ and $\lambda \colon \mathbb{R}^n \to \mathrm{End}([0,\infty)^n)$ given as follows. First, define $\theta \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\theta(v)_i = \|a_{i,*} \wedge \mathrm{supp}(v)\|_1$$

Then define μ and λ by

$$\mu(v)_i = \frac{1}{1 + \theta(v)_i}$$
$$\lambda(v)_i = \begin{cases} \frac{\mu(v)_i}{\theta(v)_i} \cdot (a_{i,*} \wedge \mathrm{supp}(v)) & \text{if } \theta(v)_i \neq 0 \\ 0 & \text{if } \theta(v)_i = 0 \end{cases}$$

Proof. Choose some $w \in W$. By the definition of κ and φ in Subsection 5

$$\begin{aligned} T_{(\kappa(w),\varphi(w))}(x)_i &= \frac{w_i/(1+\theta(w)_i)}{1 + \frac{\mu(v)_i}{\theta(v)_i}\|\mathrm{supp}(w) \wedge a_{i,*} \wedge x\|_1} \\ &= \frac{w_i}{1 + \theta(w)_i + \frac{1}{\theta(w)_i}\|\mathrm{supp}(w) \wedge a_{i,*} \wedge x\|_1} \end{aligned}$$

with the convention that $\frac{1}{\theta(w)_i} = 0$ if $\theta(w)_i = 0$. Now, $\sigma_{(L^p,\lambda,\mu,A)}(w) = \mathrm{fix}(T_{(\kappa(w),\varphi(w))})$ is the limit of any $T_{(\kappa(w),\varphi(w))}$-sequence in X. We observe that since $A = (a_{i,j})$ is the adjacency matrix of $\mathcal{G}$,

$$\theta(w)_i = |\mathtt{Att}^*(i)|$$
$$\|\mathrm{supp}(w) \wedge a_{i,*} \wedge x\|_1 = \sum_{j \in \mathtt{Att}^*(i)} a_{i,j} x_j = \sum_{j \in \mathtt{Att}^*(i)} x_j.$$

It follows from the construction of $\Sigma_{\mathbb{CB}}$ in Example 2.4 that $\Sigma_{\mathbb{CB}}^{\mathcal{G},w} = \sigma_{(L^p,\lambda,\mu,A)}$. Thus, $\Sigma_{\mathbb{CB}} = \Sigma_{(L^p,\lambda,\mu)}$. □

We can conclude from items 1, 2 and 3 and Theorem 5.3. □

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