

# Spatially-averaged flows over mobile rough beds: definitions, averaging theorems, and conservation equations

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**Abstract:** The paper reports the double-averaged (in space and in time) hydrodynamic equations for mobile-boundary conditions which are derived based on the refined double-averaging theorems, modified Reynolds decomposition, and improved definitions of the spatial and time bed porosities. The obtained double-averaged conservation equations provide a mathematical framework for studying mobile-boundary flows such as gravel-bed rivers during flood events or flows over vegetated beds. These equations will help in designing measurement campaigns for obtaining mobile-bed data and their interpretation and parameterisation, eventually leading to improved and more robust predictive models.

**CE database subject headings:** Open-channel flow; Turbulence; Sediment transport; Vegetated flows

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## Introduction

Environmental flows such as overland flows, rivers, estuaries or coastal flows can often be classified as low-submergence rough-bed flows with high levels of heterogeneity in time-averaged hydrodynamic fields due to the effects of roughness elements, especially profound in the near-bed region. These flows are typically described by the Reynolds averaged Navier-Stokes equations, which deal with time- (ensemble-) averaged variables and involve no spatial averaging. Such an approach, however, is often inconvenient, to say the least, due to the complex and often mobile boundary conditions that lead to the high flow heterogeneity. The key drawbacks of the RANS-based approaches in relation to rough-bed flows have been discussed at length in Nikora et al. (2007a). It has been argued that to resolve the problem theoretically, time (or ensemble) averaging of the hydrodynamic equations should be supplemented by volume averaging or area averaging in the plane parallel to the mean (smoothed) bed surface. Conceptually, the double-averaged (in both time and space) equations relate to the time-(ensemble)-averaged equations as the time-averaged equations relate to the Navier-Stokes and advection-diffusion equations for instantaneous hydrodynamic variables. The development of this methodology for rough-bed flows was initiated by atmospheric scientists for describing turbulent flows within and above terrestrial canopies such as forests or bushes (Wilson and Shaw 1977; Raupach and Shaw 1982; Finnigan 1985, 2000; Poggi et al. 2004), and later it was adopted in studies of water flows (e.g., Gimenez-Curto and Corniero Lera 1996; Lopez and Garcia 2001; Nikora et al. 2001, 2007a,b; Pokrajac et al. 2008; Nepf 2012).

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43 Applications of the double-averaging approach to a wide range of flows, from porous media  
 44 flows to rough-bed open-channel flows to atmospheric boundary layers, have recently been discussed  
 45 in a special issue of *Acta Geophysica* (Nikora and Rowinski 2008) that highlights the main  
 46 advantages of this methodology, i.e.: (a) rigor and self-consistency; (b) refined definitions for rough-  
 47 bed flows such as flow uniformity, two-dimensionality, and the bed shear stress; (c) a consistent link  
 48 between spatially-averaged roughness parameters, bed shear stress, and double-averaged flow  
 49 variables; (d) explicit accounting for the viscous drag, form drag and form-induced stresses and  
 50 substance fluxes as a result of rigorous derivation rather than intuitive reasoning; (e) framework for  
 51 scaling considerations and parameterizations based on double-averaged variables; and (f) the  
 52 possibility for the rigorous scale partitioning of the roughness parameters and flow properties. These  
 53 advantages underpin use of the double-averaged hydrodynamic equations in developing numerical  
 54 models and associated closures for environmental rough-bed flows; designing laboratory, field, and  
 55 numerical experiments; data analysis and interpretation; and guiding conceptual developments and  
 56 parameterizations.

57 The main achievements to date in this research area relate to fixed-bed flows while mobile-bed  
 58 flows still represent a major challenge in terms of both theoretical frameworks and experimental  
 59 data. To address this issue, Nikora et al. (2007a) presented the double-averaged hydrodynamic  
 60 equations and introduced two parameters characterising mobile bed conditions: the space and time  
 61 bed porosities. Since publication of that paper, the authors have received feedback from colleagues  
 62 interested in modelling mobile-bed flows that highlights the need for clarification of the double-  
 63 averaging methodology for mobile-bed conditions.

64 The goal of this paper is therefore to refine the double-averaged hydrodynamic equations for  
 65 mobile-bed conditions by clarifying key ingredients involved in the derivation (Nikora et al., 2007a):  
 66 (1) averaging operators and space and time bed porosities; (2) equations that link double-averaged  
 67 derivatives to derivatives of the double-averaged variables, known as the averaging theorems; and  
 68 (3) modified Reynolds decomposition of instantaneous variables. The derivation starts with  
 69 presenting instantaneous variables in the hydrodynamic equations using the modified Reynolds  
 70 decomposition, followed by applying averaging procedures for each term of the equation, similar to  
 71 how the Reynolds-averaged equations are obtained. The double-averaging theorems play the role of  
 72 the Reynolds averaging rules in this derivation. There are a number of options for derivation of the  
 73 double-averaging theorems (e.g., Raupach and Shaw 1982; Finnigan 1985; Lien et al. 2005; Kono et  
 74 al. 2010). Nikora et al. (2007a) in their derivation of the double-averaging theorems and double-  
 75 averaged equations have employed an approach suggested by Gray and Lee (1977) for local volume  
 76 averaging of instantaneous variables of multiphase systems. Nikora et al. (2007a) extended this  
 77 approach to cover double-averaging, considering both superficial averaging (over the whole  
 78 averaging domain) and intrinsic averaging (over the sub-domain occupied by fluid only; see next  
 79 section for more specific definitions). According to Gray and Lee (1977), superficial ( $\langle\theta\rangle_s$ ) and  
 80 intrinsic ( $\langle\theta\rangle$ ) spatial averages of a variable  $\theta$  are expressed as:

81

$$82 \quad \langle\theta\rangle_s(x_i, t) = \frac{1}{V_o} \int_{V_o} \theta(x_i + \xi_i, t) \gamma(x_i + \xi_i, t) dV = \frac{1}{V_o} \int_{V_f} \theta(x_i + \xi_i, t) dV \quad (1a)$$

$$83 \quad \langle\theta\rangle(x_i, t) = \frac{1}{V_f} \int_{V_o} \theta(x_i + \xi_i, t) \gamma(x_i + \xi_i, t) dV = \frac{1}{V_f} \int_{V_f} \theta(x_i + \xi_i, t) dV \quad (1b)$$

84

85 where  $\gamma(x_i, t)$  = a ‘clipping’ or ‘distribution’ function equal to 1 in the fluid and 0 otherwise;

86  $V_f = \int_{V_o} \gamma(x_i + \xi_i, t) dV$  = fluid volume within the total averaging domain of volume  $V_o$ ; and angular

87 brackets denote spatial averaging. In equations (1a) and (1b), the integration domains are centered at  
 88 position  $x_i$  and a local co-ordinate system  $\xi_i$  is used for integration (Fig. 1); resulting averaged

89 variables are assigned to the centres of the averaging domains. In studies of turbulent spots  
 90 embedded in non-turbulent environments (e.g., Antonia and Atkinson 1974) and in micromechanics  
 91 (e.g., Torquato 2002), the function  $\gamma(x_i, t)$  is also known as the intermittency function  $I(x_i, t)$ , while  
 92 an average of  $I(x_i, t)$  is often called the intermittency fraction (e.g., Field and Grigoriu, 2010). Gray  
 93 and Lee (1977) demonstrated that  $\partial\gamma/\partial x_i = n_i\delta(x_i - x_{si})$  and  $\partial\gamma/\partial t = -v_i\partial\gamma/\partial x_i = -v_in_i\delta(x_i - x_{si})$ ,  
 94 where  $\delta$  = three-dimensional analogue of the Dirac delta-function,  $x_{si}$  = the coordinates of the flow-  
 95 solid interface,  $n_i$  = unit vector normal to the bed surface and directed into fluid, and  $v_i$  = velocity  
 96 vector of the fluid-solid interface. Gray and Lee's (1977) relationships were reinforced and further  
 97 generalised by Kinnmark and Gray (1984). Operators similar to (1a) and (1b) can also be introduced  
 98 for intrinsic and superficial time-averaging as well as for space-time averaging. These operators will  
 99 be employed in our derivations below.

100 We first introduce definitions related to forms of averaging, bed porosities, and their  
 101 interrelations. These considerations are required to eliminate potential confusion between different  
 102 forms of averaging and bed porosities. Then, we consider the averaging operators, double-averaging  
 103 theorems, modified Reynolds decomposition, and hydrodynamic equations for mobile-bed  
 104 conditions, followed by a brief outline of potential applications of the equations.

## 106 **Averaging Procedures, Bed Porosities, Intrinsic and Superficial Quantities**

### 107 **Forms of Averaging**

108 The forms of averaging used in this paper include:

109 *Superficial averaging*: averaging of a variable over the whole domain (spatial, time or both). This  
 110 averaging form is indicated by the index  $s$ ; e.g., as in equation (1).  
 111

112 *Intrinsic averaging*: averaging of a variable over the sub-domain (spatial, time or both) within which  
 113 this variable is appropriately defined or marked with a special feature (e.g., presence of water within  
 114 a sub-domain).  
 115

116 *Time averaging*: averaging over time; defined with an overbar (e.g., superficial  $\bar{\theta}^s$  and intrinsic  $\bar{\theta}$  ).  
 117

118 *Spatial averaging*: averaging over space; defined with angular brackets (e.g., superficial  $\langle\theta\rangle_s$  and  
 119 intrinsic  $\langle\theta\rangle$  ).  
 120

121 *Space-time averaging*: simultaneous averaging over space and time; defined with rectangular  
 122 brackets (e.g., superficial  $[\theta]_s$  and intrinsic  $[\theta]$  ). This is probably the most robust form of double-  
 123 averaging.  
 124

125 *Consecutive time-space averaging*: first averaging over time and then over space (e.g., superficial  
 126  $\langle\bar{\theta}^s\rangle_s$  and intrinsic  $\langle\bar{\theta}\rangle$  ). This is another form of the double-averaging.  
 127

128 *Consecutive space-time averaging*: first averaging over space and then over time (e.g., superficial  
 129  $\overline{\langle\theta\rangle_s}$  and intrinsic  $\overline{\langle\theta\rangle}$  ). This is an alternative to the consecutive time-space averaging defined  
 130 above.  
 131

132 The averaging forms defined above are equally applicable to any hydrodynamic variable such as  
 133 flow velocity, pressure, or substance concentration. The selection of the shape and dimensions of the  
 134  
 135

136 spatial averaging domain and the averaging time depends on the roughness geometry and the  
 137 turbulence structure (particularly on their characteristic scales) as well as on the magnitudes and  
 138 scales of the spatial and temporal gradients of the mean flow. These issues have been discussed in  
 139 detail in Nikora et al. (2007a). Here we only mention that in most environmental flows there are  
 140 strong gradients in flow properties in the vertical direction, especially near the bed, and therefore the  
 141 volume-averaging domain should be designed as a thin slab parallel to the average bed (Fig. 1),  
 142 being much thinner than the roughness elements. The dimensions of the averaging domain in the  
 143 plane parallel to the average bed should be much larger than the dominant roughness scales, but  
 144 much smaller than the large-scale features in bed topography. For example, for gravel-bed rivers it  
 145 should be much larger than gravel particles, but much smaller than sizes of riffles or pools. The  
 146 averaging time should well exceed the integral time scales of turbulence, still being much shorter  
 147 than the duration of hydrological events such as floods. The exact shape and dimensions of the  
 148 averaging domain and the averaging time may vary depending on the problem under consideration.

149 Specific mathematical definitions for superficial and intrinsic quantities used in this paper will  
 150 be given in the following sections. Note that to derive the double-averaged equations, Nikora et al.  
 151 (2007a) essentially used *consecutive time-space averaging* that accounts for the long-term tradition  
 152 in data collection and analysis which have been largely motivated by the RANS methodology. In  
 153 general, however, the appearance of the averaging theorems and the double-averaged hydrodynamic  
 154 equations, as well as the physical meaning of the double-averaged quantities, may depend on the  
 155 averaging approach, i.e., simultaneous *space-time averaging*; *consecutive time-space averaging*; or  
 156 *consecutive space-time averaging*, as discussed later in the paper.

157

### 158 **Forms of 'Bed Porosity' (Roughness Geometry Function)**

159 The bed porosity or roughness geometry function (as introduced in Nikora et al. 2001) can be defined  
 160 in a number of ways, depending on the problem under consideration. In relation to double-averaging,  
 161 the bed porosity appears in the double-averaged hydrodynamic equations as a parameter representing  
 162 the bed geometry. A selected set of definitions for bed porosities, relevant to our considerations, is  
 163 introduced below. Fig. 2 provides sketches that illustrate porosities introduced in this and the  
 164 following sections.

165

166 *Space-time porosity*  $\phi_{VT}(x_i, t)$  represents the ratio of the part of the total averaging domain occupied  
 167 by fluid (i.e.,  $\int_{T_o} \int_{V_o} \gamma(x_i + \xi_i, t + \tau) dV d\tau$ ) to the size of this domain  $V_o T_o$ , i.e.:

168

$$169 \quad \phi_{VT}(x_i, t) = [\gamma]_s = \langle \overline{\gamma^s} \rangle_s = \overline{\langle \gamma \rangle_s} = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \gamma(x_i + \xi_i, t + \tau) dV d\tau \quad (2)$$

170

171 where  $T_o$  = the averaging period; and  $V_o$  = the size of the spatial averaging domain, i.e., a spatial  
 172 component of the total averaging domain  $V_o T_o$ . Similar to the spatial averaging (see Eq. (1)), the  
 173 integration time domain of Eq. (2) is centered at position  $t$  and a local time co-ordinate  $\tau$  is used for  
 174 integration; resulting time-averaged variables are assigned to the centers of the averaging domains.  
 175 In equations that follow, we omit  $\xi_i$  and  $\tau$  for brevity. Equation (2) shows that all three superficial  
 176 forms of double-averaging for  $\gamma$  (i.e., *space-time*, *consecutive time-space*, and *consecutive space-*  
 177 *time*) are identical while the intrinsic *space-time* average is always  $[\gamma] \equiv 1$ . The meaning of the  
 178 quantity  $\int_{T_o} \int_{V_o} \gamma(x_i, t) dV d\tau$  will be clarified at the end of next section.

179

180 *Local time porosity*  $\phi_T(x_i, t)$  is defined as the ratio of the total period of time  $T_f = \int_{T_o} \gamma(x_i, t) dt$  (not  
 181 necessarily continuous) when water passes through a fixed point  $x_i$ , to the total duration  $T_o$  of  
 182 observation, i.e.:

$$184 \quad \phi_T(x_i, t) = \bar{\gamma}^s = \frac{1}{T_o} \int_{T_o} \gamma(x_i, t) dt \quad (3)$$

185  
 186 Note that the quantity  $\phi_T(x_i, t) = \bar{\gamma}^s$  involves no spatial averaging. In the case of the fixed rough bed,  
 187 we have either  $\bar{\gamma}^s = 0$  when a point is embedded in the solid phase, or  $\bar{\gamma}^s = 1$  when the point is in the  
 188 region filled with water. For mobile beds where a fixed point is intermittently occupied by solids or  
 189 water, we have  $0 \leq \bar{\gamma}^s \leq 1$ . Note also that  $\bar{\gamma} \equiv 1$ . As mentioned above, in turbulence studies and  
 190 micromechanics, an analogue of  $\bar{\gamma}^s$  is known as the intermittency fraction. Recently, a similar  
 191 parameter has been successfully used to study intermittency in bed particle motion (Radice and  
 192 Ballio 2008).

193  
 194 *Instantaneous space porosity*  $\phi_V(x_i, t)$  represents the ratio of the fluid volume  $V_f = \int_{V_o} \gamma(x_i, t) dV$  to  
 195 the total volume of the averaging domain  $V_o$  at a moment in time  $t$ , i.e.:

$$197 \quad \phi_V(x_i, t) = \langle \gamma \rangle_s = \frac{1}{V_o} \int_{V_o} \gamma(x_i, t) dV \quad (4)$$

198  
 199 Note that the intrinsic spatial average of the clipping function is always 1, i.e.,  $\langle \gamma \rangle \equiv 1$ . In addition to  
 200 the space-time porosity  $\phi_{VT}(x_i, t)$ , the local time porosity  $\phi_T(x_i, t)$ , and the instantaneous space  
 201 (volumetric) porosity  $\phi_V(x_i, t)$ , introduced above, one may also employ a ‘plane’ porosity  $\phi_A(x_i, t)$ ,  
 202 which is the ratio of the area  $A_f$  occupied by fluid within a plane averaging domain to its total area  
 203  $A_o$ , i.e.:

$$205 \quad \phi_A(x_i, t) = \langle \gamma \rangle_{As} = \frac{1}{A_o} \int_{A_o} \gamma(x_i, t) dA \quad (5)$$

206  
 207 where  $A_f = \int_{A_o} \gamma(x_i, t) dA = \text{area occupied by fluid}$ . In Nikora et al. (2001), the parameter  $\phi_A(x_i, t)$   
 208 was defined as the roughness geometry function  $A$  to describe the flow - ‘rough’ bed interface. As in  
 209 previous cases,  $\langle \gamma \rangle_A \equiv 1$ .

### 210 **Some Interrelations between Time, Space, and Space-Time Porosities**

211 The physical meanings of the porosities introduced in the previous section can be better seen through  
 212 relationships between them. The space-time porosity  $\phi_{VT}(x_i, t)$ , introduced by Eq. (2), can be  
 213 expressed as:

$$216 \quad \phi_{VT} = [\gamma]_s = \langle \bar{\gamma}^s \rangle_s = \overline{\langle \gamma \rangle_s} = \frac{1}{V_o} \frac{1}{T_o} \int_{V_o} \int_{T_o} \gamma(x_i, t) dt dV = \frac{1}{V_o} \int_{V_o} \bar{\gamma}^s dV = \frac{1}{V_o} \int_{V_m} \bar{\gamma}^s dV + \frac{1}{V_o} \int_{V_o - V_m} \bar{\gamma}^s dV$$

217

$$218 \quad = \frac{1}{V_o} \int_{V_m} \bar{\gamma}^s dV = \frac{V_m}{V_o} \frac{1}{V_m} \int_{V_m} \bar{\gamma}^s dV = \phi_{V_m} \langle \bar{\gamma}^s \rangle = \phi_{V_m} \langle \phi_T \rangle \quad (6)$$

219

220 where  $V_m$  = the part of  $V_o$  that has been ‘visited’ by fluid, even briefly, within  $T_o$ ;  $\frac{1}{V_o} \int_{V_o - V_m} \bar{\gamma}^s dV = 0$

221 as  $\bar{\gamma}^s = 0$  everywhere within  $(V_o - V_m)$ ; and  $\phi_{V_m} = V_m / V_o$  is the space porosity based on non-zero  $\bar{\gamma}^s$ .  
 222 The quantity  $(V_o - V_m)$  is the remaining part of  $V_o$  that has never been ‘visited’ by fluid, even briefly,  
 223 within  $T_o$ , and thus represents the total volume of permanent solid ‘islands’ within  $V_o$ . For fixed  
 224 (‘frozen’) beds,  $\langle \phi_T \rangle \equiv 1$  and therefore  $\phi_{V_T} = \phi_{V_m} \langle \phi_T \rangle = \phi_{V_m} = \phi_V$ . For ‘spatially homogeneous’ mobile  
 225 beds (i.e., with no permanent solid ‘islands’ within the averaging domain),  $\bar{\gamma}^s \neq 0$  everywhere within  
 226  $V_o$ , and thus  $V_m = V_o$  and  $\phi_{V_m} = 1$ , giving  $\phi_{V_T} = \phi_{V_m} \langle \phi_T \rangle = \langle \phi_T \rangle$ . The general case of  $\phi_{V_T} = \phi_{V_m} \langle \phi_T \rangle$  is  
 227 intermediate between these two extremes.

228 In addition, we can consider  $\phi_{V_T}$  in terms of a porosity that reflects potential existence of ‘solid  
 229 islands’ in time, i.e.:

230

$$231 \quad \phi_{V_T} = [\gamma]_s = \langle \bar{\gamma}^s \rangle_s = \overline{\langle \gamma \rangle}_s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt = \frac{1}{T_o} \int_{T_o} \langle \gamma \rangle_s dt = \frac{1}{T_o} \int_{T_m} \langle \gamma \rangle_s dt + \frac{1}{T_o} \int_{T_o - T_m} \langle \gamma \rangle_s dt$$

232

$$233 \quad = \frac{1}{T_o} \int_{T_m} \langle \gamma \rangle_s dt = \frac{T_m}{T_o} \frac{1}{T_m} \int_{T_m} \langle \gamma \rangle_s dt = \phi_{T_m} \overline{\langle \gamma \rangle}_s = \phi_{T_m} \bar{\phi}_V \quad (7)$$

234

235 where  $T_m$  = time period (not necessarily continuous) within the total averaging period  $T_o$  when a  
 236 part (even very small) of  $V_o$  contained fluid;  $\frac{1}{T_o} \int_{T_o - T_m} \langle \gamma \rangle_s dt = 0$  as  $\langle \gamma \rangle_s = 0$  within  $(T_o - T_m)$ ; and

237  $\phi_{T_m} = T_m / T_o$  is the time porosity based on non-zero  $\langle \gamma \rangle_s$ . The quantity  $(T_o - T_m)$  is the remaining  
 238 part of  $T_o$  when there is no fluid anywhere within the domain  $V_o$ . For fixed (‘frozen’) beds,  $\phi_{T_m} \equiv 1$   
 239 and therefore  $\phi_{V_T} = \phi_{T_m} \bar{\phi}_V = \bar{\phi}_V = \phi_V$ , as expected. At the other extreme of temporarily homogeneous  
 240 mobile beds (i.e., with no solid ‘islands’ in time or, in other words, with no instance of a completely  
 241 ‘solid’ domain),  $\phi_{T_m} \equiv 1$  and therefore  $\phi_{V_T} = \phi_{T_m} \bar{\phi}_V = \bar{\phi}_V$ .

242 To summarise, in general the space-time porosity  $\phi_{V_T}$  can be expressed as  
 243  $\phi_{V_T} = \langle \bar{\gamma}^s \rangle_s = \phi_{V_m} \langle \phi_T \rangle$  and  $\phi_{V_T} = \overline{\langle \gamma \rangle}_s = \phi_{T_m} \bar{\phi}_V$ , giving  $\phi_{V_m} \langle \phi_T \rangle = \phi_{T_m} \bar{\phi}_V$ . For the special case of  
 244 homogeneously mobile beds (with no ‘solid islands’ in time or in space), we have  $\phi_{V_T} = \langle \bar{\gamma}^s \rangle_s = \langle \phi_T \rangle$   
 245 and  $\phi_{V_T} = \overline{\langle \gamma \rangle}_s = \bar{\phi}_V$ , giving  $\phi_{V_T} = \langle \phi_T \rangle = \bar{\phi}_V$ .

246 With the above definitions, another useful relationship is  $\int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt = \bar{V}_f T_m = V_m \langle T_f \rangle$   
 247 that clarifies the meaning of the quantity  $\int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt$  in the definition of  $\phi_{V_T}$  in Eq. (2). The  
 248 forms of bed porosity introduced in this subsection will be used below in establishing relations  
 249 between different forms of averaging as well as in the double-averaging theorems and in the double-  
 250 averaged hydrodynamic equations.

251

## 252 **Specific Definitions for Superficial and Intrinsic Averages**

253

254 *Superficial averaging*

255

256 The superficial double-averaged quantity can be defined as a generalisation of  $\langle \theta \rangle_s$  in Eq. (1), i.e.:

257

$$258 \quad [\theta]_s = \langle \bar{\theta}^s \rangle_s = \overline{\langle \theta \rangle_s}^s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt \quad (8)$$

259

260 As can be seen in Eq. (8), all three forms of double-averaging (i.e., *space-time*, *consecutive time-*  
261 *space*, and *consecutive space-time*) are identical for superficial averages. They may not be identical  
262 for intrinsic averages, however, as shown below.

263

264 *Intrinsic space-time averaging*

265

266 This form of averaging is defined as:

267

$$268 \quad [\theta] = \frac{1}{\int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{1}{V_f T_m} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{1}{V_m \langle T_f \rangle} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt \quad (9)$$

269

270 Then the relation between  $[\theta]_s$  and  $[\theta]$  follows:

271

$$272 \quad [\theta]_s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{\int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt}{T_o V_o} \frac{1}{\int_{T_o} \int_{V_o} \gamma(x_i, t) dV dt} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \phi_{VT} [\theta] \quad (10)$$

273

274 *Intrinsic consecutive time-space averaging*

275

276 By applying first time averaging and then spatial averaging, the quantity  $\langle \bar{\theta} \rangle$  can be defined as:

277

$$278 \quad \langle \bar{\theta} \rangle = \frac{1}{V_m} \int_{V_o} \frac{1}{T_f} \int_{T_o} \theta \gamma(x_i, t) dt dV = \frac{1}{V_m} \int_{V_o} \bar{\theta} dV \quad (11)$$

279

280 The relation between  $\langle \bar{\theta}^s \rangle_s = [\theta]_s = \overline{\langle \theta \rangle_s}^s$  and  $\langle \bar{\theta} \rangle$  is then derived:

281

$$282 \quad \langle \bar{\theta}^s \rangle_s = [\theta]_s = \overline{\langle \theta \rangle_s}^s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{1}{V_o} \int_{V_o} \frac{1}{T_o} \int_{T_o} \theta \gamma(x_i, t) dt dV$$

283

$$284 \quad = \frac{V_m}{V_o} \frac{1}{V_m} \int_{V_o} \frac{1}{T_o} \int_{T_o} \theta \gamma(x_i, t) dt dV = \frac{V_m}{V_o} \frac{1}{V_m} \int_{V_o} \frac{1}{T_o} \bar{\theta} dV = \phi_{Vm} \langle \phi_T \bar{\theta} \rangle \quad (12)$$

285

286 *Intrinsic consecutive space-time averaging*

287

288 We can define  $\langle \bar{\theta} \rangle$  similar to Eq. (11) but with the reverse averaging order, i.e.:

289

$$\langle \bar{\theta} \rangle = \frac{1}{T_m} \int_{T_o} \frac{1}{V_f} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{1}{T_m} \int \langle \theta \rangle dt \quad (13)$$

291

292 The relation between  $\overline{\langle \theta \rangle}_s = \langle \bar{\theta}^s \rangle_s = [\theta]_s$  and  $\langle \bar{\theta} \rangle$  can be obtained as:

293

$$\overline{\langle \theta \rangle}_s = \langle \bar{\theta}^s \rangle_s = [\theta]_s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \gamma(x_i, t) dV dt = \frac{1}{T_o} \int \frac{1}{V_o} \int \theta \gamma(x_i, t) dV dt$$

295

$$= \frac{T_m}{T_o} \frac{1}{T_m} \int \frac{V_f}{V_o} \frac{1}{V_f} \int \theta \gamma(x_i, t) dV dt = \frac{T_m}{T_o} \frac{1}{T_m} \int \frac{V_f}{V_o} \langle \theta \rangle dt = \phi_{Tm} \overline{\langle \theta \rangle} \quad (14)$$

297

298 To summarise, from the above relationships it follows that:

299

$$[\theta]_s = \langle \bar{\theta}^s \rangle_s = \overline{\langle \theta \rangle}_s = \phi_{VT} [\theta] = \phi_{Vm} \langle \phi_T \rangle [\theta] = \phi_{Tm} \bar{\phi}_V [\theta] = \phi_{Vm} \langle \phi_T \bar{\theta} \rangle = \phi_{Tm} \bar{\phi}_V \overline{\langle \theta \rangle} \quad (15)$$

301

302 where we use the equality  $\phi_{VT} = \phi_{Vm} \langle \phi_T \rangle = \phi_{Tm} \bar{\phi}_V$  derived in Eqs. (6) and (7). Equations (15) show that  
 303 the three different forms of intrinsic averages relate to each other as  
 304  $\phi_{Vm} \langle \phi_T \rangle [\theta] = \phi_{Tm} \bar{\phi}_V [\theta] = \phi_{Vm} \langle \phi_T \bar{\theta} \rangle = \phi_{Tm} \bar{\phi}_V \overline{\langle \theta \rangle}$ , i.e., in general, they are not identical. For spatially  
 305 non-correlated  $\phi_T$  and  $\bar{\theta}$ , and for time non-correlated  $\phi_V$  and  $\langle \theta \rangle$ , equations (15) simplify to:

306

$$\phi_{Vm} \langle \phi_T \rangle [\theta] = \phi_{Tm} \bar{\phi}_V [\theta] = \phi_{Vm} \langle \phi_T \rangle \langle \bar{\theta} \rangle = \phi_{Tm} \bar{\phi}_V \overline{\langle \theta \rangle} \quad (16)$$

308

309 from which we can conclude that for the uncorrelated pairs ( $\phi_T$  and  $\bar{\theta}$ ) and ( $\phi_V$  and  $\langle \theta \rangle$ ) the three  
 310 forms of intrinsic averages are identical, i.e.,  $[\theta] = \langle \bar{\theta} \rangle = \overline{\langle \theta \rangle}$ . The same applies, of course, for the  
 311 fixed bed conditions, for which  $\langle \phi_T \rangle \equiv 1$ ,  $\phi_{Vm} = \phi_V$ ,  $\phi_{Tm} \equiv 1$ ,  $\bar{\phi}_V = \phi_V$ , and thus  $[\theta] = \langle \bar{\theta} \rangle = \overline{\langle \theta \rangle}$  again.

312

## 313 Double-Averaging Theorems

314

### 315 *Double-Averaging Theorems for Superficial Quantities*

316 We first review the derivation of the double-averaging theorems for the superficial variables, refining  
 317 Nikora et al.'s (2007a) approach. Using Eq. (8) and Gray and Lee's (1977) relationships as outlined  
 318 in the Introduction, we can obtain the averaging theorem for the time derivative as:

319

$$\left[ \frac{\partial \theta}{\partial t} \right]_s = \left\langle \frac{\partial \bar{\theta}^s}{\partial t} \right\rangle_s = \overline{\left\langle \frac{\partial \theta}{\partial t} \right\rangle}_s = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \frac{\partial \theta}{\partial t} \gamma(x_i, t) dV dt = \frac{1}{T_o} \frac{1}{V_o} \int \int \frac{\partial \theta \gamma}{\partial t} dV dt - \frac{1}{T_o} \frac{1}{V_o} \int \int \theta \frac{\partial \gamma}{\partial t} dV dt$$

321

$$= \frac{\partial}{\partial t} \left( \frac{1}{T_o} \frac{1}{V_o} \int \int \theta \gamma dV dt \right) + \frac{1}{T_o} \frac{1}{V_o} \int \left( \int \theta v_i n_i \delta(x_i - x_{si}) dV \right) dt$$

323

$$= \frac{\partial [\theta]_s}{\partial t} + \frac{1}{T_o} \frac{1}{V_o} \int \left( \int_{S_{int}} \theta v_i n_i dS \right) dt = \frac{\partial [\theta]_s}{\partial t} + \frac{1}{V_o} \overline{\int \int_{S_{int}} \theta v_i n_i dS} \quad (17)$$



325  
326 For the spatial derivative, it follows, similarly, that:  
327

$$\begin{aligned}
328 \quad \left[ \frac{\partial \theta}{\partial x_i} \right]_s &= \left\langle \frac{\partial \theta}{\partial x_i} \right\rangle_s = \overline{\left\langle \frac{\partial \theta}{\partial x_i} \right\rangle_s} = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \frac{\partial \theta}{\partial x_i} \gamma(x_i, t) dV dt = \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \frac{\partial \theta \gamma}{\partial x_i} dV dt - \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \frac{\partial \gamma}{\partial x_i} dV dt \\
329 & \\
330 &= \frac{\partial}{\partial x_i} \left( \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \int_{V_o} \theta \gamma dV dt \right) - \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \left( \int_{V_o} \theta n_i \delta(x_i - x_{si}) dV \right) dt \\
331 & \\
332 &= \frac{\partial [\theta]_s}{\partial x_i} - \frac{1}{T_o} \frac{1}{V_o} \int_{T_o} \left( \iint_{S_{int}} \theta n_i dS \right) dt = \frac{\partial [\theta]_s}{\partial x_i} - \frac{1}{V_o} \overline{\iint_{S_{int}} \theta n_i dS} \quad (18)
\end{aligned}$$

333  
334 Thus, we arrive at the following general double-averaging theorems for superficial derivatives:  
335

$$336 \quad \left[ \frac{\partial \theta}{\partial t} \right]_s = \frac{\partial [\theta]_s}{\partial t} + \frac{1}{V_o} \overline{\iint_{S_{int}} \theta v_i n_i dS} \quad \text{and} \quad \left[ \frac{\partial \theta}{\partial x_i} \right]_s = \frac{\partial [\theta]_s}{\partial x_i} - \frac{1}{V_o} \overline{\iint_{S_{int}} \theta n_i dS} \quad (19)$$

337  
338 which are suitable for both fixed-bed and mobile-bed conditions. Note that although Nikora et al.  
339 (2007a) used consecutive time-space averaging, equations (19) are identical to equations (6a) in  
340 Nikora et al. (2007a), as a consequence of  $[\theta]_s = \langle \bar{\theta} \rangle_s = \overline{\langle \theta \rangle_s}$ .

### 341 **Double-Averaging Theorems for Intrinsic Quantities**

342  
343 Using the relation  $[\theta]_s = \phi_{VT} [\theta]$  from Eq. (10), we can obtain from (19) the general double-  
344 averaging theorems for intrinsic derivatives as:

$$345 \quad \left[ \frac{\partial \theta}{\partial t} \right]_s = \frac{1}{\phi_{VT}} \frac{\partial \phi_{VT} [\theta]}{\partial t} + \frac{1}{\phi_{VT} V_o} \overline{\iint_{S_{int}} \theta v_i n_i dS} \quad \text{and} \quad \left[ \frac{\partial \theta}{\partial x_i} \right]_s = \frac{1}{\phi_{VT}} \frac{\partial \phi_{VT} [\theta]}{\partial x_i} - \frac{1}{\phi_{VT} V_o} \overline{\iint_{S_{int}} \theta n_i dS} \quad (20)$$

347  
348 Taking into account that  $\phi_{VT} / \phi_{Vm} = \langle \phi_T \rangle$  and  $[\theta] = \langle \phi_T \bar{\theta} \rangle / \langle \phi_T \rangle$  that follow from equations (6) and  
349 (15), one can also obtain from (20):  
350

$$\begin{aligned}
351 \quad \left\langle \phi_T \frac{\partial \theta}{\partial t} \right\rangle &= \frac{1}{\phi_{Vm}} \frac{\partial \phi_{Vm} \langle \phi_T \bar{\theta} \rangle}{\partial t} + \frac{1}{\phi_{Vm} V_o} \overline{\iint_{S_{int}} \theta v_i n_i dS} \quad \text{and} \\
352 \quad \left\langle \phi_T \frac{\partial \theta}{\partial x_i} \right\rangle &= \frac{1}{\phi_{Vm}} \frac{\partial \phi_{Vm} \langle \phi_T \bar{\theta} \rangle}{\partial x_i} - \frac{1}{\phi_{Vm} V_o} \overline{\iint_{S_{int}} \theta n_i dS} \quad (21)
\end{aligned}$$

353  
354 Equations (21) are consistent with equations (6b) in Nikora et al. (2007a) that were derived using,  
355 implicitly, *consecutive time-space averaging*. Although being algebraically identical to those in  
356 Nikora et al. (2007a, Eq 6b), equations (21) now involve spatial and time bed porosities that have  
357 been explicitly derived in this paper for mobile-bed conditions. Thus, equations (20) and (21) update  
358 those given in Nikora et al. (2007a) by refining the meanings of variables and parameters involved in  
359 the averaging procedures, thereby making their use and parameterisations for mobile-bed conditions  
360 clarified. It should be highlighted that the spatial averaging theorems of (19) and (20), as well as

361 those that follow from them such as (21), are equally applicable for fixed- and mobile-bed  
362 conditions.

### 363 **Modified Reynolds Decomposition**

364 As outlined in the Introduction, derivation of the double-averaged hydrodynamic equations involves  
365 three components, two of which (the averaging operators and theorems) have been revisited in the  
366 previous sections. The remaining component, modified Reynolds decomposition of instantaneous  
367 variables, can be defined for the discussed forms of double averaging as:

$$368 \theta = [\theta] + \ddot{\theta} = \langle \phi_T \bar{\theta} \rangle / \langle \phi_T \rangle + \ddot{\theta} \quad \text{or} \quad \text{for space-time averaging}$$

$$369 \theta = [\theta] + \ddot{\theta} = \overline{\phi_V \langle \theta \rangle} / \overline{\phi_V} + \ddot{\theta}$$

$$370 \theta = \langle \bar{\theta} \rangle + \tilde{\tilde{\theta}} + \theta' = \bar{\theta} + \theta' \quad \text{for consecutive time-space averaging}$$

$$371 \theta = \overline{\langle \theta \rangle} + \langle \theta \rangle' + \tilde{\theta} = \langle \theta \rangle + \tilde{\theta} \quad \text{for consecutive space-time averaging}$$

372 where  $\ddot{\theta}$  = the deviation of the instantaneous variable  $\theta$  from its double-averaged value  $[\theta]$ ,  $\tilde{\theta}$  =  
373 the deviation of the instantaneous variable  $\theta$  from its spatially-averaged instantaneous value  $\langle \theta \rangle$ ,  $\tilde{\tilde{\theta}}$   
374 = the deviation of the time-averaged variable  $\bar{\theta}$  from its spatially-averaged value  $\langle \bar{\theta} \rangle$ , and prime  
375 indicates deviation from a time-averaged value. Note that to improve consistency in symbols for  
376 different forms of averaging we use here  $\tilde{\tilde{\theta}}$  and  $\tilde{\theta}$  instead of  $\tilde{\tilde{\theta}}$  and  $\hat{\theta}$  employed in Nikora et al.  
377 (2007a,b), respectively. For fixed beds or in the case of the time and space porosities being  
378 uncorrelated with hydrodynamic variables for mobile beds, the decomposition  $\theta = [\theta] + \ddot{\theta}$  merges  
379 with the decompositions  $\theta = \langle \bar{\theta} \rangle + \tilde{\tilde{\theta}} + \theta'$  and  $\theta = \overline{\langle \theta \rangle} + \langle \theta \rangle' + \tilde{\theta}$  (since  $[\theta] = \langle \bar{\theta} \rangle = \overline{\langle \theta \rangle}$ ), which can  
380 be linked through the double-decomposition of Pedras and de Lemos (2000), as discussed in Nikora  
381 et al. (2007a) and explored in Pokrajac et al. (2008), i.e.:

$$382 \theta = \langle \bar{\theta} \rangle + \ddot{\theta} = \langle \bar{\theta} \rangle + \tilde{\tilde{\theta}} + \theta', \quad \theta = \overline{\langle \theta \rangle} + \ddot{\theta} = \overline{\langle \theta \rangle} + \langle \theta \rangle' + \tilde{\theta} \quad \Rightarrow \quad \langle \theta \rangle' + \tilde{\theta} = \tilde{\tilde{\theta}} + \theta' = \ddot{\theta}$$

### 383 **Double-Averaged Hydrodynamic Equations**

384 For the general case of mobile-bed flows, the appearance of the double-averaged equations will  
385 depend on an averaging form and associated decomposition of flow variables. For consistency with  
386 RANS, and to take advantage of already-available data (which in most cases were collected within a  
387 RANS framework), the *consecutive time-space* form of double averaging and the associated  
388 decomposition  $\theta = \langle \bar{\theta} \rangle + \tilde{\tilde{\theta}} + \theta'$  are adopted in the following discussion. In the derivation of the  
389 equations below, it is assumed that  $\overline{\bar{\theta}} = \bar{\theta}$ ,  $\langle \langle \theta \rangle \rangle = \langle \theta \rangle$ ,  $\overline{\tilde{\tilde{\theta}}} = \tilde{\tilde{\theta}}$ ,  $\langle \tilde{\tilde{\theta}} \rangle = 0$ , and  $\overline{\theta'} = 0$ , similar to the  
390 Reynolds averaging rules. Thus, using equation (12), i.e.,  $[\theta]_s = \langle \bar{\theta}^s \rangle_s = \phi_{Vm} \langle \phi_T \bar{\theta} \rangle$ , the  
391 decomposition  $\theta = \langle \bar{\theta} \rangle + \tilde{\tilde{\theta}} + \theta'$ , and the spatial averaging theorems (21), the following double-  
392 averaged equations can be derived from their counterparts for instantaneous variables as shown  
393 below.

400 *Double-averaged continuity equation*

401

402  
403 Starting with:  
404

$$405 \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \quad (22a)$$

406  
407 the following double-averaged continuity equation can be obtained (for  $\rho = \text{constant}$  and  $u_i = v_i$  on  
408 the interfacial surface):  
409

$$410 \quad \frac{\partial \phi_{Vm} \langle \phi_T \rangle}{\partial t} + \frac{\partial \phi_{Vm} \langle \phi_T \bar{u}_i \rangle}{\partial x_i} = 0 \quad (22b)$$

411  
412 For spatially uncorrelated local time porosity and flow velocities (i.e.,  $\langle \phi_T \bar{u}_i \rangle = \langle \phi_T \rangle \langle \bar{u}_i \rangle$ ), equation  
413 (22b) can be simplified, using a space-time porosity  $\phi_{VT} = \phi_{Vm} \langle \phi_T \rangle$  of Eq. (6), as:  
414

$$415 \quad \frac{\partial \phi}{\partial t} + \frac{\partial \phi \langle \bar{u}_i \rangle}{\partial x_i} = 0 \quad (22c)$$

416  
417 where we use  $\phi_{VT} = \phi$  for brevity.  
418

#### 419 *Double-averaged momentum equation*

420

422 Using the Navier-Stokes equation as a starting point, i.e.:

$$423 \quad \frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu \frac{\partial u_i}{\partial x_j} \right) \quad (23a)$$

424  
425 the following double-averaged momentum equation is obtained:  
426

$$427 \quad \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \bar{u}_i \rangle}{\partial t}}_1 + \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \rangle \langle \bar{u}_i \rangle \langle \bar{u}_j \rangle}{\partial x_j}}_2 = \underbrace{\phi_{Vm} \langle \phi_T g_i \rangle}_3 - \underbrace{\frac{1}{\rho} \frac{\partial \phi_{Vm} \langle \phi_T \bar{p} \rangle}{\partial x_i}}_4 - \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \overline{u'_i u'_j} \rangle}{\partial x_j}}_5 - \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \tilde{u}_i \tilde{u}_j \rangle}{\partial x_j}}_6$$

$$428 \quad + \underbrace{\frac{\partial}{\partial x_j} \left( \phi_{Vm} \left\langle \phi_T \nu \frac{\partial \bar{u}_i}{\partial x_j} \right\rangle \right)}_7 - \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \tilde{u}_i \rangle \langle \bar{u}_j \rangle}{\partial x_j}}_8 - \underbrace{\frac{\partial \phi_{Vm} \langle \phi_T \tilde{u}_j \rangle \langle \bar{u}_i \rangle}{\partial x_j}}_9$$

$$429 \quad + \underbrace{\frac{1}{\rho} \frac{1}{V_o} \iint_{S_{\text{int}}} \overline{pn}_i dS}_{10} - \underbrace{\frac{1}{V_o} \iint_{S_{\text{int}}} \left( \nu \frac{\partial \bar{u}_i}{\partial x_j} \right) n_j dS}_{11} \quad (23b)$$

430  
431 Eq. (23b) is derived by applying an operation of superficial double-averaging to each term of the  
432 initial momentum equation (23a), and then transforming these terms using the double-averaging  
433 theorems (Eqs. 21) supplemented with the decomposition of the instantaneous velocities as  
434  $u_i = \langle \bar{u}_i \rangle + \tilde{u}_i + u'_i$ . Terms 1 and 2 in equation (23b) represent local and convective accelerations,

435 respectively. The third term is the gravity term; the fourth term is the pressure gradient; the fifth,  
 436 sixth and seventh terms are contributions from turbulent ( $\langle \phi_T \overline{u'_i u'_j} \rangle$ ), form-induced ( $\langle \phi_T \tilde{u}_i \tilde{u}_j \rangle$ ), and  
 437 viscous fluid stresses, respectively; the eighth and ninth terms represent momentum fluxes (stresses)  
 438 due to potential spatial correlations between the local time porosity and time-averaged velocities; and  
 439 the final two terms, i.e., tenth and eleventh, are pressure and viscous drag terms. For the case when  
 440 spatial correlations between the local time porosity and time-averaged flow parameters can be  
 441 neglected, equation (23b) can be simplified, using the continuity equation (22c) and the space-time  
 442 porosity  $\phi = \phi_{VT} = \phi_{Vm} \langle \phi_T \rangle$ , as:

$$443 \frac{\partial \langle \bar{u}_i \rangle}{\partial t} + \langle \bar{u}_j \rangle \frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} = g_i - \frac{1}{\rho} \frac{1}{\phi} \frac{\partial \phi \langle \bar{p} \rangle}{\partial x_i} - \frac{1}{\phi} \frac{\partial \phi \langle \overline{u'_i u'_j} \rangle}{\partial x_j} - \frac{1}{\phi} \frac{\partial \phi \langle \tilde{u}_i \tilde{u}_j \rangle}{\partial x_j} + \frac{1}{\phi} \frac{\partial}{\partial x_j} \left( \phi \left\langle v \frac{\partial u_i}{\partial x_j} \right\rangle \right) \\ 444 + \frac{1}{\rho} \frac{1}{\phi} \frac{1}{V_o} \overline{\iint_{S_{\text{int}}} p n_i dS} - \frac{1}{\phi} \frac{1}{V_o} \overline{\iint_{S_{\text{int}}} \left( v \frac{\partial u_i}{\partial x_j} \right) n_j dS} \quad (23c)$$

### 446 *Double-averaged advection-diffusion equation*

447  
 448  
 449 Using the advection-diffusion equation for instantaneous variables as a starting point, i.e.:

$$451 \frac{\partial C}{\partial t} + \frac{\partial C u_j}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \chi_m \frac{\partial C}{\partial x_j} \right) + F \quad (24a)$$

452  
 453 the following double-averaged advection-diffusion equation can be similarly derived:

$$454 \frac{\partial \phi_{Vm} \langle \phi_T \bar{C} \rangle}{\partial t} + \frac{\partial \phi_{Vm} \langle \phi_T \rangle \langle \bar{C} \rangle \langle \bar{u}_j \rangle}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \phi_{Vm} \left\langle \phi_T \chi_m \frac{\partial C}{\partial x_j} \right\rangle \right) - \frac{\partial \phi_{Vm} \langle \phi_T \overline{C' u'_j} \rangle}{\partial x_j} - \frac{\partial \phi_{Vm} \langle \phi_T \tilde{C} \tilde{u}_j \rangle}{\partial x_j} \\ 455 - \frac{\partial \phi_{Vm} \langle \phi_T \tilde{C} \rangle \langle \bar{u}_j \rangle}{\partial x_j} - \frac{\partial \phi_{Vm} \langle \phi_T \tilde{u}_j \rangle \langle \bar{C} \rangle}{\partial x_j} - \frac{1}{V_o} \overline{\iint_{S_{\text{int}}} \left( \chi_m \frac{\partial C}{\partial x_j} \right) n_j dS} + \underbrace{\phi_{Vm} \langle \phi_T \bar{F} \rangle}_9 \quad (24b)$$

457  
 458 where  $C$  = passive substance concentration;  $\chi_m$  = molecular diffusion coefficient; and  $F$  =  
 459 source/sink of substance  $C$ . Terms 1 and 2 in equation (24b) represent local change of concentration  
 460 and convective transport. The third, fourth, and fifth terms are due to the molecular diffusion,  
 461 turbulent transport ( $\langle \phi_T \overline{C' u'_j} \rangle$ ), and form-induced transport ( $\langle \phi_T \tilde{C} \tilde{u}_j \rangle$ ), respectively; the sixth and  
 462 seventh terms represent substance fluxes due to potential spatial correlations between the local time  
 463 porosity and time-averaged velocities and concentrations; the final two terms, i.e., eighth and ninth,  
 464 are an interfacial flux term (i.e., heterogeneous reaction rate) and a homogeneous reaction rate,  
 465 respectively. For the case when spatial correlations between the local time porosity and time-  
 466 averaged flow parameters can be neglected, equation (24b) can be simplified, using the continuity  
 467 equation (22c) and the space-time porosity  $\phi = \phi_{VT} = \phi_{Vm} \langle \phi_T \rangle$ , as:

468

$$\begin{aligned}
469 \quad \frac{\partial \langle \bar{C} \rangle}{\partial t} + \langle \bar{u}_j \rangle \frac{\partial \langle \bar{C} \rangle}{\partial x_j} &= \frac{1}{\phi} \frac{\partial}{\partial x_j} \left( \phi \left\langle \chi_m \frac{\partial \bar{C}}{\partial x_j} \right\rangle \right) - \frac{1}{\phi} \frac{\partial \phi \langle \bar{C}' u'_j \rangle}{\partial x_j} - \frac{1}{\phi} \frac{\partial \phi \langle \bar{C} \tilde{u}_j \rangle}{\partial x_j} \\
470 \quad &\quad - \frac{1}{\phi} \frac{1}{V_o} \overline{\iint_{S_{\text{int}}} \left( \chi_m \frac{\partial C}{\partial x_j} \right) n_j dS} + \langle \bar{F} \rangle \quad (24c)
\end{aligned}$$

471

472 Advection-diffusion equations similar to (24a)-(24c) can also be derived for fine suspended  
473 sediments at low concentrations at which the advection-diffusion approximation is appropriate.

474 Compared to the conventional RANS equations, Eqs. (22)-(24) contain some additional terms  
475 such as dispersive or form-induced stresses  $\langle \phi_T \tilde{u}_i \tilde{u}_j \rangle$  and fluxes  $\langle \phi_T \bar{C} \tilde{u}_j \rangle$  due to spatial correlations  
476 of the respective time-averaged velocities and concentration fields; momentum and substance fluxes  
477 due to potential spatial correlations between the local time porosity and time-averaged velocities and  
478 concentrations; the form drag per unit fluid mass  $f_{pi} = -(1/\rho V_o) \overline{\iint_{S_{\text{int}}} p n_i dS}$ ; the viscous drag per

479 unit fluid mass  $f_{vi} = (1/V_o) \overline{\iint_{S_{\text{int}}} (v \partial u_i / \partial x_j) n_j dS}$ ; and the diffusive flux at the water - bed surface

480 interface  $J = (1/V_o) \overline{\iint_{S_{\text{int}}} (\chi_m \partial C / \partial x_j) n_j dS}$  (including biological surfaces when relevant). The

481 quantities  $\langle \phi_T \tilde{u}_i \tilde{u}_j \rangle$  and  $\langle \phi_T \bar{C} \tilde{u}_j \rangle$  in equations (23) and (24) follow from double-averaging, similar to

482  $\overline{u'_i u'_j}$  and  $\overline{u'_j C'}$  in the time-averaged equations that appear due to time averaging of the Navier-Stokes  
483 and advection-diffusion equations for instantaneous variables. Some details on these unconventional  
484 terms for fixed-bed flows can be found in Nikora et al. (2007a,b) and Nikora and Rowinski (2008).

485 When required, the double-averaged hydrodynamic equations can also be formulated within  
486 *space-time* and *consecutive space-time* averaging frameworks, which are analytically linked to  
487 equations (22) to (24) obtained based on *consecutive time-space* averaging. The appearance of the  
488 double-averaged equations for the fixed-bed conditions are equivalent to (22c), (23c), and (24c)  
489 where  $\phi = \phi_{VT} = \phi_{vm} \langle \phi_T \rangle = \phi_v$ , as  $\langle \phi_T \rangle \equiv 1$  and  $\phi_{vm} = \phi_v$ .

490

## 491 Discussion

492

493 Equations (22c), (23c), and (24c) are identical in appearance to the corresponding equations  
494 presented in Nikora et al. (2007a). However, their justification and use for data analysis,  
495 interpretation, and modelling of the mobile-bed flows should now be clearer as the meanings of the  
496 time and spatial porosities and the potential roles of spatial correlations between the local time  
497 porosity and time-averaged velocities and concentrations are now unambiguously defined and  
498 explained. It should also be highlighted that equations (22c), (23c), and (24c) are simplified versions  
499 of the more general equations (22b), (23b), and (24b) that include the potential effects of spatial  
500 correlations between the local time porosity and time-averaged velocities and concentrations (they  
501 are neglected in Eqs. (22c), (23c), and (24c)).

502 The correlations  $\langle \phi_T \tilde{u}_i \rangle$  and  $\langle \phi_T \bar{C} \rangle$  in Eqs. (22) to (24) are introduced in this paper for the first

503 time and thus information about them and their gradients is not yet available. These terms have to be  
504 quantitatively assessed as there may be situations where they cannot be neglected and thus full  
505 equations (22b), (23b), and (24b) have to be employed. Indeed, channel beds of most natural rivers  
506 are characterised by roughness patchiness generated by a variety of mechanisms. Examples include  
507 particle clusters in gravel-bed rivers (e.g., Papanicolaou et al. 2011), ripple/dune patches in sand-bed  
508 rivers (e.g., Aberle et al. 2010), and vegetation patches in low-order rivers (e.g., Nepf 2012). These

roughness patches (or clusters) introduce some spatial heterogeneity in flow velocity and concentration fields as well as certain spatial variability in bedload and/or vegetation waviness that define the local time porosity  $\phi_T$ . Thus,  $\tilde{u}_i$  and/or  $\tilde{C}$  can well be spatially correlated with  $\phi_T$ . For instance, considering velocity change within and around a roughness patch on a gravel bed, it is likely that flow velocity above/within the patch is lower than outside it. However, the local time porosity  $\phi_T$  can be expected to be lower away from a patch centre where flow velocity and thus bedload intensity are enhanced. As a result, the spatial correlation  $\langle \phi_T \tilde{u}_i \rangle$  within an averaging window that includes a patch of increased roughness should be non-zero and negative. Vegetation patches may exhibit an opposite effect as both velocity and time porosity are likely to be minimised within the patch, leading to the positive correlation moment  $\langle \phi_T \tilde{u}_i \rangle$ . As river beds often exhibit large-scale heterogeneity (e.g., bars, meanders), the spatial gradients of  $\langle \phi_T \tilde{u}_i \rangle$  and  $\langle \phi_T \tilde{C} \rangle$  can be predicted to be non-zero too. These qualitative speculations require, however, proper quantitative assessments utilising reliable data sets from numerical simulations, laboratory experiments, and field measurements. Until very recently such data sets have been unavailable. However, latest advancements in instrumentation and computational techniques make such assessments in the nearest future realistic. The estimates of  $\langle \phi_T \tilde{u}_i \rangle$  and  $\langle \phi_T \tilde{C} \rangle$  and their spatial gradients for a range of conditions will provide a base for developing physically-driven parameterisations suitable for applied hydraulic models.

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## 528 **Conclusions**

Double-averaged conservation equations (22) to (24) provide a mathematical framework for studying turbulent mobile-bed flows such as gravel-bed rivers during flood events or flows over vegetated beds. The data on such flows, especially within moving roughness elements, are currently very limited due to both measurement difficulties and the remaining uncertainty of what exactly to measure, interpret, and model. The measurement techniques (e.g., refractive index matching Particle Image Velocimetry or those based on Magnetic Resonance Imaging) and modelling capabilities (e.g., Large Eddy Simulation method) have been improved in recent years and it is likely that extensive data on hydrodynamic variables within mobile roughness elements will appear very soon. Equations (22) to (24) will help in designing measurement and simulation campaigns for obtaining such data and for their interpretation and parameterisation, eventually leading to improved and more robust predictive models.

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## 541 **Acknowledgements**

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607 **Figure captions for the paper**

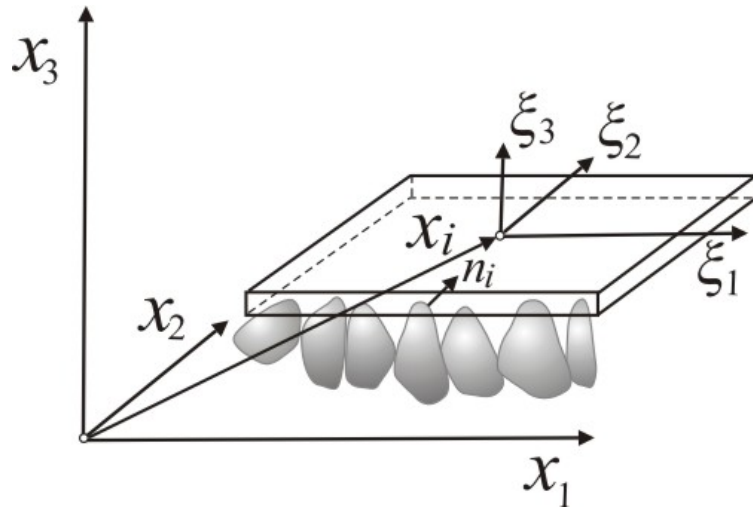
608 **“Spatially-averaged flows over mobile rough beds: definitions, averaging theorems, and**  
609 **conservation equations”**

610 by V. Nikora, F. Ballio, S. Coleman, D. Pokrajac  
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613 **Fig. 1.** A sketch showing a local co-ordinate system and a spatial averaging domain. Note that below  
614 the roughness tops the averaging domain includes both fluid and fixed or mobile bed surface.  
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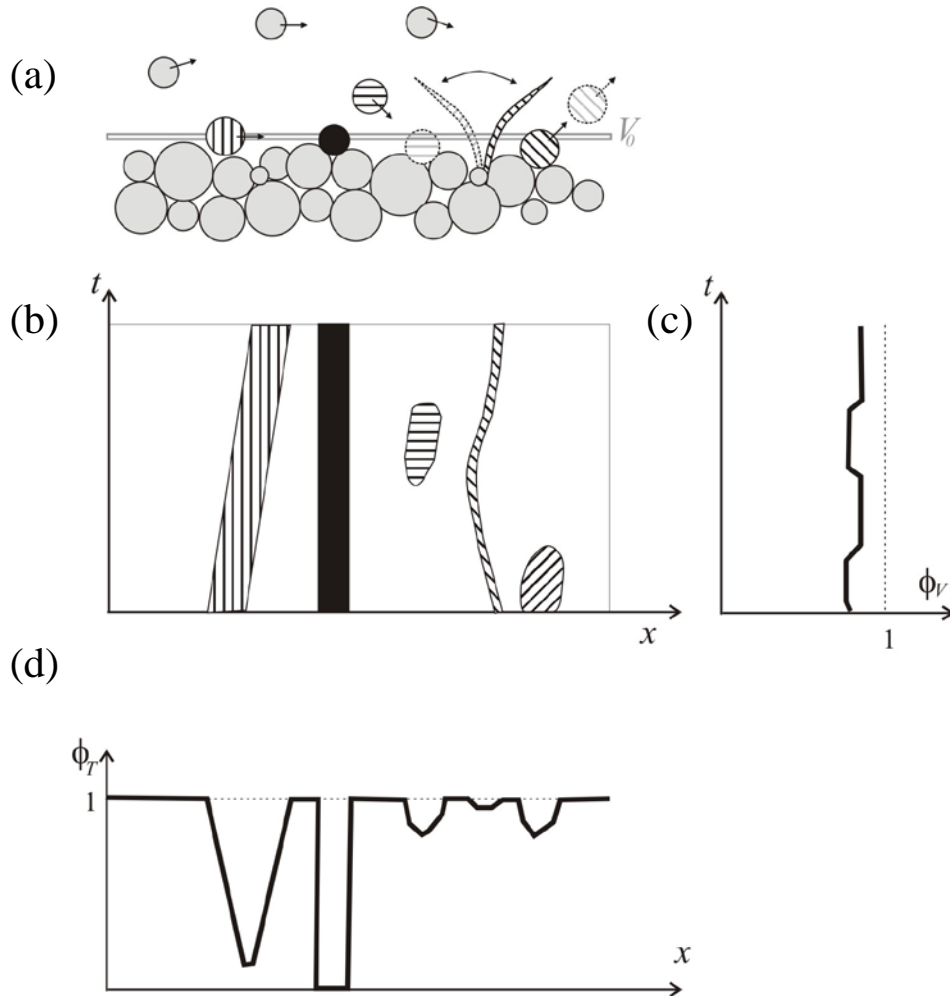
616 **Fig. 2.** A sketch for porosities for mobile-bed conditions (sediment transport and a moving plant),  
617 showing an averaging domain and embedded mobile and fixed objects (a), time evolution of solid  
618 object positions within the averaging domain: an example for the  $x$ -axis (b), spatial porosity changing  
619 in time (c), and time porosity changing along the flow (d). Black colour defines a particle that does  
620 not move within  $T_o$  (i.e., a ‘solid island’ within the spatial averaging domain  $V_o$ ); grey colour defines  
621 mobile and fixed particles that do not cross the averaging domain; and patterned objects define  
622 mobile particles and a waving plant that move through the averaging domain. In this example  $\phi_{Tm} \equiv 1$   
623 (i.e., a part of  $V_o$  is occupied by fluid at any time), while  $\phi_{Vm} < 1$  (the black particle represents a solid  
624 island within  $V_o$ ).  
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**Fig. 1.** A sketch showing a local co-ordinate system and a spatial averaging domain. Note that below the roughness tops the averaging domain includes both fluid and fixed or mobile bed surface.

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