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On the nuclear dimension of strongly purely infinite C*-algebras

Gábor Szabó [1]

Fraser Noble Building, Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, Scotland, UK

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ABSTRACT

We show that separable, nuclear and strongly purely infinite C*-algebras have finite nuclear dimension. In fact, the value is at most three. This exploits a deep structural result of Kirchberg and Rørdam on strongly purely infinite C*-algebras that are homotopic to zero in an ideal-system preserving way.

0. Introduction

Over the last decade, our understanding of simple, nuclear C*-algebras has advanced enormously. This has largely been spurred by the interplay between certain topological and algebraic regularity properties, such as finite topological dimension, tensorial absorption of suitable strongly self-absorbing C*-algebras and order completeness of

E-mail address: gabor.szabo@abdn.ac.uk.

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homological invariants, see [6] for an overview. Indeed, this reflects the Toms–Winter regularity conjecture [6,20,21], which has spawned an investigation of simple C*-algebras that has lead to deep results at a fast and furious pace. We refer to [2] and its excellent introduction for an overview of the current state-of-the-art concerning the Toms–Winter regularity conjecture. The most recent highlight of the Elliott classification program is certainly the combination of papers [7,5,4,16] (by many hands), which have completed the classification of separable, unital, simple C*-algebras with finite nuclear dimension and satisfying the UCT. See in particular [16, Section 6] for a comprehensive summary.

Apart from applications in classification, it now stands to reason (see [17,13,1]) that the interplay between finite nuclear dimension and $\mathcal{Z}$-stability goes beyond the simple case and should indeed extend to the realm of non-simple, nuclear C*-algebras that are sufficiently non-type I. In particular, it is tempting to conjecture that all separable, nuclear and $\mathcal{Z}$-stable C*-algebras have finite nuclear dimension. Given that Kirchberg has established a deep classification result [8] for separable, nuclear, strongly purely infinite C*-algebras long ago, a natural step toward this conjecture is to handle the $\mathcal{O}_\infty$-absorbing case first. Note that a separable, nuclear C*-algebra A is strongly purely infinite if and only if $A \cong A \otimes \mathcal{O}_\infty$ (see [18, 3.2] and [11, 5.11(iii), 8.6]), and if and only if $A \cong A \otimes \mathcal{Z}$ and A is traceless (see [9, 3.12]).

In this note, we take this first step towards the above conjecture and verify that all separable, nuclear and $\mathcal{O}_\infty$-stable C*-algebras have finite nuclear dimension, and in fact the nuclear dimension is at most three. In [1], it was shown that a nuclear dimension estimate

$$\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_2)$$

holds for all C*-algebras A. As an application, one could see that separable, nuclear, $\mathcal{O}_\infty$-absorbing $\mathcal{C}(X)$-algebras with simple fibres have finite nuclear dimension. Extending this technique in the first section, we show that in fact, a more general nuclear dimension estimate

$$\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\mathrm{nuc}}^{+1}(A \otimes B)$$

holds for all C*-algebras A and all non-zero, separable C*-algebras B with $B \cong B \otimes \mathcal{O}_\infty$. This will be an application of Voiculescu's observation [19] that cones over C*-algebras are quasidiagonal, combined with a trick from [1] involving positive elements with full spectrum in simple, purely infinite C*-algebras. In particular, the above estimate implies that $\mathcal{O}_\infty$ has a dimension-reducing effect on separable, nuclear C*-algebras upon tensorial stabilization if and only if there is some non-zero, separable C*-algebra with this property. The deep structural results by Kirchberg and Rørdam [12] on C*-algebras homotopic to zero in an ideal-system preserving way then show that a certain purely infinite AH algebra constructed by Rørdam in [15] has such a dimension-reducing effect. This yields the main result.

1. 2-Coloured embeddings into purely infinite ultrapowers

The following is contained in the proof of [1, 3.3] and was originally observed by Winter:

Lemma 1.1. *Let A be a unital, simple and purely infinite* C^**-algebra. Let $h \in A$ be a positive contraction with full spectrum $[0,1]$. For every $\varepsilon > 0$, there is some $n \in \mathbb{N}$, numbers $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1$ and pairwise orthogonal projections $p_0, \ldots, p_n \in A$ such that $h =_\varepsilon \sum_{j=0}^n \lambda_j p_j$. Moreover, all the projections p_j may be chosen to represent the trivial class in $K_0(A)$.*

Proof. Since A has real rank zero by [14, 4.1.1], the only thing left to show is that the projections may be chosen to be trivial in K-theory. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$, numbers $0 \leq \lambda_0 \leq \cdots \leq \lambda_n \leq 1$ and pairwise orthogonal projections $q_0, \ldots, q_n \in A$ as above without the K-theory assumption.

By [3, Section 1], we can find a non-trivial projection $\tilde{q}_n \leq q_n$ with $[\tilde{q}_n]_0 = 0 \in K_0(A)$, and set $p_n = \tilde{q}_n$. Now assume that $j \in \{1, \ldots, n\}$ is a number for which we have already constructed projections $\tilde{q}_n, \ldots, \tilde{q}_{n-j+1}$ and $p_n, \ldots, p_{n-j+1}$. Apply [3, Section 1] again to find a non-trivial projection $\tilde{q}_{n-j} \leq q_{n-j}$ such that $[\tilde{q}_{n-j}]_0 = -[q_{n-j+1} - p_{n-j+1}]_0 \in K_0(A)$, and set $p_{n-j} = \tilde{q}_{n-j} + q_{n-j+1} - p_{n-j+1}$. After n steps, we have constructed pairwise orthogonal projections $p_0, \ldots, p_n$, all being trivial in $K_0(A)$, such that

$$\sum_{j=0}^{n} \lambda_n (q_n - p_n) = \lambda_0 (q_0 - \tilde{q}_0) + \sum_{j=1}^{n} (\lambda_j - \lambda_{j-1})(q_j - \tilde{q}_j).$$

Since h has full spectrum, we necessarily have $\lambda_0 \leq \varepsilon$ and $\lambda_j - \lambda_{j-1} \leq \varepsilon$ for all $j \in \{1, \ldots, n\}$. It follows that the above sum has norm at most ε. Hence $h =_{2\varepsilon} \sum_{j=0}^n \lambda_n p_n$. □

As a consequence, we get:

Lemma 1.2. *Let A be a unital, simple and purely infinite* C^**-algebra. Let $h_0, h_1 \in A$ be two positive contractions with full spectrum $[0,1]$. For every $\varepsilon > 0$, there is a unitary $u \in A$ with $h_1 =_\varepsilon u h_0 u^*$.*

Proof. Let $\varepsilon > 0$. By 1.1, we can find numbers $k, n \in \mathbb{N}$, numbers $0 \leq \lambda_0 \leq \cdots \leq \lambda_k \leq 1$ and $0 \leq \mu_0 \leq \cdots \leq \mu_n \leq 1$, pairwise orthogonal projections $p_0, \ldots, p_k$ and $q_0, \ldots, q_n$ in A that are all trivial in $K_0(A)$, such that

$$\sum_{i=0}^{k} \lambda_i p_i =_\varepsilon h_0 \quad \text{and} \quad \sum_{j=0}^{n} \mu_j q_j =_\varepsilon h_1.$$

By breaking up projections, if necessary, we may assume that $k = n$ and that $|\lambda_i - \mu_i| \leq \varepsilon$ for all $i = 0, \ldots, n$. By breaking up the projections p_0, q_0, if necessary, we may moreover assume that $\sum_{i=0}^n p_i \neq \mathbf{1} \neq \sum_{i=0}^n q_i$.

It follows from [3, Section 1] that there exist partial isometries $v_0, \ldots, v_n, v \in A$ with

$$v^*v = \mathbf{1}_A - \sum_{j=0}^n p_j, \ vv^* = \mathbf{1}_A - \sum_{j=0}^n q_j$$

and

$$v_j^*v_j = p_j, \ v_jv_j^* = q_j \quad \text{for all } j = 0, \ldots, n.$$

The element $u = v + v_0 + \cdots + v_n \in A$ then defines a unitary. We have

$$\begin{aligned} uh_0u^* &=_\varepsilon \quad u\Big(\sum_{j=0}^n \lambda_j p_j\Big)u^* &&= \sum_{j=0}^n \lambda_j u p_j u^* \\ &= \sum_{j=0}^n \lambda_j v_j \cdot v_j^* v_j \cdot v_j^* &&= \sum_{j=0}^n \lambda_j q_j \\ &=_\varepsilon \quad \sum_{j=0}^n \mu_j q_j &&=_\varepsilon h_1. \end{aligned}$$

This finishes the proof. □

Lemma 1.3. *Let A be a unital, simple and purely infinite* C*-*algebra. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Let $h_0, h_1 \in A_\omega$ be two positive contractions with full spectrum* $[0, 1]$*. Then h_0 and h_1 are unitarily equivalent.*

Proof. Since A_ω is again simple and purely infinite, this follows directly from 1.2 and a standard reindexation argument. □

The main technical observation of this section is the following:

Proposition 1.4. *Let A be a separable* C*-*algebra with a positive element $e \in A$ of norm 1. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Then there exist two c.p.c. order zero maps $\varphi_0, \varphi_1 : A \to (\mathcal{O}_\infty)_\omega$ with $\varphi_0(e) + \varphi_1(e) = \mathbf{1}$.*

Proof. By a result of Voiculescu [19], the cone over A is quasidiagonal. Hence we can find a $*$-monomorphism $\kappa : \mathcal{C}_0\big((0, 1], A\big) \to \mathcal{Q}_\omega$, where $\mathcal{Q}$ is the universal UHF algebra. By choosing some (necessarily non-unital) embedding $\mu : \mathcal{Q} \to \mathcal{O}_\infty$,[2] the composition

[2] For example, this can be a composition of some non-unital embedding $\mathcal{O}_2 \longrightarrow \mathcal{O}_\infty$ and some unital embedding $\mathcal{Q} \longrightarrow \mathcal{O}_2$, which exists by [10, 2.8].

$\psi_0 = \mu_\omega \circ \kappa : \mathcal{C}_0\big((0,1], A\big) \to (\mathcal{O}_\infty)_\omega$ is a $*$-monomorphism. The positive element $h = \psi_0(\mathrm{id}_{[0,1]} \otimes e) \in (\mathcal{O}_\infty)_\omega$ then has full spectrum $[0,1]$.

Applying 1.3, we find a unitary $u \in (\mathcal{O}_\infty)_\omega$ with $uhu^* = \mathbf{1} - h$. Then $\psi_1 = \mathrm{Ad}(u) \circ \psi_0 : \mathcal{C}_0\big((0,1], A\big) \to (\mathcal{O}_\infty)_\omega$ is also a $*$-monomorphism. This gives rise to two c.p.c. order zero maps $\varphi_0, \varphi_1 : A \to (\mathcal{O}_\infty)_\omega$ via $\varphi_i(x) = \psi_i(\mathrm{id}_{[0,1]} \otimes x)$ for all $x \in A$ and $i = 0, 1$. We then have

$$\varphi_0(e) + \varphi_1(e) = h + uhu^* = \mathbf{1},$$

as required. □

As a consequence, we get a general nuclear dimension estimate for $\mathcal{O}_\infty$-stable C*-algebras that generalizes [1, 3.3]:

Corollary 1.5. *Let A be a* C*-*algebra and B a non-zero, separable, $\mathcal{O}_\infty$-stable* C*-*algebra. Then*

$$\dim^{+1}_{\mathrm{nuc}}(A \otimes \mathcal{O}_\infty) \leq 2 \dim^{+1}_{\mathrm{nuc}}(A \otimes B).$$

Proof. Let $d = \dim_{\mathrm{nuc}}(A \otimes B)$. Since B is assumed to be $\mathcal{O}_\infty$-absorbing, we may also assume without loss of generality that A is $\mathcal{O}_\infty$-absorbing. We may certainly also assume that A is nuclear, as there is otherwise nothing to show. By [17, 2.5, 2.6], it suffices to find an estimate for the nuclear dimension of the embedding $(\mathrm{id}_A \otimes \mathbf{1})_\omega : A \to (A \otimes \mathcal{O}_\infty)_\omega$.

So let $F \Subset A$ and $\varepsilon > 0$ be arbitrary. As B is non-zero, we can find some positive contraction $e \in B$ of norm 1. Apply 1.4 to find two c.p.c. order zero maps $\varphi_0, \varphi_1 : B \to (\mathcal{O}_\infty)_\omega$ with $\varphi_0(e) + \varphi_1(e) = \mathbf{1}$. By the definition of d, we can find a finite-dimensional C*-algebra $\mathcal{F}$, a c.p.c. map $\mu : A \otimes B \to \mathcal{F}$ and c.p.c. order zero maps $\kappa^{(0)}, \dots, \kappa^{(d)} : \mathcal{F} \to A \otimes B$ with

$$x \otimes e =_\varepsilon \sum_{j=0}^{d} \kappa^{(j)} \circ \mu(x \otimes e) \quad \text{for all } x \in F.$$

Define the c.p.c. map $\Psi : A \to \mathcal{F}$ via $\Psi(x) = \mu(x \otimes e)$. Moreover, define for all $i = 0, 1$ and $j = 0, \dots, d$ the c.p.c. order zero map $\Phi^{(i,j)} : \mathcal{F} \to (A \otimes \mathcal{O}_\infty)_\omega$ via $\Phi^{(i,j)} = (\mathrm{id}_A \otimes \varphi_i) \circ \kappa^{(j)}$. Then we have

$$\begin{aligned} \sum_{i=0,1} \sum_{j=0}^{d} \Phi^{(i,j)} \circ \Psi(x) &= \sum_{i=0,1} (\mathrm{id}_A \otimes \varphi_i) \left(\sum_{j=0}^{d} \kappa^{(j)} \circ \mu(x \otimes e) \right) \\ &=_{2\varepsilon} \sum_{i=0,1} x \otimes \varphi_i(e) \;=\; x \otimes \mathbf{1} \end{aligned}$$

for all $x \in F$. This yields a $2(d+1)$-coloured decomposition of the map $(\mathrm{id}_A \otimes \mathbf{1})_\omega : A \to (A \otimes \mathcal{O}_\infty)_\omega$ through finite-dimensional C*-algebras and finishes the proof. □

2. $\mathcal{O}_\infty$-stable C*-algebras have finite nuclear dimension

In this section, we prove our main result. This will be a consequence of the estimate 1.5, together with a deep structural result of Kirchberg and Rørdam from [12]. Let us now recall some of these results:

Definition 2.1 *(see [15]).* Let $\{t_n\}_{n\in\mathbb{N}} \subset [0,1)$ be a dense sequence. For every n, define the $*$-homomorphism

$$\varphi_n : \mathcal{C}_0\big([0,1), M_{2^n}\big) \to \mathcal{C}_0\big([0,1), M_{2^{n+1}}\big)$$

via

$$\varphi_n(f)(t) = \mathrm{diag}\Big(f(t), f\big(\max(t,t_n)\big)\Big) \quad \text{for all } t \in [0,1).$$

Set $\mathcal{A}_{[0,1]} = \varinjlim \big\{\mathcal{C}\big([0,1), M_{2^n}\big), \varphi_n\big\}$.

In [12], Kirchberg and Rørdam have shown a deep structural result for separable, nuclear, strongly purely infinite C*-algebras that are homotopic to zero in an ideal-system preserving way. An interesting consequence of this observation is that tensoring a given nuclear C*-algebra with $\mathcal{A}_{[0,1]}$ yields an AH_0 algebra with low dimension:

Theorem 2.2 *(see [12, 5.12, 6.2]). For every non-zero, separable, nuclear* C*-*algebra B, the* C*-*algebra $B \otimes \mathcal{A}_{[0,1]}$ is an* AH_0 *algebra of topological dimension one. In particular, the decomposition rank of $B \otimes \mathcal{A}_{[0,1]}$ is one.*

Combining this structural result with our nuclear dimension estimate from the previous section, we obtain our main result:

Theorem 2.3. *Let A be a separable, nuclear* C*-*algebra. Then the nuclear dimension of $A \otimes \mathcal{O}_\infty$ is at most three.*

Proof. The C*-algebra $\mathcal{A}_{[0,1]}$ is $\mathcal{O}_\infty$-stable by [15, 5.3]. So combining 1.5 and 2.2, we get

$$\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \le 2\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{A}_{[0,1]}) \le 4.$$

This shows our claim. □

Remark 2.4. At first glance, it might appear that 2.2 (and therefore 2.3) depends on Kirchberg's classification [8] of non-simple, strongly purely infinite C*-algebras. However, Kirchberg's classification theorem is only used in [12] to deduce that $\mathcal{A}_{[0,1]}$ (or more generally any separable, nuclear, strongly purely infinite C*-algebras homotopic to zero in an ideal system-preserving way) is in fact $\mathcal{O}_2$-absorbing. While this is an important ingredient in verifying the AH_0-structure of the C*-algebras in question, one can refrain from appealing to classification by weakening the statement in 2.2 to $\mathrm{dr}(B \otimes \mathcal{A}_{[0,1]} \otimes \mathcal{O}_2) = 1$ for all non-zero, separable, nuclear C*-algebras B. In particular, this weaker statement can be proved exclusively with the methods from [12], without classification, and is sufficient to deduce our main result 2.3.

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