

Quasiperiodicity and suppression of multistability in nonlinear dynamical systems

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Abstract. It has been known that noise can suppress multistability by dynamically connecting coexisting attractors in the system which are otherwise in separate basins of attraction. The purpose of this mini-Review is to argue that quasiperiodic driving can play a similar role in suppressing multistability. A concrete physical example is provided where quasiperiodic driving was demonstrated to eliminate multistability completely to generate robust chaos in a semiconductor superlattice system.

1 Introduction

Ulrike has made significant contributions, among many others, to two subfields of nonlinear dynamics: quasiperiodicity [1–15] and multistability [16–25]. In particular, Ulrike was among the first [1–6] to study strange nonchaotic attractors [26], attractors with a fractal geometry but without sensitive dependence on initial conditions, which arise commonly in quasiperiodically driven systems. While dynamical systems with multistability, i.e., systems with multiple coexisting attractors (each with a distinct basin of attraction), had been studied earlier in terms of the fractal structure of the basin boundaries and the physical consequences [27,28], Ulrike was among the first to discover that even low-dimensional nonlinear dynamical systems can possess a large number of coexisting attractors [16]. The main purpose of this mini-Review is to argue that there is a natural connection between the two subjects in the sense that quasiperiodicity tends to suppress multistability. A physical example from the semiconductor superlattice system is quoted to support the argument.

The line of reasoning leading to our proposition that quasiperiodicity can suppress or even diminish multistability is illustrated in Fig. 1. It has been known that noise can induce transition to a chaotic attractor with distinct dynamical phenomena such as on-off intermittency in the size of the snapshot attractors [29,30] formed by an ensemble of independent trajectories [31], and the linear scaling law of the largest Lyapunov exponent when it passes zero from the negative side [32–34]. The key observation underlying our argument is that these behaviors are shared by quasiperiodically driven systems about the transition to chaos [7]. In fact, in terms of transition

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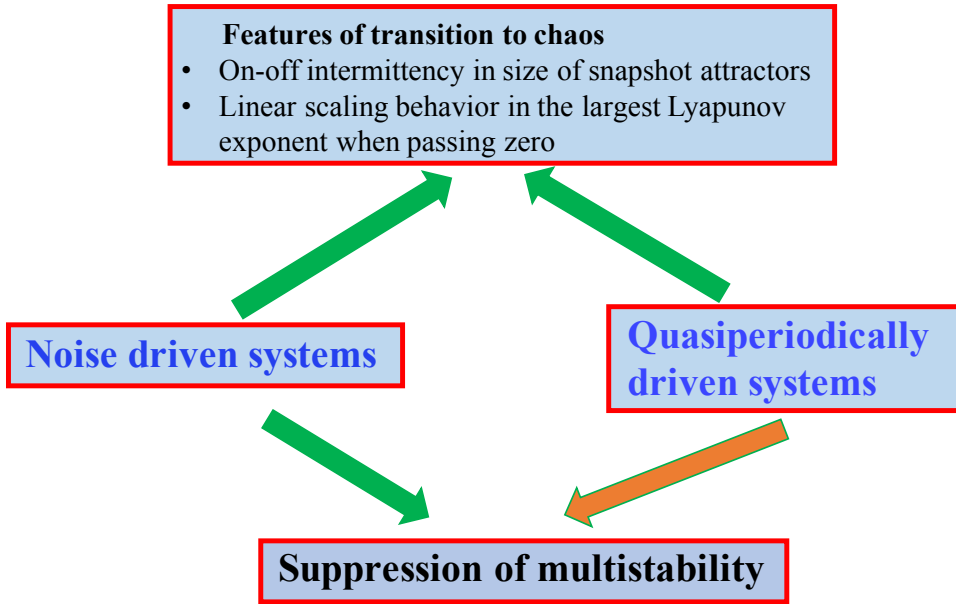


Fig. 1. Reasoning that quasiperiodicity can suppress multistability. Both random noise and quasiperiodic driving can induce a transition to chaos with similar scaling behaviors. Since noise can suppress multistability, so should quasiperiodic driving. The green arrows indicate previously established scenarios. The orange arrow represents the proposition articulated in this mini-Review.

to strange nonchaotic attractors, there is also a striking resemblance in the scaling behaviors between situations where the transition is due to quasiperiodic driving [35–37] or noise [38]. Since noise can effectively suppress the number of attractors or stable states in the system by dynamically linking previously isolated attractors [17, 39, 18, 19, 22], we expect that quasiperiodic driving can also be exploited to diminish multistability. Indeed, a recent study of the nonlinear dynamics of energetic electrons in a semiconductor superlattice system has revealed that multistability, which was present when the system is driven sinusoidally, can be effectively eliminated when an additional driving of non-commensurate frequency is introduced [40].

2 Transition to chaos

2.1 Transition to a chaotic attractor in random dynamical systems

A typical situation where noise-induced chaos can arise is periodic windows. In such a window, there is a periodic attractor and a non-attracting chaotic set that leads to transient chaos [41]. Noise can cause a trajectory to visit both the original attractor and the non-attracting chaotic set, giving rise to an extended noisy chaotic attractor. In a smooth dynamical system that exhibits chaos, in the absence of noise a chaotic attractor is structurally unstable [42–44], whereby, the periodic windows are dense and occupy open sets in the parameter space. As a result, an arbitrarily small perturbation can place the system in such a window, destroying the chaotic attractor. When noise is present, chaos is enhanced in the sense that noise-induced chaotic attractors can even occur in periodic windows. That is, chaotic attractor can now occur in open sets

in the parameter space. Noise-induced chaos thus provides the reason for observing chaotic attractors in realistic systems. This phenomenon is, of course, not restricted to periodic windows. Insofar as a nonchaotic attractor coexists with a non-attracting chaotic set in the phase space, a chaotic attractor can arise due to noise. In an opposite scenario, a chaotic attractor can be destroyed by noise through the mechanism of noise-induced crises [45]. There are also situations where a chaotic attractor, which is typically nonhyperbolic, is robust against noise [43].

A standard approach to defining a chaotic attractor under noise is the sensitive dependence on initial conditions, as characterized by the existence of at least one positive Lyapunov exponent [46, 29]. This is because the Lyapunov exponents are the time-averaged stretching or contracting rates of infinitesimal vectors along a typical trajectory in the phase space, which can be defined for both deterministic and stochastic dynamical systems. In particular, in the absence of noise, if the attractor is not chaotic, the largest Lyapunov exponent of the asymptotic attractor is negative for maps (zero for flows). As noise is turned on and its strength becomes sufficiently large, there is a nonzero probability that a trajectory originally on the attracting set escapes it and wanders near the coexisting non-attracting chaotic set. In this case, the largest Lyapunov exponent λ_1 becomes positive, indicating that the asymptotic attractor of the system is chaotic for trajectories starting from random initial conditions.

A basic issue concerns the scaling of the largest Lyapunov exponent about the transition [32–34]. Consider, for example, a continuous-time dynamical system. When the system is in a periodic window and exhibits a periodic attractor, in the deterministic case the largest Lyapunov exponent λ_1 is zero. As the noise amplitude σ is increased from zero and passes through a critical point σ_c , λ_1 becomes positive. It has been argued and supported by numerical simulations [32] that λ_1 obeys the following scaling law with the variation in the noise amplitude beyond the critical value:

$$\lambda_1 \sim (\sigma - \sigma_c)^\alpha, \quad (1)$$

for $\sigma > \sigma_c$, where $\alpha = 1$. There were heuristic arguments [32, 33] leading to (1), which were based on analyzing the overlaps between the natural measure of the noise-enlarged periodic attractor and that of the stable manifold of the non-attracting chaotic set. A theoretical approach to deriving (1) based on the concept of *quasipotentials* [20, 21, 47–54, 44] was also developed [34].

In terms of geometry, without noise, a chaotic attractor typically exhibits a fractal structure caused by the underlying dynamics. Under the influence of small random perturbations, if one examines a long trajectory produced by the dynamics, one usually observes that the fractal structure is smeared up to a distance scale proportional to the strength of the perturbations. In order to observe a clear fractal structure, a remedy is to examine the snapshot pattern formed by an ensemble of trajectories subject to the *same* random perturbation [31]. The details of the fractal structure differ from time to time, but the fractal dimensions *remain invariant* [46, 31]. The idea of *snapshot attractors* [31] was useful for visualizing and characterizing fractal patterns arising in physical situations such as passive particles advected on the surface of fluid [30, 55]. It was found [29, 30] that slightly after the transition to chaos, the size of the snapshot attractor can exhibit an extreme type of intermittency - on-off intermittency [56–72]. Snapshot attractors can also be used to study nonstationary dynamical systems [73, 74].

2.2 Transition to a chaotic attractor in quasiperiodically driven systems

In a quasiperiodically driven dynamical system, it was demonstrated [7] that, similar to the transition to chaos in random dynamical systems, the largest nontrivial Lyapunov exponent passes through zero *linearly* with the parameter about the transition.

In fact, near the transition, the tangent vector along a typical trajectory experiences both time intervals of expansion and time intervals of contraction. On the nonchaotic side, the Lyapunov exponent is slightly negative and, hence, contraction dominates over expansion. On the chaotic side where the Lyapunov exponent is slightly positive, expansion dominates over contraction. A consequence is that the collective behavior of an ensemble of trajectories observed at different instants of times exhibits an extreme type of intermittency on the chaotic side of the transition. During the expansion time intervals, the trajectories burst out by separating from each other, but during the contraction time intervals the trajectories merge together. Therefore, if one examines the snapshot attractors of this ensemble of trajectories at different times, one finds that the size of the chaotic attractor varies drastically in time in an intermittent fashion. The average size of the snapshot attractor scales *linearly* with a parameter above but near the transition. In addition, the average interval between bursts also scales linearly with the parameter above the transition.

In Ref. [7], the following quasiperiodically forced damped pendulum system [75] was considered:

$$\frac{d^2\theta}{dt^2} + \nu \frac{d\theta}{dt} + \sin\theta = K + V[\cos(\omega_1 t) + \cos(\omega_2 t)], \quad (2)$$

where θ is the angle of the pendulum with the vertical axis, ν is the dissipation rate, K is a constant, V is the forcing amplitude, ω_1 and ω_2 are the two incommensurate frequencies. Introducing two new variables, $t \rightarrow \nu t$ and $\phi \equiv \theta + \pi/2$, Eq. (2) becomes

$$\frac{1}{p} \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} - \cos\phi = K + V[\cos(\omega_1 t) + \cos(\omega_2 t)],$$

where $p = \nu^2$ is a new parameter, ω_1 and ω_2 have been rescaled accordingly: $\omega_1 \rightarrow \omega_1 \nu$ and $\omega_2 \rightarrow \omega_2 \nu$. In terms of the dynamical variables ϕ , $v \equiv d\phi/dt$, and $z \equiv \omega_2 t$, one has

$$\begin{aligned} \frac{d\phi}{dt} &= v, \\ \frac{dv}{dt} &= p[K + V(\cos(\frac{\omega_1}{\omega_2} z) + \cos z) + \cos\phi - v], \\ \frac{dz}{dt} &= \omega_2. \end{aligned} \quad (3)$$

The quasiperiodically driven system Eq. (3) exhibits rich dynamical phenomena [75, 76]. For example, for different parameters in the $K - V$ plane, one finds two- and three-frequency quasiperiodic attractors, strange nonchaotic attractors and chaotic attractors emerging from two-frequency quasiperiodic attractors. In the strong damping limit $p \rightarrow \infty$, Eq. (3) reduces to a first-order equation which is isomorphic to the Schrödinger equation with a quasiperiodic potential [77].

In Ref. [7], numerical results were presented for $K = 0.8$, $V = 0.55$, $\omega_1 = (\sqrt{5} - 1)/2$ (the inverse golden mean), $\omega_2 = 1.0$, with p as the bifurcation parameter. For large values of p ($p > 1.0$), the damping is strong so that the motion is typically periodic or quasiperiodic. As p is decreased, say, $p < p_c \approx 1.0$, both strange nonchaotic and chaotic attractors exist. Figure 2 shows the largest nontrivial Lyapunov exponent Λ for $p \in [0.969, 0.972]$. The transition occurs at $p = p_c \approx 0.9707$ where $\Lambda > 0$ for $p < p_c$ and $\Lambda < 0$ for $p > p_c$. The variables ϕ and v on the stroboscopic surface of section defined by $z = n(2\pi)$, $n = 0, 1, \dots$ can be used to visualize the attractors. Figure 3(a) shows a single long trajectory on the chaotic attractor for $p = 0.9702 < p_c$ ($\Lambda \approx 0.002$). Examination of the attractors for p slightly above p_c indicates that they

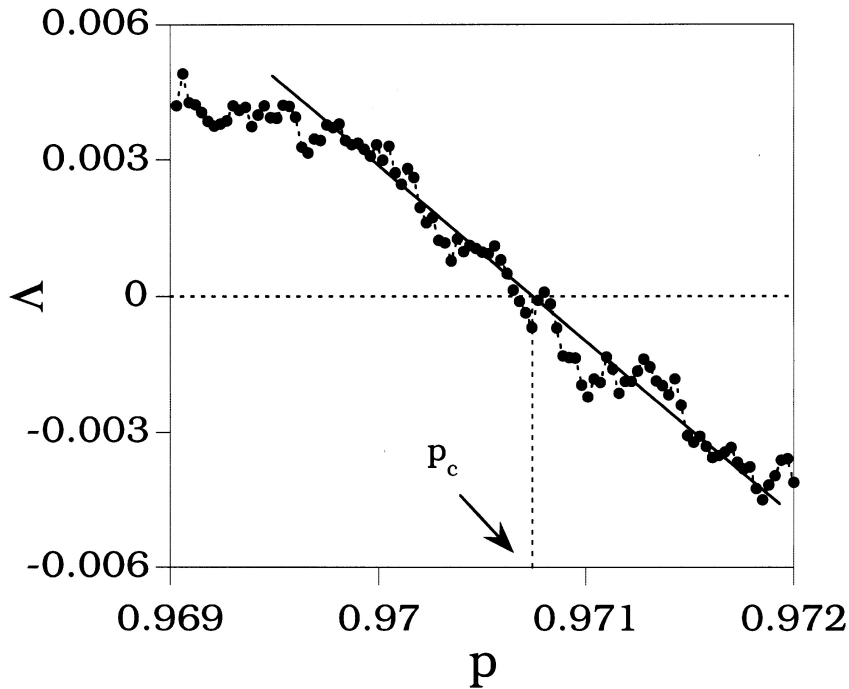


Fig. 2. Linear behavior of Lyapunov exponent about the transition to chaos in the quasiperiodically driven pendulum system. For system described by Eq. (3), the largest nontrivial Lyapunov exponent Δ versus the bifurcation parameter p (damping rate) for $0.969 \leq p \leq 0.972$. Other parameter values are $V = 0.55$, $K = 0.8$, $\omega_2 = 1$ and ω_1 is the inverse golden mean. *From Ref. [7] with permission.*

are strange nonchaotic [7]. One feature about the transition is that the Lyapunov exponent Δ passes through zero linearly, in spite of fluctuations caused by finite length of trajectories used in numerical computation. The mechanism behind such a smooth transition can be understood by examining the relative weights of the phase-space regions where a typical trajectory experiences expansion and contraction [35].

The time evolution of the snapshot attractor was also studied [7]. Specifically, an ensemble of initial conditions on the $[\phi, v]$ plane was chosen with the same $z(0) = 0$ (so they start to evolve at the same time). Snapshot attractors, *i.e.*, the distribution of the trajectories resulting from these initial conditions in the phase space at fixed subsequent instants of time, were examined. It was found that the properties of the snapshot attractors are qualitatively different for p slightly above p_c (Δ slightly negative) and p slightly below p_c (Δ slightly positive). In particular, for p slightly above p_c on the nonchaotic side, the trajectories resulting from these initial conditions eventually converge to a single trajectory. At any instant of time (after sufficiently long transient time), the snapshot attractor of these trajectories consists of only one point in the phase space. As time progresses, the single point for all trajectories moves in the phase space, tracing out a trajectory which lies on the strange nonchaotic attractor. The time required for the ensemble of trajectories to converge to a single trajectory scales as $\tau \sim 1/|\Delta| \sim 1/|p - p_c|$. However, an intermittency behavior occurs on the chaotic side for p slightly below p_c with Δ being slightly positive. In this case, the snapshot attractors are no longer a single point even after a long transient time. There are time intervals during which the snapshot attractor consists of points

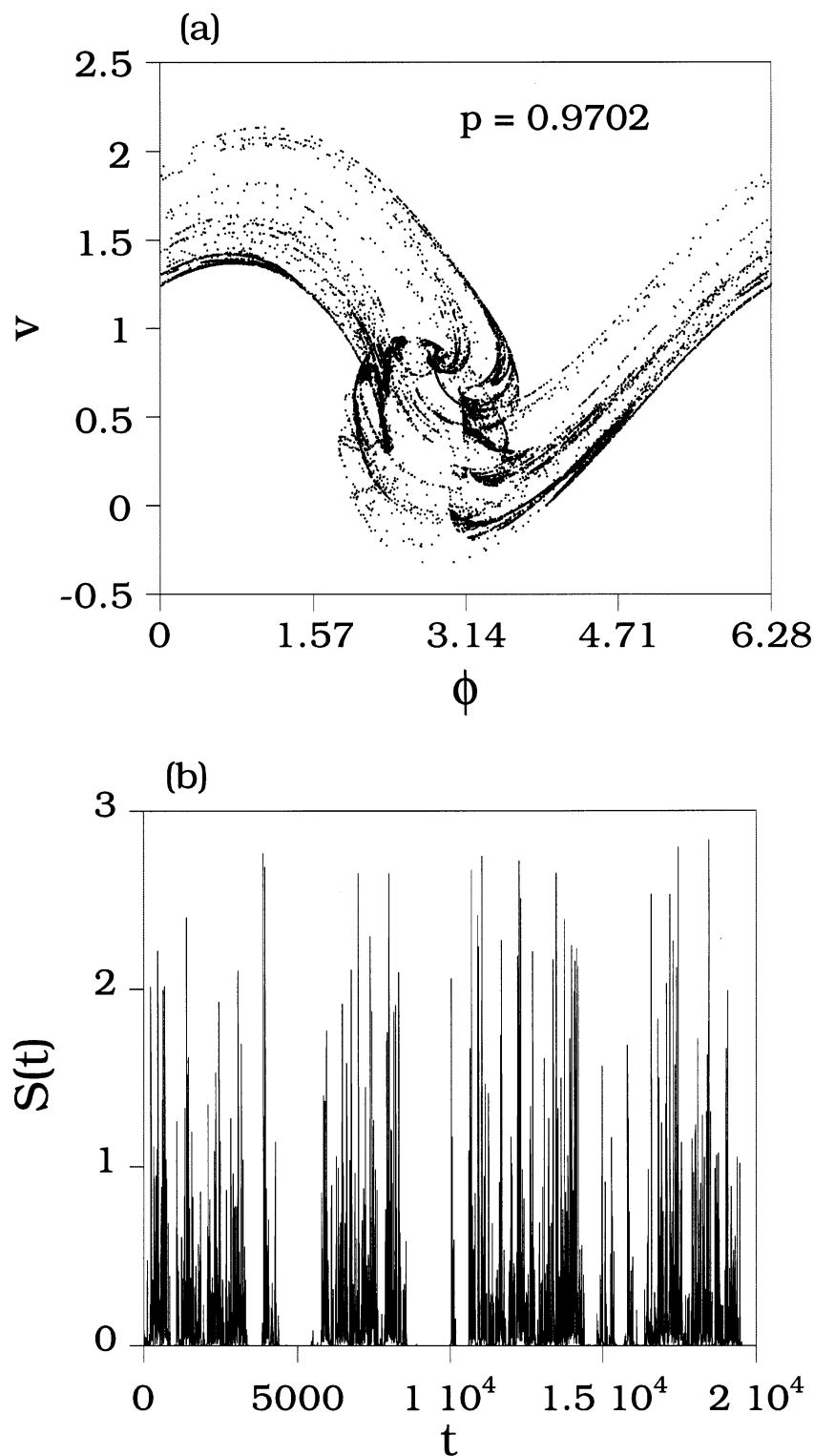


Fig. 3. Behavior of the snapshot attractor about the transition to chaos in the quasiperiodically driven pendulum system. For system described by Eq. (3), (a) a single trajectory on the chaotic attractor for $p = 0.9702$ ($\Lambda \approx 0.002$). (b) $S(t)$, the size of the snapshot attractors computed using 128 trajectories, versus time t for $p = 0.9702$. From Ref. [7] with permission.

spread over the entire chaotic attractor. There are also time intervals during which the snapshot attractor consists of points concentrated on extremely small regions in the phase space. The *size* of the snapshot attractors, therefore, changes drastically with time in an intermittent fashion.

To quantify intermittency, the following time-dependent size of the snapshot attractors can be examined:

$$S(t) = \sqrt{\frac{1}{N} \sum_{i=1}^N \{[\phi_i(t) - \langle \phi(t) \rangle]^2 + [v_i(t) - \langle v(t) \rangle]^2\}}, \quad (4)$$

where N is the number of points on the snapshot attractors, $[\langle \phi(t) \rangle, \langle v(t) \rangle]$ defines the geometric center of these points at a given time: $\langle \phi(t) \rangle \equiv \frac{1}{N} \sum_{i=1}^N \phi_i(t)$ and $\langle v(t) \rangle \equiv \frac{1}{N} \sum_{i=1}^N v_i(t)$. Figure 3(b) shows $S(t)$ versus t for $p = 0.9702$, where t is the integer time measured on the surface of section corresponding to the real time $t(2\pi/\omega_2)$, and the snapshot attractors are computed from 128 initial conditions uniformly chosen along the diagonal line of the rectangle defined by $\phi(0) \in [0, 2\pi]$ and $v(0) \in [-1, 1]$. It can be seen that the size of the snapshot attractors exhibits an extreme type of intermittent behavior, the so-called ‘‘on-off’’ intermittency [56–72]. There are time intervals during which the snapshot attractors are concentrated on a region with extremely small size ($< 10^{-14}$). The time averaged size of the snapshot attractors *on the chaotic side* near the transition, defined as $\langle S(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t) dt$, obeys the following scaling relation [7]:

$$\langle S(t) \rangle \sim \Lambda \sim |p - p_c|. \quad (5)$$

In fact, for p slightly above p_c on the nonchaotic side, the averaged size of the snapshot attractor is quite small, yet nonzero. This is due to the finite transient time used in the computation. As the transient time increases, the averaged size decreases towards zero.

3 Destruction of multistability by quasiperiodic driving in a semiconductor superlattice system

We demonstrated in Sec. 2 some common features and scaling behaviors associated with transition to chaos in noise and quasiperiodically driven systems. The implication is that, since noise can suppress multistability, quasiperiodic driving should have a similar effect. To support this idea, we present a concrete physical system - a semiconductor superlattice, to demonstrate that quasiperiodic driving can eliminate multistability and lead to robust chaos.

A semiconductor superlattice consists of a periodic sequence of thin layers of different types of semiconductor materials [78] so that the electronic properties of the structure can be engineered. Specifically, a superlattice is a periodic structure of coupled quantum wells, where at least two types of semiconductor materials with different band gaps are stacked on top of each other along the so-called growth direction in an alternating fashion [79, 80]. An effective approach to modeling transport dynamics in the superlattice system is through the force-balance equation [81–90], which can be derived either from the classical Boltzmann transport equation [84, 85] or from the Heisenberg equation of motion [91, 92]. In spite of a quantum system’s being fundamentally linear, the self-consistent field caused by the combined effects of the external bias and the intrinsic many-body mean field becomes effectively nonlinear [93, 94]. In the high field transport regime, various nonlinear phenomena including chaos can

arise [95]. In the past two decades, there were a host of theoretical and computational studies of chaotic dynamics in semiconductor superlattices [95, 94, 96, 97, 93, 98–108]. The effects of magnetic field on the nonlinear dynamics in superlattices were also investigated [109–111]. Experimentally, a number of nonlinear dynamical behaviors were observed and characterized [112, 80, 113–115].

A key application of semiconductor superlattices is to fill the so-called “THz” gap, i.e., to develop radiation sources, amplifiers and detectors [116–120] from 0.1 to 10 THz, the frequency range in which convenient radiation sources are not readily available [121–124]. In particular, below 0.1 THz electron transport based devices are typical, and above 10 THz devices based on optical transitions (e.g., solid state lasers) are commonly available. Since in general, chaotic systems can be used as random number generators [125–133], ubiquity of chaos in semiconductor superlattices implies that such systems may be exploited for random signal generation in the frequency range corresponding to the THz gap.

In a recent paper [40], the dynamics of energetic or “hot” electrons in semiconductor superlattices were investigated. Specifically, the setting was studied where the system is subject to strong *dc* and *ac* fields so that dynamical resonant tunneling occurs effectively in a quasi-one-dimensional superlattice. Due to the strong driving field, a space charge field is induced, which contains two nonlinear terms in the equation of motion. In particular, using the force-balance equation [134] for an *n*-doped semiconductor quantum-dot superlattice, the dynamical equation for the electron center-of-mass velocity $V_c(t)$ can be written as

$$\frac{dV_c(t)}{dt} = -[\gamma_1 + \Gamma_c \sin(\Omega_c t)] V_c(t) + \frac{e}{M(\mathcal{E}_e)} [E_0 + E_1 \cos(\Omega_1 t) + E'_1 \cos(\Omega'_1 t) + E_{sc}(t)], \quad (6)$$

where γ_1 is the momentum-relaxation rate constant, Γ_c comes from the channel-conductance modulation with Ω_c being the modulation frequency, $M(\mathcal{E}_e)$ is the energy-dependent averaged effective mass of an electron in the superlattice, $\mathcal{E}_e(t)$ is the average energy per electron, E_0 is the applied *dc* electric field, E_1 and E'_1 are the amplitudes of the two external *ac* fields with frequencies Ω_1 and Ω'_1 , respectively, and $E_{sc}(t)$ is the induced space-charge field due to the excitation of plasma oscillation. Here, the statistical resistive force [134] has been approximated by the momentum relaxation rate. Based on the energy-balance equation, one can show [135] that $\mathcal{E}_e(t)$ satisfies the following dynamical equation

$$\frac{d\mathcal{E}_e(t)}{dt} = -\gamma_2 [\mathcal{E}_e(t) - \mathcal{E}_0] + eV_c(t) [E_0 + E_1 \cos(\Omega_1 t) + E'_1 \cos(\Omega'_1 t) + E_{sc}(t)], \quad (7)$$

where γ_2 is the energy-relaxation rate constant and \mathcal{E}_0 is the average electron energy at the thermal equilibrium, and the thermal energy exchange of the electrons with the crystal lattice [135] is approximately described by the γ_2 term. Applying the Kirchoff’s theorem to a resistively shunted quantum-dot superlattice [93], one obtains [136] the dynamical equation for the induced space-charge field $E_{sc}(t)$ as

$$\frac{dE_{sc}(t)}{dt} = -\gamma_3 E_{sc}(t) - \left(\frac{en_0}{\epsilon_0 \epsilon_b} \right) V_c(t), \quad (8)$$

where γ_3 , which is inversely proportional to the product of the system resistance and the quantum capacitance, is the dielectric relaxation rate constant [136], n_0 is the electron concentration at the thermal equilibrium, and ϵ_b is the relative dielectric constant of the host semiconductor material. The exact microscopic calculations of γ_1 and γ_2 in the absence of space-charge field were carried out previously [137] based on the semiclassical Boltzmann transport equation and the coupled force-energy balance

equations [101], respectively. Equivalent quantum calculations of γ_1 and γ_2 can also be done through the coupled force balance and the Boltzmann scattering equations [134].

Within the tight-binding model, the single-electron kinetic energy ε_k in a semiconductor quantum-dot superlattice can be written as

$$\varepsilon_k = (\Delta/2) [1 - \cos(kd)], \quad (9)$$

where k ($|k| \leq \pi/d$) is the electron wave number along the superlattice growth direction, Δ is the miniband width, and d is the spatial period of the superlattice. This energy dispersion relation gives [134]

$$\frac{1}{M(\mathcal{E}_e)} = \left\langle \frac{1}{\hbar^2} \frac{d^2 \varepsilon_k}{dk^2} \right\rangle = \frac{1}{m^*} \left[1 - \left(\frac{2}{\Delta} \right) \mathcal{E}_e(t) \right], \quad (10)$$

where $m^* = 2\hbar^2/\Delta d^2$ and $|1/M(\mathcal{E}_e)| \leq 1/m^*$.

For numerical calculations, it is convenient to use dimensionless quantities: $v(\tau) = (m^*d/\hbar) V_c$, $w(\tau) = [(2/\Delta) \mathcal{E}_e - 1]$, $f(\tau) = (ed/\hbar\omega_0) E_{sc}$, and $\tau = \omega_0 t$ with $\omega_0 = 1$ THz being the frequency scale. In terms of the dimensionless quantities, the dynamical equations of the resonantly tunneling electrons in the superlattice become

$$\begin{aligned} \frac{dv(\tau)}{d\tau} &= -b_1 v(\tau) [1 + a_2 \sin(\bar{\Omega}\tau)] \\ &\quad - [a_0 + a_1 \cos(\Omega\tau) + a'_1 \cos(\Omega'\tau) + f(\tau)] w(\tau), \\ \frac{dw(\tau)}{d\tau} &= -b_2 [w(\tau) - \bar{w}_0] + [a_0 + a_1 \cos(\Omega\tau) + a'_1 \cos(\Omega'\tau) + f(\tau)] v(\tau), \\ \frac{df(\tau)}{d\tau} &= -b_3 f(\tau) - a_3 v(\tau), \end{aligned} \quad (11)$$

where $\bar{w}_0 = [(2/\Delta) \mathcal{E}_0 - 1] = -1$, $b_1 = \gamma_1/\omega_0$, $b_2 = \gamma_2/\omega_0$, $b_3 = \gamma_3/\omega_0$, $a_0 = \omega_B/\omega_0$, $a_1 = \omega_s/\omega_0$, $a'_1 = \omega'_s/\omega_0$, $a_2 = \Gamma_c/\gamma_1$ and $a_3 = (\Omega_c/\omega_0)^2$ are all positive real constants. The field related parameters are $\omega_B = eE_0 d/\hbar$, $\omega_s = eE_1 d/\hbar$, $\omega'_s = eE'_1 d/\hbar$, $\Omega = \Omega_1/\omega_0$, $\Omega' = \Omega'_1/\omega_0$, $\bar{\Omega} = \Omega_c/\omega_0$, and $\Omega_c = \sqrt{e^2 n_0/m^* \epsilon_0 \epsilon_b}$, where the last quantity is the bulk plasma frequency. The fields are assumed to be turned on at $t = 0$. The initial conditions for Eq. (11) are $v(0) = v_0$, $f(0) = f_0$ and $w(0) = w_0$.

The issue addressed [40] was that of reliability and robustness, i.e., for a given parameter setting, what is the probability to generate chaos from a random initial condition? It was found that, for the common case of a single ac driving field, onset of chaos is typically accompanied by the emergence of multistability in the sense that there are coexisting attractors in the phase space which are not chaotic. Using the ensemble method to calculate the maximum Lyapunov exponent, the motions on the regular and chaotic attractors can be distinguished. The probability for a random initial condition to lead to chaos is finite but in general is not close to unity. Due to the simultaneous creation of the basin of attraction of the chaotic attractor, the transition to multistability with chaos, as a system parameter passes through a critical point, is necessarily abrupt. Likewise, the disappearance of multistability is abrupt, as the typical scenario for a chaotic attractor to be destroyed is through a boundary crisis [138], which is sudden with respect to parameter variations. These behaviors are illustrated in Fig. 4.

From the point of view of random signal generation, multistability is undesired. It was found [40], however, that an additional driving field, e.g., of an incommensurate frequency, can effectively eliminate multistability to guarantee the existence of open parameter regions in which the probability of generating chaos from random initial conditions is unity. Such a behavior is illustrated in Fig. 5. It was also found [40] that noise plays a similar role in suppressing multistability in the sense that weak noise can suppress chaos but strong noise can lead to chaos with probability one.

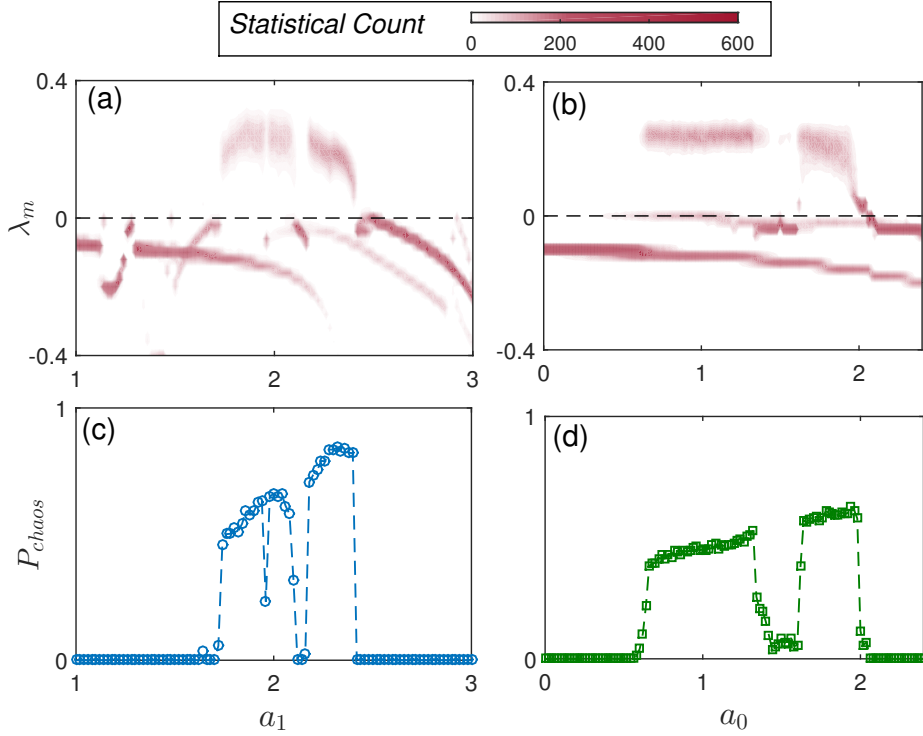


Fig. 4. Transition to chaos and multistability. (a) For fixed $a_0 = 2.23$, the values of the maximum Lyapunov exponent λ_m calculated from an ensemble of initial conditions versus a_1 for $1.0 \leq a_1 \leq 3.0$. (b) A similar plot but for fixed $a_1 = 2.13$ and a_0 varying in the range $[1.0, 2.4]$. (c) For $a_0 = 2.23$, the probability versus a_1 for a random trajectory to land in a chaotic attractor. (d) A plot similar to that in (c) but for fixed $a_1 = 2.13$ and varying a_0 . Other parameters are $a'_1 = a_2 = 0$, $a_3 = 7.48$, $b_1 = 0.28$, $b_2 = b_3 = 2.85 \times 10^{-2}$, and $\Omega = 1.34$. From (a) and (c), abrupt emergence of chaos at $a_1 \approx 1.65$ and abrupt disappearance of chaos at $a_1 \approx 2.45$ can be seen (see text for the reason of the “abruptness”). The dips in the probability curve of chaos for $a_1 \approx 2.0$ and $a_1 \approx 2.15$ are due to periodic windows. Abrupt emergence and disappearance of multistability associated with chaos also occur for fixed $a_1 = 2.13$ and varying a_0 , as shown in (b) and (d). *From Ref. [40] with permission.*

4 Conclusion

In nonlinear dynamical systems multistability is ubiquitous, where multiple attractors coexist, each with its own basin of attraction. The basin boundaries separating the distinct basins of attraction can be fractal [27, 28] or even riddled [139–150], dynamically leading to transient chaos [41]. In applications such as random signal generation in engineering systems where robust chaos is required, mitigating or even removing multistability so that the system possesses a single chaotic attractor is desired. We have laid out in this mini-Review the proposition that quasiperiodicity can be exploited to suppress and eliminate multistability, and presented an experimentally realizable physical system to support the proposition.

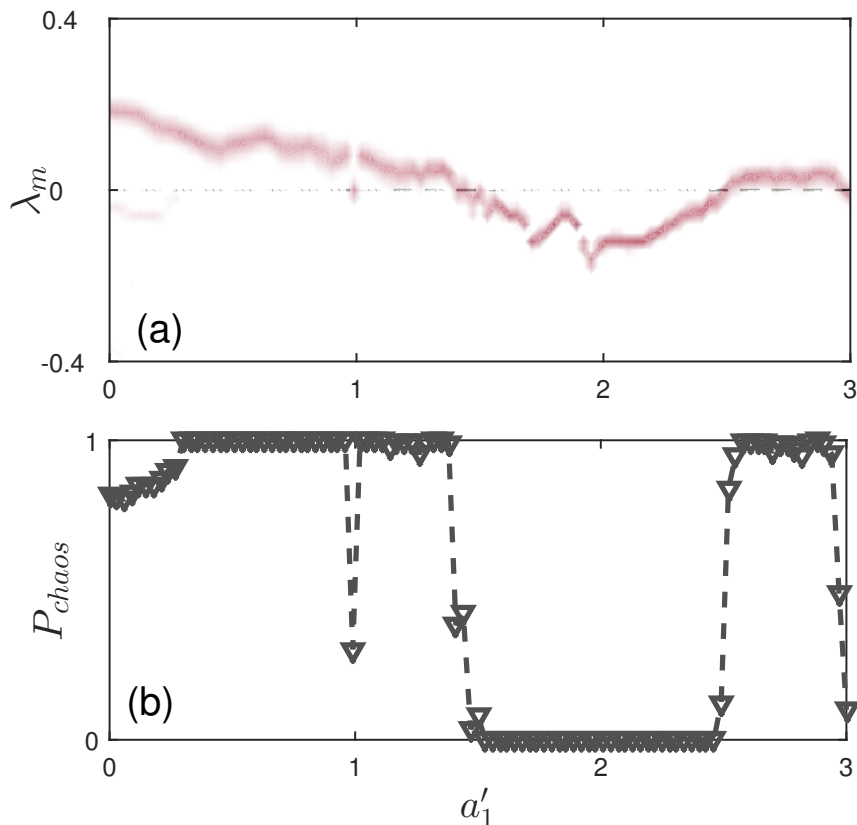


Fig. 5. Occurrence of reliable and robust chaos with probability one under quasiperiodic driving. When a second ac driving field of amplitude a'_1 and frequency $\Omega' = \sqrt{2}$ is applied to the superlattice system, open parameter intervals emerge in which the probability of generating chaos from a random initial condition is unity. (a) Statistical counts of the maximum Lyapunov exponent and (b) probability of generating chaos versus a'_1 . Other parameters are $a_0 = 2.23$, $a_1 = 2.3$, $a_2 = 0$, $a_3 = 7.48$, $b_1 = 0.28$, $b_2 = b_3 = 2.85 \times 10^{-2}$, and $\Omega = 1.34$. From Ref. [40] with permission.

Acknowledgement

YCL would like to acknowledge support from the Vannevar Bush Faculty Fellowship program sponsored by the Basic Research Office of the Assistant Secretary of Defense for Research and Engineering and funded by the Office of Naval Research through Grant No. N00014-16-1-2828. This work was also supported by ONR through Grant No. N00014-15-1-2405.

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