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Gödel Fuzzy Argumentation Frameworks

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Abstract. In this paper we combine fuzzy set theory and argumentation to facilitate the use of fuzzy arguments and attacks. Unlike many existing approaches, our work does not require the use of any parameters, bringing it closer to Dung's work in spirit. We begin by introducing Fuzzy Argumentation Frameworks, and specialise them using the Gödel t-norm. We then examine this framework's properties and show that the standard Dung extensions are obtained, though the stable semantics coincide with the preferred. Finally, we examine the relationship between our framework and Dung's original system, as well as the existing fuzzy frameworks, describing where they overlap and differ.

Keywords. Fuzzy argumentation, abstract argumentation, semantics

1. Introduction

Following on from Dung's seminal paper [5], a variety of abstract argumentation frameworks have been proposed. These extensions to Dung's original work seek to identify a subset of arguments which is considered justified under a variety of inter-argument interactions, including support [2,12]; attacks which are joint [10] or recursive [3]; and preferences over arguments [1]. The properties assigned to arguments and argument interactions in such systems are typically binary (e.g., an attack is, or is not present), or qualitative (e.g., one argument is preferred to another). Such approaches can be contrasted with work on weighted argument frameworks [6], probabilistic argument frameworks [9,13] and multi-valued or fuzzy frameworks [4,11,7,8], where quantitative properties are considered.

Unlike qualitative approaches, which identify a justified set of arguments according to some semantics, quantitative approaches (with few exceptions) provide a justified set of arguments together with some additional information. For example, weighted argumentation frameworks determine justified arguments with respect to some inconsistency budget; probabilistic argumentation frameworks compute the likelihood that some set of arguments is justified, and fuzzy frame-

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works compute acceptability with regards to some parameter. Our goal within this paper is to more closely align fuzzy argumentation frameworks with classical argumentation approaches, unequivocally identifying a justified set of arguments with no reference to parameters. We do so by considering the notion of *sufficient attacks* and *weakening defends*, which we use to determine when one argument is sufficiently strong to defeat another. Before doing so, we first justify the importance of fuzziness in argumentation, contrasting it with uncertainty as found in probabilistic argumentation frameworks. Sections 3 and 4 provides our main contribution, formalising our framework and examining its properties. In Section 5 we compare our approach with existing work, before concluding.

2. Why Fuzziness in Argumentation?

To understand why fuzzy reasoning is necessary, we consider the following example [4] that considers whether a batch of tomatoes should be eaten:

B: The tomatoes are rotten.
C: The tomatoes can be eaten.
B attacks C: If a tomato is rotten, it should not be eaten.

Within a standard argumentation formalism, argument B would be justified: one could not conclude that the tomatoes can be eaten. Argument B may, however, be partially true — a tomato may have mold on one side, but the remaining half could be consumed. Similarly, some people may have differing judgments of how rotten the fruit is, and one should be able to aggregate these judgments to make a final decision. Probabilistic frameworks could, conceivably, capture the latter case, but would not help us in the former — such frameworks are designed to deal with uncertainty rather than fuzziness. When treated as fuzzy sets, arguments B and C could potentially both be considered justified — in situations where the tomatoes are only very slightly rotten, they can still be eaten.

Graduation or strength of this type is captured by associating a fuzziness degree to each argument. Different fuzziness degrees then result in different outcomes. For example, giving B a degree of 0.8 (i.e., most of the tomatoes are rotten), together with a belief[2] that most of the tomatoes can be eaten (e.g., associating 0.9 to C), it is clear that the two arguments are in conflict. On the other hand, giving B a degree of 0.1 while maintaining C at 0.9 should result in the two arguments being justified together. In the first instance, we may view the attack from B on C as *sufficient* to cause them to be judged in conflict, while in the second case, the attack can be tolerated by the system. We refer to such a situation as a *tolerable* attack.

Consider an additional argument and attack:

A: The tomatoes are stored well.
A attacks B: If tomatoes are stored well, they will not go rotten.

[2]We will utilise the term "degree of belief" interchangeablly with "degree of fuziness". Such degrees, rather than representing uncertainty, capture the belief in the level of fuzziness of the concept under consideration.

We may assign a degree of belief to the attack here such as 0.9, as we know (for example) that in most cases this relationship holds. Now if the tomatoes are stored well (e.g., assigning A a high fuzziness degree), then we should expect that most will be edible — a high degree of belief in A defends a high degree of belief in C by weakening the degree of belief in B. Similarly, a low degree of belief in A should weaken the defense it provides to C from an attack by B[3].

Within this paper we will formalise these concepts in order to construct a framework whose outputs are similar to standard argumentation frameworks: given a set of arguments, attacks between arguments, and an appropriate semantics, we will identify sets of justified arguments.

3. Fuzzy Argumentation

Our work builds on both fuzzy set theory [15] and abstract argumentation [5], in the spirit of de Costa Pereira et al. [11][4]. We begin this section by providing an overview of fuzzy set theory and abstract argumentation.

3.1. Fuzzy set theory

Let X be a nonempty set. A fuzzy set (X, S) is determined by its membership function $S\colon X \to [0,1]$, such that for each $x \in X$ the value $S(x)$ is interpreted as the grade of membership of x within X. Given some constant set X, we may denote a fuzzy set (X, S) as S for convenience.

A fuzzy set S is contained in another fuzzy set S', if $\forall x \in X, S(x) \leq S'(x)$, which is denoted by $S \subseteq S'$.

The set $\{x \in X \mid S(x) > 0\}$ is called the *support* of (X, S) and the set $\{x \in X \mid S(x) = 1\}$ is called its *kernel*, or core.

A fuzzy set S is called a *fuzzy point* if its support is a single point $x \in X$, and is denoted by $(x, S(x))$. A fuzzy point $(x, S(x))$ is contained in a fuzzy set S if it is a subset of S.

3.2. Abstract argumentation frameworks

An abstract argumentation framework (AF) [5] contains a set of arguments and an attack relation:

Definition 1. *An AF is a pair $(Args, \mathcal{R})$ where Args is a set of arguments and $\mathcal{R} \subseteq Args \times Args$ is a set of attacks. An argument A attacks an argument B iff $(A, B) \in \mathcal{R}$.*

Dung defines a number key concepts and various types of *extension* or ways to interpret an argument graph. In this paper, we build upon the following:

Defends A set $S \subseteq Args$ defends[5] an argument $A \in Args$, if for every $B \in Args$ such that $(B, A) \in \mathcal{R}$, there is some $C \in S$ such that $(C, B) \in \mathcal{R}$.

[3]There are some clear similarities between this principle and reinstatement.
[4]A detailed comparison with this work is provided in Section 5.
[5]Dung introduced this concept as *acceptability* [5].

Conflict-free A set $S \in Args$ is conflict-free if there are no arguments $A, B \in S$ such that $(A, B) \in \mathcal{R}$.

Admissibility A conflict-free set S is admissible if it defends each argument in S.

Characteristic function The characteristic function of an AF $(Args, \mathcal{R})$ is a function $F\colon 2^{Args} \rightarrow 2^{Args}$, where $\forall S \subseteq Args$, $F(S) = \{A\colon S \text{ defends } A\}$.

Grounded extension The grounded extension is the least fixed point of F.

Complete extension A conflict-free set, S, is complete if $S = F(S)$.

Preferred extension A preferred extension is a maximal admissible set.

Stable extension A stable extension is a conflict-free set, S, that attacks each argument in $Args \setminus S$.

3.3. Fuzzy Argumentation Frameworks

Existing fuzzy argumentation models (such as [7,11]) consider either fuzzy arguments or fuzzy attacks between arguments. In this work we create a system with both fuzzy arguments and attacks. Furthermore, unlike work such as [7,9,6], our work takes an objective view to fuzzy extensions, not requiring a budget-like parameter to be specified. We begin by describing our approach, and then analyze its properties in Section 4. We refer to argumentation frameworks within our approach as *fuzzy argumentation frameworks*, abbreviated FAF.

Definition 2. (Fuzzy Argumentation Framework) *A fuzzy argumentation framework is a tuple $(\mathcal{A}, \rho)$ where $\mathcal{A} : Args \rightarrow [0,1]$ and $\rho : Args \times Args \rightarrow [0,1]$ are total functions. We refer to $\mathcal{A}$ as a fuzzy set of arguments, and ρ as a fuzzy set of attacks, while Args is a set of* crisp *arguments.*

A valid fuzzy argument can be encoded by the tuple (A, a) where $A \in Args$ and $a \in [0,1]$, subject to the constraint that $a \leq \mathcal{A}(A)$. Similarly, a valid fuzzy attack can be written as $((A,B), \rho_{AB})$ if $\rho_{AB} \leq \rho(A,B)$.

It is important to differentiate between the value of a within (A, a) and $\mathcal{A}(A)$. The function $\mathcal{A}$ identifies the *maximum* degree of belief associated with every argument that the system can permit. Therefore, any degree of belief smaller than this can also be accepted by the system. A similar argument applies to ρ. Given that our goal is to provide a means to select arguments from the FAF with some associated maximum degree (i.e., upper bound), any selected argument with a lesser degree of belief will also be accepted. It should also be noted that attacks within FAFs are between arguments; i.e., *Args*, rather than fuzzy arguments, $\mathcal{A}$. This is because attacks are determined by the relations between arguments, rather than the degree of belief in those arguments[6].

Since ρ is a total function, we assume that if $\rho((A,B))$ is not specified, then $\rho((A,B)) = 0$. Returning to the rotten tomato example, where $A =$"The tomatoes are stored well", and $B =$"The tomatoes are rotten".

[6] Additionally, if the domain of ρ was $\mathcal{A} \times \mathcal{A}$, the system could be represented as a standard Dung argument system with infinite arguments of the form $Args' = \{(A, a) : A \in Args, a \in [0,1]\}$, together with attacks between these arguments based on their different strengths.

Example 1. *Assume that* $(\mathcal{A}, \rho) = (\{(A, 0.7), (B, 0.8)\}, \{((A, B), 0.9)\})$. *Here* $\mathcal{A}(A) = 0.7$, *and we could accept that "The tomatoes are stored well" with a degree of belief 0.6 (that is,* $(A, 0.6)$*), but doing so with a degree of belief of 0.9 (i.e.,* $(A, 0.9)$*) would be counter-intuitive.*

To capture the above intuitions, we introduce the concepts of *sufficient attacks* and *weakening defends.*

3.4. Sufficient Attacks and Weakening Defends

Given Example 1, we may identify two types of attacks: tolerable and sufficient. A tolerable attack is one for which the target of the attack may be included within an extension without considering reinstatement (i.e., the attack is too "weak" to succeed in some sense), while a sufficient attack has sufficient strength to cause its target to be excluded from the extension. These terms are taken from Da Costa Pereira *et al.* [11], and they argue that these two types of attacks can be distinguished through the following principle:

> "Suppose an argument A attacks an argument B. If we strongly believe A, then we hardly believe B, and if we strongly believe the negation of A, we should believe B strongly. Additionally, the belief of B should be no more than the belief of the negation of A."

Formally, given a fuzzy argument (A, a) attacking another fuzzy argument (B, b) requires that the degree of belief b in B be no more than the value of $\neg(A, a)$[7]. One simple assignment for the strength of belief of $\neg(A, a) = 1 - a$, which therefore requires that $b \leq 1 - a$.

We can extend this idea to frameworks containing both fuzzy arguments and fuzzy attacks. Suppose that A attacks B with degree ρ_{AB}. Following Janssen *et al.* [7], the degree of belief associated with B given such an attack is based on the composition of the degree of belief in A (before considering B), together with the degree of belief placed in the attack itself. The degree of belief placed in B should, therefore, be no more than the negation of the composition of these two factors. In other words, if (A, a) (fuzzily) attacks another argument (B, b) with an attack of degree ρ_{AB}, then the following inequality must be satisfied.

$$b \leq 1 - a \star \rho_{AB} \tag{1}$$

Here, $\star$ is a composition operator, and the question immediately arises as to what desirable properties are for such an operator.

Following [7], we believe it is reasonable for $\star$ to satisfy the following conditions.

1. If, for some $(A, a) \in \mathcal{A}$ and $((A, B), \rho_{AB}) \in \rho$, $a = 1$ and $\rho_{AB} = x$, or $\rho_{AB} = 1$ and $a = x$, the value of the composition should be x; i.e., $x \star 1 = 1 \star x = x$;

[7]In an ASPIC-like system, one could interpret this as the degree of belief b being no more than the contrary of (A, a).

2. If, for some $(A, a) \in \mathcal{A}$ and $((A, B), \rho_{AB}) \in \rho$, $a = 0$ or $\rho_{AB} = 0$ (A is selected out or ρ_{AB} disappears), then the composition should be 0; and
3. Operator $\star$ should be monotone on both sides.

These conditions mean that $\star$ is a non-commutative t-norm; the simplest such operator is the Gödel t-norm: $a \star \rho_{AB} = min\{a, \rho_{AB}\}$[8]. Substituting this operator into Equation 1 yields:

$$min\{a, \rho_{AB}\} + b \leq 1 \tag{2}$$

We refer to a fuzzy argumentation framework using the Gödel t-norm as a Gödel Fuzzy Argumentation Framework (or GFAF).

Note that if $\rho_{AB} = 1$, Equation 2 reduces to $a + b \leq 1$, which is that used by the system of Da Costa Pereira *et al.* Furthermore, if the degree of belief in all attacks is 0 or 1, the model reduces to one in which all attacks are crisp. In such cases, our system is consistent with Da Costa Pereira *et al.*'s method for distinguishing between tolerable and sufficient attacks.

We are now in a position to formalise tolerable and sufficient attacks for GFAFs.

Definition 3. *Given two arguments, (A, a) and (B, b) as well as an attack $((A, B), \rho_{AB}))$, if Equation 2 is satisfied, then the attack is* tolerable*, otherwise it is* sufficient.

Example 2. *Returning to the rotten tomato example, assume that $(A, 0.1)$, $(B, 0.8)$ and $((A, B), 0.9)$. In this situation, the attack is tolerable. However, if instead we have that $(A, 0.7)$, then the attack becomes sufficient.*

As mentioned above, a tolerable attack has no influence on (B, b). However, a sufficient attack *weakens* the attacked argument.

Definition 4. *Given an attack $((A, B), \rho_{AB})$ from (A, a) to (B, b) within a GFAF $(\mathcal{A}, \rho)$, (A, a) weakens (B, b) to (B, b') by the attack $((A, B), \rho_{AB})$, thus:*

$$b' = min\{1 - min\{a, \rho_{AB}\}, b\}$$

Note that this definition captures both tolerable and sufficient attacks, with the latter resulting in $b' = b$.

Example 3. *Returning to Example 2, if the degree of belief of A is 0.1, $(B, 0.8)$ is weakened to $(B, 0.8)$ by the attack $((A, B), 0.9)$. However, if we have $(A, 0.7)$, $(B, 0.8)$ is weakened to $(B, 0.3)$ by this attack. Given the attack $((A, B), 0.6)$, $(A, 0.7)$ weakens $(B, 0.8)$ to $(B, 0.4)$.*

[8]We concentrate on the Gödel t-norm in this work, but other operators, such as the product t-norm could also be utilized; an investigation of the properties of such operators is an avenue for future research.

For convenience, we may say that (A, a) weakens (B, b) to (B, b') without referring to the degree of belief in the attack. In this case, we mean that b' is minimal — A weakens B by the maximal value of the argument A (i.e., a) or the attack (ρ_{AB}).

Since tolerable attacks do not change the degree of belief in the attacked argument, such attacks are ignored when computing a conflict-free set of fuzzy arguments.

Definition 5. *Given a GFAF $(\mathcal{A}, \rho)$, a fuzzy set of arguments $S \subseteq \mathcal{A}$ is conflict-free (abbreviated Cf) if all attacks between the fuzzy arguments in S are tolerable.*

The conflict-freeness of a set is, therefore, determined by considering the maximum degree of belief of the fuzzy attacks between arguments.

Example 4. *Consider the GFAF $(\{(A, 0.7), (B, 0.8)\}, ((A, B), 0.9))$. Both fuzzy sets $\{(A, 0.7), (B, 0.3)\}$ and $\{(A, 0.4), (B, 0.5)\}$ are conflict-free. In contrast, neither $\{(A, 1), (B, 0)\}$ nor $\{(A, 0.7), (B, 0.8)\}$ are conflict-free. This is because $\{(A, 1), (B, 0)\}$ is not a fuzzy subset of $\mathcal{A}$, and the attack $((A, B), 0.9)$ with $\{(A, 0.7), (B, 0.8)\}$ is* sufficient.

Having defined tolerable and sufficient attacks and introduced the concept of weakening, we are now in a position to define how a fuzzy set provides a weakening defense of a fuzzy argument.

Definition 6. *Given a GFAF $(\mathcal{A}, \rho)$, a fuzzy set $S \subset \mathcal{A}$* weakening defends *a fuzzy argument $(C, c) \in \mathcal{A}$ if for any $(B, b) \in \mathcal{A}$ there is some $(A, a) \in S$ such that (A, a) weakens (B, b) to (B, b') and (B, b') tolerably attacks (C, c).*

Theorem 1. *Given a GFAF, $(\mathcal{A}, \rho)$, a set $S \subset \mathcal{A}$ weakening defends $(C, c) \in \mathcal{A}$, iff $\forall (B, b) \in \mathcal{A}$,*

$$\min_{A \in Args}\{1 - \min\{S(A), \rho((A, B))\}, b, \rho((B, C))\} + c \leq 1. \tag{3}$$

Proof. ($\Leftarrow$) Suppose Equation 3 is satisfied. For a finite set $Args$, there will be some $A \in Args$ such that

$$\min\{\min\{1 - \min\{S(A), \rho((A, B))\}, b\}, \rho((B, C))\} + c \leq 1,$$

which means $(A, S(A)) \in S$ weakens (B, b) to (B, b'), where, by Definition 6, $b' = \min\{1 - \min\{S(A), \rho((A, B))\}, b\}$, and (B, b') does not sufficiently attack (C, c); i.e. S weakening defends (C, c).

($\Rightarrow$) Suppose S weakening defends (C, c). Then, for any $(B, b) \in \mathcal{A}$, there is some $(A, a) \in S$ such that (A, a) weakens (B, b) to (B, b'), and (B, b') tolerably attacks (C, c); i.e., $\min\{\min\{1 - \min\{a, \rho((A, B))\}, b\}, \rho((B, C))\} + c \leq 1$. Because $a \leq S(A)$, we have

$$\min\{\min\{1 - \min\{S(A), \rho((A, B))\}, b\}, \rho((B, C))\} + c \leq 1,$$

which immediately reduces to Equation 3. □

Example 5. *Suppose we have a GFAF* $(\{(A, 0.7), (B, 0.8), (C, 0.9)\}, \{((A, B), 0.9), ((B, C), 0.7)\})$*. The fuzzy argument* $(A, 0.6)$ *weakening defends* $(C, 0.6)$*, but* $(A, 0.8)$ *does not weakening defend* $(C, 0.9)$*.*

Using Theorem 1, we may extend Definition 6 to utilise sets.

Definition 7. *Suppose* $S \subset \mathcal{A}$ *and* $(B, b) \in \mathcal{A}$ *for GFAF* $(\mathcal{A}, \rho)$*. The set* S *weakens* (B, b) *to* (B, b')*, such that*

$$b' = \min_{A \in Args} \{1 - \min\{S(A), \rho(A, B)\}\}$$

In other words, (B, b) *is weakened by every argument in* S*, and the minimum value due to attacks from* S *is* b'*.*

The following proposition follows naturally from this definition.

Lemma 1. *If* S *weakens* (A, a) *to* (A, a')*, then* S *tolerably attacks* (A, a')*.*

4. Semantics of GFAFs

In this section we define various argumentation semantics within GFAFs, namely the grounded, complete, preferred and stable extensions. Following this, we examine the relationships between them. In defining these semantics, we utilise the concepts of an admissible set and the characteristic function of a GFAF.

Definition 8. *A conflict-free set of fuzzy arguments,* $S \in \mathcal{A}$*, in a GFAF* $(\mathcal{A}, \rho)$ *is* admissible *(abbreviated AE), if* S *weakening defends each element in* S*.*

Example 6. *Consider again our fuzzy argumentation framework with arguments* A*,* B *and* C*:* $(\{(A, 0.7), (B, 0.8), (C, 0.9)\}, \{((A, B), 0.9), ((B, C), 0.7)\})$*. Here, both* $\{(A, 0.6), (B, 0.3), (C, 0.6)\}$ *and* $\{(A, 0), (B, 0), (C, 0)\}$ *(the empty set) are admissible sets of fuzzy arguments. In contrast,* $\{(A, 0.4), (B, 0.2), (C, 0.6)\}$ *is not an admissible set, because* $(A, 0.4)$ *is not strong enough to defend* $(C, 0.6)$*; i.e.* $(A, 0.4)$ *can only weaken* $(B, 0.8)$ *to* $(B, 0.6)$*, which still sufficiently attacks* $(C, 0.6)$*. Similarly,* $\{(A, 0.4), (B, 0.4), (C, 0.4)\}$ *is not an admissible set, because* $(B, 0.4)$ *is sufficiently attacked by* $(A, 0.7)$*, which is not weakened by any other fuzzy argument in this set.*

Definition 9. *The characteristic function of a GFAF* $(\mathcal{A}, \rho)$ *is a function* $\mathcal{F}$ *from the set of all the subsets of* $\mathcal{A}$ *to itself, such that* $\forall S \subseteq \mathcal{A}$*,* $\mathcal{F}(S) = \{(A, a) \colon S$ *weakening defends* $(A, a)\}$*.*

From this definition, $\mathcal{F}$ is monotonic with respect to set inclusion; i.e., if $S_1 \subset S_2$, then $\mathcal{F}(S_1) \subset \mathcal{F}(S_2)$.

Given our formulation of fuzzy argumentation frameworks and the definitions presented, the definitions of different semantics for FAFs follow those for Dung argumentation frameworks.

Definition 10. *The grounded extension (GE) is the least fixed point of the characteristic function $\mathcal{F}$.*

Definition 11. *A conflict-free set S is a complete extension (CE) if it contains all the fuzzy arguments in $\mathcal{A}$ that S weakening defends; i.e., $\mathcal{F}(CE) = CE$.*

Example 7. *Consider the GFAF*

$$(\{(A, 0.7), (B, 0.8), (C, 0.9)\}, \{((A, B), 0.9), ((B, C), 0.7)\})$$

The sets of fuzzy arguments $\{(A, 0.6), (B, 0.3), (C, 0.6)\}$ and $\{(A, 0), (B, 0), (C, 0)\}$ are both admissible, but neither is complete. The reason for this is that the empty set defends $(A, 0.7)$, which is not within either set. In this case, there is a single complete extension: $\{(A, 0.7), (B, 0.3), (C, 0.7)\}$.

Definition 12. *An admissible extension is a preferred extension (PE) if it is maximal.*

A preferred extension E is a maximal self-defended conflict-free set of fuzzy arguments. Unlike Dung-like systems, conflict here arises due to changes in the degree of belief placed in arguments, rather than simply from the presence of arguments.

Example 8. *Consider the GFAF $(\{(A, 0.7), (B, 0.8), (C, 0.9), (D, 0.7)\}, \{((A, B), 0.9), ((B, C), 0.7), ((C, D), 0.8), ((D, C), 0.8)\})$. Here, $\{(A, 0.7), (B, 0.3), (C, 0.5), (D, 0.4)\}$ is complete but not preferred, since both the complete extension $\{(A, 0.7), (B, 0.3), (C, 0.5), (D, 0.5)\}$ and $\{(A, 0.7), (B, 0.3), (C, 0.6), (D, 0.4)\}$, which are preferred, strictly contains it.*

Definition 13. *A conflict-free extension E is* stable *(abbreviated SE) if it sufficiently attacks every elements in $\mathcal{A}$ not in E.*

The stable extensions E is both the maximal conflict-free set and the minimal set that can attack all other arguments.

Example 9. *Suppose a GFAF is given as $(\{(A, 1)\}, \{((A, A), 1)\})$. Then the extension $\{(A, 0.5)\}$ is preferred and stable.*

Next, we consider the relationship between the different extensions. These relationships are identical to those found in Dung frameworks, with the exception that an extension is preferred if and only if it is stable (i.e., preferred and stable extensions coincide).

Theorem 2.

$$PE = SE \Rightarrow CE \Rightarrow AE \Rightarrow Cf, \; GE \Rightarrow CE.$$

The converse is not valid.

Proof. (sketch) The examples above show that the converse of the implications are invalid.

$AE \Rightarrow Cf$ and $GE \Rightarrow CE$ are trivially valid.

$CE \Rightarrow AE$: From the definition of the characteristic function, CE weakening defends each element in CE. Thus, it is admissible.

$PE \Rightarrow CE$: Consider $\mathcal{F}(PE)$. From the definition of $\mathcal{F}$, $\mathcal{F}(PE)$ contains all the fuzzy arguments that are weakening defended by PE. Since PE is admissible, $PE \subset \mathcal{F}(PE)$, that is $\mathcal{F}(PE)$ is also admissible. Additionally, because PE is maximal in all the admissible extensions, $\mathcal{F}(PE)$ is not a superset of PE; i.e., $PE = \mathcal{F}(PE)$. Thus, PE is complete.

$SE \Rightarrow PE$: Consider an argument (A, a) which sufficiently attacks SE. Such an argument is not in SE, as SE is conflict-free. By Lemma 1, SE weakens (A, a) to (A, a'), such that (A, a') is not sufficiently attacked by SE, which means (A, a') does not sufficiently attack SE. Thus, SE weakening defends any argument in SE; i.e., SE is admissible.

Obviously, SE is maximal. Therefore, SE is preferred.

$PE \Rightarrow SE$: Suppose PE weakens every argument (A, a), which is sufficiently attacked by PE, to $\mathcal{A}'(A)$, i.e.

$$\mathcal{A}'(A) = \min_{B \in Args} \{1 - \min\{PE(B), \rho(B, A)\}, \mathcal{A}(A)\}.$$

Obviously, any elements not in $\mathcal{A}'$, are sufficiently attacked by PE.

Additionally, it is not difficult to show that $\mathcal{A}'$ is just $\mathcal{F}(PE)$, with $\mathcal{F}$ the characteristic function, by Definitions 6 and 9. For PE is CE, we have $\mathcal{A}' = PE$. This means PE sufficiently attacks all the other arguments not in PE; i.e., PE is stable. □

5. Discussion

In this section we explore the relationships between GFAFs and Dung argument systems, between GFAFs and the system of Da Costa Pereira *et al.* [11], and between GFAFs and Jannsen's approach *et al.* [7]. We note that a variety of other fuzzy approaches to argumentation have been proposed (e.g., [14], which considers a fuzzy instantiated argumentation system), but these are not closely related to our work, and therefore omitted due to space constraints. Additionally, we state only our main results, with proofs provided in a technical report[9].

5.1. Dung Argument Frameworks

A DAF can be viewed as a *crisp* GFAF where the degree of belief of arguments and attacks is 1 or 0 (1 for those attacks present in the DAF, and 0 for absent attacks).

Theorem 3. *A conflict-free; stable; or admissible extension within a DAF is also a conflict-free; stable; or admissible extension within the corresponding crisp GFAF.*

[9] http://homepages.abdn.ac.uk/n.oren/pages/CS2016-01.pdf

As mentioned above, the stable and preferred extensions coincide within GFAFs. A preferred, but non-stable extension within a DAF will not form a preferred extension within a GFAF.

5.2. Da Costa Pereira et al.

The model proposed by Da Costa Pereira *et al.* [11] can be seen as a FAF with crisp attacks, which utilises the following (convergent) function to provide a fuzzy labelling for arguments [11, Definition 12, page 5]:

$$\alpha_{t+1}(A) = \frac{1}{2}\alpha_t(A) + \frac{1}{2}\min\{\mathcal{A}(A),\ 1 - \max_{B:\ B \to A} \alpha_t(B)\}, \tag{4}$$

Theorem 4. *Given a fuzzy set E, defined as $E(A) = \alpha(A), \forall A \in Args$, E is a preferred/stable extension of the GFAF obtained using crisp attacks and fuzzy arguments with degree of belief obtained using α.*

5.3. Janssen et al.

We consider a restricted form of Janssen et al.'s framework [7] (referred to as JAFs) where: the truth lattice $\mathcal{L}$ is binary; the tnorm $\wedge$ is the Gödel t-norm, meaning that the implication can be defined by the residual (i.e., for any $a, b, c \in [0, 1], a \leq b \rightsquigarrow c$ iff $\min\{a, b\} \leq c$); and $\neg a = 1 - a$.

Even with these restrictions, the argumentation frameworks differ. In our work, a GFAF contains a given fuzzy subset of *Args* with some associated upper bound on degree of belief of the arguments, and a fuzzy set of attacks between arguments. Within a JAF, the sets of arguments are crisp, and the extensions are fuzzy subsets of a crisp set. Attacks in GFAFs are based on Dung's notion of attack, while in a JAF, they are from a fuzzy set of arguments to an argument or another fuzzy set of arguments. Given this, basic concepts such as conflict-freeness differ between JAFs and GFAFs, meaning that extensions typically also differ. However, JAFs and GFAFs coincide in the following situation (based on Proposition 2 of [7]).

Theorem 5. *Let $(Args, \rho)$ be a JAF and a GFAF, with $\mathcal{A} = Args$. A fuzzy set S is stable in GFAF, iff it is conflict-free in GFAF and 1-stable in JAF using the Gödel t-norm.*

6. Conclusions

In this paper we introduced Gödel Fuzzy Argumentation Frameworks, for which extensions are defined in a manner that is consistent with those defined for Dung's abstract argument frameworks, while utilising fuzzy arguments and attacks. Using the notions of sufficient attacks and weakening defends enables us to rigorously model reasoning over arguments and attacks that have degrees of belief associated with them. When restricted to crisp arguments and attacks, the extensions obtained are similar to those of a Dung system. The main thrust of our future research is to further investigate the properties of GFAFs, such as in the context of the semi-stable semantics, and to examine the properties of other t-norms.

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