QUANTUM INTEGRABILITY AND GENERALISED QUANTUM SCHUBERT CALCULUS

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Abstract. We introduce and study a new mathematical structure in the generalised (quantum) cohomology theory for Grassmannians. Namely, we relate the Schubert calculus to a quantum integrable system known in the physics literature as the asymmetric six-vertex model. Our approach offers a new perspective on already established and well-studied special cases, for example equivariant K-theory, and in addition allows us to formulate a conjecture on the so-far unknown case of quantum equivariant K-theory.

1. Introduction

Generalised complex oriented cohomology first appeared in the work of Novikov [54] and Quillen [58] who realised that formal groups naturally enter in algebraic topology. Such a theory is known to be completely characterised by the isomorphism $h^*(\mathbb{C}P^\infty) \cong h^*(\mathrm{pt})[x]$, where x is the first Chern class of the canonical line bundle over the infinite complex projective space $\mathbb{C}P^\infty$, and the Künneth formula, $h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong h^*(\mathrm{pt})[x,y]$, which implies that the first Chern class of the tensor product of two line bundles obeys a formal group law [1]. There are three known types of formal group laws which come from the one-dimensional connected algebraic groups, the additive group, the multiplicative group, and elliptic curves, describing respectively (ordinary) cohomology, K-theory and elliptic cohomology.

On the other hand to each of the mentioned groups one can associate rational, trigonometric and elliptic solutions of the Yang-Baxter equation which are linked to the appropriate quantum groups. It was first suggested in [19] that there should be a connection between the latter and the mentioned generalised cohomology theories.

The study of solutions of the Yang-Baxter equation is at the heart of the area of quantum integrable systems. Based on earlier pioneering works of Hans Bethe [8] and Rodney Baxter [4], the Faddeev School [17] developed the *algebraic Bethe ansatz* or *quantum inverse scattering method*, where starting from a solution of the Yang-Baxter equation one constructs the quantum integrals of motion of the physical system as a commutative subalgebra, now often called the *Bethe algebra*, within a larger non-commutative *Yang-Baxter algebra*. Historically, Yang-Baxter algebras were the origin for the later definition of quantum groups by Drinfeld [16] and Jimbo [30]. Using the commutation relations of the Yang-Baxter algebra the Bethe ansatz culminates in the derivation of a coupled set of – in our setting – polynomial equations, whose solutions describe the spectrum of the commuting transfer matrices which generate the Bethe algebra. In general solving these equations analytically is

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regarded as an intractable problem within the integrable systems community except for a few special cases.

The first instance were quantum integrability was used in the study of quantum cohomology of full flag varieties and quantum K-theory was in works of Givental and Kim and Givental and Lee; see [20, 22, 31, 32] and [21, 41]. In recent work of Nekrasov and Shatashvili [53] which was further developed mathematically by Braverman, Maulik and Okounkov [9, 46] it was established that the Bethe ansatz equations of some well known integrable systems related to the quantum groups known as Yangians describe the quantum cohomology and quantum K-theory for a large class of algebraic varieties, the Nakajima varieties. Particular examples are the cotangent spaces of partial flag varieties, see the work [25], the simplest case being the contangent space of the Grassmannian. This opens up an exciting new perspective on the connection made in [19].

In this article we shall instead investigate the above connection for the Grassmannians $\mathrm{Gr}_{n,N} = \mathrm{Gr}_n(\mathbb{C}^N)$ themselves rather than their cotangent spaces based on the earlier findings in [39], [37] and [24]; see also the work on non-quantum $GL(N)$-equivariant cohomology in [59]. The difficulty here is, that it is initially not clear which quantum group to expect. So instead we start out with special solutions to the Yang-Baxter equation which are tied to certain exactly solvable or quantum integrable lattice models in statistical mechanics and consider their associated Yang-Baxter algebras as our "quantum group". Despite the models being physically motivated, they are special degenerations of the asymmetric six-vertex model which describes ferroelectrics such as ice, their resulting Bethe algebras – for certain special cases – describe rings which have been defined previously in the setting of algebraic topology and geometry where they are of great mathematical interest. Specialising the parameters of the quantum integrable model in different ways, we are able to identify them as the *quantum equivariant cohomology* [48] and the (non-equivariant) *quantum K-theory* [13] of the Grassmannians using the results in *loc. cit.*

These special cases prompt us to conjecture that our main result, the description of a complex oriented generalised quantum cohomology and its equivariant version for the Grassmannians, also covers the so far unknown case of *equivariant quantum K-theory.* At the same time this description can be seen as solving the well-posed mathematical problem of finding the solution to the Bethe ansatz equations: we state the coordinate ring defined by the equations, identify a special basis in it and explicitly describe the multiplication of two basis elements in terms of a generalised Schubert calculus within the framework of Goresky-Kottwitz-MacPherson theory which we show to extend to the quantum case.

1.1. **Statement of results.** Denote by $\mathrm{Gr}_{n,N} = \mathrm{Gr}_n(\mathbb{C}^N)$ the Grassmannian of n-dimensional hyperplanes in $\mathbb{C}^N$ with $N \geq 3$ and fix a maximal torus $\mathbb{T} \subset GL(N)$. We describe generalised $\mathbb{T}$-equivariant quantum cohomology rings $qh_n^* = qh^*(\mathrm{Gr}_{n,N};\beta)$ for $n = 0, 1, \ldots, N$ using the theory of exactly solvable lattice models in statistical mechanics [4]. While the latter appear in theoretical physics, we shall use them here as abstract combinatorial objects – they define a weighted counting of non-intersecting lattice paths as described for $\beta = 0$ in [37] – which can be rigorously defined in purely mathematical terms using Yang-Baxter algebras. The weights or probabilities attached to the lattice models depend on

- a variable β (the anisotropy parameter of the six-vertex model) entering the multiplicative formal group law [58], [14] and its inverse,

$$x \oplus y = x + y + \beta xy \quad \text{and} \quad x \ominus y = \frac{x-y}{1+\beta y}\ , \tag{1.1}$$

- a "quantum" parameter q (the twist parameter related to quasiperiodic boundary conditions on the lattice) as well as
- the equivariant parameters $t = (t_1, \ldots, t_N)$ (so-called inhomogeneities in the lattice) which are connected to the natural $\mathbb{T}$-action on $\mathrm{Gr}_{n,N}$.

The case $\beta = 0$, which corresponds to the additive group law and in physics terminology to the so-called *free fermion* point of the lattice models, has been treated previously for the homogeneous case ($t_j = 0$) in [37] and recently been extended to the equivariant setting in [24].

Our approach does not require any background knowledge in statistical mechanics, the lattice models are constructed in terms of special solutions to the quantum (as opposed to classical) Yang-Baxter equation, hence they are called *quantum integrable*, and their description is purely algebraic. However, we find it noteworthy that they are degenerations of the asymmetric six-vertex model – as mentioned previously – and their combinatorial description analogous to the one in [37] provides a powerful computational tool. For the latter to work we require the previously mentioned restriction $N \geq 3$.

From these special solutions of the Yang-Baxter equation we construct Yang-Baxter algebras, which in our case are bi-algebras only and not full Hopf algebras. The so-called *row-to-row transfer matrices* of the lattice model generate a commutative subalgebra within the larger non-commutative Yang-Baxter algebra which decomposes into the direct sum $\bigoplus_{n=0}^{N} qh_n^*$ of rings, which have the following presentation.

Set $\mathcal{R}(\mathbb{T}) = \mathcal{R}(t_1, \ldots, t_N)$ where $\mathcal{R}$ is the ring of rational functions in β which are regular at $\beta = 0$ and $\beta = -1$. Define qh_n^* to be the polynomial algebra generated by $\{e_r\}_{r=1}^{n}$, $\{h_r\}_{r=1}^{N-n}$ over $\mathcal{R}(\mathbb{T}, q) = \mathbb{Z}[\![q]\!] \otimes \mathcal{R}(\mathbb{T})$ subject to the relations obtained by expanding the following functional relation in the variable x,

$$h(x)e(\ominus x) = \left(\prod_{i=1}^{n} t_i \ominus x\right)\left(\prod_{i=1}^{N-n} x \ominus t_{i+n}\right)(1 + \beta h_1) + q, \tag{1.2}$$

where 1 is the unit element and, setting $h_0 = e_0 = 1$, $h_{N+1-n} = e_{n+1} = 0$,

$$h(x) = \sum_{r=0}^{N-n} (h_r + \beta h_{r+1}) \prod_{i=1}^{N-n-r} (x \ominus t_{N+1-i}) \tag{1.3}$$

$$e(x) = \sum_{r=0}^{n} (e_r + \beta e_{r+1}) \prod_{i=1}^{n-r} (x \oplus t_i)\ . \tag{1.4}$$

For the non-experts we recall that the Grothendieck K-functor assigns to each smooth compact manifold $\mathcal{X}$ a ring which is built out of complex vector bundles on $\mathcal{X}$ [2]. It is the value of this functor and its quantum analogue QK for $\mathcal{X} = \mathrm{Gr}_{n,N}$ which we shall simply refer to as (quantum) "K-theory" of the Grassmannians throughout this article.

Denote by $\{e^{\varepsilon_j}\}_{j=1}^{N}$ the (formal) exponentials generating the character ring of $\mathfrak{gl}(N)$.

Theorem 1.1. *We have the following special cases:*

(i) $qh_n^*/\langle\beta\rangle$ *is isomorphic to the equivariant quantum cohomology* $QH_{\mathbb{T}}^*(\mathrm{Gr}_{n,N})$ *in the presentation given by Mihalcea* [48, Thm 1.1].

(ii) $qh_n^*/\langle\beta+1,t_1,\ldots,t_N\rangle$ *is isomorphic to* $QK(\mathrm{Gr}_{n,N})$ *as studied in* [13].

(iii) $qh_n^*/\langle\beta+1,t_j+e^{\varepsilon_{N+1-j}}-1,q\rangle$ *is isomorphic to* $K_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ *where* $K_{\mathbb{T}}$ *denotes the equivariant K-functor.*

Each of the cases (i)–(iii) is interesting in its own right and we compare our findings against existing presentations of these rings in the literature. In particular, in case (i) our results are linked to previous (unpublished) work by Peterson [55] and the affine nil-Hecke ring of Kostant and Kumar [35]: we explicitly construct a family of operators whose matrix elements give the structure constants of qh_n^* and which for $\beta=0$ can be identified with Peterson's basis; see [24] for details. The other cases can then be seen as a generalisation of this construction to K-theory.

To establish (ii) we compare our ring structure against the Pieri rules derived by Lenart [42] for $q=0$ and the quantum Pieri and Giambelli formulae of Buch and Mihalcea [13] for $q\neq 0$. The new result in our article is the coordinate ring presentation which follows from (1.2).

Finally, we show (iii) by defining a generalisation of Goresky-Kottwitz-MacPherson theory [26]: we identify McNamara's factorial Grothendieck polynomials [47] with localised Schubert classes using the Bethe ansatz of quantum integrable models. We also derive expressions for the localised opposite Schubert classes and identify the partition functions of our lattice models with classes for Richardson varieties.

Based on the above special cases we have the following:

Conjecture 1.2. $qh_n^*/\langle\beta+1,t_j+e^{\varepsilon_{N+1-j}}-1\rangle$ *describes the value* $QK_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ *of the equivariant quantum K-functor for the Grassmannians.*

Remark 1.3. *We note that we can define* qh_n^* *also over the ring of Laurent polynomials in* β *instead, which would introduce a natural* $\mathbb{Z}$*-grading. This suggests that our framework might also be used to describe the actual* $\mathbb{Z}$*-graded quantum equivariant K-theory which is obtained from the K-functor in conjunction with the Bott Periodicity Theorem. However, there is currently not sufficient evidence available to further substantiate this claim, hence we state this here as a mere observation and not as a conjecture.*

Besides providing a complete description of $QK_{\mathbb{T}}(\mathrm{Gr}_{n,N})$, which has so far been missing in the literature, the new aspects in our approach are

(1) that our ring is defined for general β which allows us to treat all these special cases at once in a unified setting of a quantum generalised cohomology theory as first defined in [15] and
(2) that we reveal an underlying quantum group structure in terms of Yang-Baxter algebras which we show to commute with the natural symmetric group action on the idempotents of these rings.

As a byproduct of our investigation we also derive new combinatorial results such as a generalised Jacobi-Trudy formula and Cauchy identity for factorial Grothendieck polynomials.

1.2. Outline of the article.

Section 2: We introduce the necessary combinatorial objects and notations we will use throughout the article. In particular, we review McNamara's definition of factorial Grothendieck polynomials which play a central role in our approach and derive several new results which we need to describe our generalised cohomology ring for the Grassmannian.

Section 3: Starting from special solutions to the Yang-Baxter equation, so-called L-operators, we define the Yang-Baxter algebra in terms of endomorphisms over some vector space $\mathcal{V}$ which will be identified with the direct sum of the generalised cohomology rings $\bigoplus_{n=0}^{N} qh_n^*$. We describe the commutation relations of the Yang-Baxter algebra and define the transfer matrices which generate a commutative subalgebra, the so-called Bethe algebra. The action of the latter on $\mathcal{V}$ is described combinatorially using toric skew diagrams. We also show that the transfer matrices obey the functional relation (1.2).

Section 4: We derive the spectrum of the Bethe algebra by constructing their eigenvectors and computing their eigenvalues using the algebraic Bethe ansatz. Both, eigenvectors and eigenvalues, are described in terms of the solutions of a set of coupled equations, called the Bethe ansatz equations, which we show can be solved in terms of formal power series in the quantum deformation parameter q of qh_n^*. We then initially define the generalised cohomology ring by identifying the eigenbasis of the transfer matrices as the primitive, central orthogonal idempotents of qh_n^*. We also define a bilinear form which turns qh_n^* into a Frobenius algebra. Having identified the eigenvectors of the transfer matrices as idempotents, we then fix the analogue of the Schubert basis and describe the product in this geometrically motivated basis instead. This allows us to state a residue formula for the structure constants in the Schubert basis in terms of the solutions of the Bethe ansatz equations and show that they obey a recurrence formula which is derived from an equivariant quantum Pieri-Chevalley formula for qh_n^*.

Section 5: Employing the description of qh_n^* in terms of its idempotents leads to a formulation of the ring in terms of column vectors whose components can be thought of as generalised localised Schubert classes where the localisation points are identified with the solutions of the Bethe ansatz equations. We show that these generalised Schubert classes obey generalised Goresky-Kottwitz-MacPherson conditions which derive from an action of the symmetric group. Interestingly, the latter emerges naturally from solutions of the Yang-Baxter equation discussed in Section 3, gives rise to a representation of a generalised Iwahori-Hecke algebra and commutes with the action of the Yang-Baxter algebra. Using this framework of GKM theory we prove the special cases mentioned in the introduction, that is we show that our ring qh_n^* can be specialised to equivariant quantum cohomology and quantum K-theory. This section also gives the proof of the presentation of qh_n^* as polynomial algebra modulo the relations (1.2).

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2. Preliminaries

This section introduces the combinatorial notions needed in the description of Schubert calculus in the rest of this paper. We also collect known as well as a number of new results on factorial Grothendieck polynomials.

2.1. **Minimal coset representatives.** Denote by $\mathbb{S}_N$ the symmetric group in N-letters and choose $k, n \in \mathbb{N}_0$ such that $N = n + k$. A set of minimal length coset representatives w for classes $[w]$ in $\mathbb{S}_N/\mathbb{S}_n \times \mathbb{S}_k$ is given by the permutations for which $w(1) < w(2) < \cdots < w(n)$ and $w(n+1) < w(n+2) < \cdots < w(N)$. For instance, the coset representative with $w^0(n+1) = 1$, $w^0(n+2) = 2, \ldots, w^0(N) = k$ and $w^0(1) = k+1$, $w^0(2) = k+2, \ldots, w^0(n) = N$ is given by

$$w^0 = s_n s_{n+1} \cdots s_{N-1} \ \cdots \ s_2 s_3 \cdots s_{k+1} \ s_1 s_2 \cdots s_k \ . \tag{2.1}$$

2.2. **Binary strings.** The w's are in bijection with 01-words or binary strings $b(w) = b_1 b_2 \cdots b_N \in \{0,1\}^N$ of length N, where $n = |b| := \sum_j b_j$ is the number of 1-letters, $k = N - |b|$ the number of 0-letters and

$$b_j(w) = \begin{cases} 1, & j \in I_w \\ 0, & j \in [N]\backslash I_w \end{cases} \tag{2.2}$$

with $I_w := \{w(1), \ldots, w(n)\}$ and $[N] = \{1, 2, \ldots N\}$. So, in the case of w^0 we have $I_{w^0} = \{k+1, \ldots, N\}$ and $b(w^0) = 0 \cdots 01 \cdots 1$ is the binary string with k 0-letters in front, followed by n 1-letters. The identity $w = 1$ on the other hand corresponds to the binary string $b(1) = 1 \cdots 10 \cdots 0$ instead. Note that under the above bijection the natural $\mathbb{S}_N$-action on $\mathbb{S}_N/\mathbb{S}_n \times \mathbb{S}_k$ via $s_j \cdot [w] = [s_j w]$ coincides with the natural $\mathbb{S}_N$-action on binary strings, where s_j permutes the jth and $(j+1)$th letter in $b(w)$.

2.3. **Boxed partitions.** Each binary string $b(w)$, and thus each minimal length representative w, is in one-to-one correspondence with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ which has at most n parts and for which $\lambda_1 \leq k$. That is, the corresponding Young diagram lies in a bounding box of height n and width k and we will denote this by writing $\lambda \subset (k^n)$. The bijection is given by the relation

$$w(i) = \lambda_{n+1-i} + i, \qquad i = 1, 2, \ldots, n \ . \tag{2.3}$$

For $w = w^0$ we have that $\lambda = (k^n)$, the partition with n parts equal to k, and for $w = 1$ we obtain the empty partition denoted by $\lambda = \emptyset$. N.B. the bijection is defined for fixed n, k, so each partition comes with a bounding box of fixed dimensions. For different n, k one may obtain the same partition λ but the dimensions of the bounding box will then be different. We therefore refer to λ as a *boxed partition* as the dimensions of the bounding box enter in the bijection (2.3).

Throughout this article we will use these various labellings of the same coset $[w]$ interchangeably writing $b(w)$, $\lambda(w)$ for the images under the above bijection and $b(\lambda), w(\lambda)$ for the pre-images. By abuse of notation we shall also write $s_j b$ and $s_j \lambda$ for the binary string and partition corresponding to the coset $[s_j w]$.

2.4. **Cylindric loops and toric skew diagrams.** We briefly recall the definition of cylindric loops $\lambda[r]$ associated with a partition λ and *toric skew diagrams*; see [18], [57].

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in (n, k)$, then the associated *cylindric loop* $\lambda[r]$ for any $r \in \mathbb{Z}$ is defined as

$$\lambda[r] := (\dots, \underset{r}{\lambda_n + r + k}, \underset{r+1}{\lambda_1 + r}, \dots, \underset{r+n}{\lambda_n + r}, \underset{r+n+1}{\lambda_1 + r - k}, \dots) \ . \tag{2.4}$$

For $r = 0$ the cylindric loop can be visualised as a path in $\mathbb{Z} \times \mathbb{Z}$ determined by the outline of the Young diagram of λ which is periodically continued with respect to the vector $(n, -k)$. For $r \neq 0$ this line is shifted r times in the direction of the lattice vector $(1, 1)$.

Given two boxed partitions $\lambda, \mu \subset (k^n)$ denote by $\lambda/d/\mu$ the set of squares between the two lines $\lambda[d]$ and $\mu[0]$ modulo integer shifts by $(n, -k)$,

$$\lambda/d/\mu := \{\langle i, j\rangle \in \mathbb{Z} \times \mathbb{Z}/(n, -k)\mathbb{Z} \mid \lambda[d]_i \geq j > \mu[0]_i\} \ . \tag{2.5}$$

We shall refer to $\theta = \lambda/d/\mu$ as a *cylindric skew-diagram* of degree $d = d(\theta)$. Postnikov introduced [57] the terminology *toric skew-diagram* for those θ where the number of boxes in each row does not exceed k. Note that $\lambda/0/\mu = \lambda/\mu$, that is cylindric or toric skew-diagrams contain ordinary skew diagrams as special cases.

A cylindric skew diagram θ which has at most one box in each column will be called a *toric horizontal strip* and one which has at most one box in each row a *toric vertical strip*. The *length* of such strips will be the number of boxes within the skew diagram, where we identify squares modulo integer shifts by $(n, -k)$ and choose as representatives those squares $s = \langle i, j\rangle$ with $1 \leq j \leq n$. In what follows this identification is always understood implicitly if we talk about a square in a toric strip.

2.5. **Bases in equivariant cohomology and K-theory.** We are interested in describing equivariant quantum cohomology ($\beta = 0$) and K-theory ($\beta = -1$) as special cases of our generalised cohomology theory for $\mathrm{Gr}_{n,N}$. The equivariant cohomology [35] and K-theory [36] of flag varieties – of which Grassmannians are a special case – was studied by Kostant and Kumar. The equivariant quantum cohomology of flag varieties was computed in [20], [31], [22], [32] and quantum K-theory in [21], [41], [23] and since then has been discussed by numerous authors.

Specialising $\beta = 0$ we identify $\mathcal{R}(\mathbb{T})$ with the equivariant cohomology $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{Z}(t_1, \dots, t_N)$ of a point by mapping each $f_\beta \in \mathcal{R}(\mathbb{T})$ to its value at $\beta = 0$. Let X_λ and X^λ denote the *Schubert* and *opposite Schubert varieties* where $\lambda \subset (k^n)$. We also recall the definition of the *Richardson variety* $X^\lambda_\mu = X_\mu \cap X^\lambda$. All three varieties are left invariant under the torus action. The corresponding Schubert classes $\{[X_\lambda]\}_{\lambda \subset (k^n)}$ and $\{[X^\lambda]\}_{\lambda \subset (k^n)}$ form dual bases over $\mathbb{Z}[q] \otimes \mathbb{Z}(t_1, \dots, t_N)$. Both bases are related by $X^\lambda = w_0 \cdot X_{\lambda^\vee}$ where $\lambda^\vee$ is obtained by reversing the binary string $b(\lambda)$ and w_0 is the *long element* in $\mathbb{S}_N$. One is interested in the computation of the *3-point genus 0 equivariant Gromov-Witten invariants* $C^\nu_{\lambda\mu}(t, q)$ which appear in the product

$$[X_\lambda][X_\mu] = \sum_{\nu \subset (k^n)} C^\nu_{\lambda\mu}(t, q)[X_\nu] \tag{2.6}$$

and for the Grassmannian are monomials in q, i.e. $C^{\nu}_{\lambda\mu}(t,q) = q^d C^{\nu,d}_{\lambda\mu}(t)$. The invariants for $d=0$ also appear in the expansion

$$[X^{\lambda}_{\mu}] = \sum_{\nu} C^{\lambda,0}_{\mu\nu}(t)[X^{\nu}] , \tag{2.7}$$

and, thus, $C^{\lambda;0}_{\mu\nu}(t) = c^{\nu}_{\lambda\mu}(t)$ are the analogue of Littlewood-Richardson coefficients for factorial Schur functions [51].

In the case of K-theory we specialise $\beta = -1$ and set $t_j = 1 - e^{\varepsilon_{N+1-j}}$ where the (formal) exponentials $\{e^{\varepsilon_j}\}_{j=1}^{N}$ generate the character ring of $\mathfrak{gl}(N)$. Mapping each $f_\beta \in \mathcal{R}(\mathbb{T})$ to its value at $\beta = -1$ then gives us $K_{\mathbb{T}}(\mathrm{pt}) = \mathrm{Rep}(\mathbb{T})$, the representation ring of $\mathbb{T}$ which is canonically isomorphic to the group algebra of the free abelian group of characters e^{ε_i}. The ring $K_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ is generated by the classes $[\mathcal{O}_\lambda]$ and $[\mathcal{O}^\lambda]$ of the *structure sheaves* $\mathcal{O}_\lambda$ and $\mathcal{O}^\lambda$ of the Schubert and opposite Schubert varieties within the Grothendieck group of coherent sheaves on the Grassmannian. Their product expansions

$$[\mathcal{O}_\lambda][\mathcal{O}_\mu] = \sum_{\nu\subset(k^n)} c^{\nu}_{\lambda\mu}(t)[\mathcal{O}_\nu], \qquad [\mathcal{O}_\mu][\mathcal{O}^\lambda] = \sum_{\nu\subset(k^n)} d^{\lambda}_{\mu\nu}(t)[\mathcal{O}^\nu] \tag{2.8}$$

define the K-theoretic Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}(t)$ where in case of the Grassmannian $d^{\nu}_{\lambda\mu}(t) = c^{\nu}_{\lambda\mu}(t)$; see e.g. [34] and references therein. There are known positivity statements for these structure constants, see [28] and [3, Sec 5] as well as references therein.

We shall refer to the K-classes $\{[\mathcal{O}^\lambda]\}$ as Schubert basis or simply Schubert classes. In contrast to the case $\beta = 0$ the classes $[\mathcal{O}_\lambda]$ and $[\mathcal{O}^\lambda]$ do not form dual bases in $K_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ but instead one has to introduce additional classes $[\xi^\lambda]$ which can also be defined in terms of sheaves (see [27, Prop 2.1]). For the non-equivariant case $K(\mathrm{Gr}_{n,N}) = K_{\mathbb{T}}(\mathrm{Gr}_{n,N})/\langle t_1,\dots,t_N\rangle$ one has the relation [12, Sec 8]

$$[\xi^\lambda] = (1 - [\mathcal{O}_1])[\mathcal{O}_{\lambda^\vee}] \tag{2.9}$$

where $[\mathcal{O}_1]$ is the K-class of the Schubert divisor. We shall state the analogue of this relation for the equivariant case in a subsequent section.

2.6. **Discrete symmetries.** Throughout this article we will make use of several involutions and a natural $\mathbb{Z}_N$-action defined on the set of cosets in $\mathbb{S}_N/\mathbb{S}_n \times \mathbb{S}_k$ where $k = N - n$ as before. These will induce mappings between elements in the Schubert basis, in some cases from different rings, and since they in turn lead to non-trivial transformation properties of the structure constants of qh^*_n, we refer to them as "symmetries".

2.6.1. *Poincaré Duality.* Define an involution $\vee : qh^*_n \to qh^*_n$ by reversing a binary string, i.e. $b^\vee_i = b_{N+1-i}$. We shall denote the corresponding permutation and partition by $w^\vee$ and $\lambda^\vee$, respectively. One easily verifies that the Young diagram of $\lambda^\vee$ is the complement of the Young diagram of λ in the $n \times k$ bounding box.

2.6.2. *Level-Rank Duality.* Define an involution $* : qh^*_n \to qh^*_{N-n}$ by swapping 0 and 1-letters in binary strings, i.e. $b^*_i = 1 - b_i$. The corresponding partition λ^* is obtained by taking first the conjugate partition λ' and then its complement in the bounding box or vice versa, i.e. $\lambda^* = (\lambda')^\vee = (\lambda^\vee)'$. So, in particular we can define the composite involution $qh^*_n \to qh^*_{N-n}$ by $\lambda \mapsto \lambda'$ and shall denote the corresponding binary string and permutation respectively by b' and w'.

2.7. **Set-Valued Tableaux and Grothendieck polynomials.** We recall some of the necessary combinatorial objects and the definition of factorial Grothendieck polynomials. This is based on earlier work by Buch [12] and McNamara [47], but we shall also derive several new results which are not contained in the latter works.

Let n be some non-negative integer. We will use the notation $[n] = \{1, \ldots, n\}$ and $\mathbb{P}_n = \mathbb{P}([n])$ for the power set of $[n]$, the set of all subsets of $[n]$. Denote by θ a skew Young diagram with at most $|\theta| \leq n$ boxes which we identify with a subset of $\mathbb{Z}^2$.

Definition 2.1 (Buch)**.** *A set-valued tableau is a map $T : \theta \to \mathbb{P}_n$ such that the following conditions hold*

$$\max T(i,j) \leq \min T(i,j+1) \quad \textit{and} \quad \max T(i,j) < \min T(i+1,j) \; . \tag{2.10}$$

Denote by $|T|$ the sum over the cardinalities of all the subsets in the image of T and let $\mathrm{SVT}(\theta)$ be the set of all set-valued tableau of shape θ. Then we have the following definition of factorial Grothendieck polynomials due to McNamara [47] which is an extension of Buch's earlier realisation [12] of ordinary (skew) Grothendieck polynomials as sum over set-valued tableaux.

Definition 2.2. *The factorial (skew) Grothendieck polynomial is the weighted sum*

$$G_\theta(x|t) = \sum_T \beta^{|T|-|\theta|} \prod_{\substack{(i,j)\in\theta \\ r\in T(i,j)}} x_r \oplus t_{r+j-i} \tag{2.11}$$

over all set-valued tableaux $T \in \mathrm{SVT}(\theta)$.

N.B. the factorial Grothendieck polynomials are in general defined for an *infinite* sequence $(t_j)_{j\in\mathbb{Z}}$ of parameters. For this section only we shall assume these parameters to be nonzero for all j but then set $t_j = 0$ unless $1 \leq j \leq N$ and identify them with the equivariant parameters mentioned in the introduction.

Employing (1.1) define the β-deformed factorial power

$$(x_j|t)^r := \prod_{i=1}^{r} x_j \oplus t_i \; . \tag{2.12}$$

The following determinant formula is stated in [29, Eqn (2.12)]. Its proof follows along similar lines as indicated in *loc. cit.* where the focus is on the symplectic case.

Proposition 2.3 (Ikeda-Naruse)**.**

$$G_\theta(x|t) = \frac{\det\left[(x_j|t)^{\theta_i+n-i}(1+\beta x_j)^{i-1}\right]_{1\leq i,j\leq n}}{\det[x_j^{n-i}]_{1\leq i,j\leq n}} \tag{2.13}$$

where the denominator is the Vandermonde determinant $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

We recall the following known specialisations of factorial Grothendieck polynomials.

Setting $t_j = 0$ for all j one recovers the (ordinary) *Grothendieck polynomial* which has the following determinant presentation,

$$G_\theta(x) = \frac{\det\left(x_j^{\theta_i+n-i}(1+\beta x_j)^{i-1}\right)_{1\leq i,j\leq n}}{\det\left(x_j^{n-i}\right)_{1\leq i,j\leq n}} \; . \tag{2.14}$$

Setting $\beta = 0$ one obtains the *factorial Schur function* (see e.g. [44] and [45, Cap I.3, Ex. 20] as well as references in *loc. cit.*),

$$s_\theta(x|t) = \frac{\det[(x_j|t)^{\theta_i+n-i}]_{1\le i,j\le n}}{\det[(x_j|t)^{n-i}]_{1\le i,j\le n}}\ , \qquad (x_j|t)^r \overset{\beta=0}{=} \prod_{i=1}^{r}(x_j+t_i)\ . \tag{2.15}$$

We collect further properties of factorial Grothendieck polynomials which we will use throughout this article.

We use the determinant formula (2.13) to derive the following equation which is a generalisation of the known straightening rule for Schur functions s_θ, $s_{\ldots,\theta_i,\theta_{i+1},\ldots} = -s_{\ldots,\theta_{i+1}-1,\theta_i+1,\ldots}$ [45, Ch I.3]. The latter – through repeated application – allows one to express a Schur function indexed by a composition in terms of Schur functions indexed by partitions. We will use the straightening rule for factorial Grothendieck polynomials for the same purpose.

Corollary 2.4 (straightening rule)**.** *We have the following relations*

$$\begin{aligned} &G_{\ldots,\theta_i,\theta_{i+1},\ldots} = \\ &\quad -\beta G_{\ldots,\theta_i+1,\theta_{i+1},\ldots} - \frac{1+\beta t_{n+\theta_i-i+1}}{1+\beta t_{n+\theta_{i+1}-i}}\ (G_{\ldots,\theta_{i+1}-1,\theta_i+1,\ldots} + \beta G_{\ldots,\theta_{i+1},\theta_i+1,\ldots}), \end{aligned} \tag{2.16}$$

where $G_\theta = G_\theta(x|t)$ with $\theta = (\theta_1,\ldots,\theta_n)$.

Proof. Without difficulty one verifies the identity

$$(1+\beta t_{m+1})(x|t)^m(1+\beta x)^r = (x|t)^m(1+\beta x)^{r-1} + \beta(x|t)^{m+1}(1+\beta x)^{r-1}\ .$$

Applying the latter first to the ith and then to the $(i+1)$th row of the determinant in the numerator of (2.13), the assertion follows. □

Given a boxed partition $\lambda \subset (k^n)$ we introduce the shorthand notations

$$\begin{aligned} t_\lambda &= (t_{\lambda_n+1},\ldots,t_{\lambda_i+n+1-i},\ldots,t_{\lambda_1+n}) \\ \ominus t_\lambda &= (\ominus t_{\lambda_n+1},\ldots,\ominus t_{\lambda_i+n+1-i},\ldots,\ominus t_{\lambda_1+n}) \end{aligned} \tag{2.17}$$

where $\ominus x := 0 \ominus x = -x/(1+\beta x)$ for any formal variable x; compare with (1.1). The following is a generalisation of the Vanishing Theorem for factorial Schur functions [51] to factorial Grothendieck polynomials; see [47, Thm 4.4].

Theorem 2.5 (McNamara)**.** *Let λ,μ be partitions with at most n parts then*

$$G_\lambda(\ominus t_\mu|t) = \begin{cases} 0, & \lambda \not\subseteq \mu \\ \prod_{\langle i,j\rangle\in\lambda} t_{n+j-\lambda'_j} \ominus t_{\lambda_i+n+1-i}, & \lambda = \mu \end{cases} \tag{2.18}$$

and in general $G_\lambda(\ominus t_\mu|t)$ will be non-zero if $\lambda \subset \mu$.

Following [47] we introduce for simplicity the notation

$$\Pi(x) = \prod_{i=1}^{n}(1+\beta x_i)\ . \tag{2.19}$$

We recall the following results [47, Ex 4.2 and Prop 4.8].

Lemma 2.6 (McNamara)**.** *We have the identity*

$$1 + \beta G_1(x|t) = \sum_{i=0}^{n} \beta^i e_i(x \oplus t) = \Pi(x)\Pi(t_\emptyset)\ , \tag{2.20}$$

where the e_i's denote the elementary symmetric polynomials.

Proposition 2.7 (McNamara)**.** *We have the expansion*

$$\Pi(x)G_\lambda(x|t) = \Pi(\ominus t_\lambda)\sum_{\lambda\rightrightarrows\mu}\beta^{|\mu/\lambda|}G_\mu(x|t), \tag{2.21}$$

where the notation $\lambda \rightrightarrows \mu$ indicates that the sum runs over all partitions μ which contain λ and for which the skew diagram μ/λ has at most one box in each column or row.

Denote by $\Lambda_n \otimes \mathbb{Z}(\beta, t_1, \dots, t_N)$ the linear space spanned by the monomial symmetric functions $\{m_\lambda\}_{\lambda\subset(k^n)}$, then the following result is [47, Thm 4.6].

Theorem 2.8 (McNamara)**.** *The set $\{G_\lambda(x|t)\}$ with λ having at most n parts is a basis of $\Lambda_n \otimes \mathbb{Z}(\beta, t_1, \dots, t_N)$.*

2.7.1. *New results for factorial Grothendieck polynomials.* We expect the Grothendieck polynomials indexed by partitions which either consist of a single column, $\lambda = 1^r$, or row, $\lambda = r$, to be the elementary building blocks for general λ. The following lemma states a generating function for the $G_{1^r}(x|t)$'s.

Proposition 2.9. *We have the equality*

$$\Pi(t_\emptyset)\prod_{i=1}^n(u-x_i) = (u|t)^n + \sum_{r=1}^n(-1)^r G_{1^r}(x|t)(u|t)^{n-r}\prod_{i=1}^{r-1}(1+\beta u\oplus t_{n+1-i}) \tag{2.22}$$

and the identity

$$G_{1^r}(x|t) = \sum_{j=1}^{n+1-r}\frac{\prod_{i=1}^n x_i\oplus t_j}{\prod_{i=1,i\neq j}^{n+1-r} t_j\ominus t_i}\ , \tag{2.23}$$

where $r = 1, \dots, n$.

Proof. First one derives the following equality involving the Vandermonde determinant via induction,

$$a_n(x|t) = \det[(x_j|t)^{n-i}]_{1\le i,j\le n} = \Delta(x)\prod_{i=1}^n(1+\beta t_i)^{n-i}\ . \tag{2.24}$$

Then it follows that

$$\frac{a_{n+1}(u,x_1,\dots,x_n|t)}{a_n(x_1,\dots,x_n|t)} = \Pi(t_\emptyset)\prod_{i=1}^n(u-x_i)\ .$$

Expanding the determinant $a_{n+1}(u, x_1, \dots, x_n|t)$ with respect to the first column one obtains the first formula. Setting $u = \ominus t_i$ with $i = 1, 2, \dots, n$ results in a linear system with lower triangular matrix which can be solved to obtain the second formula. □

Using the last result we now derive an alternative generating function for the $G_{1^r}(x|\ominus t)$'s which will play an important role in what follows.

Corollary 2.10. *We have*

$$\prod_{i=1}^n(u\oplus x_i) = (u|t)^n + \sum_{r=1}^n(u|t)^{n-r}(1+\beta\ u\oplus t_{n+1-r})G_{1^r}(x|\ominus t)\,. \tag{2.25}$$

Setting $t_j = 0$ for all j this becomes

$$\prod_{i=1}^{n}(u \oplus x_i) = u^n + (1+\beta u)\sum_{r=1}^{n} u^{n-r} G_{1^r}(x_1,\dots,x_n) \tag{2.26}$$

which implies for $r = 1,2,\dots,n$ the identities

$$e_r(x_1,\dots,x_n) = \sum_{s=r}^{n}(-\beta)^{s-r}\binom{s-1}{s-r} G_{1^s}(x_1,\dots,x_n)\ , \tag{2.27}$$

where $e_r(x_1,\dots,x_n)$ are the elementary symmetric polynomials.

Proof. Let $f(u) = \prod_{i=1}^{n}(u \oplus x_i)$. This is a polynomial in u of degree n with the coefficient of u^n being $\Pi(x)$. Setting successively $u = \ominus t_1, \ominus t_2, \dots, \ominus t_n$ one finds

$$\begin{aligned} f(u) &= (u|t)^n + (1+\beta u)\sum_{r=1}^{n}(u|t)^{n-r}(1+\beta t_{n+1-r})f_r \\ &= (u|t)^n(1+\beta f_1) + (u|t)^{n-1}(f_1+\beta f_2) + \cdots + f_n \end{aligned}$$

with

$$f_{n+1-r} = \sum_{i=1}^{r}\frac{f(\ominus t_i)}{\prod_{j=1, j\neq i}^{r} t_j \ominus t_i}\ , \quad r = 1,2,\dots,n\ .$$

The identity (2.25) then follows from (2.20) and (2.23). Setting $t_1 = \cdots = t_n = 0$ in (2.25) we arrive at (2.26).

Finally, we have

$$\begin{aligned} \prod_{i=1}^{n}(u - x_i) &= (1+\beta u)^n(-1)^n f(\ominus u) \\ &= u^n + \sum_{r=1}^{n}(-1)^r(1+\beta u)^{r-1}u^{n-r}G_{1^r}(x_1,\dots,x_n) \end{aligned}$$

and comparing powers of u on both sides of the equality sign the last assertion now follows. □

As in the case of factorial Schur functions [45, Chap I.3, Ex. 20] define a *shift operator* τ by

$$(x|\tau^m t)^n = \prod_{j=1}^{n}(x \oplus t_{j+m}), \qquad m \in \mathbb{Z}\,. \tag{2.28}$$

We wish to derive an analogue of the Jacobi-Trudy identity for factorial Schur functions. To this end we require the following result first.

Lemma 2.11. *We have the expression*

$$G_r(x|t) = \sum_{i=1}^{n}(x_i|t)^{n+r-1}\prod_{j\neq i}\frac{1}{x_i \ominus x_j} \tag{2.29}$$

and the following equality between determinants,

$$\det[G_{\lambda_i - i + j}(x|\tau^{1-j}t)]_{1\le i,j\le n} = \frac{\det[(x_j|t)^{n+\lambda_i - i}]_{1\le i,j\le n}}{\det[(x_j|t)^{n-i}]_{1\le i,j\le n}} \tag{2.30}$$

where $(x|t)^m$ is defined in (2.12).

Proof. The proof follows along the same steps as the proof for the analogous identities in the case of factorial Schur functions; see e.g. the section on the "6th variation" in [44] and [44, Chap I.3, Ex. 20]. We therefore omit the details. □

While it would be desirable to have a single determinant in the G_r's expressing the Grothendieck polynomial G_λ, this seems in general not possible. Instead we obtain an expression in terms of sums of determinants which involve the polynomials in (2.30)

$$F_\lambda(x|t)=\frac{\det[(x_j|t)^{n+\lambda_i-i}]_{1\le i,j\le n}}{\det[(x_j|t)^{n-i}]_{1\le i,j\le n}} \tag{2.31}$$

Note that $F_\lambda(x|t)=s_\lambda(x|t)$ for $\beta=0$ and $F_\lambda(x|0)=s_\lambda(x)$, that is the F_λ's do not specialise to the ordinary (non-factorial) Grothendieck polynomial for $t_j=0$. We shall therefore treat this case separately.

Before we can state the expansion formula of G_λ into F_λ's we require the following technical result.

Lemma 2.12.

$$(1+\beta u)^r(u|\ominus t)^{n-r}=\sum_{i=0}^{r}(u|\ominus t)^{n-i}\Gamma_i(r,n) \tag{2.32}$$

where the coefficients are given by

$$\Gamma_i(r,n)=\beta^{r-i}\prod_{j=i}^{r-1}(1+\beta t_{n-j})\sum_{i-1\le j_1\le\cdots\le j_i\le r-1}\prod_{l=1}^{i}(1+\beta t_{n-j_l}) \tag{2.33}$$

Explicitly,

$$\begin{aligned}
\Gamma_0(r,n) &= \beta^r\prod_{j=0}^{r-1}(1+\beta t_{n-j})\\
\Gamma_1(r,n) &= \beta^{r-1}\prod_{j=1}^{r-1}(1+\beta t_{n-j})\sum_{j=0}^{r-1}(1+\beta t_{n-j})\\
\Gamma_2(r,n) &= \beta^{r-2}\prod_{j=2}^{r-1}(1+\beta t_{n-j})\sum_{j=1}^{r-1}(1+\beta t_{n-j})\sum_{i=1}^{j}(1+\beta t_{n-i})\\
&\ \ \vdots\\
\Gamma_r(r,n) &= (1+\beta t_{n+1-r})^r
\end{aligned}$$

Proof. Use the simple identity

$$(1+\beta u)(u|\ominus t)^n=(1+\beta t_{n+1})[(u|\ominus t)^n+\beta(u|\ominus t)^{n+1}]$$

to find the recurrence relation

$$\Gamma_i(r,n)=(1+\beta t_{n+1-r})(\Gamma_{i-1}(r-1,n-1)+\beta\Gamma_i(r-1,n))\ .$$

Here $\Gamma_i=0$ for $i<0$. Defining

$$\Gamma_i(r,n)=\gamma_i(r,n)\beta^{r-i}\prod_{j=i}^{r-1}(1+\beta t_{n-j})$$

The recurrence relation simplifies to

$$\gamma_i(r,n) = (1+\beta t_{n+1-r})\gamma_{i-1}(r-1,n-1) + \gamma_i(r-1,n))$$

and can now be successively solved starting from $\gamma_0(r,n) = 1$. □

We now state a generalised Jacobi-Trudy identity for factorial Grothendieck polynomials which simplifies for $\beta = 0$ to the known Jacobi-Trudy identity for factorial Schur functions. We state it for the parameters $\ominus t$ as it is in this form that we will use the identity later on in this article, but making the replacement $t \to \ominus t$ in the formula and the coefficients (2.33) is straightforward.

Proposition 2.13. *Let $x = (x_1, \ldots, x_n)$ and λ a partition with at most n parts. Then*

$$G_\lambda(x|\ominus t) = \sum_\alpha \beta^{|\alpha|}\phi_\alpha(\lambda)F_{\lambda+\alpha}(x|\ominus t), \tag{2.34}$$

where the sum runs over all compositions $\alpha = (0, \alpha_2 \ldots, \alpha_n)$ *with* $0 \le \alpha_i \le i-1$ *and*

$$\phi_\alpha(\lambda) = \frac{\prod_{i=2}^n \varphi_{\alpha_i}(\lambda_i)}{\prod_{i=1}^n (1+\beta t_i)^{n-i}}, \qquad \beta^{\alpha_i}\varphi_{\alpha_i}(\lambda_i) = \Gamma_{i-1-\alpha_i}(i-1, n+\lambda_i-1)\ . \tag{2.35}$$

N.B. the determinant formula (2.31) for F_α is well-defined for *compositions* α by which we mean finite sequences of non-negative integers which are not necessarily weakly decreasing. Any such F_α can be expressed in terms of F_λ's indexed by partitions λ using the same straightening rules which hold for Schur functions,

$$F_{(\ldots,a,b,\ldots)} = -F_{(\ldots,b-1,a+1,\ldots)} \qquad \text{and} \qquad F_{(\ldots,a,a+1,\ldots)} = 0\ . \tag{2.36}$$

Both rules should be obvious from (2.31), the first rule follows from exchanging two rows in the determinant in the numerator of (2.31), while the second is simply a result of two rows being linearly dependent.

Proof. Employ the previous lemma and the formula (2.13) to find

$$\begin{aligned}(x_j|t)^{\lambda_i+n-i}(1+\beta x_j)^{i-1} &= \sum_{\alpha_i=0}^{i-1}(x_j|\ominus t)^{n+\lambda_i-1-\alpha_i}\Gamma_{\alpha_i}(i-1,n+\lambda_i-1) \\ &= \sum_{\alpha_i=0}^{i-1}\beta^{\alpha_i}\varphi_{\alpha_i}(\lambda_i)(x_j|\ominus t)^{n+\lambda_i-i+\alpha_i}\ .\end{aligned}$$

The assertion now follows from row-linearity of the determinant and (2.24). □

Example 2.14. *Let $n = 2$. Then the compositions α in the sum in (2.34) are $\alpha = (0,0)$ and $\alpha = (0,1)$. We find from (2.33) and (2.35) that $\Gamma_0(0,\lambda_1+1) = 1$, $\Gamma_1(1,\lambda_2+2) = 1+\beta t_{\lambda_2+1}$ and $\Gamma_0(1,\lambda_2+2) = \beta(1+\beta t_{\lambda_2+1})$. Hence, we arrive at*

$$G_{\lambda_1,\lambda_2}(x|\ominus t) = \frac{1+\beta t_{\lambda_2+1}}{1+\beta t_1}\ (F_{\lambda_1,\lambda_2} + \beta F_{\lambda_1,\lambda_2+1})\ . \tag{2.37}$$

The analogous expansion of G_λ for the non-factorial case corresponds to an expansion into Schur functions instead.

Proposition 2.15. *Set $t_j = 0$ then*

$$G_\lambda(x) = \sum_\alpha \beta^{|\alpha|} \prod_{i=1}^{n-1} \binom{i}{\alpha_i} s_{\lambda+\alpha}(x), \tag{2.38}$$

where the sum runs over all compositions $\alpha = (0, \alpha_1, \ldots, \alpha_{n-1})$ *with* $0 \le \alpha_i \le i$ *and* $s_\alpha(x) = \det(x_j^{n+\alpha_i-i})/\det(x_j^{n-i})$ *is the (generalised) Schur function with* α *being a composition.*

Proof. Use the binomial theorem and row-linearity of the determinant. □

Using the Yang-Baxter algebra we will prove below the following special case of a Cauchy identity.

Proposition 2.16. *Let* $\mu \subset (k^n)$ *be a partition inside the* $n \times k$ *bounding box, then*

$$\prod_{i=1}^{n} \prod_{j \in I_{\mu^*}} x_i \ominus t_j = \sum_\lambda G_\lambda(x|\ominus t) G_{\lambda^\vee}(t_\mu|\ominus t') \frac{\Pi(t_\mu)}{\Pi(t_\lambda)} \tag{2.39}$$

$$= \sum_\lambda G_\lambda(x|\ominus t) G_{\lambda^*}(\ominus t_{\mu^*}|t) \frac{\Pi(t_{\lambda^*})}{\Pi(t_{\mu^*})} \tag{2.40}$$

where the second equality follows from the stronger identity

$$G_{\lambda^\vee}(\ominus t_\mu|t') = G_{\lambda^*}(t_{\mu^*}|\ominus t), \qquad t' = w_0 t\ . \tag{2.41}$$

Proof. See Corollary 4.5 and Prop 4.9. □

3. Yang-Baxter Algebras

This section contains the main algebraic setup for the definition of the hierarchy of generalised equivariant quantum cohomologies qh_n^*. As explained earlier these are realised as commutative subalgebras of a larger non-commutative algebra, the Yang-Baxter algebra, which then naturally acts on the direct sum $\bigoplus_{n=0}^N qh_n^*$.

3.1. Quantum space and spin bases.

One of the main ingredients of our approach is to realise the Schubert basis as vectors in an N-fold tensor product of fundamental $sl(2)$-modules by taking the direct sum $\bigoplus_{n=0}^N qh_n^*$. Let $V = \mathbb{Z}v_0 \oplus \mathbb{Z}v_1$ and $\sigma^- = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\sigma^+ = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ the fundamental representation of sl_2, the Pauli matrices, acting on V via $\sigma^- v_1 = v_0$, $\sigma^+ v_0 = v_1$ and $\sigma^z v_\alpha = (-1)^\alpha v_\alpha$, $\alpha = 0, 1$. We introduce the abbreviations $V(t_j) := \mathcal{R}(t_j) \otimes V$ etc. Here we have dropped the dependence on β in the notation to simplify formulae. We will identify (as vector spaces) $\bigoplus_{0 \le n \le N} qh_n^*$ with the following tensor product

$$\mathcal{V} = \bigotimes_{j=1}^N V(t_j) \cong \mathcal{R}(t_1, \ldots, t_N) \otimes V^{\otimes N}\ . \tag{3.1}$$

On the latter space, which is called the *quantum space* in the area of quantum integrable systems, we will define the action of the Yang-Baxter algebra. Define the following "spin basis" $\{v_{\lambda(b)}\} \subset V^{\otimes N}$ where b runs over all binary strings of length N and

$$v_{\lambda(b)} = v_{b_1} \otimes v_{b_2} \otimes \cdots \otimes v_{b_N}\ . \tag{3.2}$$

We will also need its dual basis which we shall denote by $\{\tilde v_\lambda\} \subset \tilde V^{\otimes N}$, with $\tilde V$ being the dual space of V, and use the familiar bracket notation $\langle \tilde v_\lambda | v_\mu \rangle = \delta_{\lambda\mu}$.

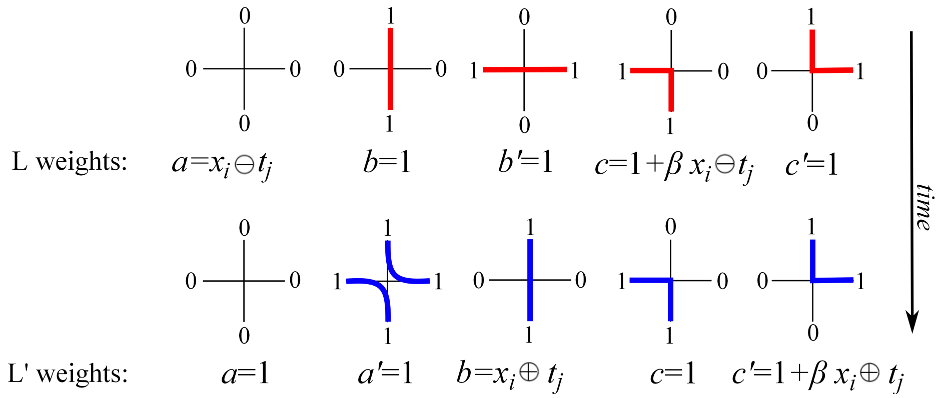

FIGURE 3.1. Graphical depiction of the matrix elements of the L-operators (3.3) and (3.4) as weighted vertex configurations.

There is a natural $U(sl_2)$-action on $V^{\otimes N}$. Fix a Cartan subalgebra $\mathfrak{h}$ then we have the decomposition $V^{\otimes N} = \bigoplus_{0\leq n\leq N} V_n$ into $U(\mathfrak{h})$-weight spaces where $V_n \subset V^{\otimes N}$ denotes the subspace which is spanned by $\{v_{\lambda(b)}\}_{|b|=n}$, i.e. the basis vectors indexed by binary strings with n 1-letters. This induces an analogous decomposition of the quantum space $\mathcal{V}$ into the subspaces $\mathcal{V}_n = \mathcal{R}(\mathbb{T}) \otimes V_n$. Below we shall identify for each subspace $\mathcal{V}_n$ the basis (3.2) with the Schubert basis in qh_n^*. We now define so-called exactly solvable lattice models in End $\mathcal{V}$ following the analogous steps as in [37, Sec 3].

3.2. **Solutions to the Yang-Baxter equation.** Let $x = (x_1, x_2, \ldots)$ be a set of commuting indeterminates which we call *spectral parameters* and as in the case of the equivariant parameters t_j set $V(x_i) := \mathcal{R}(x_i) \otimes V$. Define the following L-operators $V(x_i) \otimes V(t_j) \to V(x_i) \otimes V(t_j)$ by setting

$$L(x_i|t_j) = \begin{pmatrix} \sigma^+\sigma^- + x_i \ominus t_j\ \sigma^-\sigma^+ & (1+\beta x_i \ominus t_j)\sigma^+ \\ \sigma^- & \sigma^-\sigma^+ \end{pmatrix} \tag{3.3}$$

and

$$L'(x_i|t_j) = \begin{pmatrix} \sigma^-\sigma^+ + x_i \oplus t_j\ \sigma^+\sigma^- & \sigma^+ \\ (1+\beta x_i \oplus t_j)\sigma^- & \sigma^+\sigma^- \end{pmatrix} \tag{3.4}$$

where the matrix notation is to be read as the decomposition $L = \sum_{a,b=0,1} e_{ab} \otimes L_{ab}$ with respect to the first factor in $V(x_i) \otimes V(t_j)$ and e_{ab} denote the 2×2 unit matrices.

The matrix elements of these L-operators can be identified with weights for vertex configurations using the same conventions as in [37, Sec 3]. Namely, define $(L)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$ and $(L')_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$ via the expansion $Lv_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon_1',\varepsilon_2'=0,1} (L)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} v_{\varepsilon_1'} \otimes v_{\varepsilon_2'}$ with $\varepsilon_i, \varepsilon_i' = 0, 1$. Then the coefficients can be explicitly computed from (3.3), (3.4). They are the weights of the vertex configurations given in Figure 3.1 where $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ are the values of the W, N, E and S edge of the vertex.

Proposition 3.1. *The L, L'-operators satisfy Yang-Baxter equations of the type*

$$R_{12}(x_i \ominus x_{i'}) L_{13}(x_i|t_j) L_{23}(x_{i'}|t_j) = L_{23}(x_{i'}|t_j) L_{13}(x_i|t_j) R_{12}(x_i \ominus x_{i'}) \tag{3.5}$$

and

$$r_{23}(t_j \ominus t_{j'})L_{12}(x_i|t_j)L_{13}(x_i|t_{j'}) = L_{13}(x_i|t_{j'})L_{12}(x_i|t_j)r_{23}(t_j \ominus t_{j'}) \tag{3.6}$$

where R, r *can be identified with* 4×4 *matrices respectively in* $V(x_i) \otimes V(x_{i'})$ *and* $V(t_j) \otimes V(t_{j'})$ *with respect to the basis* $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ *and are of the form*

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c' & b' & 0 \\ 0 & 0 & 0 & a' \end{pmatrix} . \tag{3.7}$$

The matrix entries are given in the following table for each of the respective cases with r', R' *denoting the respective matrices for* L'*:*

(3.8)

	a	b	c	c'	b'	a'
R	1	0	1	$1 + \beta x_{i'} \ominus x_i$	$x_{i'} \ominus x_i$	1
R'	1	$x_i \ominus x_{i'}$	1	$1 + \beta x_i \ominus x_{i'}$	0	1
$r = r'$	1	0	$1 + \beta t_j \ominus t_{j'}$	1	$t_j \ominus t_{j'}$	1

Proof. A straightforward but rather tedious and lengthy computation which we omit. □

Remark 3.2. *The Lax operators (3.3) and (3.4) are 5-vertex degenerations of the asymmetric 6-vertex model which is used to model ferroelectrics in external electromagnetic fields* [4]*. The solutions (3.3), (3.4) and (3.8) of the Yang-Baxter equation are special cases of this more general model. It is known that solutions of the form (3.7) exist if the Boltzmann weights* (a, a', b, b', c, c') *for each R-matrix in the Yang-Baxter equation yield constant values for the following two ratios,*

$$\Delta = \frac{aa' + bb' - cc'}{2ab}, \qquad \Gamma = \frac{a'b'}{ab} . \tag{3.9}$$

This statement is originally due to Baxter [4] *but can also be found in e.g.* [11]*. For (3.4) we find* $\Delta = -\beta/2$ *and* $\Gamma = 0$ *and the same values apply also to (3.3) after "spin-reversal", i.e. exchanging 0 and 1-letters. The special point* $\beta = 0$ *corresponds to the so-called free fermion point, while* $\beta = -1$ *is the value where connections with the alternating sign matrix conjecture and counting of plane partitions have been made in the literature; see e.g.* [10] *and* [61] *as well as references therein.*

In what follows we concentrate on the solutions of (3.5) but when discussing Goresky-Kottwitz-MacPherson theory towards the end of this article, the solutions of (3.6) will become important.

3.3. **Monodromy matrices.** We will now consider products of the L and L'-operators where the number of factors is linked to the dimension of the ambient space $N = n + k$ of $\mathrm{Gr}_{n,N}$, the dimension n of the hyperplanes and their co-dimension k. Namely, we consider the so-called *auxiliary spaces*

$$W_r = \bigotimes_{i=1}^{r} V(x_i) \cong \mathcal{R}(x_1, \ldots, x_r) \otimes V^{\otimes r}, \qquad r = n, k \tag{3.10}$$

and associate the tensor product $W_n \otimes \mathcal{V}$ with an $n \times N$ square lattice (the vicious walker model) and $W_k \otimes \mathcal{V}$ with an $k \times N$ square lattice (the osculating walker

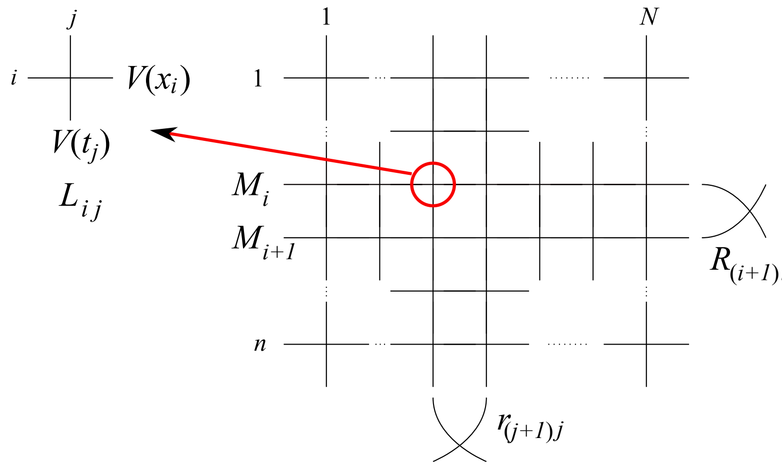

FIGURE 3.2. Graphical depiction of the L-operators and the monodromy matrices. Each operator L_{ij} is represented by a vertex in the ith row and jth column. The square lattice on the right then represents the operator (3.11) over the tensor product $W_n \otimes \mathcal{V}$ obtained by reading out the lattice rows right to left, $M_n \cdots M_2 M_1$. Braiding two lattice rows or two lattice columns leads to the matrices $R_{i+1,i}$ and $r_{j+1,j}$, respectively.

model); compare with [37]. Consider the following operator $\mathcal{Z} : W_n \otimes \mathcal{V} \to W_n \otimes \mathcal{V}$

$$\mathcal{Z} = M_n \cdots M_2 M_1, \qquad M_i = L_{iN} \cdots L_{i2} L_{i1}\ , \tag{3.11}$$

where $M_i = M(x_i|t) : V(x_i) \otimes \mathcal{V} \to V(x_i) \otimes \mathcal{V}$ is called *row-monodromy matrix* and the $L_{ij} = L_{ij}(x_i|t_j)$ operators in (3.11) only act non-trivially in the ith row and jth column of the lattice, i.e. the ith factor in the tensor product W_n and the jth factor in $\mathcal{V}$.

Corollary 3.3. *The row monodromy matrices also obey the Yang-Baxter equation*

$$R_{12}(x_i \ominus x_{i'}) M_{13}(x_i|t) M_{23}(x_{i'}|t) = M_{23}(x_{i'}|t) M_{13}(x_i|t) R_{12}(x_i \ominus x_{i'}) \tag{3.12}$$

where $i, i' = 1, \ldots, n$ and $j, j' = 1, \ldots, N$. The analogous identity holds for M'.

Proof. The Yang-Baxter equations for the monodromy matrices are obtained by repeated applying (3.5) in the definition (3.11). □

The equation in (3.12) can be seen as definition of a subalgebra in $\operatorname{End}\mathcal{V}$. Namely, for any i we can decompose the row monodromy matrix $M = M(x_i|t)$ defined in (3.11) over the auxiliary space $V(x_i)$ as follows,

$$M = \sum_{a,b=0,1} e_{ab} \otimes M_{ab}, \qquad (M_{ab}) = \begin{pmatrix} A(x_i|t) & B(x_i|t) \\ C(x_i|t) & D(x_i|t) \end{pmatrix} \tag{3.13}$$

where e_{ab} are the 2×2 unit matrices and the matrix entries A, B, C, D are elements in $\operatorname{End}\mathcal{V}$. The latter generate the so-called *row Yang-Baxter algebra* $\subset \operatorname{End}\mathcal{V}$ with the commutation relations of A, B, C, D given in terms of the matrix elements (3.8) of R via (3.12). The row Yang-Baxter algebra A', B', C', D' for the monodromy

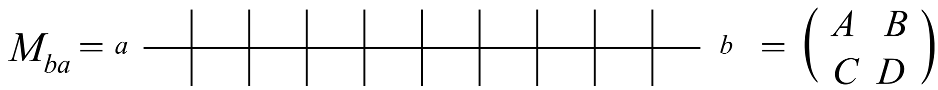

FIGURE 3.3. Graphical depiction of the monodromy matrix. Its matrix elements correspond to weighted sums over the vertex configurations defined by the L-operator. To obtain the A, B, C, D-operators set respectively $(a, b) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

matrix M' associated with L' is defined analogously. Similar to the L, L'-operators A, B, C, D also have a graphical representation: they can be depicted as a single lattice row with N vertices where the values of the outer horizontal edges are fixed; see Figure 3.3. Matrix elements of the Yang-Baxter algebra generators then correspond to weighted sums over all possible vertex configurations of Figure 3.1 in such a lattice row subject to the respective boundary condition.

3.4. **Commutation relations of the Yang-Baxter algebras.** We state a number of important commutation relations of the Yang-Baxter algebra elements which we will employ in subsequent sections. The first will be used when deriving the spectrum of the Bethe algebra and the Bethe ansatz equations.

Lemma 3.4. *Let $y = (y_1, \ldots, y_n)$ be an n-tuple of pairwise distinct variables and $x \neq y_i$ for all $i = 1, \ldots, n$. Then we have the identities*

$$A(x)B(y_1)\cdots B(y_n) = \frac{B(y_1)\cdots B(y_n)A(x)}{(x \ominus y_1)\cdots(x \ominus y_n)} - \sum_{i=1}^{n} \frac{B(y_1)\cdots \overset{i}{B(x)} \cdots B(y_n)A(y_i)}{(x \ominus y_i)\prod_{j\neq i} y_i \ominus y_j} \tag{3.14}$$

and

$$D(x)B(y_1)\cdots B(y_n) = \frac{B(y_1)\cdots B(y_n)D(x)}{(y_1 \ominus x)\cdots(y_n \ominus x)} + \sum_{i=1}^{n} \frac{B(y_1)\cdots \overset{i}{B(x)} \cdots B(y_n)D(y_i)}{(x \ominus y_i)\prod_{j\neq i} y_j \ominus y_i} \tag{3.15}$$

Analogous identities hold for the Yang-Baxter algebra elements A', B', D'.

Proof. Induction in n. The case $n = 1$ follows from (3.5), namely one deduces via (3.13) the commutation relations

$$\begin{aligned} (x \ominus y)A(x)B(y) &= B(y)A(x) - B(x)A(y) \\ (y \ominus x)D(x)B(y) &= B(y)D(x) - (1 + \beta\, y \ominus x)\ B(x)D(y)\ . \end{aligned}$$

For the induction step use is made of the fact that $B(x)B(y) = B(y)B(x)$, which again is a consequence of (3.5) and implies that the result is symmetric in the y_i's. □

The next lemma will be used to compute the bilinear form of our generalised cohomology ring.

Lemma 3.5. *Let $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ be some mutually pairwise distinct sets of variables. Then*

$$C(x_1)\cdots C(x_n)B(y_n)\cdots B(y_1)= \frac{1}{\Pi(x)}\sum_w w\left(\frac{\Pi(x)D(y_1)\cdots D(y_n)A(x_1)\cdots A(x_n)}{\prod_{1\le i,j\le n}x_i\ominus y_j}\right) \tag{3.16}$$

where the sum runs over the minimal length coset representatives w of $\mathbb{S}_{2n}/\mathbb{S}_n\times\mathbb{S}_n$ which act in the obvious manner on the alphabet $\{x_1,\ldots,x_n,y_1,\ldots,y_n\}$.

Proof. By induction in n. The case $n=1$ follows from the commutation relation

$$C(x)B(y)=\frac{D(y)A(x)-D(x)A(y)}{x\ominus y}=\frac{A(x)D(y)-A(y)D(x)}{x\ominus y}$$

which is a direct consequence of (3.5). For the induction step one uses the commutation relations

$$O(x)O(y)=O(y)O(x),\qquad O=A,B,C,D$$

and

$$\begin{aligned} C(x)D(y) &= \frac{D(y)C(x)-D(x)C(y)}{x\ominus y}\\ A(x)B(y) &= \frac{B(y)A(x)-B(x)A(y)}{x\ominus y}\end{aligned}$$

all of which follow once more from (3.5). Note that these commutation relations again imply that the result must be symmetric in the x_i's and symmetric in the y_i's. This greatly simplifies the computation. □

Analogous commutation relations hold for the monodromy matrix M' and the generators A',B',C',D'. These can be derived easily from the following result which relates both Yang-Baxter algebras in a simple manner.

Lemma 3.6. *Let $\Theta:\mathcal{V}\to\mathcal{V}$ be the linear extension of the involution $v_\lambda\mapsto v_{\lambda'}$. Then we have the identity*

$$\Theta M_{ab}(x_i|t)=M'_{ba}(x_i|\ominus t')\Theta \tag{3.17}$$

where $\ominus t'=(\ominus t_N,\ldots,\ominus t_2,\ominus t_1)$.

Proof. Recall the definition of the matrix elements of the L-operator (3.3) via the expansion $Lv_{\varepsilon_1}\otimes v_{\varepsilon_2}=\sum_{\varepsilon_1',\varepsilon_2'=0,1}(L)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}v_{\varepsilon_1'}\otimes v_{\varepsilon_2'}$ and similarly define $(L'_{ij})_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$. The matrix elements are the weights of the vertex configurations given in Figure 3.1 where $\varepsilon_1,\varepsilon_2,\varepsilon_1',\varepsilon_2'$ are the values of the W, N, E and S edge of the vertex as explained earlier. Interchanging 0 with 1-letters attached to the vertical lines going through the vertex configurations displayed in Figure 3.1 we find

$$L(x_i|t_j)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}=L'(x_i|\ominus t_j)_{\varepsilon_1'(1-\varepsilon_2)}^{\varepsilon_1(1-\varepsilon_2')} \tag{3.18}$$

for all $\varepsilon_i,\varepsilon_i'=0,1$ with $i=1,2$.

The assertion for the row monodromy matrix is now an immediate consequence of the definition (3.11) and the identity (3.18). □

3.5. **Transposed Yang-Baxter algebras.** We will also need to consider the action of the Yang-Baxter algebra in the dual quantum space. The transposed monodromy matrices can be explicitly computed.

Define another pair of L-operators

$$L^\vee(x_i|t_j) = \begin{pmatrix} \sigma^+\sigma^- + x_i \ominus t_j\ \sigma^-\sigma^+ & \sigma^+ \\ (1+\beta x_i \ominus t_j)\sigma^- & \sigma^-\sigma^+ \end{pmatrix} \tag{3.19}$$

and

$$L^*(x_i|t_j) = \begin{pmatrix} \sigma^-\sigma^+ + x_i \oplus t_j\ \sigma^+\sigma^- & (1+\beta x_i \oplus t_j)\sigma^+ \\ \sigma^- & \sigma^+\sigma^- \end{pmatrix}. \tag{3.20}$$

Employing the latter we define *dual row monodromy matrices* $M_i^\vee = M^\vee(x_i)$as follows

$$M_i^\vee = L_{i1}^\vee L_{i2}^\vee \cdots L_{iN}^\vee \tag{3.21}$$

and, similarly, M_i^* where we use the L^*-operators instead. The dual monodromy matrices also obey the Yang-Baxter equation, where the R-matrix elements are given by similar expressions as in (3.8). As we shall not need their explicit form we omit them here.

Lemma 3.7. *Recall the definitions of the row monodromy matrices $M_i, M_i', M_i^\vee, M_i^*$ as maps $V(x_i)\otimes\mathcal{V}\to V(x_i)\otimes\mathcal{V}$ for some i. Then*

$$M_i^{1\otimes t} = (M_i^\vee)^{t\otimes 1} \qquad \text{and} \qquad (M_i')^{1\otimes t} = (M_i^*)^{t\otimes 1}, \tag{3.22}$$

where the upper indices $1\otimes t$ and $t\otimes 1$ indicate the transpose in respectively the quantum space $\mathcal{V}$ and the auxiliary space $V(x_i)$ with respect to the spin basis $\{v_\lambda\}$.

Proof. Recall the definition of $L_{ij} : V(x_i)\otimes V(t_j)\to V(x_i)\otimes V(t_j)$ and take the transpose in the second factor to find that

$$L_{ij}^{1\otimes t} = \begin{pmatrix} \sigma_j^-\sigma_j^+ + x_i\oplus t_j\ \sigma_j^+\sigma_j^- & (1+\beta x_i\oplus t_j)\sigma_j^- \\ \sigma_j^+ & \sigma_j^+\sigma_j^- \end{pmatrix}_i = (L_{ij}^\vee)^{t\otimes 1}.$$

Thus, we can deduce for the monodromy matrix $M_i : V(x_i)\otimes\mathcal{V}\to V(x_i)\otimes\mathcal{V}$

$$\begin{aligned} M_i^{1\otimes t} &= L_{iN}^{1\otimes t}\cdots L_{i2}^{1\otimes t}L_{i1}^{1\otimes t} \\ &= (L_{iN}^\vee)^{t\otimes 1}\cdots (L_{i2}^\vee)^{t\otimes 1}(L_{i1}^\vee)^{t\otimes 1} = (M_i^\vee)^{t\otimes 1} \end{aligned}$$

The proof of the other identity is completely analogous. □

3.6. **Quantum deformation.** We discuss a slight generalisation of the previous results which will allow us to introduce additional (invertible) "quantum parameters" $q_1,\ldots,q_N$ in the monodromy matrices. Consider the extension $\mathbb{Z}[\![q_1,\ldots,q_n]\!]\otimes\mathcal{V}$ as quantum space and $\mathbb{Z}[\![q_1^{-1},\ldots,q_n^{-1}]\!]\otimes W_r$ as auxiliary space.

Lemma 3.8. *We have the q-deformed Yang-Baxter equation*

$$r_{23}(q)L_{12}(x_i;t_j)\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)_i L_{13}(x_i;t_{j'}) = L_{13}(x_i;t_{j'})\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)_i L_{12}(x_i;t_j)r_{23}(q), \tag{3.23}$$

where

$$r(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\beta t_j}{1+\beta t_{j'}} & 0 \\ 0 & 1 & q^{-1}\frac{t_j - t_{j'}}{1+\beta t_{j'}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

Using this result we can generalise our previous formulae for the monodromy matrices by setting

$$M_i(q_1,\ldots,q_N) := L_{iN}\left(\begin{smallmatrix}1&0\\0&q_N\end{smallmatrix}\right)_i\cdots L_{i2}\left(\begin{smallmatrix}1&0\\0&q_2\end{smallmatrix}\right)_i L_{i1}\left(\begin{smallmatrix}1&0\\0&q_1\end{smallmatrix}\right)_i \tag{3.25}$$

and

$$m_j(q_j) := L_{nj}\left(\begin{smallmatrix}1&0\\0&q_j\end{smallmatrix}\right)_n\cdots L_{2j}\left(\begin{smallmatrix}1&0\\0&q_j\end{smallmatrix}\right)_2 L_{1j}\left(\begin{smallmatrix}1&0\\0&q_j\end{smallmatrix}\right)_1 . \tag{3.26}$$

Employing the same type of arguments as in our previous discussion, one shows that these deformed monodromy matrices satisfy the same type of Yang-Baxter relations (3.23) as the non-deformed ones, the only difference lies in the braid matrix r which is now replaced by $r(q)$. For discussing the q-deformation of the cohomology and K-theory of the Grassmannian we need to choose $q_1 = q$ and $q_2 = \cdots = q_N = 1$. We shall henceforth denote $\mathbb{Z}[\![q]\!]\otimes\mathcal{V}$ by $\mathcal{V}^q$ and, similarly, $\mathbb{Z}[\![q]\!]\otimes\mathcal{V}_n$ by $\mathcal{V}_n^q$.

3.7. **Row-to-row transfer matrices.** We now introduce periodic boundary conditions in the horizontal direction of the lattice by taking the partial trace of the operator (3.11) over the auxiliary space $V^{\otimes n}$. We obtain the following operator $Z_n : \mathcal{R}[x_1,\ldots,x_n]\otimes\mathcal{V}^q \to \mathcal{R}[x_1,\ldots,x_n]\otimes\mathcal{V}^q$,

$$Z_n(x|t) = \operatorname*{Tr}_{V^{\otimes n}}\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)_n M_n\cdots\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)_2 M_2\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)_1 M_1 . \tag{3.27}$$

The matrix elements of the latter are partition functions of our lattice models on a cylinder. We also define an operator Z'_k using instead the L'-operators and replacing $n \to k$ everywhere.

Lemma 3.9. *Denote by $H = Z_1 = A + qD$ and $E = Z'_1 = A' + qD'$. We have the relations*

$$Z_n(x|t) = H(x_n|t)\cdots H(x_2|t)H(x_1|t) \tag{3.28}$$

and

$$Z'_k(x|t) = E(x_k|t)\cdots E(x_2|t)E(x_1|t) . \tag{3.29}$$

The operators H, E are called the row-to-row transfer matrices*.*

Proof. This is immediate from the definitions (3.11), (3.27) and the fact that the L-operators $L_{ij}, L_{i'j'}$ commute if $i \neq i'$ and $j \neq j'$. □

Corollary 3.10. *We have the following identity for the row-to-row transfer matrices, $\Theta H(x|t)\Theta = E(x|\ominus t')$.*

Proof. Employ (3.17) and the defining relations $H = A + qD$, $E = A' + qD'$. □

The following statement shows that the transfer matrix generate a commutative subalgebra - the so-called Bethe algebra - within the Yang-Baxter algebra which we will identify with our generalised cohomology ring. Because of the existence of this commutative subalgebra, which should be thought of as the analogue of integrals of motion of a classical integrable system described in terms of differential equations, the models are called (quantum) *integrable*.

Proposition 3.11 (Integrability)**.** All *the row-to-row transfer matrices commute, that is we have that*

$$H(x_i)H(x_{i'}) = H(x_{i'})H(x_i), \qquad E(x_i)E(x_{i'}) = E(x_{i'})E(x_i) \tag{3.30}$$

as well as

$$H(x_i)E(x_{i'}) = E(x_{i'})H(x_i)\ . \tag{3.31}$$

In particular ,the operators Z_n, Z'_k are symmetric in the x-variables.

Proof. The last assertion is a direct consequence of the Yang-Baxter equation (3.12):

$$\begin{aligned}
Z_n(x_1,\dots,x_n|t) &= \operatorname*{Tr}_{V^{\otimes n}}(R_{i,i+1}M_n\cdots M_1R^{-1}_{i,i+1})\\
&= \operatorname*{Tr}_{V^{\otimes n}}(M_n\cdots R_{i,i+1}M_iM_{i+1}\cdots M_1R^{-1}_{i,i+1})\\
&= \operatorname*{Tr}_{V^{\otimes n}}(M_n\cdots M_{i+1}M_iR_{i,i+1}\cdots M_1R^{-1}_{i,i+1})\\
&= \operatorname*{Tr}_{V^{\otimes n}}(M_n\cdots M_{i+1}M_i\cdots M_1)\\
&= Z_n(x_1,\dots,x_{i+1},x_i,\dots,x_n|t)
\end{aligned}$$

The proof for Z'_k follows along the same lines. Setting $n=k=2$ we obtain (3.30).

To prove (3.31) one establishes the existence of additional solutions of the Yang-Baxter equation,

$$R''_{12}(x_i\oplus x_{i'})M_{13}(x_i|t)M'_{23}(x_{i'}|t) = M'_{23}(x_{i'}|t)M_{13}(x_i|t)R''_{12}(x_i\oplus x_{i'}) \tag{3.32}$$

where R'' is again of the form (3.7) with

(3.33)

	a	b	c	c'	b'	a'
R''	$x_i\oplus x_{i'}$	1	$1+\beta\ x_i\oplus x_{i'}$	1	1	0

Note that R'' is singular. However, from the Yang-Baxter equations (3.32) one derives the commutation relations

$$\begin{aligned}
A(x|t)A'(y|t) &= A'(y|t)A(x|t)\\
A(x|t)D'(y|t)-A'(y|t)D(x|t) &= D'(y|t)A(x|t)-D(x|t)A'(y|t)
\end{aligned}$$

for the row Yang-Baxter algebras. Employing the graphical calculus in terms of the vertex configurations in Figure 3.1 one obtains the additional relations

$$D(x|t)D'(y|t) = D'(y|t)D(x|t) = 0\ .$$

From these equalities we then easily deduce that $H(x|t)E(y|t) = E(y|t)H(x|t)$. $\square$

3.8. **Combinatorial description of the transfer matrices.** We now describe the action of the transfer matrices, that is the action of the commutative subalgebra, in the *spin basis* $\{v_\lambda\}\subset\mathcal{V}_n$ for $n\le N/2$ using toric horizontal and vertical strips; see the earlier section on preliminaries. For $n>N/2$ the action can be deduced employing Cor 3.10.

We interpret partitions and their associated cylindric loops as subsets of $\mathbb{Z}^2$. Given a toric horizontal strip $\theta=\lambda/d/\mu$ of degree d denote by

- $\mathcal{R}_\theta$ the set which contains all squares $s=\langle i,j\rangle\in\mathbb{Z}^2$, $1\le i\le n$ such that the square immediately left to it, $s'=\langle i,j-1\rangle$, is the rightmost square in a row of $\lambda[d]$ intersecting θ;
- $\bar{\mathcal{C}}_\theta$ the set which contains all the bottom squares $s=\langle i,j\rangle$, $1\le j\le k$ from each column of $\mu[0]$ which does not intersect θ as well as the squares $s=\langle 1,j\rangle$ in empty columns if $\lambda_1+n<j\le N$ and $\mu\subset\lambda$.

Likewise, given a toric vertical strip $\theta=\lambda/d/\mu$ denote by

- $\bar{\mathcal{R}}_\theta$ the set which contains the square $s = \langle i, j\rangle$ next to the rightmost square $s' = \langle i, j-1\rangle$ in each row of μ not intersecting θ. This includes squares $s = \langle i, 1\rangle$ in empty rows for which $1 \le i < n$;
- $\mathcal{C}_\theta$ the set which contains the bottom squares from each column of $\lambda[d]$ which intersects θ.

Proposition 3.12. *We have the following combinatorial action of the transfer matrices on $\mathcal{V}_n^q$ in the basis $\{v_\lambda\}$,*

$$H(x|t)v_\mu = \sum_{\substack{\theta=\lambda/d/\mu \\ hor\ strip}} q^d \left(\prod_{s\in\bar{\mathcal{C}}_\theta} x \ominus t_{n+c(s)}\right)\left(\prod_{s\in\mathcal{R}_\theta}(1+\beta x\ominus t_{(n+c(s))\bmod N}\right) v_\lambda$$

$$E(x|t)v_\mu = \sum_{\substack{\theta=\lambda/d/\mu \\ ver\ strip}} q^d \left(\prod_{s\in\bar{\mathcal{R}}_\theta} x \oplus t_{n+c(s)}\right)\left(\prod_{s\in\mathcal{C}_\theta}(1+\beta x\oplus t_{n+c(s)}\right) v_\lambda$$

where the degree d of the toric strips is either zero or one and $c(s) = j - i$ is the content of the square $s = \langle i, j\rangle$ in the Young diagram of λ or μ.

Proof. The proof of these formulae follows along similar lines as in [37] and we therefore only sketch the main argument. Consider a fixed matrix element $\langle\lambda|H(x|t)|\mu\rangle := \langle\tilde{v}_\lambda|H(x|t)v_\mu\rangle$ which is simply the partition function for a single lattice row where the values of the upper and lower vertical edges have been fixed in terms of the binary strings $b(\mu)$ and $b(\lambda)$, respectively. Using the bijection between binary strings b and boxed partitions $\lambda(b)$ from Section 2.3 one can translate the various vertex configurations in Figure 3.1, which represent matrix elements of the L and L'-operators, into the operation of adding boxes to the Young diagram of μ. For example, the first and second vertex configuration in the top row of Figure 3.1 leave the Young diagram of μ unchanged, the fourth and fifth vertex configurations signal respectively the end and start of a horizontal strip being added to μ, while the third vertex in the top row corresponds to two boxes being added in the same row. Similarly, the first two vertex configurations in the bottom row of 3.1 do not add any boxes to μ, the fourth and fifth signal the start and end of a vertical strip, while the third vertex in the bottom row indicates that two boxes are added in the same column. Using these equivalences between horizontal (vertical) strips the above formulae follow from the weights fixed via the definitions (3.3) and (3.4). □

Example 3.13. *Consider the simplest non-trivial case $\mathrm{Gr}_{1,3} = \mathbb{P}^2$, i.e. we set $N = 3$ and $n = 1$. In terms of binary strings $\mathcal{V}_1$ is spanned by $\{v_{100}, v_{010}, v_{001}\}$. We consider the matrix elements of $H(x)$ in this basis, which can be visualised as a sum over all the possible vertex configurations shown in Figure 3.1 occurring in a single lattice row of length $N = 3$. Drawing all these allowed lattice configurations with fixed binary strings* 010 *and* 001 *on the top edges, we arrive at Figures 3.4 and 3.5 with the product of the respective vertex weights shown below. We now convert the binary strings into partitions with bounding box 1×2 to obtain toric horizontal strips; see Section 2.4.*

Starting from the left in Figure 3.4 the first lattice configuration is the matrix element $\langle\tilde{v}_{010}|H(x)v_{010}\rangle$. The binary string 010 *is the partition with one square at position $\langle 1, 1\rangle$ and we have $\lambda = \mu = (1)$, that is an empty horizontal strip where no*

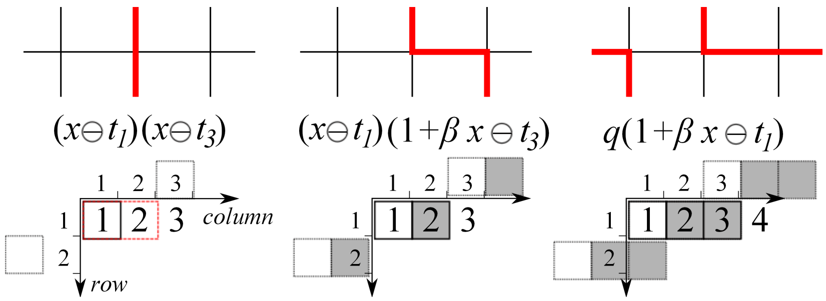

FIGURE 3.4. Lattice configurations for the projective space $\mathbb{P}^2$ and their weights; see Figure 3.1. The values of the top edges of the vertices are fixed by the binary string 010. Below the weights are the corresponding toric skew diagrams; see Proposition 3.12 and Example 3.13. The dotted boxes are the cylindric continuation (2.4) of the solid Young diagrams; see Section 2.4

box is added and $d = 0$*. Thus,* $\mathcal{R}_{\lambda/\mu} = \emptyset$ *and* $\bar{\mathcal{C}}_{\lambda/\mu} = \{\langle 1,1\rangle, \langle 1,3\rangle\}$ *where the last square in* $\bar{\mathcal{C}}_{\lambda/\mu}$ *belongs to an empty column with the column number* j *obeying the stated condition* $1 < j = 3 \leq N$*. According to Prop 3.12 we arrive at the weight*

$$\langle \tilde{v}_{010}|H(x)v_{010}\rangle = (x\ominus t_1)(x\ominus t_3)\,.$$

The next lattice configuration is the matrix element $\langle \tilde{v}_{001}|H(x)v_{010}\rangle$ *with* $\lambda = (2), \mu = (1)$*. Thus, we have the horizontal strip* $\theta = \lambda/\mu$ *with one square at* $\langle 1,2\rangle$ *and* $d = 0$*. The sets appearing in the formula of Prop 3.12 are* $\mathcal{R}_{\lambda/\mu} = \{\langle 1,3\rangle\}$ *and* $\bar{\mathcal{C}}_{\lambda/\mu} = \{\langle 1,1\rangle\}$*, since the square* $\langle 1,3\rangle$ *is adjacent to the square* $\langle 1,2\rangle$ *which appears in a row intersecting* λ/μ *while the square* $\langle 1,1\rangle$ *is the bottom square in a column not intersecting* λ/μ*. Hence,*

$$\langle \tilde{v}_{001}|H(x)v_{010}\rangle = (x\ominus t_1)(1+\beta x\ominus t_3)\,.$$

The last lattice configuration in the top row is the matrix element $\langle \tilde{v}_{100}|H(x)v_{010}\rangle$ *with* $\lambda = (0), \mu = (1)$*. Now, we have a toric strip with* $d = 1$*, that is* $\lambda/1/\mu = \{\langle 1,2\rangle, \langle 1,3\rangle\}$*. The first column with the square at* $\langle 1,1\rangle$ *now intersects* $\lambda/1/\mu$*, because the square at* $\langle 2,1\rangle$ *is in the cylindric loop* $\lambda[1]$*. Therefore,* $\mathcal{R}_{\lambda/1/\mu} = \{\langle 1,4\rangle\}$*,* $\bar{\mathcal{C}}_{\lambda/1/\mu} = \emptyset$ *and*

$$\langle \tilde{v}_{100}|H(x)v_{010}\rangle = q(1+\beta x\ominus t_1)\,.$$

In summary, we have the action (compare with Prop 3.12),

$$H(x)v_{010} = (x\ominus t_1)(x\ominus t_3)v_{010} + (x\ominus t_1)(1+\beta x\ominus t_3)v_{001} + q(1+\beta x\ominus t_1)v_{100}\,.$$

We leave the verification of the weights in Figure 3.5 to the reader.

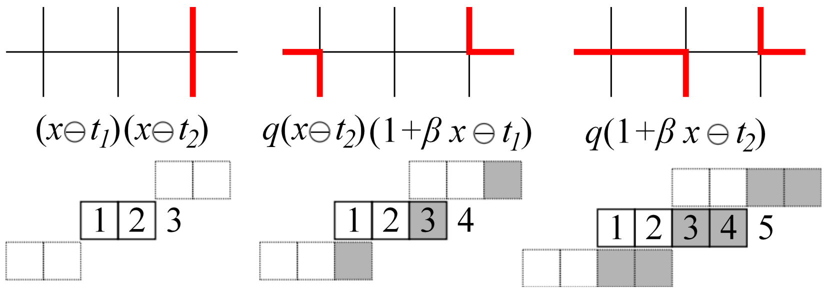

FIGURE 3.5. Lattice configurations for $\mathbb{P}^2$ and binary string 001, their weights and corresponding toric strips; see Figure 3.1.

Let $t^* = (\ominus t_N, \ldots, \ominus t_2, \ominus t_1)$ and on each $\mathcal{V}_n^q$ define operators $\{H_r\}_{r=1}^k$ and $\{E_r\}_{r=1}^n$ through the expansions

$$(3.34)\quad H(x|t)|_{\mathcal{V}_n^q} = (x|t^*)^k \cdot \mathbf{1}_{\mathcal{V}_n^q} + (1+\beta x)\sum_{r=1}^k H_r\, \frac{(x|t^*)^{k-r}}{1+\beta t_{n+r}},$$

$$(3.35)\quad E(x|t)|_{\mathcal{V}_n^q} = (x|t)^n \cdot \mathbf{1}_{\mathcal{V}_n^q} + (1+\beta x)\sum_{r=1}^n E_r\, (1+\beta t_{n+1-r})(x|t)^{n-r}\ ,$$

where $(x|t)^r = \prod_{j=1}^r (x\oplus t_j)$ are the factorial powers (2.12) with respect to the group law (1.1). Below we will relate the operator coefficients in these expansions to the Pieri rules in qh_n^*. Setting $\beta = 0$ they correspond to the generators in Mihalcea's coordinate ring representation of equivariant quantum cohomology [48, Thm 1.1].

Corollary 3.14. *The operators $\{E_r\}_{r=1}^n \cup \{H_r\}_{r=1}^k$ generate a commutative subalgebra $\subset \operatorname{End}\mathcal{V}_n^q$ and we have the formulae ($t'_j = t_{N+1-j}$)*

$$(3.36)\qquad H_{k+1-i} = \sum_{j=1}^i \frac{H(t'_j)}{\prod_{1\le \ell\ne j\le i} t'_j \ominus t'_\ell}, \qquad i = 1,\ldots,k$$

$$(3.37)\qquad E_{n+1-i} = \sum_{j=1}^i \frac{E(\ominus t_j)}{\prod_{1\le \ell\ne j\le i} t_\ell \ominus t_j}, \qquad i = 1,\ldots,n\ .$$

In particular, for $i=1$ we have $H_k = H(t_N)$ and $E_n = E(t_1)$.

Proof. Setting $x = t_i$ in (3.34) and $x = \ominus t_i$ in (3.35) we obtain a linear system of equations expressing $H(t_i)$ and $E(\ominus t_i)$ in terms of the (operator) coefficients H_r and E_r respectively. The corresponding matrices are lower triangular and therefore can be easily inverted to produce the stated expressions.

It follows from Prop 3.11 that all these operators commute. □

Together with Prop 3.12 the last result allows one to compute the action of H_r and E_r in the spin-basis $\{v_\lambda\} \subset \mathcal{V}_n$.

Example 3.15. *We continue Example 3.13 with $\mathrm{Gr}_{1,3} = \mathbb{P}^2$. It follows from (3.36) that*

$$H_1 = \frac{H(t_2)}{t_2\ominus t_3} + \frac{H(t_3)}{t_3 \ominus t_2}, \qquad H_2 = H(t_3)\ .$$

Employing the weights shown in Figure 3.4,

$$\begin{aligned}\langle\tilde v_{010}|H(x)v_{010}\rangle &= (x\ominus t_1)(x\ominus t_3),\\ \langle\tilde v_{001}|H(x)v_{010}\rangle &= (x\ominus t_1)(1+\beta x\ominus t_3)\\ \langle\tilde v_{100}|H(x)v_{010}\rangle &= q(1+\beta x\ominus t_1)\end{aligned}$$

we arrive at the matrix elements

$$\begin{aligned}\langle\tilde v_{010}|H_1v_{010}\rangle &= \frac{(t_2\ominus t_1)(t_2\ominus t_3)}{t_2\ominus t_3}+0=t_2\ominus t_1\\ \langle\tilde v_{001}|H_1v_{010}\rangle &= \frac{(t_2\ominus t_1)(1+\beta t_2\ominus t_3)}{t_2\ominus t_3}+\frac{t_3\ominus t_1}{t_3\ominus t_2}=1+\beta\ t_2\ominus t_1\\ \langle\tilde v_{100}|H_1v_{010}\rangle &= q\frac{(1+\beta t_2\ominus t_1)}{t_2\ominus t_3}+q\frac{(1+\beta t_3\ominus t_1)}{t_3\ominus t_2}=0\end{aligned}$$

From these we obtain,

$$H_1v_1=t_2\ominus t_1v_1+(1+\beta t_2\ominus t_1)v_2\,. \tag{3.38}$$

In an analogous fashion one finds,

$$H_2v_1=t_3\ominus t_1v_2+q(1+\beta t_1\ominus t_1)v_\emptyset$$

and using the weights in Figure 3.5

$$H_1v_2=t_3\ominus t_1v_2+q(1+\beta t_3\ominus t_1)v_\emptyset \tag{3.39}$$

$$H_2v_2=(t_3\ominus t_2)(t_3\ominus t_1)v_2+q(t_3\ominus t_2)(1+\beta t_3\ominus t_1)v_\emptyset+q(1+\beta t_3\ominus t_2)v_1$$

Below we will define a product by $v_r\circledast v_s=H_rv_s$. Upon setting $\beta=-1$ and $t_{4-i}=1-e^{\varepsilon_i}$ with $i=1,2,3$ the above formulae then match the product expansions for quantum equivariant K-theory of $\mathbb{P}^2$ stated by Buch and Mihalcea in [13, Sec 5.5].

3.8.1. *Functional relation & quantum Pieri-Chevalley rule.* The coefficients (3.36) and (3.37) of the transfer matrices are algebraically dependent. We now derive the functional relation (1.2) which allows one to deduce this dependence and as a byproduct of our computation we give an explicit formula for the action of H_1 in the spin basis.

Let $u_j=\sigma_j^-\sigma_{j+1}^+$ for $j=1,\dots,N-1$ and $u_N=q\sigma_1^+\sigma_N^-$. Define the following operator on $\mathcal{V}^q$,

$$\bar H_1=\sum_{j=1}^N u_j+\beta\sum_{|j_1-j_2|\bmod N>1}u_{j_1}u_{j_2}+\beta^2\sum_{|j_a-j_b|\bmod N>1}u_{j_1}u_{j_2}u_{j_3}+\cdots \tag{3.40}$$

as a formal power series in β. Note that the sums only run over indices where $|j_a-j_b|\bmod N>1$ which ensures that all the u_j's in each monomial commute. Obviously, only finitely many terms act non-trivially for finite N and the series therefore terminates.

Lemma 3.16. *Acting with $\bar H_1$ on a spin basis vector $v_\mu\in\mathcal{V}_n$ one obtains*

$$\bar H_1v_\mu=\sum_{\substack{\mu\rightrightarrows^*\lambda[d]\\ d=0,1}}q^d\beta^{|\lambda/d/\mu|-1}v_\lambda, \tag{3.41}$$

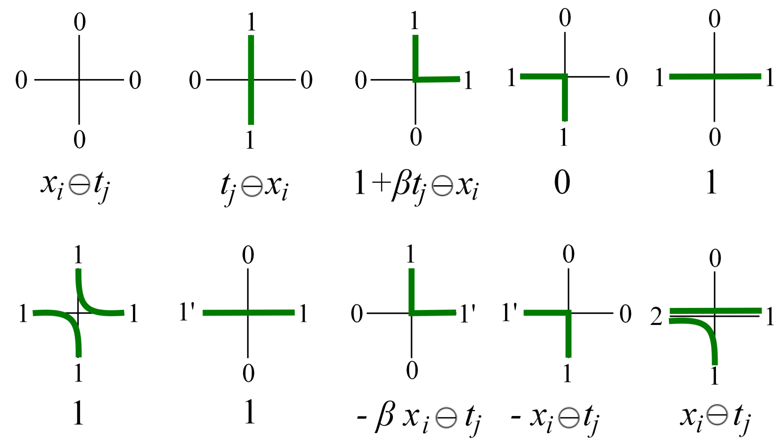

FIGURE 3.6. The vertex configurations corresponding to the operator $L'_{i+1j}(\ominus x_i)L_{i,j}(x_i)$.

where the sum runs over all boxed partitions $\lambda \subset (k^n)$ *such that either* $\lambda/0/\mu = \lambda/\mu$ *or* $\lambda/1/\mu$ *are toric diagrams which contain at most one box in each column and row and* $\lambda \neq \mu$.

Proof. Using the bijection between binary strings and partitions detailed in Section 2.3 and the definition of cylindric loops in Section 2.4, one proves that either $u_j v_\mu = q^d v_\lambda$ where one adds a box with coordinates (x, y) and $j = n + y - x$ to obtain λ (or $\lambda[1]$ if $d = 1$ and $j = N$) or $u_j v_\mu = 0$. The assertion then easily follows from the fact that all u_j's in each monomial term commute. $\square$

Proposition 3.17. *The transfer matrices obey the following functional operator identity*

$$H(x|t)E(\ominus x|t) = (1 + \beta \bar{H}_1) \prod_{j=1}^{N} (t_j \ominus x)^{\sigma_j^+ \sigma_j^-} (x \ominus t_j)^{\sigma_j^- \sigma_j^+} + q \cdot 1 \ . \tag{3.42}$$

In particular, we have that $H(t_j|t)E(\ominus t_j|t) = q \cdot 1$ *for all* $j = 1, \ldots, N$ *which amount to non-trivial identities between the coefficients* $\{H_r\}$ *and* $\{E_r\}$ *defined in (3.34), (3.35).*

Proof. A computation along similar lines as in [37]. The idea is to analyse the action of $\mathcal{L}_j = L'_{1j}(\ominus x)L_{2j}(x) : W(x) \otimes V(t_j) \to W(x) \otimes V(t_j)$ where $W(x) = V(\ominus x) \otimes V(x) = \mathcal{R}(x) \otimes V^{\otimes 2}$ with respect to the basis vectors

$$\begin{aligned} w_0 &= v_0 \otimes v_0, \quad w_1 = v_0 \otimes v_1 + v_1 \otimes v_0, \\ w_{1'} &= v_0 \otimes v_1, \quad w_2 = v_1 \otimes v_1 \ . \end{aligned}$$

We find that

$$
\begin{aligned}
\mathcal{L}_j w_0 \otimes v_0 &= x \ominus t_j \; w_0 \otimes v_0 \\
\mathcal{L}_j w_0 \otimes v_1 &= t_j \ominus x \; w_0 \otimes v_1 + (1 + \beta t_j \ominus x) w_1 \otimes v_0 - \beta t_j \ominus x \; w_{1'} \otimes v_0 \\
\mathcal{L}_j w_1 \otimes v_0 &= w_1 \otimes v_0 \\
\mathcal{L}_j w_1 \otimes v_1 &= w_1 \otimes v_1 \\
\mathcal{L}_j w_{1'} \otimes v_0 &= w_1 \otimes v_0 - x \ominus t_j \; w_0 \otimes v_1 \\
\mathcal{L}_j w_{1'} \otimes v_1 &= 0 \\
\mathcal{L}_j w_2 \otimes v_0 &= (1 + \beta x \ominus t_j) w_1 \otimes v_1 - \beta x \ominus t_j \; w_{1'} \otimes v_1 \\
\mathcal{L}_j w_2 \otimes v_1 &= 0
\end{aligned}
$$

This action of the $\mathcal{L}_j$ in the spin basis (3.2) can be encoded in terms of the vertex configurations shown in Figure 3.6 with labels $0, 1, 1', 2$ similarly as we did deduce the action of L and L' from the vertex configurations in Figure 3.1. Thus, the operator product $H(x)E(\ominus x)$ can be written as the partial trace

$$
E(\ominus x)H(x) = \operatorname*{Tr}_{V \otimes V} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix} \mathcal{L}_N \cdots \mathcal{L}_2 \mathcal{L}_1
$$

and its matrix elements in the quantum space $\mathcal{V}_n^q$ are sums over the possible vertex configurations of Figure 3.6 in a single lattice row of length N. This lattice row is closed and forms a circle of circumference N, since the partial trace together with the matrix containing the deformation parameter q imposes quasi-periodic boundary conditions. Due to these periodicity conditions, one finds the following constraints:

- the last vertex in the bottom row of Figure 3.6 cannot occur;
- the 2nd and 3rd vertex from the right in the top row always have to come as a pair, but since one of them has weight zero their contribution can be discarded;
- configurations involving the second vertex from the left in the bottom row do not contribute as they eventually lead to a vertex configuration shown at the 2nd position from the right in the top row which has weight zero;
- the 2nd and 3rd vertex from the right in the bottom row always have to come as an adjacent pair and it are these vertices which give rise to the term involving $\beta \bar{H}_1$ as they correspond to shifting a 1-letter in a binary string to the right.

From these conditions, which can be checked graphically, one then deduces the asserted identity (3.44) as only a very restricted number of vertices in Figure 3.6 remain. $\square$

Corollary 3.18 (equivariant quantum Pieri-Chevalley rule)**.** *We have the following explicit action of H_1 in terms of the basis $\{v_\lambda\} \subset \mathcal{V}_n$,*

$$
(1 + \beta H_1) v_\mu = \frac{\Pi(t_\mu)}{\Pi(t_\emptyset)} \sum_{\substack{\mu \rightrightarrows \lambda[d] \\ d=0,1}} q^d \beta^{|\lambda/d/\mu|} v_\lambda, \tag{3.43}
$$

where the sum runs over all $\lambda \subset (k^n)$ such that either λ/μ or $\lambda/1/\mu$ is a skew diagram which contains at most one box in each column or row. Moreover, the *identity (3.42) can be rewritten as*

$$H(x|t)E(\ominus x|t)=\prod_{j=1}^{n}(t_j\ominus x)\prod_{j=n+1}^{N}(x\ominus t_j)\,(1+\beta H_1)+q\cdot 1\;. \tag{3.44}$$

Proof. Acting with the first term on the right hand side of (3.42) on a basis vector v_λ we obtain

$$\begin{aligned}(1+\beta\bar{H}_1)\prod_{j=1}^{N}(t_j\ominus x)^{\sigma_j^+\sigma_j^-}(x\ominus t_j)^{\sigma_j^-\sigma_j^+}v_\lambda = \\ \left(\textstyle\prod_{j\in I_\lambda}t_j\ominus x\right)\left(\textstyle\prod_{j\in I_{\lambda^*}}x\ominus t_j\right)\,(1+\beta\bar{H}_1)v_\lambda = \\ \left(\textstyle\prod_{j=1}^{n}t_j\ominus x\right)\left(\textstyle\prod_{j=n+1}^{N}x\ominus t_j\right)\,\frac{\Pi(t_\lambda)}{\Pi(t_\emptyset)}(1+\beta\bar{H}_1)v_\lambda\end{aligned}$$

On the other hand using the expansions (3.34) and (3.35) we see that the coefficients of the leading factorial powers are

$$\begin{aligned}H(x|t) &= (x|\ominus t')^k(1+\beta H_1)+\cdots\\ E(x|t) &= (x|t)^n(1+\beta E_1)+\cdots\end{aligned}$$

from which we deduce the desired identities with help of the left hand side of (3.42) and (2.32). Namely, we have

$$\begin{aligned}(-1)^n\frac{(1+\beta x)^n}{\Pi(t_\emptyset)}\;E(\ominus x) &= (x|\ominus t)^n\sum_{r=0}^{n}(-1)^r\beta^r(E_r+\beta E_{r+1})+\;\dots\\ &= (x|\ominus t)^n\cdot 1+\;\dots\end{aligned}$$

where the omitted terms involve factorial powers $(x|\ominus t)^p$ with $p<n$ and we have set $E_0=1$, $E_{n+1}=0$. Thus,

$$(-1)^n\frac{(1+\beta x)^n}{\Pi(t_\emptyset)}\;E(\ominus x)H(x)=(x|\ominus t)^N(1+\beta H_1)+\;\dots$$

and the assertion follows. □

Example 3.19. *We consider once more the example* $\mathrm{Gr}_{1,3}=\mathbb{P}^2$*. It follows from (3.38) and (3.39) in Example 3.15 that*

$$\begin{aligned}(1+\beta H_1)v_{010} &= (1+\beta t_2\ominus t_1)(v_{010}+v_{001})\\ (1+\beta H_1)v_{001} &= (1+\beta t_3\ominus t_1)(v_{001}+qv_{100})\;.\end{aligned}$$

We compare this against the quantum Pieri-Chevalley rule (3.43). The binary string 010 *corresponds to the partition* $\mu=(1)$ *with a single box and* 001 *to the partition* $\mu=(2)$*. Thus, in the first case the only partitions* λ *for which* λ/μ *contains at most a single box in each row and column are* $\lambda=(1)$ *and* $\lambda=(2)$*. For* $\mu=(2)$ *we obtain* $\lambda=(2)$ *and* $\lambda=\emptyset$*, since in the latter case the cylindric loop* $\lambda[1]$ *contains 3 boxes and* $\lambda/1/\mu$ *has one box in one row.*

Let us now consider the functional relation (3.44). For $N=3$ *and* $n=1$ *expand the transfer matrices into factorial powers as follows*

$$\begin{aligned}H(x) &= (x\ominus t_2)(x\ominus t_3)(1+\beta H_1)+(x\ominus t_3)(H_1+\beta H_2)+H_2\\ -\frac{1+\beta x}{1+\beta t_1}E(\ominus x) &= (x\ominus t_1)(1+\beta E_1)-\frac{1+\beta x}{1+\beta t_1}E_2=(x\ominus t_1)-E_1\end{aligned}$$

The left hand side of (3.44) yields

$$
\begin{aligned}
-\frac{1+\beta x}{1+\beta t_1}E(\ominus x)H(x) = (x|\ominus t)^3(1+\beta H_1)\\
-(x\ominus t_2)(x\ominus t_3)[(1+\beta H_1)E_1-(1+\beta t_2\ominus t_1)(H_1+\beta H_2)]\\
+(x\ominus t_3)[(1+\beta t_3\ominus t_1)H_2+(t_2\ominus t_1-E_1)(H_1+\beta H_2)]\\
-E_1H_2(1-t_3\ominus t_1)
\end{aligned}
$$

while the right hand side reads

$$
\begin{aligned}
-\frac{1+\beta x}{1+\beta t_1}E(\ominus x)H(x) &= (x|\ominus t)^3(1+\beta H_1)-\frac{1+\beta x}{1+\beta t_1}\,q\\
&= (x|\ominus t)^3(1+\beta H_1)-(1+\beta t_3\ominus t_1)q[1+\beta(x\ominus t_3)]
\end{aligned}
$$

Comparing the coefficients of each factorial power we obtain

$$
\begin{gathered}
E_1H_2(1-t_3\ominus t_1)=(1+\beta t_3\ominus t_1)q\\
(E_1-t_2\ominus t_1)(H_1+\beta H_2)-(1+\beta t_3\ominus t_1)H_2=\beta(1+\beta t_3\ominus t_1)q\\
(1+\beta H_1)E_1=(1+\beta t_2\ominus t_1)(H_1+\beta H_2)
\end{gathered}
$$

We will see in a subsequent section that there is an easier way to describe the ideal which avoids these rather complicated looking relations. However, in the non-equivariant limit where $t_j=0$ for $j=1,2,3$, they simplify to

$$E_1=H_1,\qquad E_1^2=H_2,\qquad E_1^3=q\;.$$

4. Bethe vectors as idempotents

We now consider the eigenvalue problem of the transfer matrices. Eigenvalues and eigenvectors can be explicitly contructed using the Yang-Baxter algebra, this general approach is known as *quantum inverse scattering method* or *algebraic Bethe ansatz*. Using the eigenvectors, called *Bethe vectors* in the quantum integrable systems literature, we then define for each subspace $\mathcal{V}_n^q$ a generalised matrix ring qh_n^* by identifying appropriate renormalised versions of the Bethe vectors as its idempotents.

4.1. Bethe vectors & factorial Grothendieck polynomials.

Let $y=(y_1,\dots,y_n)$ and $z=(z_1,\dots,z_k)$ be some indeterminates. Recall McNamara's definition of factorial Grothendieck polynomials from Section 2.7 and the definition of the Yang-Baxter algebra (3.13).

Proposition 4.1. *Let $\lambda\subset(k^n)$. Then we have the following equalities for the C and B'-operators,*

$$C(y_1)\cdots C(y_n)v_\lambda \;=\; G_\lambda(y|\ominus t)\;v_0\otimes\cdots\otimes v_0 \tag{4.1}$$

$$B'(z_1)\cdots B'(z_k)v_\lambda \;=\; G_{\lambda'}(z|t')\;v_1\otimes\cdots\otimes v_1 \tag{4.2}$$

when acting on the basis vector v_λ in $\mathcal{V}_n$.

Proof. We only sketch the proof leaving technical details to the reader to verify. Since $\lambda\subset(k^n)$ the corresponding binary string $b(\lambda)$ contains n 1-letters. From the definition (3.13) it follows that $C(x):\mathbb{Z}[x]\otimes\mathcal{V}_n\to\mathbb{Z}[x]\otimes\mathcal{V}_{n-1}$ and $B':\mathbb{Z}[x]\otimes\mathcal{V}_n\to\mathbb{Z}[x]\otimes\mathcal{V}_{n+1}$. This implies that $C(y_1)\cdots C(y_n)v_\lambda$ is a multiple of

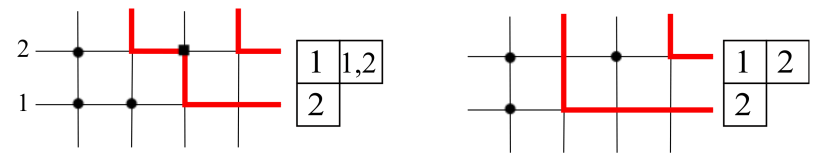

FIGURE 4.1. The lattice configurations corresponding to the C-operator.

$v_0 \otimes \cdots \otimes v_0$ and $B'(z_1)\cdots B'(z_k)v_\lambda$ a multiple of $v_1 \otimes \cdots \otimes v_1$. Denote the proportionality factors, i.e. the respective matrix elements, by $\langle 0|C(y_n)\cdots C(y_1)|\lambda\rangle$ and $\langle N|B'(z_k)\cdots B'(z_1)|\lambda\rangle$. Each can be identified with a weighted sum $\sum_\gamma \mathrm{wt}(\gamma)$ over vertex configurations γ on a lattice with certain fixed boundary conditions; see Figure 4.1 and 4.2 for simple examples with $N = 4$, $n = k = 2$. Here $\mathrm{wt}(\gamma) = \prod_{\mathrm{v}\in\gamma} \mathrm{wt}(\mathrm{v})$ with v being one of the vertex configurations in Figure 3.1 and the respective weight wt(v) takes the values a, a', b, b', c, c' as specified in the figure or zero if it is none of the allowed vertices. We now identify lattice configurations γ with certain *sets* of set-valued tableaux.

Define a surjection $\mathrm{SVT}(\lambda) \twoheadrightarrow \mathrm{SST}(\lambda)$ which assigns to each set-valued tableau $\mathcal{T}$ the semi-standard tableau $T : \lambda \to [n]$ with $T(i,j) = \min \mathcal{T}(i,j)$. Given a semistandard tableau T with entries $\leq n$, there exists a unique "maximal" set valued tableau $\mathcal{T}^{\max}$ in its pre-image that has the maximum number of entries $\leq n$, i.e. $|\mathcal{T}^{\max}| \geq |\mathcal{T}|$ for all $\mathcal{T}$ in the pre-image of T. The lattice path configurations are in one-to-one correspondence with these maximal set-valued tableaux (and therefore semi-standard tableaux) of shape λ and λ'. We state the bijections:

- *Vicious walkers.* Starting from the bottom, place for each vertex labelled with a *bullet* in lattice row i a box labelled with i in the jth row of the Young tableau where j is the total number of paths crossing the row to the right of the vertex. Vertices with a *square* in row i mean that an entry i is placed within an existing box of the jth row of the Young tableau where j is again the total number of paths crossing the row to the right of the vertex. The resulting set-valued tableau has shape λ.
- *Osculating walkers.* Consider the rightmost path and add in the first column (counting from left to right) of the Young diagram of λ' a box with the lattice row number where a vertex with a bullet occurs. If a vertex with a square occurs in row i then place an i into an existing box in this column. Then do the same for the next path writing the lattice row numbers now in the second column of the Young tableau etc. If there are no vertices with a bullet leave the column empty. The resulting set-valued tableau has shape λ'.

Let γ be a lattice configuration of the C-operator (B'-operator) and denote by T_γ the corresponding semi-standard tableau under the surjection $\mathrm{SVT}(\lambda) \twoheadrightarrow \mathrm{SST}(\lambda)$. Note that each vertex with a bullet contributes a factor $y_i \ominus t_j$ and each vertex with a square a factor $(1 + \beta y_i \ominus t_j)$; see Figure 3.1. Here i, j are the lattice row and column numbers where the vertex occurs and we number lattice rows decreasingly from top to bottom and lattice rows increasingly from left to right. This allows us to deduce the following result.

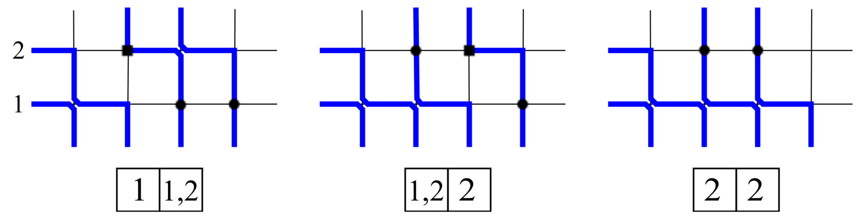

FIGURE 4.2. The lattice configurations corresponding to the B'-operator.

Lemma 4.2. *We have the following identities*

$$\begin{aligned}
\mathrm{wt}(\gamma_C) &= \prod_{\substack{\langle i,j\rangle\in\lambda\\ r=T_\gamma(i,j)}} (y_r \ominus t_{r+j-i}) \prod_{\substack{\langle i,j\rangle\in\lambda\\ r\in\mathcal{T}_\gamma^{\max}(i,j)\backslash T_\gamma(i,j)}} (1+\beta y_r \ominus t_{r+j-i}) \\
&= \sum_{\mathcal{T}} \beta^{|\mathcal{T}|-|\lambda|} \prod_{\substack{\langle i,j\rangle\in\lambda^\vee\\ r\in\mathcal{T}(i,j)}} y_r \ominus t_{r+j-i} \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{wt}(\gamma_{B'}) &= \prod_{\substack{\langle i,j\rangle\in\lambda'\\ r=T_\gamma(i,j)}} (z_r \oplus t_{r+j-i}) \prod_{\substack{\langle i,j\rangle\in\lambda'\\ r\in\mathcal{T}_\gamma^{\max}(i,j)\backslash T_\gamma(i,j)}} (1+\beta y_r \oplus t_{r+j-i}) \\
&= \sum_{\mathcal{T}} \beta^{|\mathcal{T}|-|\lambda|} \prod_{\substack{\langle i,j\rangle\in\lambda^*\\ r\in\mathcal{T}(i,j)}} z_r \oplus t'_{r+j-i}, \qquad (4.4)
\end{aligned}$$

where the sums run over all set-valued tableaux $\mathcal{T}$ of shape λ (λ') which obey the condition that

$$\min \mathcal{T}(i,j) = T_\gamma(i,j)\ .$$

Thus, it follows that

$$\begin{aligned}
\langle 0|C(y_n)\cdots C(y_1)|\lambda\rangle &= \sum_{\gamma_C} \mathrm{wt}(\gamma_C) = \sum_{\mathcal{T}\in\mathrm{SVT}(\lambda)} \beta^{|\mathcal{T}|-|\lambda^\vee|} \prod_{\substack{\langle i,j\rangle\in\lambda\\ r\in\mathcal{T}(i,j)}} y_r \ominus t_{r+j-i} \\
\langle N|B'(z_k)\cdots B'(z_1)|\lambda\rangle &= \sum_{\gamma_{B'}} \mathrm{wt}(\gamma_{B'}) = \sum_{\mathcal{T}\in\mathrm{SVT}(\lambda')} \beta^{|\mathcal{T}|-|\lambda^\vee|} \prod_{\substack{\langle i,j\rangle\in\lambda'\\ r\in\mathcal{T}(i,j)}} z_r \oplus t'_{r+j-i}
\end{aligned}$$

which proves the assertion as the last two equations are McNamara's definition (2.11) of factorial Grothendieck polynomials. □

Introduce the so-called off-shell Bethe vector in $\mathbb{Z}[y_1,\dots,y_n]\otimes\mathcal{V}_n$ and its dual

$$|y_1,\dots,y_n\rangle = B(y_n|t)\cdots B(y_1|t)|0\rangle \qquad (4.5)$$

$$\langle y_1,\dots,y_n| = \langle 0|C^\vee(y_n|t)\cdots C^\vee(y_1|t) \qquad (4.6)$$

where $|0\rangle = v_0\otimes\cdots\otimes v_0$ and $\langle 0| = \tilde v_0\otimes\cdots\otimes\tilde v_0$. Similarly, we define for a k-tuple $z=(z_1,\dots,z_k)$ the complementary Bethe vector in $\mathbb{Z}[z_1,\dots,z_k]\otimes\mathcal{V}_n$ and its dual as

$$|z_1,\ldots,z_k\rangle = C'(z_k|t)\cdots C'(z_1|t)|N\rangle \tag{4.7}$$

$$\langle z_1,\ldots,z_k| = \langle N|B^*(z_k|t)\cdots B^*(z_1|t)\ . \tag{4.8}$$

with $|N\rangle = v_1\otimes\cdots\otimes v_1$ and $\langle N| = \tilde v_1\otimes\cdots\otimes\tilde v_1$. From (3.5) one deduces that the $B, B^*, C', C^\vee$-operators each commute for different values of the spectral parameters. Hence, we can conclude that the vectors (4.5), (4.7) as well as their dual versions are symmetric in the y's and z's.

We now identify the coefficients of the off-shell Bethe vectors with factorial Grothendieck polynomials.

Proposition 4.3. *Recall the definitions of $\lambda^\vee$, λ^* from Sec 2.3 and set once more $\ominus t' = (\ominus t_{N+1},\ldots,\ominus t_2,\ominus t_1)$. Then we have the identities*

$$|y_1,\ldots,y_n\rangle = \Pi(y)\sum_{\lambda\in(k^n)}\frac{G_{\lambda^\vee}(y_1,\ldots,y_n|\ominus t')}{\Pi(t_\lambda)}\ v_\lambda \tag{4.9}$$

$$|z_1,\ldots,z_k\rangle = \Pi(z)\sum_{\lambda\in(k^n)} G_{\lambda^*}(z_1,\ldots,z_k|t)\Pi(t_{\lambda^*})v_\lambda, \tag{4.10}$$

For the dual vectors we obtain instead

$$\langle y_1,\ldots,y_n| = \sum_{\lambda\in(k^n)} G_\lambda(y_1,\ldots,y_n|\ominus t)\ \tilde v_\lambda \tag{4.11}$$

$$\langle z_1,\ldots,z_k| = \sum_{\lambda\in(k^n)} G_{\lambda'}(z_1,\ldots,z_k|t')\ \tilde v_\lambda\ . \tag{4.12}$$

Proof. The proof is very similar to the one of the previous identities with some minor changes in the bijections between lattice configurations of non-intersecting paths and maximal set-valued tableaux.

As before the matrix elements $\langle\lambda|B(y_n)\cdots B(y_1)|0\rangle$ and $\langle\lambda|C'(z_k)\cdots C'(z_1)|N\rangle$ can each be identified with a weighted sum $\sum_\gamma \mathrm{wt}(\gamma)$ over lattice path configurations γ with certain fixed boundary conditions; examples are provided in Figure 4.3 for B with $N=9$, $n=5$ and Figure 4.4 for C' with $N=9$, $k=5$. Also these configurations are in one-to-one correspondence with certain maximal set-valued tableaux (as defined previously) and are respectively mapped onto semistandard tableaux of shape $\lambda^\vee$ and $(\lambda^\vee)'$ when taking the smallest entry in each box. The bijections are now as follows:

- *Vicious walkers.* Starting now from the top, place for each vertex labelled with a bullet in lattice row i a box labelled with i in the jth row of the Young tableau where j is the total number of paths crossing the row to the left of the vertex. For a vertex with a square in row i place an additional entry i into an already existing box. The resulting tableau will now have shape $\lambda^\vee$.
- *Osculating walkers.* Consider the leftmost path and write in the first column (counting from left to right) of the Young diagram of $(\lambda^\vee)'$ the lattice row numbers where a vertex with a bullet occurs. If there is a vertex with a square in row i place an i into the last added box in the same column. Then do the same for the next path writing the lattice row numbers now in the second column etc. If there are no vertices with a bullet leave the column empty.

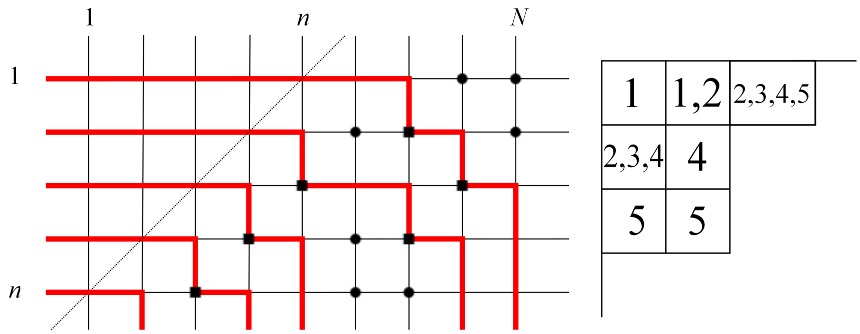

FIGURE 4.3. A lattice configuration corresponding to the B-operator. The vertex configurations above the dotted line are "frozen", i.e. there is no other choice possible which would yield a nonzero weight.

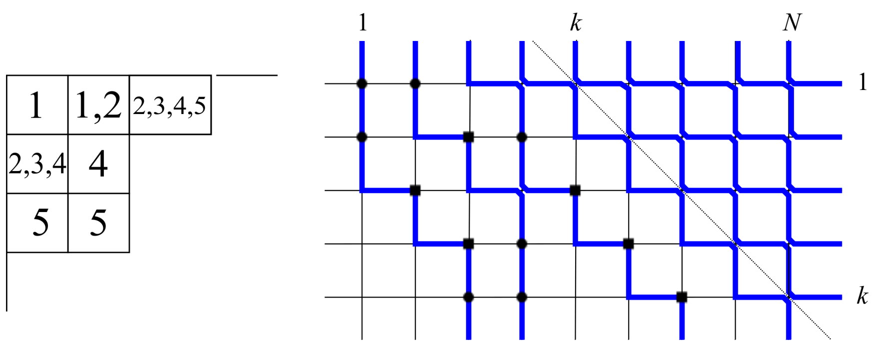

FIGURE 4.4. A lattice configuration corresponding to the C'-operator. The vertex configuration above the dotted line are "frozen".

Let γ be a lattice configuration of the B-operator (C'-operator) and denote by T_γ the corresponding semistandard tableau. If we multiply the matrix element $\langle\lambda|B(y_n)\cdots B(y_1)|0\rangle$ with $\Pi(t_\lambda)/\Pi(y)$ then according to Figure 3.1 each vertex with a bullet contributes a factor $y_i \ominus t'_j$ and each vertex with a square a factor $(1+\beta y_i \ominus t'_j)$ in the vicious walker case. In the osculating walker case we divide $\langle\lambda|C'(z_k^{-1})\cdots C'(z_1^{-1})|N\rangle$ with $\Pi(z)\Pi(t_{\lambda^*})$ to obtain respectively the factors $z_i \oplus t_j$ and $(1+\beta z_i \oplus t_j)$. Here i,j are the lattice row and column numbers where the vertex occurs. As before this implies the following summation identities for the lattice weights

$$\tfrac{\Pi(t_\lambda)}{\Pi(y)}\quad \mathrm{wt}(\gamma_B) = \sum_{\mathcal{T}} \beta^{|\mathcal{T}|-|\lambda^\vee|} \prod_{\substack{\langle i,j\rangle\in\lambda^\vee \\ r\in\mathcal{T}(i,j)}} y_r \ominus t'_{r+j-i},$$

$$\frac{\mathrm{wt}(\gamma_{C'})}{\Pi(z)\Pi(t_{\lambda^*})} = \sum_{\mathcal{T}} \beta^{|\mathcal{T}|-|\lambda^\vee|} \prod_{\substack{\langle i,j\rangle\in\lambda^* \\ r\in\mathcal{T}(i,j)}} z_r \oplus t_{r+j-i},$$

where the sums run over all set-valued tableaux $\mathcal{T}$ of shape $\lambda^\vee$ ($\lambda^* = (\lambda^\vee)'$) which obey the condition that $\min \mathcal{T}(i,j) = T_\gamma(i,j)$. The final step then uses again that the map $\mathrm{SVT}(\lambda^\vee) \twoheadrightarrow \mathrm{SST}(\lambda^\vee)$ which assigns to each set-valued tableau $\mathcal{T}$ the SST $T : \lambda^\vee \to [n]$ with $T(i,j) = \min \mathcal{T}(i,j)$ is a surjection. Thus, it follows that

$$\begin{aligned}
\langle \lambda | B(y_n) \cdots B(y_1) | 0 \rangle &= \frac{\Pi(y)}{\Pi(t_\lambda)} \sum_{\mathcal{T} \in \mathrm{SVT}(\lambda^\vee)} \beta^{|\mathcal{T}| - |\lambda^\vee|} \prod_{\substack{\langle i,j \rangle \in \lambda^\vee \\ r \in \mathcal{T}(i,j)}} y_r \ominus t'_{r+j-i} \\
\langle \lambda | C'(z_k^{-1}) \cdots C'(z_1^{-1}) | N \rangle &= \Pi(z)\Pi(t_{\lambda^*}) \sum_{\mathcal{T} \in \mathrm{SVT}(\lambda^*)} \beta^{|\mathcal{T}| - |\lambda^\vee|} \prod_{\substack{\langle i,j \rangle \in \lambda^* \\ r \in \mathcal{T}(i,j)}} z_r \oplus t_{r+j-i}
\end{aligned}$$

The identities for the dual Bethe vectors (4.11), (4.12) are obtained by a very similar argument noting from the definition (3.21) that the matrix elements of the transposed operators are obtained by reversing binary strings and swapping $t \leftrightarrow t'$. □

We are now in the position to proof a generalised Cauchy identity for factorial Grothendieck polynomials; compare with [52, Thm 5.3] and [40, Thm 9] for the non-factorial case which we obtain as a special case.

Corollary 4.4. *Setting* $e(x,y) = \langle 0 | C(x_1) \cdots C(x_n) B(y_n) \cdots B(y_1) | 0 \rangle$ *we have*

$$\begin{aligned}
e(x,y) &= \Pi(y) \sum_{\lambda \subset (k^n)} \frac{G_\lambda(x | \ominus t) G_{\lambda^\vee}(y | \ominus t')}{\Pi(t_\lambda)} \\
&= \frac{1}{\Pi(x)} \sum_w w \left(\Pi(x) \frac{\prod_{i=1}^n \prod_{j=1}^N x_i \ominus t_j}{\prod_{1 \le i,j \le n} x_i \ominus y_j} \right)
\end{aligned} \tag{4.13}$$

where - as in Lemma 3.5 - the sum runs over the minimal length coset representatives w *of* $\mathbb{S}_{2n}/\mathbb{S}_n \times \mathbb{S}_n$ *which act on* (x,y) *in the obvious manner.*

Proof. Noting that

$$A(x)|0\rangle = \left(\prod_{j=1}^N x \ominus t_j \right) |0\rangle \qquad \text{and} \qquad D(x)|0\rangle = |0\rangle \tag{4.14}$$

the assertion is immediate from Lemma 3.5 and the formulae (4.1), (4.9). □

Note that the limit $\lim_{x_i \to y_i} e(x,y)$ is well-defined as can be seen from the definition of $e(x,y)$ as matrix element $\langle 0 | C(x_1) \cdots C(x_n) B(y_n) \cdots B(y_1) | 0 \rangle$ and (4.13). The poles in the last line of Equation (4.13) cancel against zeroes in the numerator as $x_i \to y_i$ after the sum over the w's is taken.

Corollary 4.5. *Setting* $y = t_\mu$ *we obtain*

$$\prod_{i=1}^n \prod_{j \in I_{\mu^*}} (x_i \ominus t_j) = \sum_{\lambda \subset (k^n)} \frac{\Pi(t_\mu)}{\Pi(t_\lambda)} \; G_{\lambda^\vee}(t_\mu | \ominus t') G_\lambda(x | \ominus t) \; . \tag{4.15}$$

This proves in particular (2.39) and, thus, we obtain after setting also $x = t_\mu$*,*

$$e(t_\mu, t_\mu) = \prod_{i \in I_\mu} \prod_{j \in I_{\mu^*}} (t_i \ominus t_j) \; . \tag{4.16}$$

Proof. Specialising $y = t_\mu$ in (4.13) one easily sees that only the term with w being the identity survives in the last sum. □

4.2. **The Bethe ansatz equations.** We call the Bethe vectors (4.5), (4.7) "on-shell" if the indeterminates $y = (y_1, \ldots, y_n)$ are *pairwise distinct* solutions to the following set of coupled Bethe ansatz equations with $\Pi(y)$ defined in (2.19),

$$(-1)^n \frac{\Pi(y)}{(1+\beta y_i)^n} \prod_{j=1}^{N} y_i \ominus t_j + q = 0, \qquad i = 1, \ldots, n\,. \tag{4.17}$$

We define a second set of equations for the indeterminates $z = (z_1, \ldots, z_k)$ in (4.7),

$$(-1)^k \frac{\Pi(z)}{(1+\beta z_i)^k} \prod_{j=1}^{N} z_i \oplus t_j + q = 0, \qquad i = 1, \ldots, k\,. \tag{4.18}$$

The origin of these equations is Lemma 3.4 from which we deduce that if (4.17) holds the Bethe vector (4.5) is an eigenvector of $H = A + qD$; see below. By the same argument one shows that (4.7) is an eigenvector of $E = A' + qD'$ if (4.18) hold. Obviously, one set of equations transforms into the other under the substitution $t = (t_1, \ldots, t_N) \to \ominus t' = (\ominus t_N, \ldots, \ominus t_1)$ and exchanging n with k. This substitution is related to level-rank duality (3.17) which we will use below to relate the Bethe vectors (4.5) with the vectors (4.7). We shall therefore focus on the equations (4.17) only.

Lemma 4.6. *The set of equations (4.17) has* $\binom{N}{n}$ pairwise distinct *solutions*

$$y_\lambda = (y_{\lambda_n+1}, \ldots y_{\lambda_2+n-1}, y_{\lambda_1+n}) \in \mathbb{Z}[\![q]\!] \otimes \mathbb{Z}(t_1, \ldots, t_N), \tag{4.19}$$

where $\lambda \subset (k^n)$ *and up to first order in* q *we have*

$$y_i = t_i + q\ (-1)^{n-1} \frac{(1+\beta t_i)^{n+1}}{\Pi(t_\lambda) \prod_{j \neq i} t_i \ominus t_j} + O(q^2)\,. \tag{4.20}$$

Proof. Make the ansatz $y_i = y_i^{(0)} + y_i^{(1)} q + y_i^{(2)} q^2 + \cdots$ and set $q = 0$ in (4.17). Because the equations are coupled setting $y_{i'}^{(0)} = -\beta^{-1}$ with $i' \neq i$ in the factor in front of the product in (4.17) is not a valid solution, since it would imply a singular term in the equations with i replaced by i'. Therefore, the only possible solution is $y_i^{(0)} = t_i$. Here the labelling in terms of the index i is a matter of choice but it will prove convenient later on. Differentiating the equations with respect to q,

$$\frac{d}{dq}\left(\frac{(-1)^{n-1}\Pi(y)}{(1+\beta y_i)^n}\right) \prod_{j=1}^{N} y_i \ominus t_j + \frac{(-1)^{n-1}\Pi(y)}{(1+\beta y_i)^n} \frac{d}{dq} \prod_{j=1}^{N} y_i \ominus t_j = 1,$$

and setting $q = 0$ afterwards we find

$$\left.\frac{d}{dq} \prod_{j=1}^{N} y_i \ominus t_j \right|_{q=0} = \frac{y_i^{(1)}}{1 - \beta t_i} \prod_{j \neq i} t_i \ominus t_j$$

and the formula (4.20) follows. Continuing in the same manner by taking the rth derivative we see that the coefficient $\Pi(t_\lambda)/(1+\beta t_i)^n$ in front of the term

$$\left.\frac{d^r}{dq^r} \prod_{j=1}^{N} y_i \ominus t_j \right|_{q=0}$$

is alway nonzero and, hence, that the equations yield a rational solution in the t_j's for any $y_i^{(r)}$. $\square$

Lemma 4.7. *Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 > k$ and a solution $y = y_\mu$ of the Bethe ansatz equations (4.17), one has the identity*

$$G_\lambda(y|\ominus t) = q \sum_{r=0}^{\lambda_1-1-k} h_{\lambda_1-1-k-r}(t_1, \dots, t_{r+1}) G_{(\lambda_2-1,\dots,\lambda_n-1,r)}(y|\ominus t) \prod_{i=1}^{r} (1+\beta t_i), \tag{4.21}$$

where the h_r's denote the complete symmetric functions and the factorial Grothendieck polynomial on the right hand side is defined via (2.16).

Proof. Recall the determinant formula (2.13) for factorial Grothendieck polynomials. Writing out the determinant in the numerator we find

$$a_\lambda = \begin{vmatrix} (y_1|\ominus t)^{n+\lambda_1-1} & \cdots & (y_n|\ominus t)^{n+\lambda_1-1} \\ (y_1|\ominus t)^{n+\lambda_2-2}(1+\beta y_1) & & (y_n|\ominus t)^{n+\lambda_2-2}(1+\beta y_n) \\ \vdots & & \vdots \\ (y_1|\ominus t)^{\lambda_n}(1+\beta y_1)^{n-1} & \cdots & (y_n|\ominus t)^{\lambda_n}(1+\beta y_n)^{n-1} \end{vmatrix}$$
$$= \frac{q}{\Pi(y)} \begin{vmatrix} (y_1|\ominus t)^{n+\lambda_2-2}(1+\beta y_1) & \cdots & (y_n|\ominus t)^{n+\lambda_2-2}(1+\beta y_n) \\ \vdots & & \vdots \\ (y_1|\ominus t)^{\lambda_n}(1+\beta y_1)^{n-1} & & (y_n|\ominus t)^{\lambda_n}(1+\beta y_n)^{n-1} \\ y_1^{\lambda_1-1-k}(1+\beta y_1)^n & \cdots & y_n^{\lambda_1-1-k}(1+\beta y_n)^n \end{vmatrix}$$

Here we have made use of (4.17), exchanged the first row with the last row in the determinant and used row linearity of the determinant to pull out the common factor in front. Note that $t_j = 0$ for $j > N$, whence the powers in the bottom row are not factorial. To rewrite them as factorial powers we use the equality

$$x^m = \sum_{r=0}^{m} (x|\ominus t)^{m-r} h_r(t_1, \dots, t_{m+1-r}) \prod_{i=1}^{m-r} (1+\beta t_i)$$

which is easily proved via induction using the known recursion relation

$$h_{r+1}(t_1, \dots, t_{m+1-r}) = h_r(t_1, \dots, t_{m+1-r}) t_{m+1-r} + h_{r+1}(t_1, \dots, t_{m-r})$$

of the complete symmetric functions. We leave this step to the reader.

Thus, after employing the above identity and column/row linearity of the determinant we arrive at

$$a_\lambda = q \sum_{r=0}^{\lambda_1-1-k} a_{(\lambda_2-1,\dots,\lambda_n-1,\lambda_1-1-k-r)} h_r(t_1, \dots, t_{\lambda_1-k-r}) \prod_{i=1}^{\lambda_1-1-k-r} (1+\beta t_i)$$

which is the asserted identity (4.21) after dividing by the Vandermonde determinant. $\square$

Theorem 4.8. *The on-shell Bethe vectors (4.5), (4.7) and (4.6), (4.8) form respectively right and left eigenbases of the transfer matrices H and E in each subspace*

$\mathcal{V}_n^q$ with eigenvalue equations

$$H(x|t)|y_\mu\rangle = \left(\frac{\prod_{j=1}^{N} x \ominus t_j + (-1)^n q \prod_{i\in I_\mu} (1+\beta x\ominus y_i)}{\prod_{i\in I_\mu} x\ominus y_i} \right) |y_\mu\rangle \tag{4.22}$$

and

$$E(x|t)|z_\mu\rangle = \left(\frac{\prod_{j=1}^{N} x \oplus t_j + (-1)^n q \prod_{i\in I_{\mu^*}} (1+\beta x\ominus z_i)}{\prod_{i\in I_{\mu^*}} x\ominus z_i} \right) |z_\mu\rangle \ . \tag{4.23}$$

Proof. Here we use the commutation relations of the Yang-Baxter algebra as per Lemma 3.4 and (4.14) from which we deduce that if (4.17) holds the Bethe vector (4.5) is an eigenvector of $H = A + qD$. The computation follows along the same lines for (4.7) and the left eigenvectors (4.6, 4.8).

One deduces that the eigenvalues must separate points and, hence, $\langle y_\lambda|y_\mu\rangle = \langle z_\lambda|z_\mu\rangle = 0$ for $\lambda \neq \mu$. That these eigenvectors form a basis then follows from the fact that there exist $\dim \mathcal{V}_n = \binom{N}{n}$ solutions to the equations (4.17); see Lemma 4.6. □

Note that the above formulae simplify if $q = 0$. Then the on-shell Bethe vectors with $y_\mu = t_\mu$ are given by

$$|t_\mu\rangle = \sum_{\lambda\subset(k^n)} G_{\lambda^\vee}(t_\mu|\ominus t')\frac{\Pi(t_\mu)}{\Pi(t_\lambda)}\ v_\lambda \tag{4.24}$$

and form an eigenbasis of the transfer matrices with eigenvalues,

$$H(x|t)|t_\mu\rangle = \left(\prod_{j\in I_{\mu^*}} x\ominus t_j\right)|t_\mu\rangle \tag{4.25}$$

$$E(x|t)|t_\mu\rangle = \left(\prod_{j\in I_{\mu}} x\oplus t_j\right)|t_\mu\rangle \tag{4.26}$$

As we will discuss below this special case describes generalised equivariant cohomology theory, $h_n^* = qh_n^*/\langle q\rangle$ and we show below that $h_n^*/\langle\beta+1\rangle \cong K_{\mathbb{T}}(\mathrm{Gr}_{n,N})$.

Proposition 4.9. *The eigenvectors of H and E coincide under the substitution $z_{\lambda'} = \ominus y_{\lambda^\vee}$ and, thus, we have the equality*

$$G_{\lambda'}(\ominus y_{\mu^*}|t) = G_\lambda(y_\mu|\ominus t') \tag{4.27}$$

for each solution y_μ of (4.17). In particular, for $q = 0$ we have $G_{\lambda'}(\ominus t_{\mu^*}|t) = G_\lambda(t_\mu|\ominus t')$.

Proof. Using the identity (3.17) when acting on the Bethe vectors we find

$$\begin{aligned}\Theta H(x|t)|y_\lambda\rangle &= E(x|\ominus t')\Theta|y_\lambda\rangle\\ \Theta E(x|t)|z_\lambda\rangle &= H(x|\ominus t')\Theta|z_\lambda\rangle\end{aligned}$$

and

$$\begin{aligned} \Theta|y_\lambda\rangle &= C'(y_1|\ominus t')\cdots C'(y_n|\ominus t')|N\rangle \\ \Theta|z_\lambda\rangle &= B(z_1|\ominus t')\cdots B(z_k|\ominus t')|0\rangle \end{aligned}$$

These identities together with the expansion (4.20) allows us to identify $z_{\lambda'} = \ominus y_{\lambda^\vee}$. □

Corollary 4.10. *The eigenvalue equation (4.23) of the E-transfer matrix simplifies in the Bethe roots (4.19) to*

$$E(x|t)|y_\mu\rangle = \prod_{i\in I_\mu}(x\oplus y_i)\;\;|y_\mu\rangle\;. \tag{4.28}$$

Proof. Setting $x = t_j$ the functional equation (3.44) together with (4.22) implies

$$q|y_\mu\rangle = E(\ominus t_j)H(t_j)|y_\mu\rangle = \frac{q}{\prod_{i=1}^n (t_j\ominus y_i)}E(\ominus t_j)|y_\mu\rangle$$

for all $j = 1, 2, \ldots, N$. Since the t_j's are arbitrary variables and the Bethe vectors form an eigenbasis the assertion follows. □

Since the Bethe vectors (4.5) and (4.6) form each an eigenbasis they give rise to a resolution of the identity $\mathbf{1} = \sum_{\alpha\in(n,k)}|y_\alpha\rangle\langle y_\alpha|$ where $|y_\alpha\rangle\langle y_\alpha|$ denotes the orthogonal projector onto the eigenspace spanned by $|y_\alpha\rangle$. This elementary fact of linear algebra translates into the following non-trivial identities for factorial Grothendieck polynomials evaluated at solutions of the Bethe ansatz equations (4.17).

Corollary 4.11 (orthogonality & completeness)**.** *For all $\lambda, \mu \subset (k^n)$ we have the identities*

$$\sum_{\alpha\subset(k^n)}\frac{\Pi(y_\lambda)}{\Pi(t_\alpha)}\frac{G_{\alpha^\vee}(y_\lambda|\ominus t')G_\alpha(y_\mu|\ominus t)}{e(y_\lambda,y_\lambda)} = \delta_{\lambda\mu} \tag{4.29}$$

and

$$\sum_{\alpha\subset(k^n)}\frac{\Pi(y_\alpha)}{\Pi(t_\lambda)}\frac{G_{\lambda^\vee}(y_\alpha|\ominus t')G_\mu(y_\alpha|\ominus t)}{e(y_\alpha,y_\alpha)} = \delta_{\lambda\mu}\;, \tag{4.30}$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta with $\delta_{\lambda\mu} = 1$ if $\lambda = \mu$ and 0 otherwise.

4.3. **Generalised matrix algebras and Frobenius structures.** Following the suggested construction in [38, Section 7] we now introduce a ring structure on each $\mathcal{V}_n^q = \mathbb{Z}[\![q]\!]\otimes\mathcal{V}_n$ by interpreting the on-shell Bethe vectors (4.5) as central orthogonal idempotents of a semisimple algebra: for each $n = 0, 1, \ldots, N$ define $qh_n^* = (\mathcal{V}_n^q, \circledast)$ by fixing the product $\circledast$ as follows,

$$Y_\lambda \circledast Y_\mu = \delta_{\lambda\mu}Y_\mu\;,\qquad Y_\lambda = e(y_\lambda,y_\lambda)^{-1}|y_\lambda\rangle\;, \tag{4.31}$$

where $e(y_\lambda, y_\lambda)$ is the matrix element defined in (4.13). Note that $e(y_\lambda, y_\lambda)$ is a power series in q with nonzero constant term (4.16) according to (4.20). The unit element is given by

$$v_\emptyset = \sum_{\lambda\subset(k^n)} Y_\lambda\;. \tag{4.32}$$

This determines qh_n^* via its Peirce decomposition [56]. We turn qh_n^* into a Frobenius algebra by introducing in addition the following symmetric bilinear form $\mathcal{V}_n^q \times \mathcal{V}_n^q \to \mathcal{R}(\mathbb{T}, q)$,

$$(Y_\lambda, Y_\mu) = e(y_\lambda, y_\lambda)^{-1} \delta_{\lambda\mu} \, . \tag{4.33}$$

By definition this bilinear form is invariant with respect to the product (4.31) and non-degenerate, since the Bethe vectors form a basis.

4.4. **A residue formula for the structure constants.** We now describe the resulting generalised matrix algebra qh_n^* in the spin basis $\{v_\lambda\}_{\lambda \subset (k^n)}$. Introduce a family of operators $\{\boldsymbol{G}_\lambda\}_{\lambda \subset (k^n)} \subset \operatorname{End} \mathcal{V}_n^q$ via the following eigenvalue equation

$$\boldsymbol{G}_\lambda Y_\mu = G_\lambda(y_\mu | \ominus t) Y_\mu \, . \tag{4.34}$$

This *defines* the operators $\boldsymbol{G}_\lambda$, since the Bethe vectors form an eigenbasis and the eigenvalues separate points. Recall from Section 2.7 that the factorial Grothendieck polynomials form a basis [47, Thm 4.6]. Below we give an explicit, basis independent construction of $\boldsymbol{G}_\lambda$ in terms of the transfer matrix $H(x)$.

Corollary 4.12. *In the spin basis (3.2) the product (4.31) is given by*

$$v_\lambda \circledast v_\mu = \boldsymbol{G}_\lambda v_\mu = \sum_{\nu \subset (k^n)} C_{\lambda\mu}^\nu(t,q) v_\nu, \tag{4.35}$$

where the structure constants $C_{\lambda\mu}^\nu(t,q) = \langle \nu | \boldsymbol{G}_\lambda | \lambda \rangle$ are obtained in terms of the Bethe roots (4.19) via the residue formula

$$C_{\lambda\mu}^\nu(t,q) = \sum_{\alpha \subset (k^n)} \frac{\Pi(y_\alpha)}{\Pi(t_\nu)} \frac{G_\lambda(y_\alpha | \ominus t) G_\mu(y_\alpha | \ominus t) G_{\nu^*}(\ominus y_{\alpha^*} | t)}{e(y_\alpha, y_\alpha)} \, . \tag{4.36}$$

Similarly, the bilinear form (4.33) can be expressed as

$$(v_\lambda, v_\mu) = \sum_{\alpha \subset (k^n)} \frac{G_\lambda(y_\alpha | \ominus t) G_\mu(y_\alpha | \ominus t)}{e(y_\alpha, y_\alpha)} \, . \tag{4.37}$$

Remark 4.13. *Our residue formula (4.36) is a generalisation of the Bertram-Vafa-Intriligator formula for Gromov-Witten invariants. It holds also true for $q = 0$, where the Bethe roots are explicitly known, $y_i = t_i$,*

$$c_{\lambda\mu}^\nu(t) = C_{\lambda\mu}^\nu(t,0) = \sum_{\alpha \subset (k^n)} \frac{\Pi(t_\alpha)}{\Pi(t_\nu)} \frac{G_\lambda(t_\alpha | \ominus t) G_\mu(t_\alpha | \ominus t) G_{\nu^*}(\ominus t_{\alpha^*} | t)}{\prod_{i \in I_\alpha, j \in I_{\alpha^*}} t_i \ominus t_j} \, . \tag{4.38}$$

The bilinear form (4.37) for $q = 0$ reads

$$(v_\lambda, v_\mu) = \sum_{\alpha \subset (k^n)} \frac{G_\lambda(t_\alpha | \ominus t) G_\mu(t_\alpha | \ominus t)}{\prod_{i \in I_\alpha, j \in I_{\alpha^*}} t_i \ominus t_j} \, . \tag{4.39}$$

Proof. According to (4.9) and (4.30) we have the inverse basis transformation

$$v_\lambda = \sum_{\mu \subset (k^n)} G_\lambda(y_\mu | \ominus t) Y_\mu \, . \tag{4.40}$$

which allows us to compute

$$\begin{aligned} v_\lambda \circledast v_\mu &= \sum_{\rho,\sigma} G_\lambda(y_\rho|\ominus t)G_\mu(y_\sigma|\ominus t)Y_\rho \circledast Y_\sigma \\ &= \sum_{\rho} G_\lambda(y_\rho|\ominus t)G_\mu(y_\rho|\ominus t)Y_\rho = \boldsymbol{G}_\lambda v_\mu = \boldsymbol{G}_\mu v_\lambda\ . \end{aligned}$$

This proves the first assertion. Continuing the computation from the second line employing (4.9) we arrive at (4.36).

The expression (4.37) is also an immediate consequence of (4.40). Insert the latter and use the definition (4.33) to find the asserted identity (4.37). □

As is to be expected from our previous results (3.18) and (3.17), the rings related by exchanging the dimension n with the codimension k of the hyperplanes in the Grassmannian are closely related.

Corollary 4.14 (level-rank duality)**.** *The involution $qh_n^* \to qh_k^*$ given by $f(t,q)v_\lambda \mapsto f(\ominus t',q)v_{\lambda'}$ is a ring isomorphism over $\mathcal{R}\otimes\mathbb{Z}[\![q]\!]$. That is,*

$$C^\lambda_{\mu\nu}(t,q) = C^{\lambda'}_{\mu'\nu'}(\ominus t',q)\ . \tag{4.41}$$

Proof. First we note that (2.20) and (4.27) imply the identity

$$\begin{aligned} \frac{\Pi(y_\lambda)}{\Pi(t_\mu)} &= \frac{\Pi(t_\emptyset)}{\Pi(t_\mu)}\ (1+\beta G_1(y_\lambda|\ominus t)) \\ &= \frac{\Pi(\ominus t'_\emptyset)}{\Pi(\ominus t'_{\mu'})}\ (1+\beta G_1(\ominus y_{\lambda^*}|t')) = \frac{\Pi(\ominus y_{\lambda^*})}{\Pi(\ominus t'_{\mu'})}\ . \end{aligned}$$

Note further that according to (4.20) the k-tuple $\ominus y_{\lambda^*}$ is obtained from solutions y_i by replacing $t=(t_1,\dots,t_N)$ with $\ominus t' = (\ominus t_N,\dots,\ominus t_1)$, i.e. the constant terms of the components of the solution $\ominus y_{\lambda^*}$ are $\ominus t'_{\lambda'}$ which identifies the solution uniquely. Using the residue formula (4.36) and (4.27) we compute

$$\begin{aligned} C^\lambda_{\mu\nu}(t,q) = \sum_{\alpha\subset(k^n)} \frac{\Pi(y_\alpha)}{\Pi(t_\nu)}\frac{G_\lambda(y_\alpha|\ominus t)G_\mu(y_\alpha|\ominus t)G_{\nu^\vee}(y_\alpha|\ominus t')}{e(y_\alpha,y_\alpha)} = \\ \sum_\alpha \frac{\Pi(\ominus y_{\alpha^*})}{\Pi(\ominus t'_{\nu'})}\frac{G_{\lambda'}(\ominus y_{\alpha^*}|t')G_{\mu'}(\ominus y_{\alpha^*}|t')G_{\nu^*}(\ominus y_{\alpha^*}|t)}{e(y_\alpha,y_\alpha)} = C^{\lambda'}_{\mu'\nu'}(\ominus t',q), \end{aligned}$$

where in the last step we have used the definition (4.13) to show that

$$\begin{aligned} e(y_\alpha,y_\alpha) &= \sum_{\lambda\subset(k^n)} \frac{\Pi(y_\alpha)}{\Pi(t_\lambda)} G_\lambda(y_\alpha|\ominus t)G_{\lambda^\vee}(y_\alpha|\ominus t') \\ &= \sum_{\lambda\subset(k^n)} \frac{\Pi(\ominus y_{\alpha^*})}{\Pi(\ominus t'_{\lambda'})} G_{\lambda'}(\ominus y_{\alpha^*}|t')G_{\lambda^*}(\ominus y_{\alpha^*}|t) = e(\ominus y_{\alpha^*},\ominus y_{\alpha^*})\ . \end{aligned}$$

□

4.5. **A recurrence formula.** We now return to the result (3.43) and show that the latter formula describes the multiplication with the class of the Schubert divisor, i.e. that (3.43) describes indeed the equivariant quantum Pieri-Chevalley rule for the generalised cohomology ring qh_n^*.

Corollary 4.15. *Let* $\lambda = (1, 0, \dots, 0)$ *then*

$$\boldsymbol{G}_1 = H_1 \tag{4.42}$$

and the product $v_1 \circledast v_\lambda = H_1 v_\lambda$ *in the spin basis is given explicitly via (3.43).*

Proof. Employing the functional equation (3.44) and (4.22), (4.28) we obtain

$$\prod_{j=1}^{n}(t_j \ominus x) \prod_{j=n+1}^{N}(x \ominus t_j)(1+\beta H_1)Y_\mu = (H(x)E(\ominus x) - q\cdot 1)Y_\mu$$

$$= (-1)^n \frac{\Pi(y_\mu)}{(1+\beta x)^n} \prod_{j=1}^{N}(x \ominus t_j)\; Y_\mu$$

$$= \frac{\Pi(y_\mu)}{\Pi(t_\emptyset)} \prod_{j=1}^{n}(t_j \ominus x) \prod_{j=n+1}^{N}(x \ominus t_j)\; Y_\mu$$

Thus, according to (2.21), (4.34) we have

$$(1+\beta H_1)Y_\mu = \frac{\Pi(y_\mu)}{\Pi(t_\emptyset)}\; Y_\mu = (1+\beta \boldsymbol{G}_1)Y_\mu$$

and the assertion follows from the fact that the Bethe vectors form a basis. □

Analogous to the case of equivariant (quantum) cohomology one derives from the quantum Pieri-Chevalley rule (3.43) the following recurrence relation for the structure constants.

Corollary 4.16 (Recurrence relation)**.** *We have the identity*

$$(\Pi(t_\nu) - \Pi(t_\lambda))\, C^\nu_{\lambda\mu} = \sum_{\tilde\lambda/d'/\lambda} \beta^{|\tilde\lambda/d/\lambda|} C^\nu_{\tilde\lambda\mu} - \sum_{\nu/d''/\tilde\nu} \beta^{|\nu/d''/\tilde\nu|}\Pi(t_{\tilde\nu})C^{\tilde\nu}_{\lambda\mu}, \tag{4.43}$$

where the sums run over all partitions $\tilde\lambda \neq \lambda, \tilde\nu \neq \nu$ *such that respectively* $\tilde\lambda/d'/\lambda$ *and* $\nu/d''/\nu$ *are toric skew-diagrams with* d', d'' *either* 0 *or* 1 *and where each row and column contains at most one box.*

Proof. The derivation follows the same idea as in ordinary (quantum) cohomology; see e.g. [33]. Since the product $\circledast$ by definition is associative we have in light of (4.42) that

$$[(1+\beta H_1)v_\lambda] \circledast v_\mu = (1+\beta H_1)(v_\lambda \circledast v_\mu)\;.$$

Applying the Pieri-Chevalley rule (3.43) on both sides of the equality sign and comparing coefficients the assertion follows. □

Example 4.17. *Consider once more the simplest non-trivial case* $\mathrm{Gr}_{1,3} = \mathbb{P}^2$*. Let* $\lambda = \mu = (2)$ *and* $\nu = \emptyset$*. Then* $\Pi(t_\nu) = 1 + \beta t_1$, $\Pi(t_\lambda) = 1 + \beta t_3$ *and* $\tilde\lambda = \emptyset$, $\tilde\nu = (2)$ *with* $d' = d'' = 1$ *are the only boxed partitions which give rise to allowed cylindric skew diagrams. Therefore, we arrive at the relation*

$$\beta(t_1 - t_3)C^\emptyset_{22} = q\beta C^\emptyset_{\emptyset 2} - q\beta(1+\beta t_3)C^2_{22} = -q\beta(1+\beta t_3)C^2_{22},$$

where we have used that $v_\emptyset$ *is the unit and we therefore must have* $C^\emptyset_{\emptyset 2} = 0$*. Similarly, setting* $\nu = 1$ *we obtain*

$$\beta(t_2 - t_3)C^1_{22} = q\beta C^1_{\emptyset 2} - \beta(1+\beta t_1)C^\emptyset_{22}\;.$$

Thus, we end up with the recursion

$$C_{22}^{\emptyset}=q\frac{1+\beta t_3}{t_3-t_1}\;C_{22}^{2},\qquad C_{22}^{1}=\frac{1+\beta t_1}{t_3-t_2}\;C_{22}^{\emptyset}$$

with $C_{22}^2=(t_3\ominus t_2)(t_3\ominus t_1)$. *Thus,*

$$C_{22}^{\emptyset}=q(t_3\ominus t_2)\frac{1+\beta t_3}{1+\beta t_1},\qquad C_{22}^{1}=q\frac{1+\beta t_3}{1+\beta t_2}$$

which is in agreement with our earlier computation and the product expansion in [13, Sec 5.5] *upon setting* $t_i=1-e^{\varepsilon_{4-i}}$ *and* $\beta=-1$.

5. Localised Schubert classes and GKM theory

An important result in (ordinary) equivariant quantum cohomology and equivariant K-theory is that the respective rings have a purely algebraic realisation by restricting Schubert classes to the fixed points under the torus action. This monomorphism becomes a ring isomorphism with respect to pointwise multiplication if one imposes the Goresky-Kottwitz-MacPherson (GKM) conditions [26, Thm 1.2.2]; see [36, Thm 3.13] for the analogous statement in K-theory. We now show that this algebraic realisation naturally emerges from our lattice model approach for our generalised cohomology theories qh_n^*.

5.1. **Generalised difference operators and Iwahori-Hecke algebras.** We recall that the ring $\mathcal{R}(\mathbb{T})=\mathcal{R}(t_1,\dots,t_N)$ is naturally endowed with an $\mathbb{S}_N$-action by permuting the equivariant parameters. By abuse of notation we will identify permutations $w\in\mathbb{S}_N$ with their operators acting on $\mathcal{R}(\mathbb{T})$. This $\mathbb{S}_N$-action can be used to define a representation of a generalised (affine) Hecke or Iwahori algebra $\mathbb{H}_N(\beta)$.

Definition 5.1. *Denote by* $\mathbb{H}_N(\beta)$ *the associative unital algebra with the following generators and relations*

$$\pi_i^2=\beta\pi_i\qquad\text{and}\qquad\begin{cases}\pi_i\pi_j=\pi_j\pi_i, & (i-j)\bmod N\neq\pm1\\ \pi_i\pi_{i+1}\pi_i=\pi_{i+1}\pi_i\pi_{i+1}, & \textit{else}\end{cases}\tag{5.1}$$

where all indices are understood modulo N. *Denote by* $\mathbb{H}_N^{\rm fin}(\beta)$ *the subalgebra generated by* $\{\pi_1,\dots,\pi_{N-1}\}$.

The subring $\mathcal{R}[t_1,\dots,t_N]\subset\mathcal{R}(\mathbb{T})$ and $\mathcal{R}(\mathbb{T})$ itself are both $\mathbb{H}_N^{\rm fin}(\beta)$-modules with respect to the following action in terms of isobaric divided difference operators

$$\partial_j=(1+\beta t_j)\;\frac{1-s_j}{t_j-t_{j+1}}\;,\tag{5.2}$$

where s_j is the simple transposition interchanging t_j and t_{j+1}. Note that setting $\beta=0$ we obtain a representation of the nil-Coxeter algebra $\mathbb{A}_N=\mathbb{H}_N(0)$ and when setting $\beta=-1$ a representation of the nil-Hecke algebra $\mathbb{H}_N=\mathbb{H}_N(-1)$.

Proposition 5.2 (braid matrices)**.** *Let* $p_j:V_n\to V_n$ *the operator which permutes vectors in the* j*th and* $(j+1)$*th factor and acts everywhere else trivially, i.e.* $p_jv_b=v_{s_jb}$. *Then the matrices* $\{\hat r_j(t_j,t_{j+1})=p_jr_{j+1,j}(t_{j+1}\ominus t_j)\}_{j=1}^N$ *act on the standard basis* $\{v_b\}_{|b|=n}$ *via*

$$\hat r_j(t_j,t_{j+1})v_b=\begin{cases}(1+\beta t_{j+1}\ominus t_j)v_b+q^{-\delta_{j,N}}t_{j+1}\ominus t_j\;v_{s_jb}, & b_j<b_{j+1}\\ v_b, & \textit{else}\end{cases}\tag{5.3}$$

Moreover, the $\hat{r}_j$'s obey the relations

$$\hat{r}_j(t_{j+1},t_{j+2})\hat{r}_{j+1}(t_j,t_{j+2})\hat{r}_j(t_j,t_{j+1}) = \hat{r}_{j+1}(t_j,t_{j+1})\hat{r}_j(t_j,t_{j+2})\hat{r}_{j+1}(t_{j+1},t_{j+2}) \tag{5.4}$$

and

$$\hat{r}_j^2 - (2+\beta t_{j+1}\ominus t_j)\ \hat{r}_j + (1+\beta t_{j+1}\ominus t_j)\ 1 = 0 \tag{5.5}$$

$$(s_j\otimes 1)\hat{r}_j = \hat{r}_j^{-1}(s_j\otimes 1)\ . \tag{5.6}$$

Here all indices are understood modulo N.

Proof. If we fix the basis $\{v_0\otimes v_0, v_0\otimes v_1, v_1\otimes v_0, v_1\otimes v_1\}$ in $V_j\otimes V_{j+1}$ then $\hat{r}_j$ reads as a matrix,

$$\hat{r}_j(t_j,t_{j+1}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1+\beta t_{j+1}\ominus t_j & 0 & 0\\ 0 & q^{-\delta_{j,N}}t_{j+1}\ominus t_j & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}_{j,j+1} \tag{5.7}$$

Using this matrix form one now verifies easily the various assertions. □

Corollary 5.3 (symmetric group action)**.** *The operators $\boldsymbol{s}_j = (s_j\otimes 1)\hat{r}_j$ for $j = 1,\dots,N-1$ define an action of the symmetric group $\mathbb{S}_N$ on the space $\mathcal{V}_n$. For q invertible, we have an action of the affine symmetric group with $\boldsymbol{s}_N = (s_N\otimes 1)\hat{r}_N(q^{-1})$ on $\mathbb{Z}[q^{\pm 1}]\otimes\mathcal{V}_n$, where s_N is the affine reflection in the level-zero representation on $\mathcal{R}(\mathbb{T})$. Explicitly, one has in the spin-basis*

$$\boldsymbol{s}_j v_b = \begin{cases} (1+\beta t_j\ominus t_{j+1})v_b + q^{-\delta_{j,N}}t_j\ominus t_{j+1}\ v_{s_j b}, & b_j < b_{j+1}\\ v_b, & else\end{cases}\ . \tag{5.8}$$

Note that the $\mathbb{S}_N$-action does not commute with the multiplicative action of $\mathcal{R}(\mathbb{T})$ on $\mathcal{V}_n$.

Proof. That the $\boldsymbol{s}_j$ yield a representation of $\mathbb{S}_N$ follows easily from our previous findings (5.4), (5.5), and (5.6). □

The next result shows that the Yang-Baxter algebra (3.13) commutes with the action of the symmetric group. For the transfer matrices this extends to the action including the affine reflection depending on the deformation parameter q; compare with 3.23.

Corollary 5.4. *The action on $\mathbb{Z}[x]\otimes\mathcal{V}$ commutes with the action of the row Yang-Baxter algebras, i.e.*

$$(1\otimes\boldsymbol{s}_j)M(x|t) = M(x|t)(1\otimes\boldsymbol{s}_j) \tag{5.9}$$

$$(1\otimes\boldsymbol{s}_j)M'(x|t) = M'(x|t)(1\otimes\boldsymbol{s}_j), \qquad j=1,2,\dots,N-1, \tag{5.10}$$

where M, M' are the monodromy matrices in (3.11) and (3.13) for L and L', respectively. In case of the transfer matrices we have the additional relations

$$\boldsymbol{s}_N H(x|t) = H(x|t)\boldsymbol{s}_N \quad \text{and} \quad \boldsymbol{s}_N E(x|t) = E(x|t)\boldsymbol{s}_N \tag{5.11}$$

with $\boldsymbol{s}_N = (s_N\otimes 1)\hat{r}_N$ and $s_N = s_1 s_2\cdots s_{N-2}s_{N-1}s_{N-2}\cdots s_2 s_1$.

Remark 5.5. *The commutation of the symmetric group action (5.8) with the action of the Yang-Baxter algebra (3.13) is reminiscent of Schur-Weyl duality and we will explore this connection in a forthcoming publication.*

Proof. The commutation relations with the monodromy and transfer matrices follow from (3.6) and (3.23). $\square$

Proposition 5.6 (generalised divided difference operators)**.** *The matrices*

$$\delta_j = \frac{1-\hat{r}_j}{t_j - t_{j+1}}\,(1+\beta t_j) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & q^{-\delta_{jN}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{j,j+1}, \tag{5.12}$$

define an action of $\mathbb{H}_N(\beta)$ *on the space* $\mathbb{Z}[q^{\pm 1}] \otimes V^{\otimes N}$.

Proof. A straightforward computation using the explicit matrix representation given which follows from (5.3). $\square$

Note that the action (5.12) commutes with the multiplicative action of $\mathcal{R}(\mathbb{T})$ on $\mathcal{V}$.

5.2. **Localised Schubert classes.** Recall that each boxed partition $\mu \subset (k^n)$ can be identified with the 01-word of length N which has one-letters at positions I_μ. Recall the natural $\mathbb{S}_N$-action on 01-words, i.e. write $s_j\mu$ for the partition obtained by exchanging the jth and $(j+1)$th letter in the corresponding 01-word for μ.

Theorem 5.7 (localised Schubert classes)**.** *The sequence* $[\mathcal{O}_\lambda] = ([\mathcal{O}_\lambda]_\mu)_{\mu\subset(k^n)}$ *in* $\mathcal{R}(\mathbb{T},q)^{\binom{N}{n}}$ *with* $[\mathcal{O}_\lambda]_\mu := G_\lambda(y_\mu|\ominus t)$ *obeys the following generalised Goresky-Kottwitz-MacPherson condition*

$$\boldsymbol{s}_j[\mathcal{O}_\lambda] - [\mathcal{O}_\lambda] = (t_j \ominus t_{j+1})\delta_j^*[\mathcal{O}_\lambda], \tag{5.13}$$

where $\boldsymbol{s}_j$ *denotes the* $\mathbb{S}_N$*-action given by* $(\boldsymbol{s}_j[\mathcal{O}_\lambda])_\mu = s_j G_\lambda(y_{s_j\mu}|\ominus t)$ *and*

$$\delta_j^*[\mathcal{O}_\lambda] = \begin{cases} \beta[\mathcal{O}_\lambda] + [\mathcal{O}_{s_j\lambda}], & \text{if } j \notin I_\lambda \text{ and } (j+1) \in I_\lambda \\ 0, & \text{else} \end{cases}. \tag{5.14}$$

To prove the theorem we require the following result first.

Lemma 5.8. *Let* $\boldsymbol{s}_j = (s_j \otimes 1)\hat{r}_j$ *be the* $\mathbb{S}_N$*-action (5.8). Then*

$$\boldsymbol{s}_j Y_b = Y_{s_j b}, \qquad j = 1, 2, \ldots, N-1\,. \tag{5.15}$$

In other words, the action (5.8) is the natural diagonal $\mathbb{S}_N$*-action on* $\mathcal{V}_n$ *in the basis of Bethe vectors.*

Proof. Consider the action of $\hat{r}_j$ on an off-shell Bethe vector. According to (5.9) we have

$$\hat{r}_j B(x_1|t)\cdots B(x_n|t)|0\rangle = B(x_1|s_j t)\cdots B(x_n|s_j t)|0\rangle\,.$$

According to Lemma 4.6 the Bethe roots are uniquely determined by the constant term, $y_\lambda = t_\lambda + O(q)$, thus, we have

$$\boldsymbol{s}_j|y_\mu\rangle = s_j(B(y_{\mu_n+1}|s_j t)\cdots B(y_{\mu_1+n}|s_j t))|0\rangle = |s_j y_\mu\rangle = |y_{s_j\mu}\rangle\,.$$

An analogous argument shows that

$$s_j e(y_\mu, y_\mu) = \langle 0|\prod_{i=1}^n C(y_i)\prod_{i=1}^n B(y_i)|0\rangle = e(y_{s_j\mu}, y_{s_j\mu})\,.$$

$\square$

We now prove the generalised GKM conditions (5.13).

Proof of Theorem 5.7. Employ the expansion (4.40) and apply $\boldsymbol{s}_j$ on both sides of the equation. Then using (5.8) on the left hand side and (5.15) on the right hand side of the equality, we obtain

$$v_\lambda + (t_j \ominus t_{j+1})\delta_j v_\lambda = \sum_{\mu \subset (k^n)} (s_j G_\lambda(y_{s_j\mu}| \ominus t))Y_\mu \ .$$

Comparing coefficients with respect to the basis of the Bethe vectors yields (5.13). □

The next result states a generating formula for localised Schubert classes using the representation (5.14) of the Iwahori-Hecke algebra. For $q = 0$ and $\beta = -1$ this statement is originally due to Kostant and Kumar [36].

Employing McNamara's Vanishing Theorem we easily find for $q = 0$ that

$$[\mathcal{O}_{(k^n)}]_\lambda = G_{(k^n)}(t_\lambda| \ominus t) = \begin{cases} \prod_{i=1}^k \prod_{j=k+1}^N t_j \ominus t_i, & \lambda = (k^n) \\ 0, & \text{else} \end{cases} \tag{5.16}$$

which gives us an explicit description for the top (localised) Schubert class. For the quantum case with $q \neq 0$ we have instead

$$[\mathcal{O}_{(k^n)}]_\lambda = G_{(k^n)}(y_\lambda| \ominus t) = \prod_{j=1}^k \prod_{i \in I_\lambda} y_i \ominus t_j \tag{5.17}$$

where y_λ is the solution (4.19) of (4.17) and the values $[\mathcal{O}_{(k^n)}]_\lambda$ at fixed points y_λ with $\lambda \neq (k^n)$ are in general nonzero.

Corollary 5.9. *Any Schubert class $[\mathcal{O}_\lambda]$ can be obtained by successive action of the generalised difference operators $\delta^*_{j_1}, \delta^*_{j_2}, \ldots, \delta^*_{j_r}$ on the top class $[\mathcal{O}_{(k^n)}]$ for some $j_1, \ldots, j_r \in [N]$ such that $w = s_{j_1} \cdots s_{j_r}$ is a reduced word with $w(k^n) = \lambda$ in terms of the natural $\mathbb{S}_N$-action on 01-words.*

Proof. A direct consequence of (5.14) and the $\mathbb{S}_N$-action on binary strings. □

Corollary 5.10. *The ring $qh_n^*/\langle q, \beta{+}1, t_j{-}1{+}e^{\varepsilon_{N+1-i}}\rangle$ is isomorphic to $K_{\mathbb{T}}(\mathrm{Gr}_{n,N})$, while the ring $qh_n^*/\langle q, \beta\rangle$ is isomorphic to $H^*_{\mathbb{T}}(\mathrm{Gr}_{n,N})$. In both cases the isomorphism is given by $v_\lambda \mapsto [\mathcal{O}_\lambda]$, that is the spin basis (3.2) is mapped onto Schubert classes.*

Proof. Working in the basis of Bethe vectors we employ once more (4.40) for $q = 0$ to find

$$v_\lambda = \sum_\mu G_\lambda(t_\mu| \ominus t)Y_\mu \ . \tag{5.18}$$

In other words each Schubert class $[\mathcal{O}_\lambda]$ is identified with the (finite) sequence $\{G_\lambda(t_\mu|\ominus t)\}_{\mu\subset(k^n)}$ where each boxed partition μ labels a fixed point under the torus action. The definition (4.31) of $\circledast$ corresponds to pointwise multiplication of these sequences which satisfy the conditions (5.13) and can be successively generated from the top class (5.16). The assertion then follows from [26, Thm 1.2.2] for $\beta = 0$ and from [36, Thm 3.13] for $\beta = -1$. □

Corollary 5.11. *The ring $qh_n^*/\langle\beta\rangle$ is isomorphic to equivariant quantum cohomology $QH^*_{\mathbb{T}}(\mathrm{Gr}_{n,N})$.*

Proof. Consider the equivariant quantum Pieri-Chevalley rule (3.43). Rewriting it as

$$H_1 v_\mu = \beta^{-1}\left(\frac{\Pi(t_\mu)}{\Pi(t_\emptyset)} - 1\right) v_\mu + \frac{\Pi(t_\mu)}{\Pi(t_\emptyset)} \sum_{\substack{\mu \rightrightarrows^* \lambda[d] \\ d=0,1}} q^d v_\lambda$$

where the sum runs over all $\lambda \subset (k^n)$ such that $\lambda \neq \mu$ and either λ/μ or $\lambda/1/\mu$ is a skew diagram which contains at most one box in each column or row. Setting $\beta = 0$ this simplifies to

$$H_1 v_\mu = \left(\sum_{i \in I_\mu} t_i - \sum_{i=1}^{n} t_i\right) v_\mu + \sum_{\substack{\lambda/d/\mu=(1) \\ d=0,1}} q^d v_\lambda = v_1 \circledast_{\beta=0} v_\mu$$

where the sum now runs over all $\lambda \subset (k^n)$ such that $\lambda \neq \mu$ and either λ/μ or $\lambda/1/\mu$ is a skew diagram which contains *exactly* one box. This is Mihalcea's equivariant quantum Pieri-Chevalley rule for $QH^*_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ which together with the usual grading, v_λ has degree $|\lambda|$ and q has degree N, fixes the ring up to isomorphism; see [50, Cor 7.1]. An alternative proof which exploits the presentation of $QH^*_{\mathbb{T}}(\mathrm{Gr}_{n,N})$ as Jacobi algebra can be found in [24]. □

5.3. **Equivariant quantum Pieri rules and Giambelli formula.** According to its definition (4.34) the operator $\boldsymbol{G}_\lambda$ is the multiplication operator which multiplies with a localised Schubert class. The following corollary states that for λ being a single row or column this operator is given by the transfer matrices (3.34), (3.35) in the spin-basis.

Corollary 5.12. *The operators $\{H_r\}_{r=1}^k$ and $\{E_r\}_{r=1}^n$ defined respectively in (3.34) and (3.35) act on the Bethe vectors $|y_\mu\rangle$ by multiplication with $G_r(y_\mu| \ominus t)$ and $G_{1^r}(y_\mu| \ominus t)$, respectively. That is,*

$$\boldsymbol{G}_r = H_r \qquad \text{and} \qquad \boldsymbol{G}_{1^r} = E_r \;. \tag{5.19}$$

Note in particular, that this implies for $q = 0$ that the matrix elements $\langle\lambda|H_r|\mu\rangle$, $\langle\lambda|E_r|\mu\rangle$ in the basis $\{v_\lambda\}$ give the coefficients in the equivariant Pieri rules for $H^*_{\mathbb{T}}(\mathrm{Gr}_{n,n+k})$ if $\beta = 0$ and for $K_{\mathbb{T}}(\mathrm{Gr}_{n,n+k})$ if $\beta = -1$.

Proof. Using (4.28) and the expansions (2.25), (3.35) we deduce that

$$E_r|y_\mu\rangle = G_{1^r}(y| \ominus t)|y_\mu\rangle \;.$$

But then (3.17) together with (4.27) gives

$$\begin{aligned} H_r(t)|y_\mu\rangle &= H_r(t)\Theta|y_{\mu^*}\rangle = \Theta E_r(\ominus t')|y_{\mu^*}\rangle \\ &= G_{1^r}(\ominus y_{\mu^*}|t')|y_\mu\rangle = G_r(y_\mu| \ominus t)|y_\mu\rangle \;. \end{aligned}$$

□

In light of the expansion (2.34) and (2.30), the last result allows us to express the operator (4.34) which corresponds to multiplication with a Schubert class, in terms of the transfer matrix coefficients H_r from (3.34). The latter, as we have just seen, correspond to multiplication with a Chern class. Such a formula expressing a general Schubert class in terms of Chern classes, is often called *Giambelli formula* in the literature on cohomology.

Corollary 5.13 (equivariant quantum Giambelli formula). *For $\lambda \subset (k^n)$ a boxed partition define the operators $\boldsymbol{F}_\lambda = \det(\tau^{1-j} H_{\lambda_i - i + j})$ where τ is the shift operator (2.28) and $\tau^p H_r$ is the coefficient in (3.34) with respect to the shifted factorial powers $(x|\tau^p t^*)^r$. Then*

$$\boldsymbol{G}_\lambda = \sum_\alpha \beta^{|\alpha|} \phi_\alpha(\lambda) \boldsymbol{F}_{\lambda+\alpha} \tag{5.20}$$

with the same conventions for α and $\phi_\alpha(\lambda)$ as in Prop 2.13 and $\boldsymbol{F}_{\lambda+\alpha}$ is defined in terms of the straightening rule (2.36).

Example 5.14. *Recall the formula (2.37) for $n = 2$. Then*

$$\boldsymbol{G}_{\lambda_1,\lambda_2} = \frac{1 + \beta t_{\lambda_2+1}}{1 + \beta t_1} \; (\boldsymbol{F}_{\lambda_1,\lambda_2} + \boldsymbol{F}_{\lambda_1,\lambda_2+1})$$

where

$$\boldsymbol{F}_{\lambda_1,\lambda_2} = \begin{vmatrix} H_{\lambda_1} & \tau^{-1} H_{\lambda_1 - 1} \\ H_{\lambda_2 - 1} & \tau^{-1} H_{\lambda_2} \end{vmatrix}$$

and H_r is given by (3.36) while the negative shifted factorial power is defined as

$$\tau^{-1} H_{k+1-i} = \sum_{j=1}^{i} \frac{H(t_{N-j})}{\prod_{1 \le \ell \ne j \le i} t_{N-j} \ominus t_{N-\ell}} \; .$$

For certain choices the formula (5.20) considerably simplifies. We already saw that for λ a single row or column we obtain $\boldsymbol{G}_r = H_r$ and $\boldsymbol{G}_{1^r} = E_r$. Setting $\lambda = (k^n)$ we find from (4.28) and (5.17) that

$$\prod_{i=1}^{k} E(\ominus t_i) \; Y_\mu = G_{(k^n)}(y_\mu | \ominus t) Y_\mu$$

and, hence, that $\boldsymbol{G}_{(k^n)} = \prod_{i=1}^{k} E(\ominus t_i)$.

Corollary 5.15 (Fusion matrices). *The matrices $\{\boldsymbol{G}_\lambda\}_{\lambda \subset (k^n)}$ yield a faithful representation of qh_n^*, that is*

$$\boldsymbol{G}_\lambda \boldsymbol{G}_\mu = \sum_{\nu \subset (k^n)} C_{\lambda\mu}^{\nu}(t, q) \boldsymbol{G}_\nu \; . \tag{5.21}$$

Proof. This is a direct consequence of $v_\lambda = v_\lambda \circledast v_\emptyset = \boldsymbol{G}_\lambda v_\emptyset$ and the fact that the v_λ's are linearly independent. Namely, assume $0 = \sum_{\lambda \subset (k^n)} c_\lambda \boldsymbol{G}_\lambda$ for some coefficients c_λ. Then $0 = \sum_{\lambda \subset (k^n)} c_\lambda \boldsymbol{G}_\lambda v_\emptyset = \sum_{\lambda \subset (k^n)} c_\lambda v_\lambda$ and, thus, we must have $c_\lambda = 0$ for all $\lambda \subset (k^n)$. The product expansion follows from (4.35), $v_\lambda \circledast v_\mu = \boldsymbol{G}_\lambda \boldsymbol{G}_\mu v_\emptyset$. □

5.4. **Coordinate ring presentation.** We now prove the presentation of qh_n^* stated in the introduction. Consider the polynomial algebra $\mathcal{A}_n$ generated by $\{e_r\}_{r=1}^n \cup \{h_r\}_{r=1}^k$ over $\mathcal{R}(\mathbb{T}, q)$ subject to the relations given by (1.2) with $e(x)$ and $h(x)$ as in (1.3) and (1.4). Define $\{g_\lambda\}_{\lambda \subset (k^n)} \subset \mathcal{A}_n$ as just explained for the $\boldsymbol{G}_\lambda$'s: set $f_\lambda = \det(\tau^{1-j} h_{\lambda_i - i + j})$, where the "shifted generators" $\tau^p h_r$ are obtained by expanding $h(x)$ into shifted factorial powers $(x|\tau^p t^*)^r$, and then introduce g_λ through the analogous expansion as in Prop 2.13 and (5.20).

Theorem 5.16. *The map $g_\lambda \mapsto v_\lambda$ constitutes an algebra isomorphism $\mathcal{A}_n \cong qh_n^*$.*

Proof. Introduce auxiliary variables $\xi = (\xi_1, \ldots, \xi_n)$ by setting

$$e(x) = \prod_{i=1}^{n} x \oplus \xi_i \ .$$

Dividing by $e(x)$ in (1.2) one obtains $h(x)$ as a rational function in x, but as $h(x)$ – by definition – is polynomial in x the residues at the poles must vanish. This implies that the ξ_i's obey the Bethe ansatz equations (4.17). Moreover, one deduces in a similar manner as we did before that $e_r = G_{1^r}(\xi| \ominus t)$ and $h_r = G_r(\xi| \ominus t)$. Thus, $g_\lambda = G_\lambda(\xi| \ominus t)$ according to Prop 2.13. This then implies that the map $g_\lambda \mapsto v_\lambda$ is an algebra homomorphism and it is also surjective. It remains to show that the dimension of $\mathcal{A}_n$ equals the dimension of qh_n^*. Recall from Section 2.7 that each G_λ can be expressed via (2.30), (2.31) and (2.34) in terms of G_r's and that the factorial Grothendieck polynomials $\{G_\lambda\}$ with λ having at most n parts form a basis of $\mathcal{R}(\mathbb{T}, q)[\xi_1, \ldots, \xi_n]^{\mathbb{S}_n}$, hence $\mathcal{R}(\mathbb{T}, q)[\xi_1, \ldots, \xi_n]^{\mathbb{S}_n} \cong \mathcal{R}(\mathbb{T}, q)[G_1, G_2, \ldots]$. Therefore, we only have to show that each $G_\lambda(\xi|\ominus t)$ with $\lambda \nsubseteq (k^n)$ can be expressed as a linear combination of the $\{g_\mu\}_{\mu\subset(k^n)}$. But since the ξ's obey (4.17), we can deduce that each $G_\lambda(\xi| \ominus t)$ with $\lambda \nsubseteq (k^n)$ can be "reduced" using multiple times (4.21) until it is indexed by a composition where no part is greater than k. Then one applies repeatedly the straightening rule (2.16) to rewrite the result as a linear combination of the g_μ's with $\mu \subset (k^n)$. $\square$

5.4.1. *A generalised rim-hook algorithm.* Our proof of the last theorem contains an algorithm for the successive computation of the structure constants $C^\nu_{\lambda\mu}(t, q)$ without making use of the explicit solutions of the Bethe ansatz equations (4.17) and the residue formula (4.36). Namely, starting from the Pieri rule (2.21) for G_1, one can use (4.21) and (2.16) to define a generalised version of the rim-hook algorithm at $\beta = 0$ [7]; see [5] for a recent extension to the equivariant case with $\beta = 0$. We shall demonstrate this only on a simple example.

Example 5.17. *Set $G_\lambda = G_\lambda(\xi|\ominus t)$ and consider the following product expansions which follow from (2.20) and (2.21),*

$$\begin{aligned} G_{1,0} \cdot G_{1,0} &= t_3 \ominus t_2 \ G_{1,0} + \frac{1 + \beta t_3}{1 + \beta t_2} \left(G_{2,0} + G_{1,1} + \beta G_{2,1}\right) \\ G_{1,0} \cdot G_{1,1} &= t_3 \ominus t_1 \ G_{1,1} + \frac{1 + \beta t_3}{1 + \beta t_1} \ G_{2,1} \ . \end{aligned}$$

For $N = 3$ and $n = 2$ employ (4.21) and (2.16) to find

$$G_{2,0} = qG_{-1,0} = -q\beta G_{0,0} = -q\beta \qquad \text{and} \qquad G_{2,1} = qG_{0,0} = q \ .$$

This yields the following product expansion in qh_2^*,

$$\begin{aligned} g_{1,0} \cdot g_{1,0} &= t_3 \ominus t_2 \ g_{1,0} + (1 + \beta t_3 \ominus t_2)\, g_{1,1} \\ g_{1,0} \cdot g_{1,1} &= t_3 \ominus t_1 \ g_{1,1} + (1 + \beta t_3 \ominus t_1)\, q \end{aligned}$$

which because of $qh_2^* \cong qh_1^*$ – see (4.41) – are equivalent to the products $g_1 \cdot g_1$ and $g_1 \cdot g_2$ in qh_1^* which we computed in Example 3.19.

5.5. **Partition functions and Richardson varieties.** We provide another concrete example where a natural link between our lattice model approach and geometry occurs. Recall the definition of Richardson varieties and the expansion (2.7) for $\beta = 0$.

Proposition 5.18. *The partition functions (3.28), (3.29) have the expansions*

$$\langle\lambda|Z_n(x|t)|\mu\rangle = \sum_{\lambda\subset(k^n)} c^\lambda_{\mu\nu}(t)G_{\nu^\vee}(x|\ominus t') \tag{5.22}$$

$$\langle\lambda|Z'_k(x|t)|\mu\rangle = \sum_{\lambda\subset(k^n)} c^\lambda_{\mu\nu}(t)G_{\nu^*}(x|t), \tag{5.23}$$

where the coefficients are explicity given by (4.38).

Proof. Employing the result (4.28) from the Bethe ansatz and the Cauchy identity (2.39) for factorial Grothendieck polynomials we find,

$$\begin{aligned}\langle\lambda|Z'_{\lambda,\mu}(x_1,\ldots,x_k|t)|\mu\rangle &= \sum_{\alpha\subset(k^n)}\langle\lambda|E(x_1)\cdots E(x_k)|y_\alpha\rangle\langle y_\alpha|\mu\rangle\\ &= \sum_{\alpha\subset(k^n)}\frac{\Pi(t_\alpha)}{\Pi(t_\lambda)}\frac{G_\mu(t_\alpha|\ominus t)G_{\lambda^\vee}(t_\alpha|\ominus t')}{e(t_\alpha,t_\alpha)}\prod_{i=1}^k\prod_{j\in I_\alpha}(x_i\oplus t_j)\\ &= \sum_{\nu\subset(k^n)}\sum_{\alpha\subset(k^n)}\frac{\Pi(t_\alpha)}{\Pi(t_\lambda)}\frac{G_\mu(t_\alpha|\ominus t)G_{\lambda^\vee}(t_\alpha|\ominus t')}{e(t_\alpha,t_\alpha)}\frac{\Pi(t_\nu)}{\Pi(t_\alpha)}G_\nu(t_\alpha|\ominus t)G_{\nu^*}(x|t)\\ &= \sum_{\nu\subset(k^n)} c^\lambda_{\mu\nu}(t)G_{\nu^*}(x|t)\end{aligned}$$

The identity for the vicious walker model now follows from level-rank duality (4.41) for $q=0$ and (3.17),

$$\begin{aligned}\langle\lambda|Z'_k(x|t)|\mu\rangle &= \langle\lambda|E(x_1|t)\cdots E(x_k|t)|\mu\rangle\\ &= \langle\lambda'|H(x_1|\ominus t')\cdots H(x_k|\ominus t')|\mu'\rangle = \langle\lambda'|Z_k(x|\ominus t')|\mu'\rangle\ .\end{aligned}$$

□

Remark 5.19. *We expect that an analogous expansion of the partition function holds also for the quantum case with $q\neq 0$. However, we are currently lacking the necessary quantum analogue of the identity (2.39).*

5.6. **Opposite Schubert varieties and their classes.** Recall the definition of the opposite Schubert class $[\mathcal{O}^\lambda]$ and the dual basis (2.9) from Section 2.5 for $q=0$. We state the analogous quantum relations for qh^*_n.

Define a "opposite Schubert basis" $\{v^\lambda\}$ by setting

$$v^\lambda = \frac{\Pi(t_\emptyset)}{\Pi(t_\lambda)}\sum_{\mu\subset(k^n)} G_{\lambda^\vee}(y_\mu|\ominus t')Y_\mu\ . \tag{5.24}$$

Employing the bilinear form (4.33) we now identify the dual spin basis $\{\tilde v_\lambda\}$ in terms of the product (4.31).

Proposition 5.20. *We have the relation*

$$(v_\lambda,(1+\beta H_1)v^\mu) = \delta_{\lambda\mu} \tag{5.25}$$

and the product expansion (c.f. (2.8))

$$v_\mu\circledast v^\lambda = \sum_{\nu\subset(k^n)} C^\lambda_{\mu\nu}(t,q)v^\nu\ . \tag{5.26}$$

Proof. From the definition (5.24), the identity (2.20) and (4.42) it follows that

$$(1+\beta H_1)v^\mu = \sum_{\alpha\subset(k^n)} \frac{\Pi(y_\alpha)}{\Pi(t_\lambda)}\, G_{\lambda^\vee}(y_\alpha|\ominus t')Y_\alpha\;.$$

The assertion (5.25) then follows from the definition (4.33) and (4.30).

To find the product expansion we make once more use of (4.40) and (5.24) to find

$$v_\mu \circledast v^\lambda = \sum_{\alpha\subset(k^n)} \frac{\Pi(t_\emptyset)}{\Pi(t_\lambda)}\, G_\mu(y_\alpha|\ominus t)G_{\lambda^\vee}(y_\alpha|\ominus t')Y_\alpha\;.$$

Using (4.29) we compute the expansion

$$Y_\alpha = \frac{\Pi(y_\alpha)}{\Pi(t_\emptyset)} \sum_{\lambda\subset(k^n)} \frac{G_\lambda(y_\alpha|\ominus t)}{e(y_\alpha,y_\alpha)}\, v^\lambda\;.$$

Inserting the latter into the previous equation we arrive at (5.26) by making use of (4.36). □

In light of the known relations (2.8) and (2.9) for the non-equivariant case, we conjecture based on (5.26) and (5.25) the following:

Conjecture 5.21. *Consider the ring* $K_{\mathbb{T}}(\mathrm{Gr}_{n,N}) = qh_n^*/\langle q,\beta+1,t_j-1+e^{\varepsilon_{N+1-i}}\rangle$. *The localised class of the sheaf* $\mathcal{O}^\lambda$ *of the opposite Schubert variety is given by the expression*

$$[\mathcal{O}^\lambda]_\mu = \frac{\Pi(t_\emptyset)}{\Pi(t_\lambda)}\, G_{\lambda^\vee}(t_\mu|\ominus t') = \frac{\Pi(t_{\lambda^*})}{\Pi(t_{(k^n)})}\, G_{\lambda^*}(\ominus t_{\mu^*}|t)\;. \tag{5.27}$$

5.7. **The homogeneous limit $t_j = 0$: quantum K-theory.** The inversion formulae (3.36), (3.37) for the expansions (3.34), (3.35) do not hold true in the homogeneous limit when $t_j=0$ for all $j=1,\dots,N$. We therefore need to discuss this case separately. We start with the Pieri formulae, i.e. the action of the transfer matrices in the spin basis.

Given a toric horizontal (vertical) strip $\theta=\nu/d/\lambda$ denote by $c(\theta)=|\mathcal{C}_\theta|$ the number of columns and by $r(\theta)=|\mathcal{R}_\theta|$ the number of rows which intersect the strip.

Corollary 5.22 (non-equivariant Pieri rules)**.** *Set* $t_j=0$ *for all* j*. Then*

$$H_\ell v_\mu = \sum_{\substack{\theta=\lambda/d/\mu\\ \text{toric hor strip}}} q^d\beta^{|\theta|-\ell}\binom{r(\theta)-1}{|\theta|-\ell}v_\lambda \tag{5.28}$$

$$E_{\ell'} v_\mu = \sum_{\substack{\theta=\lambda/d/\mu\\ \text{toric ver strip}}} q^d\beta^{|\theta|-\ell'}\binom{c(\theta)-1}{|\theta|-\ell'}v_\lambda \tag{5.29}$$

where $\ell=1,\dots,k$ *and* $\ell'=1,\dots,n$.

Proof. Setting $t_j = 0$ for all j the combinatorial action of the transfer matrices on $\mathcal{V}_n$ simplifies to

$$H(x)v_\mu = \sum_{\substack{\theta=\lambda/d/\mu \\ \text{toric hor strip}}} q^d x^{k-|\theta|}(1+\beta x)^{r(\theta)} v_\lambda \tag{5.30}$$

$$E(x)v_\mu = \sum_{\substack{\theta=\lambda/d/\mu \\ \text{toric ver strip}}} q^d x^{n-|\theta|}(1+\beta x)^{c(\theta)} v_\lambda \tag{5.31}$$

Employing in addition that the expansions (3.34), (3.35) of the transfer matrices on $\mathcal{V}_n$ in the variable x now read

$$H(x) = x^k \cdot \mathbf{1}_{\mathcal{V}_n^q} + (1+\beta x)\sum_{\ell=1}^{k} H_\ell x^{k-\ell} \tag{5.32}$$

$$E(x) = x^n \cdot \mathbf{1}_{\mathcal{V}_n^q} + (1+\beta x)\sum_{\ell=1}^{n} E_\ell x^{n-\ell} \tag{5.33}$$

the asserted formulae are easily deduced. □

We now turn to the Bethe ansatz computation. Since the matrix elements of the R-matrix in (3.12) do not depend on the t_j's the commutation relations in the row Yang-Baxter algebra, and in particular the relations in Lemma 3.4 and 3.5, are unchanged for $t_j = 0$. From this one deduces, along the same lines as before, that the Bethe ansatz equations are obtained by formally setting $t_j = 0$ in (4.17),

$$y_i^N \prod_{j\neq i} \frac{1+\beta y_j}{1+\beta y_i} = (-1)^{n-1} q, \qquad i = 1,\dots,n\ . \tag{5.34}$$

We have the following result which replaces Lemma 4.6 when $t_j = 0$. Suppose $q^{1/N}$ exists and set $\zeta = \exp(2\pi\imath/N)$ where $\imath$ is the imaginary unit.

Lemma 5.23. *The set of equations (5.34) has* $\binom{N}{n}$ pairwise distinct *solutions*

$$y_\lambda = (y_{\frac{n+1}{2}+\lambda'_n-n},\dots,y_{\frac{n+1}{2}+\lambda'_1-1}) \in \mathbb{C}[\![\beta, q^{\frac{1}{N}}]\!], \tag{5.35}$$

where $\lambda \subset (k^n)$ *and up to first order in* β *we have*

$$y_j = q^{\frac{1}{N}}\zeta^j + \beta\ (-1)^{n-1} q^{\frac{2}{N}}\zeta^j \sum_{l\neq j}(\zeta^l - \zeta^j) + O(\beta^2)\ . \tag{5.36}$$

Moreover, the r*th term in this expansion is proportional to* $q^{r/N}$ *and, thus, we can always force convergence for a given* β *provided we specialise* q *to a sufficiently small number.*

Proof. We now make the ansatz $y_j = \sum_{r\geq 0} y_j^{(r)}\beta$. Setting $\beta = 0$ in (5.34) we obtain the Bethe ansatz equations at the free fermion point which decouple. Clearly, each of the n equations has then N solutions and using the conventions from [39, Prop 10.4] we set $y_j^{(0)} = q^{1/N}\zeta^j$ with $j \in \{\frac{n+1}{2}+\lambda'_1-1,\dots,\frac{n+1}{2}+\lambda'_n-n\}$ for $\lambda \subset (k^n)$. By the analogous arguments as in the previous case when we expanded the Bethe roots with respect to q we find by differentiating with respect to β and setting $\beta = 0$ afterwards the desired expansion. In particular, when taking the rth derivative with respect to β, the coefficient $\left(\prod_{j\neq i}\frac{1+\beta y_j}{1+\beta y_i}\right)_{\beta=0} = 1$ in front of the term $\frac{d^r}{d\beta^r} y_j^N|_{\beta=0}$ is always nonzero. One then proves by induction the stated dependence on $q^{1/N}$. □

This lemma can be used to establish the completeness of the Bethe ansatz when $t_j = 0$ and to derive the results analogous to (4.21), (4.22), (4.23), (4.28) and (4.29), (4.30) by simply setting formally $t_j = 0$ in the respective formulae. Thus, extending the base field of our quantum space, $\mathcal{V}^q = V^{\otimes N} \otimes \mathbb{C}[\![\beta, q^{\frac{1}{N}}]\!]$, we can introduce an algebra structure via (4.31) as before by making use of the Bethe vectors. Note, however, that this extension to $\mathbb{C}[\![\beta, q^{\frac{1}{N}}]\!]$ is only necessary if we require the existence of idempotents. Alternatively, we can introduce the product structure via (4.35) by defining the analogue of the operator $\boldsymbol{G}_\lambda$ for $t_j = 0$ as follows.

For each $n = 0, 1, \ldots, N$ define operators $\boldsymbol{s}_\lambda \in \mathrm{End}(\mathbb{Z}[\![q]\!] \otimes V_n)$ for $\lambda \subset (k^n)$ by

$$\boldsymbol{s}_\lambda = \det(\boldsymbol{e}_{\lambda'_i - i + j}), \qquad \boldsymbol{e}_r = \sum_{j=r}^{n} (-\beta)^{j-r} \binom{j-1}{j-r} E_j \ , \tag{5.37}$$

where $r = 1, 2, \ldots, n$; compare with (2.27). We set $\boldsymbol{e}_0$ to be the identity operator. Note that since the E_j's mutually commute so do the $\boldsymbol{e}_r$'s, whence the determinant $\boldsymbol{s}_\lambda$ is well-defined.

Consider the commutative algebra $qh_n^*/\langle t_1, \ldots, t_N\rangle$ generated by $\{H_r\}_{r=1}^k \cup \{E_r\}_{r=1}^n$ with $t_j = 0$. For each $\lambda \subset (k^n)$ define in analogy with (2.38) the operators

$$\boldsymbol{G}_\lambda = \sum_\alpha \beta^{|\alpha|} \prod_{i=1}^n \binom{i-1}{\alpha_i} \boldsymbol{s}_{\lambda+\alpha} \ , \tag{5.38}$$

where $\boldsymbol{s}_{\lambda+\alpha}$ is defined in terms of the straightening rules analogous to (2.36).

Corollary 5.24. *Consider $qh_n^*/\langle t_1, \ldots, t_N\rangle$. The map $v_\lambda \mapsto [\mathcal{O}_\lambda]$ defines for $\beta = 0$ a ring isomorphism with $QH^*(\mathrm{Gr}_{n,N})$ and for $\beta = -1$ with $QK(\mathrm{Gr}_{n,N})$.*

Proof. Recall that the rings $QH^*(\mathrm{Gr}_{n,N})$ and $QK(\mathrm{Gr}_{n,N})$ are multiplicatively generated from the Chern classes (see [60] for the case of quantum cohomology and [13, Cor 5.7] for quantum K-theory) which under the above maps are identified with the coefficients H_r and E_r defined in (5.32) and (5.33), respectively. Thus, it suffices to show that the respective rings feature the same Pieri rule, i.e. that the respective expansions of the product of such a Chern class with a general class coincide.

Setting $t_j = 0$ in the functional relation (1.2), (3.44) the resulting ring is well-defined and it follows from our previous results (4.35), Cor 5.15 and Thm 5.16 that $qh_n^*/\langle t_1, \ldots, t_N\rangle$ is isomorphic to the ring with product $v_\lambda \circledast v_\mu = \boldsymbol{G}_\lambda v_\mu$ with $\boldsymbol{G}_\lambda$ given by (5.38). Here we implicitly used the fact that the transfer matrices E, H stay well-defined when setting formally $t_j = 0$, which in turn can be deduced from the explicit expressions for the L-operators (3.3), (3.4). Furthermore, from the definition (5.38) it follows that $\boldsymbol{G}_{1^r} = E_r$ and, thus, the ring structure is fixed by the Pieri rule (5.29) which for $\beta = 0$ coincides with the Pieri rule of $QH^*(\mathrm{Gr}_{n,N})$ [6, p. 293] and for $\beta = -1$ with the Pieri rule of $QK(\mathrm{Gr}_{n,N})$ [13, Thm 5.4]. $\square$

The functional relation (3.42) when setting $t_j = 0$ becomes,

$$(-1)^n (1 + \beta x)^n H(x) E(\ominus x) = x^N (1 + \beta H_1) + q(-1)^n (1 + \beta x)^n \ .$$

Using the expansions (5.32), (5.33) and comparing powers on both sides of the functional relation one arrives at the following explicit relations between the generators

$$\sum_{a+b=N-r} (-1)^a \boldsymbol{e}_a (H_b + \beta H_{b+1}) = \begin{cases} 0, & r = 1, \ldots, k-1 \\ q(-1)^n \binom{n}{N-r} \beta^{N-r}, & r = k, \ldots, N \end{cases} \ . \tag{5.39}$$

The expression (4.37) of the bilinear form and the definition of the dual basis (5.25) simplify to

$$(v_\lambda, v_\mu) = \sum_{\alpha\subset(k^n)} \frac{G_\lambda(y_\alpha)G_\mu(y_\alpha)}{e(y_\alpha, y_\alpha)} \tag{5.40}$$

and

$$(v_\lambda, (1+\beta H_1)v_{\mu^\vee}) = \delta_{\lambda\mu}, \tag{5.41}$$

because factorial Grothendieck polynomials are replaced with ordinary ones. In particular, the opposite spin basis (5.24) simply becomes $v^\lambda = v_{\lambda^\vee}$ when $t_j = 0$. Note that for $\beta = -1$ the definition (5.40) and the relation (5.41) are different from [13, Thm 5.14]. This is not a contradiction, as the invariance of the bilinear form only fixes it up to a multiplicative factor, which with respect to the form defined in *loc. cit.*, is $(1-q)$.

Remark 5.25. *In the homogeneous limit the analogue of the Littlewood-Richardson rule for stable Grothendieck polynomials is known* [12, Thm 5.4 and Cor 5.5]. *Therefore we can apply our generalised rim-hook algorithm from Section 5.4.1 also for the computation of the structure constants of the quantum K-theory ring for Grassmannians.*

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